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PHASE SPACE FACTORS IN MULTIPARTICLE PROCESSES

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ABSTRACT

General phase space theorems are discussed for the cases (A) with only energy conservation applied and (B) with energy and momentum conservation applied. It is shown that in the non-relativistic limit for N particles there is a very close relationship between the multiparticle phase space integral in case B and that for case A and N-1 particles.

1. Introduction. There are in physics many phenomena that can be described as phase space effects. By this is meant effects that are determined essentially by the number of states available to the system. Examples may be found in equilibrium statistical mechanics and in the evaluation of dynamical processes. In particular, the cross sections and rates associated with processes initiated by binary collisions are determined by the interaction that causes the process and also by the number of final states associated with the various products or channels for the reaction. Applications or examples of this kind of effect occur in electronic and atomic processes, chemical kinetics, nuclear reactions, and elementary particle processes. All of these processes can be represented as a reaction of the type

$$a + b \longrightarrow \text{products } (c + d + e + \cdots) \; . \qquad (1)$$

In the c.m. frame and for given energy (E) available to the outgoing particles, the total momentum-space volume available to these particles is, with momentum and energy conservation imposed,

$$\Phi^{(N)} = \int \cdots \int \prod_{j=1}^{N} d^3 \underset{\sim}{p}_j \; \delta^{(3)}\Big(\sum_{j=1}^{N} \underset{\sim}{p}_j\Big) \, \delta\Big(\sum_{j=1}^{N} E_j - E\Big). \qquad (2)$$

This quantity, proportional to the number of available final states, is fundamental and simple expressions for $\Phi^{(N)}$ can be derived for small N when the outgoing particles are non-relativistic (NR), extremely-relativistic (ER), or even of general energy. When all are NR or ER (or even a mixture), it is possible to derive expressions for $\Phi^{(N)}$ for general N, using various mathematical methods or tricks. One such trick will be illustrated in this paper.

In the limit where one of the outgoing particles (say, particle N) has a large mass, its energy ($p_N^2/2m_N$) is small and can be neglected. Then, the $\delta^{(3)}$ function can be eliminated by integrating over $d^3\underset{\sim}{p}_N$, and

$$\Phi^{(N)} \xrightarrow[m_N \ell g]{} \psi^{(N-1)} = \int \cdots \int \prod_{j=1}^{N-1} d^3 \underset{\sim}{p}_j \; \delta\Big(\sum_{j=1}^{N-1} E_j - E\Big); \qquad (3)$$

that is, the heavy particle just plays the role of satisfying the momentum-conservation restriction. The phase space integral (3), with only the energy-conservation restriction applied is the one appearing in statistical mechanics in the so-called microcanonical ensemble. It turns out, however, that, in the NR limit, $\Phi^{(N)}$ is very closely related to $\psi^{(N-1)}$ for an arbitrary distribution of masses $m_1, m_2, \ldots, m_N$; we shall demonstrate this in a simple way.

Before proceeding to consider the very special problem mentioned just above, a few other introductory remarks should be made. In modern particle physics the use of the invariant phase space volume $\phi^{(N)}$ is more common; this quantity is the same as $\Phi^{(N)}$ except each factor d^3p_j is divided by $2E_j$, yielding Lorentz-invariant factors; the two δ-functions are also combined to a single invariant $\delta^{(4)}$ factor. The factors ϕ are certainly more convenient than Φ in relativistic calculations, but it is not clear which is actually more fundamental. In a sense Φ is, since it is proportional to the number of states. For examples of the use of the invariant ϕ-factors the book by Perl [1] may be consulted; this work also gives references to earlier papers on the general subject. A general survey is also given in a monograph [2] in preparation by the author.

2. General Expressions: NR Limit. If we make the change of variable

$$\underset{\sim}{p}_j = (2Em_j)^{1/2}\underset{\sim}{x}_j \;, \tag{4}$$

thereby introducing dimensionless momentum variables $\underset{\sim}{x}_j$, the integral can be written in more convenient form. It is also convenient to introduce dimensionless masses in terms of the total mass M :

$$m_j = \nu_j^{\,2} M \;;\; \sum_{j=1}^{N} \nu_j^{\,2} = 1 \;. \tag{5}$$

Then, in the NR limit,

$$\Phi^{(N)} = (2M)^{3(N-1)/2}\left(\prod_{j=1}^{N} \nu_j\right)^3 E^{3(N-1)/2-1}\; I_N \;, \tag{5}$$

where

$$I_N = \int\cdots\int \prod_{j=1}^{N} d^3\underset{\sim}{x}_j \;\delta^{(3)}\left(\sum_{j=1}^{N} \nu_j\underset{\sim}{x}_j\right)\delta\left(\sum_{j=1}^{N} \underset{\sim}{x}_j^{\,2} - 1\right) \tag{7}$$

is a dimensionless integral. In terms of the variables $\underset{\sim}{x}_j$ and parameters ν_j, the phase space integral ψ [eq. (3)] is

$$\psi^{(N)} = (2M)^{3N/2}\left(\prod_{j=1}^{N} \nu_j\right)^3 E^{3N/2-1} J_N \;, \tag{8}$$

where

$$J_N = \int\cdots\int \prod_{j=1}^{N} d^3\underset{\sim}{x}_j \;\delta\left(\sum_{j=1}^{N} \underset{\sim}{x}_j^{\,2} - 1\right) \;. \tag{9}$$

In the following section we shall outline the proof that

$$I_N = J_{N-1} \tag{10}$$

This is, at first sight, a remarkable result that means, for example, that I_N is independent of the mass spectrum $\nu_1, \nu_2, \cdots, \nu_N$. The integral (9) for J_N can be evaluated easily by a number of methods (see [2] or almost any book on statistical mechanics):

$$J_N = \pi^{3N/2}\Gamma(3N/2) \ . \tag{11}$$

3. Theorem [eq. (10)] on Phase Space Integrals. The theorem is easy to prove for $N=2$ and 3, but it is of interest to prove it for general N. This can be done by considering the indices $j=1, 2, \ldots, N$ labelling the particles as designating an N-dimensional space. With $\hat{e}_j$ as a unit vector along the j^{th} axis of this space, the axes are taken to be orthogonal: $\hat{e}_j \cdot \hat{e}_k = \delta_{jk}$. In this space $\hat{\nu} = (\nu_1, \nu_2, \ldots, \nu_N)$ is a vector of unit length: $\hat{\nu} \cdot \hat{\nu} = 1$, because of (5). Also, in terms of $\hat{\underset{\sim}{x}} = (\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N)$, and

$$d^3\hat{\underset{\sim}{x}} = \prod_{j=1}^{N} d^3\underset{\sim}{x}_j = dx_1 \cdots dx_N dy_1 \cdots dy_N dz_1 \cdots dz_N \ , \tag{12}$$

the integral (7) can be written

$$I_N = \int d^3\hat{\underset{\sim}{x}} \ \delta(\hat{\underset{\sim}{x}}^2 - 1)\,\delta^{(3)}(\hat{\nu} \cdot \hat{\underset{\sim}{x}}) \tag{13}$$

But $\hat{\nu} \cdot \hat{\underset{\sim}{x}}$ is invariant to a rotation of the axes in the N-dimensional space, as is $\hat{\underset{\sim}{x}}^2$. Essentially, such a rotation corresponds to a relabelling of the particles. With this invariance, it is convenient to choose an orientation such that the vector $\hat{\nu}$ is along one axis such that, say, $\hat{\nu} = (0, 0, \ldots, 1)$ corresponding, physically, to the case $m_N \gg m_{j>N}$. The δ-function $\delta^{(3)}$ is then simply $\delta^{(3)}(\underset{\sim}{x}_N)$ which can be eliminated by integration over $d^3\underset{\sim}{x}_N$. The resulting integral is then just J_{N-1} and the identity (10) is obtained.

It is interesting how in this problem we make use of the mass spectrum $m_1, m_2, \ldots, m_N$ to prove a theorem and simplify a derivation. In the ER limit we have no such device to employ and the evaluation of the corresponding $\Phi^{(N)}$ is more complicated. However, the same trick employed to prove the result (10) can be used to derive results when we have, say, N NR particles and N' ER particles. For more details, see [2].

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References

[1] M. L. Perl, High Energy Hadron Physics (New York: John Wiley and Sons, 1974).

[2] R. J. Gould, High Energy Phenomena in Astrophysics (in preparation, Princeton Univ. Press).