View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by NASA Technical Reports Server

NASA Contractor Report 177977

ICASE REPORT NO. 85-40

NASA-CR-177977
19860003083

ICASE

AN ASYMPTOTIC INVESTIGATION OF THE STATIONARY MODES OF INSTABILITY OF THE BOUNDARY LAYER ON A ROTATING DISC

Philip Hall

LIBRARY COPY

NOV 14 1985

LANGLEY RESEARCH CENTER
LIBRARY, NASA
HAMPTON, VIRGINIA

NASA Contract No. NAS1-17070
September 1985

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

NF01007

1 Report No NASA CR-177977 ICASE Report No. 85-40	2 Government Accession No	3 Recipient's Catalog No
4 Title and Subtitle AN ASYMPTOTIC INVESTIGATION OF THE STATIONARY MODES OF INSTABILITY OF THE BOUNDARY LAYER ON A ROTATING DISC		5 Report Date September 1985
		6 Performing Organization Code
7 Author(s) Philip Hall		8 Performing Organization Report No 85-40
9 Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665		10 Work Unit No
		11 Contract or Grant No NAS1-17070
		13 Type of Report and Period Covered Contractor Report
12 Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		
		14 Sponsoring Agency Code 505-31-83-01

15 Supplementary Notes
Langley Technical Monitor: J. C. South Jr.
Final Report
Submitted to Proc. Roy. Soc.

16 Abstract

An investigation of high Reynolds number stationary instabilities in the boundary layer on a rotating disc is given. It is shown that in addition to the inviscid mode found by Gregory, Stuart, and Walker (1955) at high Reynolds numbers, there is a stationary short wavelength mode. This mode has its structure fixed by a balance between viscous and Coriolis forces and cannot be described by an inviscid theory. The asymptotic structure of the wavenumber and orientation of this mode is obtained. A similar analysis is given for the inviscid mode, the expansion procedure used is capable of taking non-parallel effects into account in a self-consistent manner. The results are compared to numerical calculations and experimental observations.

17 Key Words (Suggested by Author(s))	18 Distribution Statement
instability theory boundary layers	34 - Fluid Mechanics & Heat Transfer Unclassified - Unlimited

19 Security Classif (of this report)	20 Security Classif (of this page)	21 No of Pages	22 Price
Unclassified	Unclassified	30	A03

For sale by the National Technical Information Service, Springfield, Virginia 22161

NASA Langley, 1985

AN ASYMPTOTIC INVESTIGATION OF THE STATIONARY MODES OF INSTABILITY OF THE BOUNDARY LAYER ON A ROTATING DISC

Philip Hall
Mathematics Department, University of Exeter
Exeter, England

Abstract

We investigate high Reynolds number stationary instabilities in the boundary layer on a rotating disc. The investigation demonstrates that in addition to the inviscid mode found by Gregory, Stuart, and Walker (1955) at high Reynolds numbers, there is a stationary short wavelength mode. This mode has its structure fixed by a balance between viscous and Coriolis forces and cannot be described by an inviscid theory. The asymptotic structure of the wavenumber and orientation of this mode is obtained, and a similar analysis is given for the inviscid mode. The expansion procedure provides the capacity of taking non-parallel effects into account in a self-consistent manner. The results are compared to numerical calculations and experimental observations.

Research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-17070 while the author was in residence at the Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA 23665.

#
N86-12550

1. Introduction

In recent years there has been much interest in the manner in which three-dimensional boundary layers become unstable. Much of this work has been motivated by the need to understand the instability mechanisms which are operational in the boundary layer on a swept wing. This research has been directed towards the development of laminar flow airfoils and the possible instability of the flow to Görtler vortices and crossflow vortices, while taking Tollmien-Schlichting waves into account. Thus, Hall (1985) considered the Görtler vortex instability in a weakly three-dimensional boundary layer and found that an asymptotically small spanwise velocity component is sufficient to prevent the Görtler mechanism occurring at finite Görtler numbers.

The crossflow mechanism has also been the subject of many investigations (see, for example, Gregory, Stuart, and Walker (1955), Cebeci and Stewartson (1980), Malik, Wilkinson, and Orszag (1981), and Reed (1985)). This instability mechanism occurs when the effective velocity profile in an Orr-Sommerfeld approximation to the linear instability equations has an inflection point where the velocity field vanishes. The importance of such a profile was explained by the inviscid analysis of Gregory, Stuart, and Walker (hereafter referred to as GSW) in the context of the rotating disc problem.

The noteworthy feature of this profile is that it can support a stationary vortex pattern relative to the disc. GSW showed that the normal to the vortex boundaries made an angle ϕ of about 13^{o} to the radius vector. This was found to be in excellent agreement with their experimental observations, but the number of vortices predicted by the theory was found to be too large by a factor of about 4. The latter discrepancy has been attributed to viscous effects, but the reason why the angle ϕ should not also be significantly altered by such effects is not clear. The asymptotic investigation of the GSW mode which we will give in Section 3 will shed light on this question.

A recent parallel flow numerical investigation by Malik (1985) found that the point at infinity of GSW in the wavenumber-Reynolds number plane is connected to a curve corresponding to stationary modes at finite Reynolds number. However, the angle ϕ varies along the curve and the critical Reynolds number corresponds to $\phi \sim 11^{o}$, and there is also a lower branch on which ϕ asymptotes to about 39^{o} when the Reynolds number is larger.

The first purpose of this paper is to set up a rational framework which can take non-parallel effects into account at large Reynolds numbers. The second aim is to provide an analytical method of producing the wavenumber-Reynolds numbers dependence of the upper and lower branch modes. Since our analysis is applicable to any three-dimensional boundary layer, our calculations enable the likely stationary vortex patterns in such flows at high Reynolds numbers to be predicted analytically. We will see that the lower branch mode corresponds to the case when the effective velocity profile has zero shear stress at the wall and the disturbance takes on a triple deck structure. The development of an asymptotic theory will also enable nonlinear effects to be investigated in a self-consistent manner. Such an investigation is beyond the scope of the present paper but is clearly necessary in order to explain why the upper branch mode is apparently almost always the only one to be observed experimentally. The asymptotic theory of the lower branch mode is also relevant to short wavelength instabilities of Stokes layers. The procedure adopted in the rest of the paper is as follows: in Section 2 we formulate the instability equations; in Sections 3 and 4 we develop asymptotic theories for the upper and lower branch modes. Finally, in Section 5 we draw some conclusions.

2. Formulation of the Problem

We consider the flow of a viscous fluid of kinematic viscosity ν in the region $z > 0$. The motion of the fluid is induced by the steady rotation with angular velocity Ω of the plane $z = 0$ about the z axis. We take cylindrical polar coordinate (r, θ, z) with r and z having been made dimensionless with respect to some reference length ℓ. The Reynolds number R for the flow is defined by

$$R = \frac{\Omega \ell^2}{\nu}, \qquad (2.1)$$

and if the axes rotate with the plane, then the basic steady velocity field is

$$\underline{u} = \underline{u}_B = \ell\Omega\left(r\bar{u}(Rz^{1/2}),\ r\bar{v}(Rz^{1/2}),\ R^{-1/2}\bar{w}(Rz^{1/2})\right). \qquad (2.2)$$

Here the functions $\bar{u}$, $\bar{v}$, and $\bar{w}$ are determined by

$$\bar{u}^2 - (\bar{v} + 1)^2 + \bar{u}'\,\bar{w} - \bar{u}'' = 0, \qquad (2.3a)$$

$$2\bar{u}(\bar{v} + 1) + \bar{v}'\,\bar{w} - \bar{v}'' = 0, \qquad (2.3b)$$

$$2\bar{u} + \bar{w}' = 0, \tag{2.3c}$$

where the prime denotes differentiation with respect to z. The appropriate boundary conditions are

$$\begin{aligned} &\bar{u} = 0,\ \bar{v} = 0,\ \bar{w} = 0,\ \bar{z} = 0 \\ &\bar{u} \to 0,\ \bar{v} \to -1,\ z \to \infty. \end{aligned} \tag{2.4}$$

We now perturb the above flow by writing

$$u = \underline{u}_B + \Omega \ell U\big((r,\theta,z),\ V(r,\theta,z),\ W(r,\theta,z)\big) \tag{2.5}$$

where U, V, and W are small and steady. The expression (2.5) is then substituted into the Navier-Stokes equations in the rotating frame and linearized to give

$$\left\{r\bar{u}\frac{\partial}{\partial r} + \bar{v}\frac{\partial}{\partial\theta} + R^{-1/2}\bar{w}\frac{\partial}{\partial z}\right\}U + \bar{u}U - 2\{\bar{v}+1\}V + rW\frac{\partial\bar{u}}{\partial z}$$

$$= -\frac{\partial P}{\partial r} + \frac{1}{R}\left\{LU - \frac{2}{r^2}\frac{\partial V}{\partial\theta} - \frac{U}{r^2}\right\}, \tag{2.6a}$$

$$\left\{r\bar{u}\frac{\partial}{\partial r} + \bar{v}\frac{\partial}{\partial\theta} + R^{-1/2}\bar{w}\frac{\partial}{\partial z}\right\}V + \bar{u}V + 2\{\bar{v}+1\}U + rW\frac{\partial\bar{v}}{\partial z}$$

$$= -\frac{1}{r}\frac{\partial P}{\partial\theta} + \frac{1}{R}\left\{LV + \frac{2}{r^2}\frac{\partial U}{\partial\theta} - \frac{V}{r^2}\right\}, \tag{2.6b}$$

$$\{r\bar{u}\frac{\partial}{\partial r} + \bar{v}\frac{\partial}{\partial \theta} + R^{-1/2}\bar{w}\frac{\partial}{\partial z}\}W + R^{-1/2}\ W\frac{\partial \bar{w}}{\partial z}$$

$$= -\frac{\partial P}{\partial z} + \frac{1}{R}\{LW\}, \qquad (2.6c)$$

where

$$L \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

and P is the nondimensional pressure perturbation. The equation of continuity then becomes

$$\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{\partial W}{\partial z} = 0. \qquad (2.7)$$

Finally, we must solve (2.6), (2.7) subject to the no-slip condition at the wall, whilst sufficiently far away from the wall we insist that the disturbance decays to zero. However, we shall see below that the length scale for this decay to zero will depend on the type of disturbance under consideration.

3. The Inviscid Modes

From the inviscid theory of GSW we expect these modes to have wavelengths scaled on the boundary layer thickness. Thus, we must consider modes with a length scale of order $R^{-1/2}$ in the r and θ directions. It is convenient to define the small parameter ε by

$$\varepsilon = R^{-1/6}$$

and then we write

$$U = u(z) \exp \frac{1}{\varepsilon^3} \{ \int^r \alpha(r,\varepsilon)dr + \theta\beta(\varepsilon)\} \tag{3.1}$$

together with similar expressions for V, W, and the pressure perturbation P. The wavenumber α will, in general, be complex and is determined in terms of ε and β. However, we will restrict our attention to neutral disturbances and find α and β such that the flow is neutrally stable at the position r. We now expand α, β as

$$\alpha = \alpha_0 + \varepsilon\alpha_1 + ,\cdots, \tag{3.2a}$$

$$\beta = \beta_0 + \varepsilon\beta_1 + ,\cdots . \tag{3.2b}$$

The disturbance structure in the z direction is then fixed by the following considerations. Firstly, from the results of GSW we anticipate that there will be an inviscid zone of depth $O(\varepsilon^3)$. In order to satisfy the no-slip condition on the velocity at the wall, a viscous layer must exist. The thickness of this layer is then found to be $O(\varepsilon^4)$ by balancing the convection and diffusion terms in the disturbance equations.

In the inviscid zone we expand u, v, w, and p in the form

$$u = u_0(\zeta) + \varepsilon u_1(\zeta) + ,\cdots, \tag{3.3a}$$

$$v = v_0(\zeta) + \varepsilon v_1(\zeta) + ,\cdots, \tag{3.3b}$$

$$w = w_0(\zeta) + \varepsilon w_1(\zeta) + ,\cdots, \tag{3.3c}$$

$$p = p_0(\zeta) + \varepsilon p_1(\zeta) + ,\cdots, \tag{3.3d}$$

where $\zeta = z\varepsilon^{-3}$. The above expansions are then substituted into (2.6), (2.7) with $\frac{\partial}{\partial r}$ replaced by $\frac{\partial}{\partial r} + \frac{i}{\varepsilon^3}\{\alpha_0 + \varepsilon\alpha_1 + ,\cdots,\}$ and with $\frac{\partial}{\partial\theta}$ replaced by $\frac{i}{\varepsilon^3}\{\beta_0 + \varepsilon\beta_1 + ,\cdots,\}$. If we equate terms of order ε^{-4}, we obtain

$$i\bar{\bar{u}}u_0 + rw_0\,\bar{u}' = -i\alpha_0\,p_0 \tag{3.4a}$$

$$i\bar{\bar{u}}v_0 + rw_0\,\bar{v}' = -\frac{i\beta_0}{r}\,p_0 \tag{3.4b}$$

$$i\bar{\bar{u}}w_0 = -\,p_0' \tag{3.4c}$$

$$i\alpha_0\,u_0 + \frac{i\beta v_0}{r} + w_0' = 0, \tag{3.4d}$$

where $\bar{\bar{u}} = \alpha_0\,\overline{ur} + \beta_0\,\bar{v}$. If we eliminate u_0, v_0 and the pressure from the above equations, we find that w_0 satisfies

$$\bar{\bar{u}}[w_0'' - \gamma_0^2\,w_0] - \bar{\bar{u}}''\,w_0 = 0 \tag{3.5}$$

with $\bar{\bar{u}}$ now acting as the ´effective´ or ´equivalent´ two-dimensional velocity profile whilst $\gamma_0^2 = \alpha_0^2 + \frac{\beta_0^2}{r}$ is the effective wavenumber. Thus, w_0 satisifies Rayleigh´s equation and γ_0 is determined as an eigenvalue when (3.5) is solved subject to

$$w_0 = 0, \qquad \zeta = 0,\infty. \tag{3.6}$$

We further note that α_0/β_0 is chosen such that $\bar{\bar{u}}$ and $\bar{\bar{u}}''$ vanish at the same nonzero value of $\zeta = \bar{\zeta}$; in this case (3.5) has no singularity at $\zeta = \bar{\zeta}$. The eigenvalue problem was solved using central differences; we obtained

$$\gamma_0 = 1.16, \tag{3.7a}$$

$$\frac{\alpha_0}{\beta_0} = \frac{4.26}{\tau}, \tag{3.7b}$$

$$\bar{\zeta} = 1.46.$$

The eigenfunction w_0 normalized with $w_0' = 1$ at $\zeta = 0$ is shown in Figure 1.

Having calculated w_0 we can use (2.8), (2.9) to solve for u_0, v_0, and p_0; however, it suffices here to say that when $\zeta \to 0$

$$i\{\alpha_0 u_0 + \frac{\beta_0 v_0}{r}\} \to - w_0'(0). \tag{3.8}$$

Before proceeding to the next order in the inviscid zone, we calculate the zeroth-order solution in the wall layer. If we write

$$\xi = \varepsilon^{-4} z,$$

then in the wall layer $\bar{u}$ expands as

$$\bar{u} = \varepsilon \bar{u}_0 \xi + ,\cdots,$$

and $\bar{v}$, $\bar{w}$ are expanded in a similar manner. The disturbance velocity and pressure now are written as

$$u = U_0(\xi) + \epsilon U_1(\xi) + ,\cdots,$$

$$v = V_0(\xi) + \epsilon V_1(\xi) + ,\cdots,$$

$$w = \epsilon W_0(\xi) + \epsilon^2 W_1(\xi) + ,\cdots, \qquad (3.9)$$

$$p = \epsilon P_0(\xi) + \epsilon^2 P_1(\xi) + ,\cdots .$$

After substituting the above expansions into the disturbance equations equating the dominant terms and performing some manipulations, we find that $\alpha_0 U_0 + \frac{\beta_0 V_0}{r}$ satisfies

$$\left[\alpha_0 U_0 + \frac{\beta_0 V_0}{r}\right]''' - i\xi[\alpha_0 \bar{u}_0 r + \beta_0 \bar{v}_0]\left[\alpha_0 U_0 + \frac{\beta_0 V_0}{r}\right]' = 0 \qquad (3.10)$$

and the solution of this equation which satisfies $U_0(0) + \frac{\beta_0 V_0}{\alpha_0 r} = 0$, and (3.8) is

$$i\left[\alpha_0 U_0 + \frac{\beta_0 V_0}{r}\right] = - \frac{w_0'(0) \int_0^\xi A_i(\gamma s)d\xi}{\int_0^\infty A_i(\gamma s)d\xi} \qquad (3.11)$$

where

$$\gamma = \left\{i[\alpha_0 \bar{u}_0 r + \beta_0 \bar{v}_0]\right\}^{1/3}. \qquad (3.12)$$

For large values of ξ we can show that

$$W_0 \sim w_0' \xi + \frac{w_0'(0) A_i'(0)}{\gamma \int_0^\infty A_i(s)ds} ,$$

so that w_1 the order ϵ inviscid zone normal velocity component must satisfy

$$w_1 \to + \frac{w_0'(0)A_i'(0)}{\gamma \int_0^\infty A_i(s)ds}, \qquad \zeta \to 0. \tag{3.13}$$

We now turn to the next order problem in the inviscid zone. Thus, we now equate terms $O(\varepsilon^{-3})$ in the inviscid zone disturbance equations, and we obtain a set of equations similar to (3.4) with (u_0, v_0, w_0, p_0) replaced by (u_1, v_1, w_1, p_1) but now having inhomogeneous terms. If we repeat the manipulations carried out on (3.4) in order to get (3.5), we obtain

$$\bar{\bar{u}}[w_1'' - \gamma_0^2 w_1] - \bar{\bar{u}}'' w_1 = 2\bar{\bar{u}}\left\{\alpha_0 \alpha_1 + \frac{\beta_0 \beta_1}{r^2}\right\}w_0$$
$$+ \left\{\alpha_1 - \frac{\beta_1 \alpha_0}{\beta_0}\right\} r\left\{\bar{u}'' - \frac{\bar{\bar{u}}'' \bar{u}}{\bar{\bar{u}}}\right\}w_0. \tag{3.14}$$

The second term on the right-hand side of (3.14) causes w_1 to have a logarithmic singularly at $\zeta = \bar{\zeta}$; this can be removed in the usual way by introducing a critical layer at $\zeta = \bar{\zeta}$. We can formally write down a solution of (3.14) which satisfies $w_1(\infty) = 0$ in the form

$$w_1 = 2\,\alpha_0\left\{\alpha_1 + \frac{\beta_0 \beta_1}{r^2}\right\}w_0(\zeta)\int_{\bar{\bar{\zeta}}}^{\zeta} \frac{d\eta}{w_0^2(\eta)}\int_\infty^\eta w_0^2(\theta)d\theta$$
$$+ \left\{\alpha_1 - \frac{\alpha_0 \beta_1}{\beta_0}\right\} rw_0(\zeta)\int_{\bar{\bar{\zeta}}}^{\zeta} \frac{d\eta}{w_0^2(\eta)}\int_\infty^\eta w_0^2(\theta)\left[\frac{\bar{u}''(\theta)\bar{\bar{u}}(\theta) - \bar{\bar{u}}(\theta)\bar{u}(\theta)d\theta}{\bar{\bar{u}}^2}\right], \tag{3.15}$$

when $\bar{\bar{\zeta}}$ is a constant with $\bar{\bar{\zeta}} > \bar{\zeta}$. The above solution is valid for $\zeta < \bar{\bar{\zeta}}$

if the path of integration is deformed appropriately into the complex plane near $\zeta = \bar{\zeta}$. It can then be shown from (3.15) that

$$w_1(0) = 2\left\{\alpha_0\,\alpha_1 + \frac{\beta_1\,\beta_0}{r^2}\right\}\frac{I_1}{w_0'(0)} + \left\{\frac{\alpha_1}{\beta_0} - \frac{\alpha_0\,\beta_1}{\beta_0^2}\right\}\frac{r}{w_0'(0)}\,I_2, \qquad (3.16)$$

where

$$I_1 = \int_0^\infty w_0^2(\theta)d\theta \qquad (3.17a)$$

$$I_2 = \beta_0 \int_0^\infty w_0^2\left[\frac{\bar{u}''\,\bar{\bar{u}} - \bar{\bar{u}}''\,\bar{u}}{\bar{\bar{u}}^2}\right] \qquad (3.17b)$$

where the path of integration in (3.17b) is deformed below the singularity at $\zeta = \bar{\zeta}$. The matching condition (3.13) produces the eigenrelation,

$$\frac{A_{10}'\left(w_0'(0)^2\right)}{\gamma\int_0^\infty A_i(s)ds} = 2\left\{\alpha_0\,\alpha_1 + \frac{\beta_1\,\beta_0}{r^2}\right\}I_1 + \left\{\frac{\alpha_1}{\beta_0} - \frac{\alpha_0\,\beta_1}{\beta_0^2}\right\}r\,I_2. \qquad (3.18)$$

Our calculations showed that

$$I_1 = .094, \qquad I_2 = .058 - .029i,$$

and using the well known values for A_{10}', $\int_0^\infty A_i(s)ds$, we obtained

$$\alpha_0\,\alpha_1 + \frac{\beta_0\,\beta_1}{r^2} = -\,14\cdot r^{-1/3}\,\gamma_0,$$

$$\left\{\frac{\alpha_1}{\beta_0} - \frac{\alpha_0\,\beta_1}{\beta_0^2}\right\}r = 29\cdot r^{-1/3}.$$

The above equations can be solved for α_1, β_1; however, it is more useful to evaluate

$$\sqrt{\alpha^2 + \beta^2/r^2} = \gamma_0 + \left[\alpha_0\,\alpha_1 + \frac{\beta_0\,\beta_1}{\gamma^2}\right]\frac{\varepsilon}{\gamma_0} + ,\ldots,$$

$$= 1.16 - \frac{14.4}{R^{1/6}}\,r^{-1/3} + ,\ldots, \qquad (3.19)$$

which we interpret as the 'effective' wavenumber of the disturbance.

We now define the wave angle ϕ by

$$\tan[\tfrac{\pi}{2} - \phi] = \frac{r\alpha_0}{\beta_0} + r\left\{\frac{\alpha_1}{\beta_0} - \frac{\beta_1\,\alpha_0}{\beta_0^2}\right\}\varepsilon + ,\ldots,$$

$$= 4.26 + \frac{29\cdot r^{-1/3}}{R^{1/6}} + ,\ldots\,. \qquad (3.20)$$

Thus, we have calculated the first correction terms to the classical results of GSW. The sign of the correction term in (3.20) has some important consequences which we will discuss in Section 5.

4. The Wall Modes

We have seen in the previous section that the 'effective' velocity profile for a three-dimensional disturbance with wavenumbers α and β in the r and θ directions is $\alpha\bar{u}r + \beta\bar{v}$. The inviscid modes are such that $\alpha\bar{u}r + \beta\bar{v}$ and $\alpha\bar{u}''r + \beta\bar{v}''$ vanish simultaneously. It is easy to show that lower branch disturbances having a triple deck structure of the type discussed by Smith (1978) for Blasius flow can also exist. However, such modes are necessarily time-dependent with α, β real if the effective wall shear $\alpha\bar{u}'r + \beta\bar{v}'$ does not vanish. Therefore, we choose to look for stationary modes for which the effective wall shear vanishes at zeroth order.

It is easy to show that the appropriate triple-deck structure is based on the small parameter ε now defined by

$$\varepsilon = R^{-1/16}, \tag{4.1}$$

and the lower, main, and upper decks are of thickness ε^9, ε^8, and ε^4 respectively. The disturbances structure in the main and upper decks is essentially the same as that found by Smith (1978) who investigated lower branch disturbances to Blasius flow. The wavenumbers in the r and θ directions are now $O(\varepsilon^{-4})$; we therefore write

$$U = U(z)\exp \frac{i}{\varepsilon^4} \left\{ \int^r \alpha(r,\varepsilon)dr + \theta\beta(\varepsilon)\right\},$$

together with similar expressions for V, W, and P. We define ξ, ζ, Z by

$$\xi = \frac{y}{\varepsilon^9}, \quad \zeta = \frac{z}{\varepsilon^8}, \quad Z = \frac{z}{\varepsilon^4}. \tag{4.2a,b,c}$$

The wavenumbers then expand as

$$\alpha = \alpha_0 + \varepsilon^2 \alpha_1 + \varepsilon^3 \alpha_2 + ,\cdots, \tag{4.3a}$$

$$\beta = \beta_0 + \varepsilon^2 \beta_1 + \varepsilon^3 \beta_2 + ,\cdots . \tag{4.3b}$$

Here we have anticipated that the order ε terms are zero, and we again seek α_i, β_i, etc. such that the flow is neutrally stable at the location r. In the upper deck $\bar{u} = 0$, $\bar{v} = -1$, and U expands as

$$U = \varepsilon^3 U_0(z) + \varepsilon^4 U_1(z) + ,\cdots,$$

and V, W, and P have similar expansions. We found that the zeroth-order equations to be solved in the upper deck are

$$\beta_0 U_0 = \alpha_0 P_0, \quad \beta_0 V_0 = \frac{\beta_0 P_0}{r}, \quad i\beta_0 W_0 = \frac{dP_0}{dz},$$

$$i\alpha_0 U_0 + i \frac{\beta_0 V_0}{r} + \frac{dW_0}{d\xi} = 0,$$

and the solution of this system which decays to zero when $z \to \infty$ is

$$P_0 = Ce^{-\gamma_0 z}, \qquad U_0 = \frac{\alpha_0}{\beta_0} Ce^{-\gamma_0 z}, \tag{4.4a,b}$$

$$V_0 = \frac{C}{r} e^{-\gamma_0 z}, \qquad W_0 = \frac{i\gamma_0}{\beta_0} Ce^{-\gamma_0 z}, \tag{4.4c,d}$$

where

$$\gamma_0 = \sqrt{\alpha_0^2 + \beta_0^2/r^2},$$

and C is an unknown function of r.

In the main deck the disturbance expands as

$$U = \frac{1}{\varepsilon} u_0(\zeta) + u_1(\zeta) + ,\cdots,$$

$$V = \frac{1}{\varepsilon} v_0(\zeta) + v_1(\zeta) + ,\cdots,$$

$$W = \varepsilon^3 w_0(\zeta) + \varepsilon^2 w_1(\zeta) + ,\cdots,$$

$$P = \varepsilon^3 C + \varepsilon^2 P_1(\zeta) + ,\cdots,$$

where we have anticipated that P is independent of ξ to order ε^3 and therefore equal to C. Substituting into (2.6), (2.7) we find that u_0, v_0, w_0 satisfy

$$i\alpha_0 r\bar{u} u_0 + i\beta_0 \bar{v} u_0 + r\bar{u}' w_0 = 0,$$

$$i\alpha_0 r\bar{u} v_0 + i\beta_0 \bar{v} v_0 + r\bar{v}' w_0 = 0,$$

$$i\alpha_0 u_0 + \frac{i\beta_0}{r} v_0 + \frac{dw_0}{d\zeta} = 0,$$

and the solution of this system which matches with the upper deck solution is

$$u_0 = \frac{Cr\,\gamma_0\,\bar{u}'}{\beta_0^2}, \tag{4.5a}$$

$$v_0 = \frac{Cr\,\gamma_0\,\bar{v}'}{\beta_0^2}, \tag{4.5b}$$

$$w_0 = -\frac{Cr\,\gamma_0}{\beta_0^2}\left(\alpha_0 r\bar{u} + \beta_0 \bar{v}\right). \tag{4.5c}$$

We note from (4.5c) that w_0 in fact satisfies the no-slip condition when $\zeta \to 0$; however, unless $\bar{u}'$ and $\bar{v}'$ both vanish at $\zeta = 0$ the other velocity components are nonzero there. If we choose α_0 and β_0 such that

$$\alpha_0 \, \bar{u}'(0) + \frac{\beta_0}{r} \bar{v}'(0) = 0, \tag{4.6}$$

which gives $\frac{\alpha_0 \, r}{\beta_0} = 1.207$, then $\alpha_0 \, u_0 + \frac{\beta_0 \, v_0}{r} \to 0$ when $\zeta \to 0$. It is the imposition of the constraint (4.6) on the effective velocity profile which enables us to find stationary disturbances. If we expand $\bar{u}$, $\bar{v}$ for small ζ and write $\xi = \zeta/\varepsilon$, we have

$$\bar{u} = \varepsilon \bar{u}_0 \, \xi + \varepsilon^2 \, \bar{u}_1 \, \xi^2 + \bar{u}_2 \, \xi^3 + , \cdots, \tag{4.7a}$$

$$\bar{v} = \varepsilon \bar{v}_0 \, \xi + \varepsilon^2 \, \bar{v}_1 \, \xi^2 + \bar{v}_2 \, \xi^3 + , \cdots, \tag{4.7b}$$

when $\bar{u}_{j-1} = \frac{\bar{u}^j(0)}{j!}$, $\bar{v}_{j-1} = j \, \frac{\bar{v}^j(0)}{j!}$; for $j = 1,2,\cdots$. In order to match with the solution (4.5), written in terms of ξ using (4.7), we therefore expand the lower deck disturbance in the form

$$U = \frac{r \, \gamma_0 \, C}{\varepsilon \beta_0^2} \left[\bar{u}_0 + 2\varepsilon \bar{u}_1 \, \xi + \cdots \right] + \frac{U_{-1}(\xi)}{\varepsilon} + U_0(\xi) + \varepsilon U_1(\xi) + , \cdots, \tag{4.8a}$$

$$V = \frac{r \gamma_0 \, C}{\varepsilon \beta_0^2} \left[\bar{v}_0 + 2\varepsilon \bar{v}_1 \, \xi + \cdots \right] + \frac{V_{-1}(\xi)}{\varepsilon} + V_0(\xi) + \varepsilon V_1(\xi) + , \cdots, \tag{4.8b}$$

$$W = - \frac{i \gamma_0 \, \varepsilon^5 \, C}{\beta_0^2} \left[(\alpha_0 \, \bar{u}_1 \, r + \beta_0 \, \bar{v}_1) \xi^2 + \cdots \right] + \varepsilon^6 \, W_1(\xi) + , \cdots, \tag{4.8c}$$

$$P = \varepsilon^3 \, P_1(\xi) + \cdots \, . \tag{4.8d}$$

We must now substitute the above expansions into the disturbance equations and solve for (U_{-1}, V_{-1}), (U_0, V_0), (U_1, V_1, W_1, P_1), etc. From the continuity equation we obtain immediately that

$$V_{-1} = -\frac{\alpha_0}{\beta_0 r} U_{-1}, \tag{4.9a}$$

$$V_0 = -\frac{\alpha_0}{\beta_0 r} U_0, \tag{4.9b}$$

where $\frac{\alpha_0}{\beta_0}$ satisfies (4.6). From the radial momentum equation we obtain

$$-i[r\alpha_0 \bar{u}_1 + \beta_0 \bar{v}_1]\xi^2 U_{-1} + \frac{d^2 U_{-1}}{d\xi^2} = 0, \tag{4.10a}$$

$$-i[r\alpha_0 \bar{u}_1 + \beta_0 \bar{v}_1]\xi^2 U_0 + \frac{d^2 U_0}{d\xi^2} = -r\bar{u}_0 W_1 + [r\alpha_0 \bar{u}_2 + \beta_0 \bar{v}_2]\xi^3 U_0, \tag{4.10b}$$

which must be solved subject to

$$U_{-1} = -\frac{r\gamma_0 C\bar{u}_0}{\varepsilon\beta_0^2}, \quad U_0 = 0, \quad \xi = 0,$$

$$U_{-1}, \quad U_0 \to 0, \quad \xi \to \infty. \tag{4.11}$$

The function U_{-1} is given by

$$U_{-1} = -\frac{\bar{u}_0 \gamma_0 Cr}{\beta_0^2} \frac{U(0, \sqrt{2}\,\Delta^{1/4}\xi)}{U(0,0)}, \tag{4.12}$$

where

$$\Delta = i\{\alpha_0\, r\bar{u}_1 + \beta_0\, \bar{v}_1\} \tag{4.13}$$

and $U(0, \sqrt{2}\,\Delta^{1/4}\xi)$ is a parabolic cylinder function. The functions U_0, V_0 cannot be determined unil W_1 is calculated, the latter function can be found by considering the next order approximation to the radial and azimuthal momentum equations. If we multiply these equations by $i\alpha_0$ and $\frac{i\beta_0}{\gamma}$ respectively and add them we obtain:

$$i\left\{\alpha_0 \frac{d^2 U_1}{d\xi^2} + \frac{\beta_0}{r}\frac{d^2 U_1}{d\xi^2}\right\} + \gamma_0^2 P_0 + 2i\alpha_0 V_{-1} - \frac{2i\beta_0 U_{-1}}{r}$$

$$= i\{\alpha_0 r\bar{u}_1 + \beta_0\, \bar{v}_1\}\left\{i\alpha_0\, U_1 + \frac{i\beta_0\, V_1}{r}\right\}\xi^2 + 2i\xi\{\alpha_0\, r\bar{u}_1 + \beta_0\, \bar{v}_1\}W_1$$

$$- 2\{\alpha_1\, r\bar{u}_0 + \beta_1\, \bar{v}_0\}\frac{r\gamma_0\, C\xi^2}{\beta_0^2}\left\{\alpha_0\, \bar{u}_1 + \frac{\beta_0\, \bar{v}_1}{r}\right\}, \tag{4.14}$$

whilst the z momentum and continuity equations give

$$\frac{dP_0}{d\xi} = 0\ ,$$

$$i\left\{\alpha_0\, U_1 + \frac{\beta_0\, V_1}{r}\right\} + \frac{dW_1}{d\xi} = -\frac{ir\gamma_0\, C}{\beta_0^2}\left[\alpha_1\, \bar{u}_0 + \frac{\beta_1\, \bar{v}_0}{r}\right] - i\alpha_1\, U_{-1} - \frac{i\beta_1\, V_{-1}}{r} \tag{4.15}$$

so that $P_0 = C$. It is important to point out at this stage that the terms proportional to U_{-1}, V_{-1} in (4.14) are due to Coriolis effects; thus the structure of the neutral curve for stationary small wavenumber disturbances depends both on viscous and Coriolis effects. We can eliminate U_1, V_1 from (4.14) and (4.15) to give

$$\frac{d^3 W_1}{d\xi^3} - i\{\alpha_0 \, r\bar{u}_1 + \beta_0 \, \bar{v}_1\}\xi^2 \frac{dW_1}{d\xi} + 2i\xi\{\alpha_0 \, r\bar{u}_1 + \beta_0 \, \bar{v}_1\}W_1$$

$$= \gamma_0^2 \, C + \{\alpha_1 \, r\bar{u}_0 + \beta_1 \, \bar{v}_0\} \frac{r\gamma_0 \, \xi^2 \, C}{\beta_0^2} \left\{\alpha_0 \, \bar{u}_1 + \frac{\beta_0}{r} \bar{v}_1\right\}$$

$$+ \frac{2i\beta_0}{r} \left\{1 + \frac{\bar{v}_0^2}{\bar{u}_0^2}\right\} \bar{u}_0 \, \frac{U(0, \sqrt{2}s)}{U(0,0)} \, , \qquad (4.16)$$

where

$$s = \Delta^{1/4} \xi. \qquad (4.16)$$

We write the solution of (4.16) in the form

$$W_1 = - i\{\alpha_1 \, r\bar{u}_0 + \beta_1 \, \bar{v}_0\} \frac{\gamma_0 \, C\xi}{\beta_0^2} + \Delta^{-3/4}\left\{\gamma_0^2 \, CF_1(s)\right.$$

$$\left. + \frac{2i \, \beta_0}{r}\left[1 + \frac{\bar{v}_0^2}{\bar{u}_0^2}\right] \frac{\bar{u}_0 \, F_2(s)}{U(0,0)}\right\} + k_1 \, \xi^2, \qquad (4.17)$$

where k_1 is a constant and F_1 satisfies

$$F_1''' - s^2 \, F_1' + 2sF_1 = 1, \qquad F_1(0) = F_1(\infty) = 0,$$

whilst F_2 satisfies a similar equation with the right-hand side of the differential equation replaced by $U(0, \sqrt{2}s)$. In fact, it is straightforward to express F_1, F_2 in terms of integrals involving parabolic cylinder functions. It remains for us to satisfy $U_1 = V_1 = 0$ at $\xi = 0$; from (4.15) and (4.17) we can show that this condition leads to the eigenrelation:

$$\gamma_0^2 \, I_3 + \frac{i\beta_0 \, \bar{u}_0}{r}\left[1 + \frac{\bar{v}_0^2}{\bar{u}_0^2}\right] I_4 = i\Delta^{1/2} \{\alpha_1 \, r\bar{u}_0 + \beta_1 \, \bar{v}_0\}. \tag{4.18}$$

Here the integrals I_3, I_4 are given by

$$I_3 = \frac{\int_0^\infty \theta U(0,\theta)d\theta}{2U(0,0)} = .599,$$

$$I_4 = \frac{\int_0^\infty \theta U^2(0,\theta)d\theta}{U^2(0,0)} = .457,$$

and (4.18) can be solved to give

$$\gamma_0 = \left\{\frac{\beta_0 \, \bar{u}_0}{r \, I_3}\left(1 + \frac{\bar{v}_0^2}{\bar{u}_0^2}\right) I_4\right\}^{1/2} = 1.224 \; r^{-1/2}, \tag{4.19}$$

$$\left\{\frac{\alpha_1}{\beta_0} - \frac{\beta_1 \, \alpha_0}{\beta_0^2}\right\} = \frac{2\gamma_0^{3/2}\left(1 + \frac{\bar{v}_0^2}{\bar{u}_0^2}\right)^{-1/4}}{|\bar{v}_0 \, \bar{u}_0|^{1/2}} I_3$$

$$= 2.312 \; r^{-5/4}. \tag{4.20}$$

We see at this stage that it is still not possible to find α_1 and β_1 independently; however, it follows from above that ϕ the angle between the radius vector and the normal to the vortices is given by

$$\tan[\pi/2 - \phi] = 1.207 + 2.1312 \; \varepsilon^2 \; r^{-1/4} + ,\dots , \tag{4.21}$$

whilst the total wavenumber $\frac{1}{\varepsilon^4}\sqrt{\alpha^2 + \beta^2/r^2}$ is given by

$$\frac{1}{\varepsilon^4}\sqrt{\alpha^2 + \beta^2/r^2} = \frac{1.224\, r^{-1/2}}{\varepsilon^4} + ,\ldots . \qquad (4.22)$$

The above expansion procedure can be continued in principle to any order and can take non-parallel effects into account in a self-consistent manner. We stress that (4.21), (4.22) have been obtained by taking the Coriolis effect into account; an Orr-Sommerfeld approximation to the full equations gives incorrect values for the second term in (4.21) and the first term in (4.22). The sensitivity of the structure of the lower branch modes to a combination of viscous and Coriolis forces means that, unlike the upper branch modes, for a more general three-dimensional boundary layer this class of modes might not even exist. Finally, we note that time-dependent modes with a sufficiently slow time scale are also possible and introduce a frequency into the eigenrelation (4.18).

5. Conclusion

The Reynolds number R_Δ based on the boundary layer thickness, and the local azimuthal velocity of the disc is given by

$$R_\Delta = Rr^{1/2} .$$

The inviscid modes have local wavenumber k_Δ defined by

$$k_\Delta = \sqrt{\alpha^2 + \beta^2/r^2} ,$$

where the appropriate length scale is the boundary layer thickness. On the lower branch the local wavenumber k_Δ is defined by

$$k_\Delta = R^{-1/4}\sqrt{\alpha^2 + \beta^2/r^2} \ ,$$

so that (3.19), (4.22) are equivalent to

$$k_\Delta = 1.16 - 14.4\ R_\Delta^{-1/3} + ,\cdots, \tag{5.1}$$

and

$$k_\Delta = 1.22\ R_\Delta^{-1/2} + ,\cdots, \tag{5.2}$$

respectively. Similarly (3.20), (4.21) become

$$\tan[\pi/2 - \phi] = 4.26 + 29\ R_\Delta^{-1/3} + ,\cdots, \tag{5.3}$$

and

$$\tan[\pi/2 - \phi] = 1.21 + 2.31\ R_\Delta^{-1/4} + ,\cdots \ . \tag{5.4}$$

Thus, if the neutral values are expressed in terms of R_Δ, k_Δ, and ϕ have no explicit dependence on the radial variable r.

In Figures 2 and 3 we have compared the above asymptotic predictions with the numerical results of Malik (1985). The latter author solved the parallel flow approximation to (2.6) obtained by setting $\partial/\partial r \equiv i\alpha$, $\partial/\partial\theta = i\beta$. Such an approximation is valid only for $R \to \infty$ but to the order shown in (5.1) - (5.4); our asymptotic results apply to the system solved by Malik.

We see in Figures 2 and 3 that there is satisfactory agreement between the asymptotic theory and Malik's results. Thus, the asymptotic approach will be a useful tool in finding the structure of the possible stationary modes in other three-dimensional boundary layers rather than having to solve the full parallel flow equations numerically. Similarly, the asymptotic theory could be used to identify the stationary modes which are likely to be important in a Navier-Stokes investigation of this problem.

It is interesting to question why the lower branch modes have not been investigated earlier. The reason appears to be that in most experimental investigations of the disc problem, only the modes with $\phi \sim 13^{\circ}$ were observed. However, there is some discussion of modes with $\phi \sim 20^{\circ}$ in the paper by Federov, Plavnik, and Prokhorov (1976). These modes were found to exist closer to the center of the disc than the GSW modes and have a different vertical structure. Thus, it would appear that the lower branch modes perhaps bifurcate subcritically and therefore do not persist into the region where the GSW modes occur. Obviously, only a weakly nonlinear theory at least could settle this matter; however, it is interesting to note that Allen and Stuart* (1985) have pointed out the possible existence of a subcritical mode with azimuthal wavenumber $n = 2$.

The upper branch asymptotic results are again consistent with the results of Malik. The positive sign associated with the first correction term in (5.3) has important consequences. Thus, if (3.3) and (5.4) are to be connected at some finite Reynolds number the higher order terms in (5.3)

---

*personal communication

must be negative. This means that for some range of R_Δ, the value of ϕ along the upper branch modes will stay close to the infinite value of about 13^o. This is exactly what Malik found numerically and even at the critical Reynolds number; ϕ is still close to 13^o. The wavenumber, however, changes much more along the upper branch; this presumably explains why GSW predicted ϕ so well but not the number of waves.

Finally, we turn to the relevance of the lower branch modes in other boundary layer flows. At first sight we might think that our analysis is directly applicable to a two-dimensional boundary having zero shear stress at some position along the boundary. However, it is easily shown that the structure given in Section 3 is only applicable to boundary layers having a non-zero third normal derivative of the streamwise velocity component. This constraint effectively means that there are no neutral modes of the type found in Section 4 for spatially varying two-dimensional boundary layers. The Stokes layer velocity profile is another matter; at high Reynolds numbers the flow varies slowly in time, and the modes discussed in Section 4 are relevant to the times in a cycle when the shear stress instantaneously vanishes at the oscillating wall during the fluid motion.

For three-dimensional boundary layers we expect that the lower branch modes are directly relevant. Moreover, it is of course possible that in such flows nonlinear effects might cause them to be more important than the GSW modes in the development of crossflow vortices. This matter can, of course, only be resolved by further calculations.

References

Cebeci, T. and Stewartson, K., (1980), AIAA J. 18, p. 1485.

Gregory, N., Stuart, J. T., and Walker, W. S., (1955), Phil. Trans. Roy. Soc. (A), 248, p. 155.

Federov, B. I., Plavnik, G. Z., Prokhorov, I. V., and Zhukhovitskii, L. G., (1976), J. Engineering Physics, 31, p. 1448.

Hall, P., (1985), Proc. Roy. Soc. (A), 339, p. 135.

Malik, M., (1985), J. Fluid Mechanics, in press.

Malik, M., Wilkinson, S. P., and Orszag, S. A., (1981), AIAA J., 19, p. 1131.

Reed, H., (1984), AIAA Paper No. 84-1678.

Smith, F. T., (1978), Proc. Roy. Soc. (A), 366, p. 91.

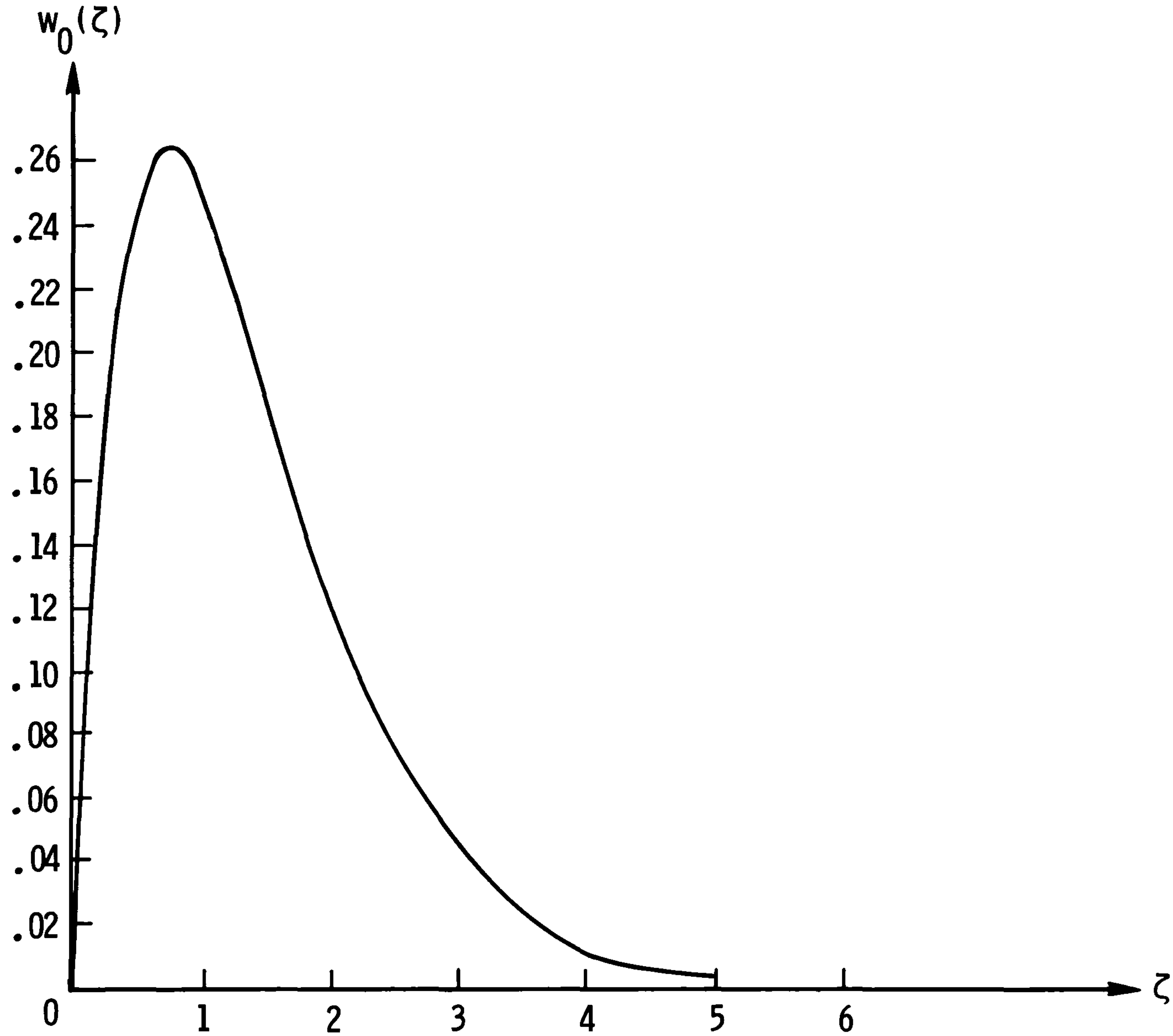

Figure 1. The inviscid motion eigenfunction.

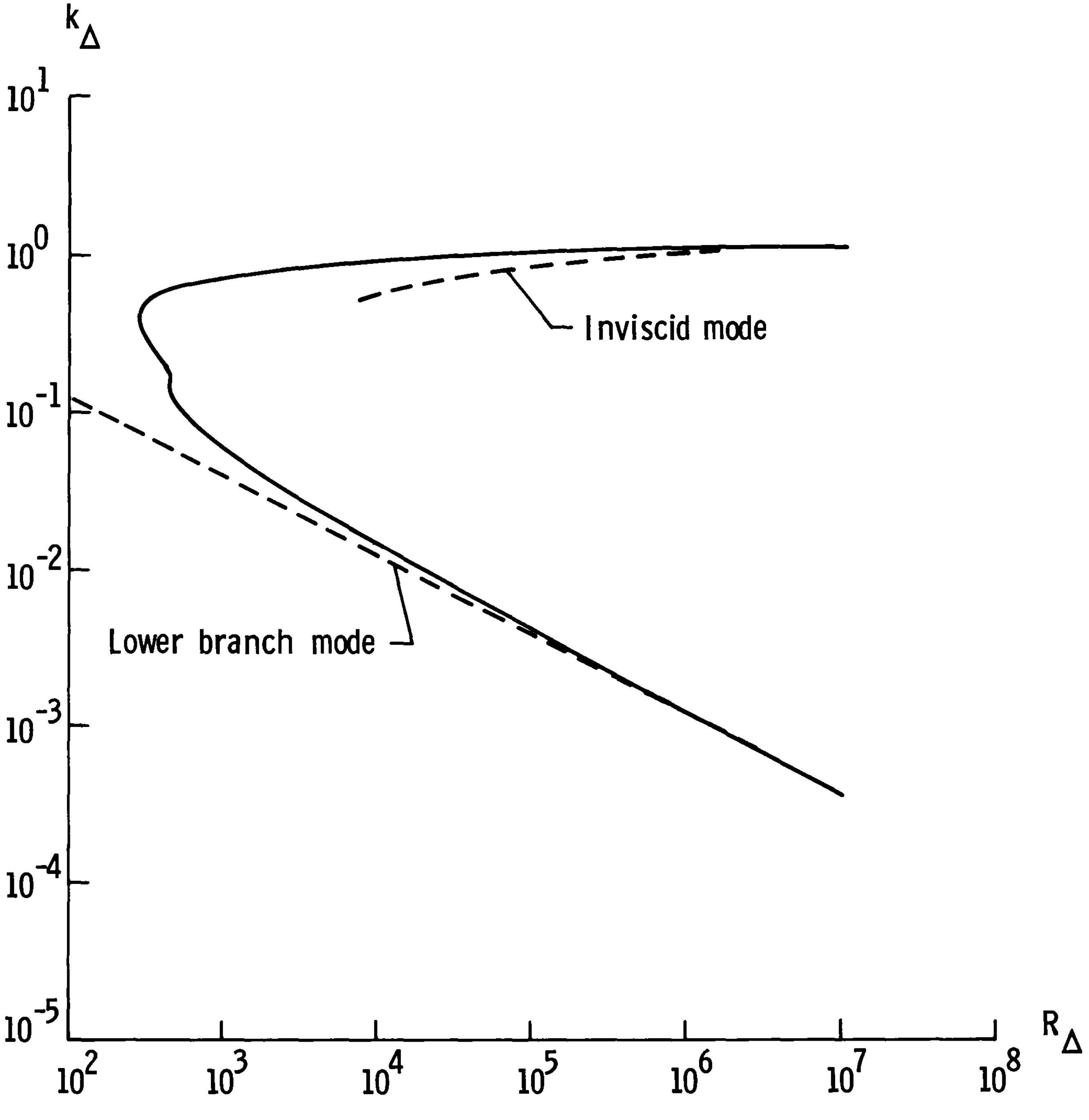

Figure 2. Comparison between the results of Malik (1985) and the asymptotic wavenumber predictions.

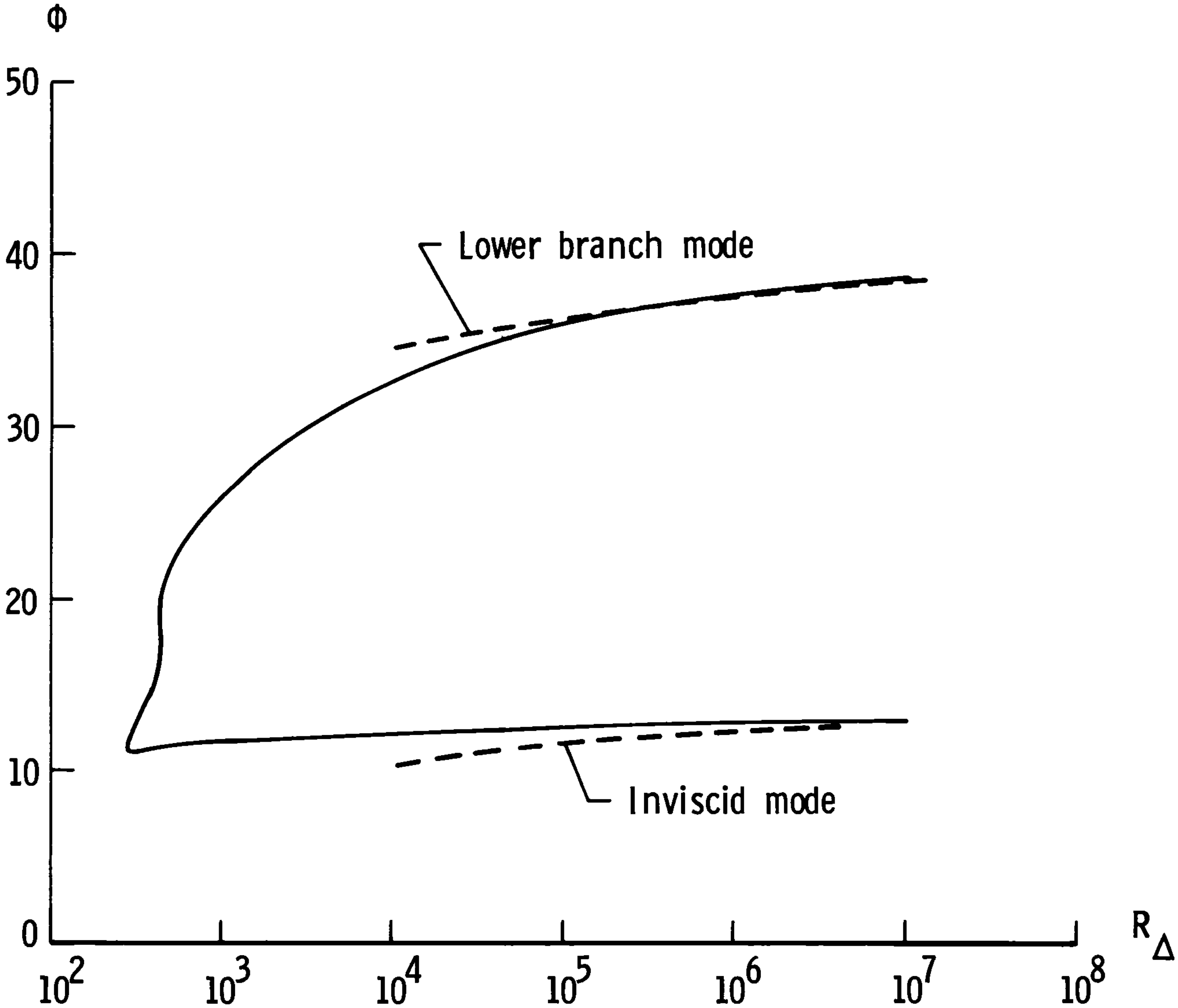

Figure 3. Comparison between the results of Malik (1985) and the asymptotic wave angle predictions.

End of Document