A NEW APPROACH TO EVALUATE GAMMA-RAY MEASUREMENTS

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ABSTRACT: Misunderstanding about the term "random samples" and its implications may easily arise. Conditions under which the phases, obtained from arrival times, do not form a random sample and the dangers involved are discussed. Watson's U^2 test for uniformity is recommended for light curves with duty cycles larger than 10%. Under certain conditions, non-parametric density estimation may be used to determine estimates of the true light curve and its parameters.

1. INTRODUCTION: Consider a series of arrival times t_i, $i=1,\ldots,N$, of γ-rays from a certain source direction. The case is studied where the data contains a periodic component of strength p (pulsed counts/total counts) and period T. In the case of detectors with low count rates, the obvious requirement is to determine the significance of p as being due to a periodic source against the possibility that it is only a statistical fluctuation from the uniform background. The deduction of a possible light curve is also important. In this paper the following points are covered: (1) the problem of "random samples", (2) tests for uniformity, (3) non-parametric density estimators of the true periodic light curve and (4) the determination of the light curve parameters from the non-parametric density estimator.

2. THE PROBLEM OF "RANDOM SAMPLES": The measured data are the arrival times with the property $t_i > t_{i-1}$ $(i=2,\ldots,N)$. Assume this process, apart from the periodic component in the data, to be time independent. It is desirable to estimate the true light curve from the arrival times. This is done by folding the t_i's modulo 2π, with respect to a known period T. This results in the "sample" $(\theta_1,\ldots,\theta_N)$, with θ_i the so called phases which are calculated as

$$\theta_i = \frac{2\pi t_i}{T} \ (\mathrm{mod}\ 2\pi) = 2\pi\left|\frac{t_i}{T} - k\right| , \quad i = 1, \ldots, N, \ k \in \mathbb{N}^+ \qquad (1)$$

The choice of 2π is to allow the application of trigonometric functions on the phases. This sample has mostly been treated as being random. This sample would be random if and only if (a) all the θ_i's are identically distributed and (b) if they are statistically independent. If the phases do not form a random sample, then no conclusions about the "true underlying light curve" can be made. The fact is that the phases do not form a random sample! This can be seen as follows:

From eq. (1) the probability density functions (p.d.f) of θ_i and t_i are related by the following wrapping process (Mardia, 1972):

$$f_{\theta_i}(\theta) = \frac{T}{2\pi} \sum_{k=0}^{\infty} f_{t_i}\left(\frac{\theta T}{2\pi} + Tk\right) \qquad (2)$$

Since $t_i > t_{i-1}$, it follows that $f_{\theta_i}(\theta) \neq f_{\theta_j}(\theta)$ for every θ and all $i \neq j$, thus proving that the θ_i's are not identically distributed. Furthermore $t_i = t_{i-1} + (t_i - t_{i-1})$, which implies that t_i is a function of t_{i-1}. Since θ_i is a function of t_i, it follows that θ_i is also a function of θ_{i-1}. This shows that the phases are not independently distributed. It should however be noted that if the time differences $v_i = t_i - t_{i-1}$ are used, a random sample would result by folding the v_i's.

From simulations of arrival times the following seems evident (let $b = E(t_i - t_{i-1}) \equiv 1/\text{count rate}$): The distributions become approximately identical when $T<b$. If $T \approx b$, then it suffices to add a constant large time to each t_i, so that $t_1 >> 0$. This will ensure almost identically distributed random phases. If the period T equals the whole period of observation ($T>>b$), then

$$\theta_i = \frac{2\pi t_i}{T} \quad \text{and} \quad f_{\theta_i}(\theta) = \frac{T}{2\pi} f_{t_i}(\theta) \qquad (3)$$

so that the phases are not identically distributed. The "runs-test" (Lindgren, 1976) was used to determine whether the phases are independently distributed: For $T<b$, the phases seem to be independent random variables and for $T>>b$ there was strong evidence for dependency, which is also clear from eq. (3). Independency can with a 10% uncertainty be accepted for $T<3b$. This result seems to be independent of the pulsed fraction.

Thus, for T>3b, the true light curve cannot be estimated. In γ-ray astronomy this problem amounts to the case of astrophysical objects with periods that is large in comparison with b.

3. TESTS FOR UNIFORMITY: Let θ be a random variable with p.d.f. f(θ), which is assumed to be unknown. An appropriate test in this case would be some non-parametric test:

$$H_o: f(\theta) = U(\theta) = 1/2\pi \text{ against } H_1: f(\theta) \neq U(\theta) \quad (4)$$

The alternative hypothesis H_1 only suggests that the unknown p.d.f. is different from uniformity. In order to compare tests, the following general form of f(θ), which covers most cases in γ-ray astronomy, was assumed:

$$f(\theta) = p_1 S(\theta;\, \mu_1,\, \delta_1) + p_2 S(\theta;\, \mu_2,\, \delta_2) + \frac{1-p_1-p_2}{2\pi} \quad (5)$$

The pulsed fraction and phase (mean position) of each peak are denoted for i=1,2 by p_i and μ_i respectively, while δ_i refers to the FWHM of each peak, divided by the period T.

The two most commonly used tests for uniformity in γ-ray astronomy are:

1) χ^2-test: The advantage of this test is that it is a non-parametric test, but its drawback is the choice of the number of bins K (=degrees of freedom+1) and their positions on the phasogram. The best choice for K is $1/\hat{\delta}$, where $\hat{\delta}$ is some estimate of δ. From simulations it was evident that the sensitivity of this test increases with decreasing duty cycles.

2) Rayleigh test (Mardia, 1972): The motivation for the use of this test is its independence of bins. It is however a parametric test that was derived for von Mises alternatives. This corresponds to $p_1=1$, $p_2=0$ and $S(\theta;\mu_1,\delta_1)$ the von Mises distribution $M(\theta;\mu,\kappa)$. In the case of bimodal data (as with certain pulsars), the phase difference is ≈0,42 and the value of the test statistic $\bar{R}$ is small when $p_1 \approx p_2$. This is the result of two nearly opposing vectors, cancelling each other, when the test statistic

$$\bar{R} = \sqrt{(\bar{C}^2 + \bar{S}^2)} \quad \text{with } \bar{C} = \frac{1}{N}\sum_{i=1}^{N} \cos\theta_i;\ \bar{S} = \frac{1}{N}\sum_{i=1}^{N} \sin\theta_i \quad (6)$$

is computed. Consequently bimodal data may be interpreted by this test as being uniform and real sources could then be discarded.

Two somewhat neglected non-parametric tests in this area of research are Kuiper's V_N test and Watson's U^2 test. Their distributions under H_o with the corresponding critical values are discussed by Mardia (1972). A brief outline of each test's algorithm is as follows:

3) Kuiper's V_N test: Let $\theta_{(1)},\ldots,\theta_{(N)}$ be the ordered phases. With $U_i=\theta_{(i)}/2\pi$, the test statistic is computed by

$$V_N = \max_i\left(U_i - \frac{i}{N}\right) - \min_i\left(U_i - \frac{i}{N}\right) + \frac{1}{N} \quad (7)$$

so that only the minimum and maximum deviations from the uniform distribution are taken into account. It can intuitively be seen that this test will be sensitive to light curves with narrow duty cycles, but insensitive to those with broad duty cycles.

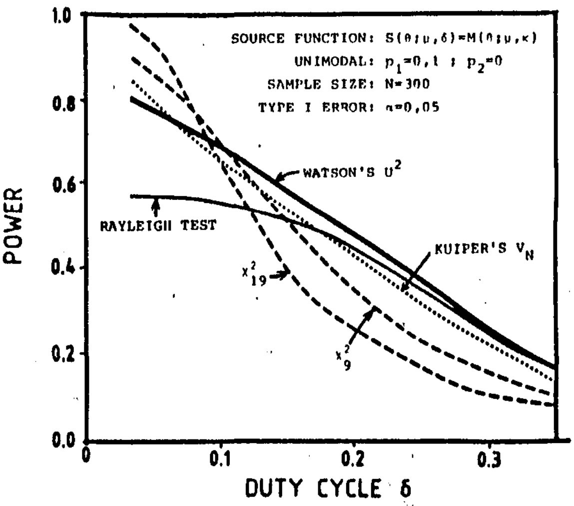

Figure 1 Power curves of the four different tests for uniformity. The light curve is assumed to be unimodal. The subscripts for the χ^2-test refer to the degrees of freedom.

4) Watson's U^2 test: With U_i as above, the statistic is computed as follows:

$$U^2 = \sum_{i=1}^{N} \left[U_i - \bar{U} - \{(2i-1)/(2N)\} + \tfrac{1}{2}\right]^2 + {}^1/12N \quad (8)$$

This is a type of a mean square error with respect to the uniform distribution, so that the information of each phase is taken directly into account in the calculation of U^2.

The procedure to determine which of these four tests is the best test, would be to find the test with the largest power. Since these tests are non-parametric, (except the Rayleigh test), one cannot expect to find a single test with the largest power for all choices of parameters in eq. (5). An indication of the relative performances of these tests are given in Figure 1, which was obtained through simulations of unimodal data. It can be seen that Watson's test is the best test for duty cycles larger than 10% and the χ^2-test is best for duty cycles less than 10%. In the latter case it can be seen that the power of the χ^2-test increases if the number of bins is increased. A good choice is $K \approx 1/\delta$. At small duty cycles it can be seen that the Rayleigh test performs badly relative to any other test. These conclusions remain independent of the pulsed fraction p_1.

The question obviously arises whether one may use these tests for uniformity when the phases are not random. The answer is yes, but it applies only to those kind of tests where the distribution of the test statistic is insensitive (robust) with respect to deviations from randomness. This has been investigated for the four discussed tests by looking for a change in the critical values as T increases with respect to b. Fortunately these values did not change, so that these tests may be used for any relation between T and b.

4. NON-PARAMETRIC DENSITY ESTIMATION OF LIGHT CURVES: Although a test for uniformity is a first step in identifying a source, the additional estimation of a light curve is very important. The usual method to display a light curve in γ-ray astronomy, is to bin the data into a histogram. The disadvantage of this method is that it is dependent on bin positions and their sizes. A more correct way to display an estimate of the true unknown p.d.f., is through the use of a non-parametric density estimator. This method assumes that the data is random. Since the light curve is a periodic one, a good estimator would be a truncated Fourier series. This estimate and its standard error can easily be computed. The application to estimation on a circle is as follows: Let the random sample be $D=(\theta_1,\ldots,\theta_N)$ with unknown p.d.f. $f(\theta)$. The characteristic function (c.f.) of $f(\theta)$ and its corresponding estimator are

$$\phi_p = \int_0^{2\pi} e^{ip\theta} f(\theta)\,d\theta = \alpha_p + i\beta_p \text{ and } \hat{\phi}_p = \hat{\alpha}_p + i\hat{\beta}_p = \left(\frac{1}{N}\sum_{i=1}^{N} \cos p\theta_i\right) + i\left(\frac{1}{N}\sum_{i=1}^{N} \sin p\theta_i\right) \quad (8)$$

Using the inversion formula (Mardia, 1972) we obtain

$$f(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^{\infty}(\alpha_p \cos p\theta + \beta_p \sin p\theta)\right)$$

The following asymptotically unbiased estimator of $f(\theta)$ is proposed:

$$\hat{f}(\theta;\ D,\ m) = \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^{m}(\hat{\alpha}_p \cos p\theta + \hat{\beta}_p \sin p\theta)\right) \quad (9)$$

where m is some "smoothing parameter". Using the method of cross-validation (Bowman, 1984), m can be estimated by $\hat{m}$, where $\hat{m}$ is that value of m which minimizes

$$\sum_{i=1}^{N}\left[\frac{1}{N}\int_0^{2\pi} \hat{f}^2_{N-1}(\theta;D_i,m)\,d\theta - \frac{2}{N}\hat{f}_{N-1}(\theta;D_i,m)\right] \text{ with } D_i=(\theta_1,\ldots,\theta_{i-1},\theta_{i+1},\ldots,\theta_N) \quad (10)$$

The approximate confidence band of $f(\theta)$ is

$$\hat{f}(\theta;\ D,\ \hat{m}) \pm s\sqrt{(\mathrm{var}\ \hat{f})} \quad (11)$$

with s=1.96 being the quantity determining the 95% confidence limit. The probability that the true p.d.f will be within the band, will be approximately 95%. Figure 2 displays an example of these bands. One can thus use these bands, in their own fashion, to determine the significance of periodic emission. For $\delta << 1$, one may encounter the problem of oversmoothing. Tabulated values of $\hat{m}$ for such cases will be presented by the authors.

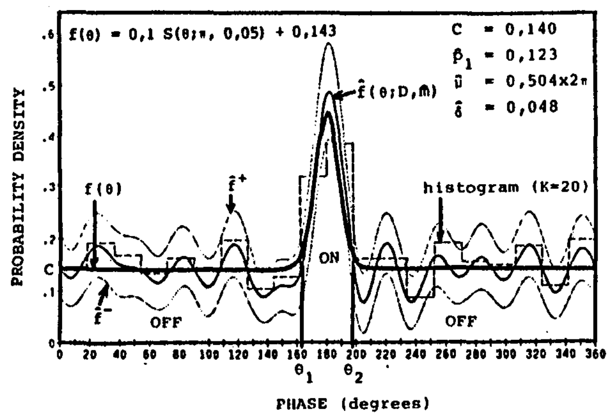

Figure 1 The density estimator $\hat{f}(\theta;D,\hat{m})$ of $f(\theta)$. The 95% confidence band is indicated by $\hat{f}^{\pm}$. The ON- and OFF-source regions are also indicated.

5. LIGHT CURVE PARAMETERS FROM THE DENSITY ESTIMATOR: Even if one does not have any knowledge of the true p.d.f $f(\theta)$, it is still desirable to know the light curve parameters. Since the estimator is asymptotically unbiased, one may estimate the desired parameters as follows:: Determine the pulsed region θ_1 and θ_2 roughly. Then determine the uniform background level c:

$$c = [\int_0^{\theta_1} f(\theta)d\theta + \int_{\theta_2}^{2\pi} f(\theta)d\theta] / [2\pi - \theta_2 + \theta_1] \quad (13)$$

Using this line of height c, determine a better estimate of the pulsed region. This may lead to a small improvement of c. Obtain the light curve parameters:

$$p = \int_{\theta_1}^{\theta_2} (\hat{f}(\theta)-c)d\theta \quad \text{and} \quad \mu = \int_{\theta_1}^{\theta_2} \theta(\hat{f}(\theta)-c)d\theta \, / \, p \quad (14)$$

The duty cycle (FWHM) can be obtained graphically or numerically from the peak of the light curve. The latter can only be done when a specific source function $S(\theta;\mu,\delta)$ is assumed:

$$\sigma^2(\delta) = \int_{\theta_1}^{\theta_2} \theta^2 \, (\hat{f}(\theta)-c)d\theta \, / \, p - \mu^2 \quad (15)$$

From these parameters one can obtain the significance of periodic emission in terms of the usual number of standard deviations NSIG from the uniform background. Using a normal distribution for $S(\theta;\mu,\delta)$ and the interval $\mu \pm 1.96\sigma$ (95% area under the normal curve for this interval) for the pulsed region, NSIG was computed for unimodal light curves with a 10% periodic signal. The results are presented in Figure 3. The latter can be used to determine the total number of events that is required to obtain a certain level of significance. From Figure 3 it can be seen that the smaller the duty cycle, the easier it is to identify a source. This method can also be applied to bimodal light curves.

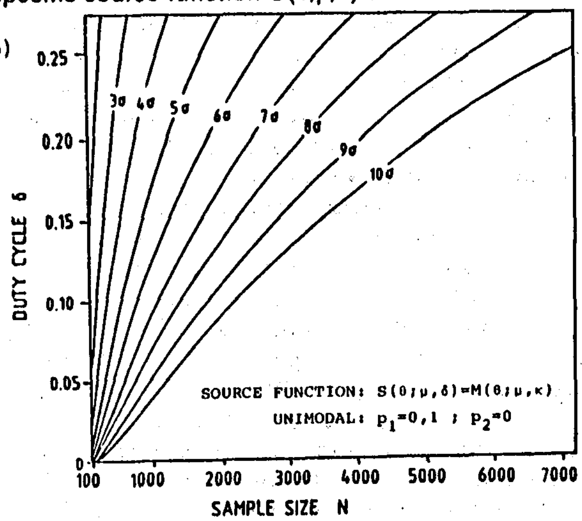

FIGURE 3 Contours of significance of periodic emission as function of the pulse-duty cycle and sample size.

6. CONCLUSIONS: When the phases are formed from the arrival times, great care should be taken if the periodic light curve and the corresponding parameters are to be estimated from the sample. In the first place analysis should be restricted to time independent processes (i.e. the form of the light curve should not change during the observation time). The next step would be to perform a test for the independency of the sample. The null-hypothesis of independency will usually be accepted for T<3b. This condition will usually also ensure that the sample variables (phases) are identically distributed if one let $t_1 >> 0$. Under these conditions the sample will be random and the p.d.f. with its corresponding parameters can be estimated. Certain tests for uniformity, like those discussed in section 3, may be used whether the sample is random or not. Watson's test seems to be the best test of those discussed for unimodal light curves with duty cycles larger than 10%, while the χ^2-test performs better at smaller duty cycles. The best choice for the number of bins in the χ^2-test is approximately $1/\delta$. The Rayleigh test is not a very dependable test since it is a parametric test that was derived for a very limited form of the light curve.

Likelihood ratio tests for uniformity are presently being investigated by the authors. This will result in the best test for light curves of the form of eq. (5). Such an analysis would automatically present the light curve parameters with their corresponding standard errors.

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