RESEARCH ON OUTPUT FEEDBACK CONTROL

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SUMMARY

The research for the current year has dealt with the synthesis of fixed order dynamic compensators for multivariable linear time invariant systems by minimizing a linear quadratic performance cost functional. In particular, attention has been given to robustness issues in terms of frequency domain specifications, namely low frequency performance, high frequency robustness to unmodeled dynamics and parasitics, and cross-over frequency sensitivity.

In designing fixed order compensators, an output feedback formulation has been adopted by suitably augmenting the system description to include the compensator states. However, the minimization of the performance index over the range of possible compensator descriptions was impeded due to the nonuniqueness of the compensator transfer function. A controller canonical form of the compensator was chosen to reduce the number of free parameters to its minimal number in the optimization. In the MIMO case, the controller form requires a prespecified set of ascending controllability indices. This constraint on the compensator structure is rather innocuous in relation to the increase in convergence rate of the optimization. Moreover, the controller form is easily relatable to a unique controller transfer function description. This structure of the compensator does not require penalizing the compensator states for a nonzero or coupled solution, a problem that occurs when following a standard output feedback synthesis formulation.

Unlike full state LQR designs, output feedback designs do not have guaranteed gain and phase margin properties. For that matter, neither does an LQG design. Recent robustness recovery techniques try to asymptotically approximate the full state LQR design by incorporating fictitious noise in the performance index. A cheap control formulation results with high frequency

dynamics appearing in the compensator. The approach taken in this research is to extend the concept of frequency-dependent cost functionals to the output feedback problem. Lead-lag performance weightings have been used to manipulate the performance index and provide a method to expand the compensator to enhance the systems closed loop robustness. Results on SISO systems confirm our hypothesis that output feedback dynamic compensators can be designed to meet phase margin requirements of the system without sacrificing control bandwidth constraints. Design examples for MIMO systems are currently underway, and similar results are expected.

This report summarizes the results to date on the SISO case. The contents of this report will be submitted for presentation at the next AIAA Conference on Guidance, Navigation and Control.

1. Introduction

It is well known that Linear Quadratic Regulator (LQR) synthesis methods have guaranteed stability margins. Unfortunately, these properties hold only in case of full state feedback. Observer based compensator design techniques exist to estimate unavailable plant states, and make the LQR design viable. However, this combination of state estimation and regulation may result in a compensator with poor stability margins, even though the separate designs are robust. It has been shown that the robustness properties of the LQR design for nonminimum phase plants can be recovered via an asymptotic method called Loop Transmission Recovery (LTR) [1]. The LTR method relies on a cheap control formulation with a subset of the compensator dynamics becoming fast. In this post analog era, finite word length, sampling rate or time delay may impede the use of high gain controllers when implemented digitally. Several other shortcomings of the LTR method that are related to plant inversion are discussed in [2]. Aside from the robustness issue, the order of the compensator when designed for large scale systems may prove unwarranted.

Optimal output feedback design of fixed order compensators introduced in the early seventies [3] has received limited attention throughout the years. Part of the difficulty rests with the inability to characterize the stability margin properties. Unlike the algebraic Ricatti equation that arises in the case of full state feedback, the necessary conditions that result from the optimal output feedback problem are not conducive to analysis in the frequency domain. The other difficulty in synthesizing optimal dynamic compensators is that the standard approach of adjoining the compensator dynamics to the plant dynamics (and reformulating as a static gain output feedback design) results in an over parameterized formulation. This is a direct consequence of the fact that the compensator lacks a predefined structure, which invariably results in difficulties with convergence to an optimal solution. This is avoided here by representing the compensator in controller canonical form.

This paper presents an LQ optimal design approach for designing fixed order compensators that satisfy frequency domain performance and robustness specifications. Only single-input single-output plants are considered; however, the approach can be extended to the multivariable case. Implicit in the system description is the presence of noise in the measurement, and to avoid control activity generated by this noise direct output feedthrough in the compensator has been eliminated.

The following section discusses the optimal output feedback formulation in controller canonical form. Next, it is shown how the concept of frequency-dependent cost functionals can easily be handled in this canonical setting. This is useful in improving the stability margins of the compensator. Finally, examples are given to elucidate the design technique.

2. Fixed-Order Dynamic Compensation

Consider the linear time invariant single-input single-output system

$$\dot{x} = Ax + \underline{b}u \qquad x(0)=x_o \qquad x \in R^n \tag{1a}$$

$$y = \underline{c}^t x \qquad y \in R^1 \tag{1b}$$

The system (1) is controlled by the fixed-order dynamic compensator without direct feedthrough of the output

$$u = -\underline{h}^t z \qquad u \in R^1 \tag{2a}$$

$$\dot{z} = -Pz - \underline{n}y \qquad z(0)=0 \qquad z \in R^{n_c} \tag{2b}$$

where $0<n_c<n$ is chosen a priori subject to the requirement that there exist a stabilizing compensator for the closed loop system defined by

$$\hat{A} = \begin{bmatrix} A & -\underline{b}\underline{h}^t \\ -\underline{n}\underline{c}^t & -P \end{bmatrix} \tag{3}$$

The dynamic compensator (2) is defined in a controller canonical form:

$$u = -\underline{h}^t z \qquad u \in R^1 \tag{4a}$$

$$\dot{z} = P^o z + \underline{n}u_c \qquad z \in R^{n_c} \tag{4b}$$

$$u_c = -\underline{p}^t z - y \qquad u_c \in R^1 \tag{4c}$$

where

$$P^o = \begin{bmatrix} 0 & 1 & . & . & . & 0 & 0 \\ 0 & 0 & & & & 0 & 0 \\ . & & & & & & \\ . & & & & & & \\ 0 & 0 & & & & 0 & 1 \\ 0 & 0 & . & . & . & 0 & 0 \end{bmatrix} \qquad \underline{n} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{h}^t = [h_1 \; . \; . \; . \; h_{n_c}] \qquad \underline{p}^t = [p_1 \; . \; . \; . \; p_{n_c}] \tag{5}$$

This canonical form reduces the number of free parameters from $n_c^2 + 2n_c$ to

$2n_c$. Furthermore, the companion form of $P = P^o - \underline{n}\underline{p}^t$ provides structure to the compensator and is a natural form for eliminating direct feedthrough of the output. This point will be further explained below.

In the spirit of an output feedback formulation, augment the plant states to include the compensator states, that is, let $\hat{x}^t = [x^t, z^t]$, $\hat{y}^t = [z^t]$ and $\hat{u}^t = [u, u_c]$. The augmented system is then given by

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{u} \tag{6a}$$

$$\hat{y} = \hat{C}\hat{x} \tag{6b}$$

$$\hat{u} = -\hat{G}\hat{y} \qquad \hat{u} \in R^2 \tag{6c}$$

where

$$\hat{A} = \begin{bmatrix} A & 0 \\ -\underline{n}\underline{c}^t & P^o \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} \underline{b} & 0 \\ 0 & \underline{n} \end{bmatrix} \qquad \hat{C} = [\ 0 \quad I_{n_c}]$$

$$\hat{G} = \begin{bmatrix} \underline{h}^t \\ \underline{p}^t \end{bmatrix} \tag{7}$$

Note that the controller form reduces the number of compensator control elements (u_c) from n_c to 1. The fixed order dynamic compensation problem has now been converted to a static output feedback problem where the compensator gains ($\underline{h},\underline{p}$) are given by $\hat{G}$. Notice that this formulation alleviates the need for zeroing those elements associated with the static feedthrough which would arise if the standard output feedback formulation is used.

The compensator transfer function

$$K(s) = \underline{h}^t(sI_n + P)^{-1}\underline{n} = h(s)/p(s)$$
$$= (h_1 + \dots + h_{n_c} s^{n_c - 1})/(p_1 + \dots + p_{n_c} s^{n_c - 1} + s^{n_c}) \quad (8)$$

directly correlates with the elements of $\tilde{G}$ in (7). Note that the controller form is easily relatable to a transfer function description, and as such any parametric optimization of individual compensator gains provides a direct transfer function counterpart. In particular by zeroing the ascending coefficients $p_1, p_2, \dots$ of the monic polynomial p(s) defined in (8), type 1,2,... feedback systems are formed. Likewise the degree of compensator rolloff can be controlled by zeroing the appropriate elements in $\underline{h}^t$.

The resulting composite system can now be cast as a parametric optimization problem. The free parameters $(\underline{h},\underline{p})$ are chosen to minimize the quadratic performance index

$$J(\underline{h},\underline{p}) = E\{\int_0^\infty [\tilde{x}^t Q \tilde{x} + \rho u^2 + \rho_c {u_c}^2]\, dt\} \quad (9)$$

where E denotes expectation over the distribution on $\tilde{x}(0)$, $Q = D^t D$ such that $(\tilde{A},D)$ is detectable, and ρ and ρ_c are positive scalars. The input $u_c = -\underline{p}^t z - y$ drives the compensator, and increasing or decreasing the penalty on u_c correlates directly with decreasing or increasing the compensator bandwidth.

The solution to the LQ optimal output feedback problem depends on the initial distribution of the states. Extending the standard LQ optimal output feedback design to include a compensator requires some knowledge of the compensator states' initial distribution. One approach is to assume that the compensator states are known and at rest, and that $E\{[x_o,z_o][x_o,z_o]^t\}$ = block diag $[X_o, Z_o]$, $Z_o = 0_{n_c}$. In practice however, the initial distribution of the compensator states may be needed for the existence of $(\tilde{C}L\tilde{C}^t)^{-1}$ in the calculation of $\tilde{G}$. This fact is stated in the following thereom, which is proven in the appendix.

Theorem

A sufficient condition for the existence of $(\hat{C}L\hat{C}^t)^{-1}$ with $(P,\underline{n},\underline{h}^t)$ in controller form is that $Z_0 \supset \underline{nn}^t$.

3. Frequency Shaped Compensation

Frequency-dependent cost functionals have been recently introduced as a way of embedding classical design concepts within the context of LQ optimal control [4,5]. These frequency-dependent cost functionals have been developed for the design of a full state controller and its dual, the state estimator. When applied to the design of observer based compensators, the resulting controller is dimensionally larger than the system model due to the introduction of the frequency shaping terms [4]. In this section we formulate the use of frequency-dependent cost functionals in the context of LQ optimal output feedback.

Modifying the performance index in (9) to include input and output frequency shaping, the cost functional becomes

$$J(\underline{p},\underline{h}) = E\{1/2\pi \int_{-\infty}^{\infty} [\tilde{x}^*(j\omega)Q\tilde{x}(j\omega) + |Q_y(j\omega)y(j\omega)|^2 + \rho|R(j\omega)u(j\omega)|^2 + \rho_c|v_c(j\omega)|^2]d\omega\} \quad (10)$$

where * denotes complex conjugate transpose, Q, ρ, and ρ_c are as before, and $Q_y(j\omega)$ and $R(j\omega)$ are frequency shaping terms with the particular lead-lag form:

$$Q_y(j\omega) = (\gamma j\omega + 1)/(\theta j\omega + 1) \quad (11a)$$

$$R(j\omega) = (\beta j\omega + 1)/(\alpha j\omega + 1) \quad (11b)$$

The weighting functions $Q_y(j\omega)$ and $R(j\omega)$ were chosen for their effectiveness in shaping the stability margins in classical designs; however, other rational functions of frequency are possible [4]. Realistically one would select $\beta > \alpha$

> 0 to penalize higher frequency control activity and $\theta > \gamma > 0$ to weight low frequency sensor information. More importantly, the frequency shaping parameters are varied to adjust the loop phase and gain margins while still optimizing the compensator.

The cost functional in (10) can be expressed in the time domain as

$$J(\underline{p},\underline{h}) = \int_0^\infty \{\hat{x}^t Q \hat{x} + y_1^2 + \rho u_1^2 + \rho_c u_c^2\} dt \tag{12}$$

with the system (6,7) augmented to include the state realization of the frequency shaping terms (11):

$$\dot{w}_1 = w_1/\theta + y \qquad w_1 \in R^1 \tag{13a}$$

$$y_1 = w_1(\theta-\gamma)/\theta^2 + y/\theta \qquad y_1 \in R^1 \tag{13b}$$

$$\dot{w}_2 = w_2/\beta + u_1 \qquad w_2 \in R^1 \qquad u_1 \in R^1 \tag{14a}$$

$$u = w_2(\beta-\alpha)/\beta^2 + u_1\alpha/\beta \tag{14b}$$

The dynamic compensator is expanded to include the states w_1 and w_2:

$$u_1 = - g_{11}w_1 - g_{12}w_2 - \underline{h}^t z \tag{15a}$$

$$\dot{z} = P^o z + \underline{n}u_c \tag{15b}$$

$$u_c = - g_{21}w_1 - g_{22}w_2 - \underline{p}^t z - y_1 \tag{15c}$$

Design via Optimal Output Feedback

The optimization problem can be reformulated using the augmented system description (6) where $\hat{x}^t = [x^t, w_1, w_2, z^t]$, $\hat{y}^t = [w_1, w_2, z^t]$, $\hat{u}^t = [u_1, u_c]$ and

$$\tilde{A} = \begin{bmatrix} A & 0 & \underline{b}(\beta-\alpha)/\beta^2 & 0_{n \times n_c} \\ \gamma\,\underline{c}^t/\theta & -1/\theta & 0 & \underline{0}^t \\ 0^t & 0 & -1/\beta & \underline{0}^t \\ -\gamma\underline{n}\underline{c}^t/\theta & \underline{n}(\gamma-\theta)/\theta\gamma & 0 & P^o \end{bmatrix}$$

$$\tilde{B}^t = \begin{bmatrix} \underline{b}^t\alpha/\beta & 0 & 1 & \underline{0}^t \\ \underline{0}^t & 0 & 0 & \underline{n}^t \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} \underline{0}^t & 1 & 0 & \underline{0}^t \\ \underline{0}^t & 0 & 1 & \underline{0}^t \\ 0_{n_c \times n} & \underline{0} & \underline{0} & I_{n_c} \end{bmatrix}$$

$$\tilde{G} = \begin{bmatrix} g_{11} & g_{12} & \underline{h}^t \\ g_{21} & g_{22} & \underline{p}^t \end{bmatrix} \qquad (16)$$

The frequency shaped fixed order compensator design is now in terms of a static gain output feedback design, and the solution $\tilde{G}$ does not require any zeroing of elements. However, as will be shown in the next section, zeroing the g_{11} and g_{22} terms in $\tilde{G}$ gives a classical interpretation of the effect of frequency shaping in the case of output feedback.

Transfer Function Description

The transfer function description of the fixed order dynamic compensator with lead-lag frequency shaping (13-15) is given by

$$u(s)/y(s) = K(s) = (\alpha s + 1)K'_c(s)/(\theta s + 1) \qquad (17)$$

where

$$K'_c(s) = [(\gamma s+1+g_{21}\gamma)K_c(s) - g_{11}\gamma]/[(\beta s+1+g_{12}\beta)-g_{22}\beta K_c(s)] \qquad (18a)$$

$$K_c(s) = \underline{h}^t(sI+P)^{-1}\underline{n} \tag{18b}$$

By zeroing g_{11} and g_{22}, the frequency shaped compensator, $K(s)$, simplifies to

$$K(s) = F_u(s)K_c(s)F_y(s) \tag{19}$$

where

$$F_u(s) = (\alpha s+1)/(\beta s+1+g_{12}\beta) \tag{20a}$$

$$F_y(s) = (\gamma s+1+g_{21}\gamma)/(\theta s+1) \tag{20b}$$

which is just the nominal compensator (with its internal parameters reoptimized) cascaded fore and aft with the lead-lag type first order filters, $F_u(s)$ and $F_y(s)$, respectively.

Both the input and output frequency shaping cost functions (10) were introduced to point out the benefits of each. Notice from (17) that by shaping the output of the system with a lead-lag cost function $Q_y(j\omega)$ in (11a), an pole is placed in the compensator at $-1/\theta$. The location of the zero in the compensator is influenced by the gains g_{21} and g_{11}. Conversely, by shaping the input of the system with a lead-lag cost function $R(j\omega)$ in (11b), a zero is placed in the compensator at $-1/\alpha$. The location of the pole in the compensator is influenced by the gains g_{12} and g_{22}. Hence, because of the commutative property of SISO systems, the only difference between shaping the output or input of the system is that either a pole or zero of the resulting compensator is fixed by the weighting parameters θ or α.

4. Example

Consider the simple 2nd order SISO system with state description:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -3x_1 - 4x_2 + u \\ y &= 2x_1 + x_2 \end{aligned} \tag{21}$$

and loop transfer function:

$$G(s) = (s+2)/[(s+1)(s+3)] \qquad (22)$$

The performance specifications call for 20 dB of gain for frequencies less than 1 rad/s and a cross-over frequency 10 rad/s.

A first order compensator ($n_c=1$) was designed to meet the performance specifications with associated cost penalties

$$Q = \text{block diag}[Q_x, 10^3] \qquad Q_x = \underline{d}\underline{d}^t, \quad \underline{d}^t = 4/5[\sqrt{35},1]10^{-2}$$

$$\rho = 0.0035 \quad \rho_c = 25.0 \qquad (23)$$

and

$$[X_o, Z_o] = \text{diag}[1,1,1] \qquad (24)$$

The example and plant weighting matrix, Q_x, were taken from [6], where an LTR design was illustrated. The algorithm described in [7] was used to solve the optimal output feedback problem. This algorithm has guaranteed convergence properties. It also permits a constraint of the form $\gamma(\hat{G}) = 0$ to be approximated using a penalty function approach. This can be used to zero selected elements in $\hat{G}$, or to place selected poles or zeros of the compensator as will be illustrated below. The above performance weightings resulted in the following compensator design

$$K(s) = 124.9/(s+4.86) \qquad (25)$$

Next, a second order compensator was designed to exhibit type 1 behavior and have a zero at 1 rad/s. This design was carried out by zeroing the p_1 element in $\hat{G}$ of (7) and by constraining h_1 to equal h_2. The cost penalties were the same as above with Q and Z_o expanded to include the extra compensator state

$$Q = \text{block diag}[Q_x, 0, 10^3] \text{ and } [X_o, Z_o] = \text{diag}[1,1,0,1] \qquad (26)$$

The performance index cost increased by 15% and resulted in the compensator

$$K(s) = 129.9(s+1)/s(s+5) \qquad (27)$$

The magnitude and phase responses of the loop transfer function G(s)K(s) for the first order compensator (25) and the type 1 second order compensator (27) are illustrated on a Bode diagram in figure 1.

Lead-Lag Augmentation

The preceeding first order compensator design (25) meets the performance specification defined earlier in this example and compares favorably with the full state feedback controller designed in [6]. The first order compensator results in 35 degrees of phase margin; while, the full state design gives 85 degrees. Manipulating the input and output performance weightings to improve the phase margin of the first order compensator resulted in designs which strayed from the original specifications with no real improvement in the phase margin. The performance index was next augmented to include a lead-lag term at the output

$$Q_y(j\omega) = (0.2j\omega + 1)/(.02j\omega + 1) \quad (28)$$

which from classical compensation techniques should add about 50 degrees of phase margin near 10 rad/s. The compensator was then redesigned with the associated cost penalties

$$Q = \text{block diag}[Q_x, 0, 10^3], \quad \rho = 0.30 \quad \rho_c = 40.0$$

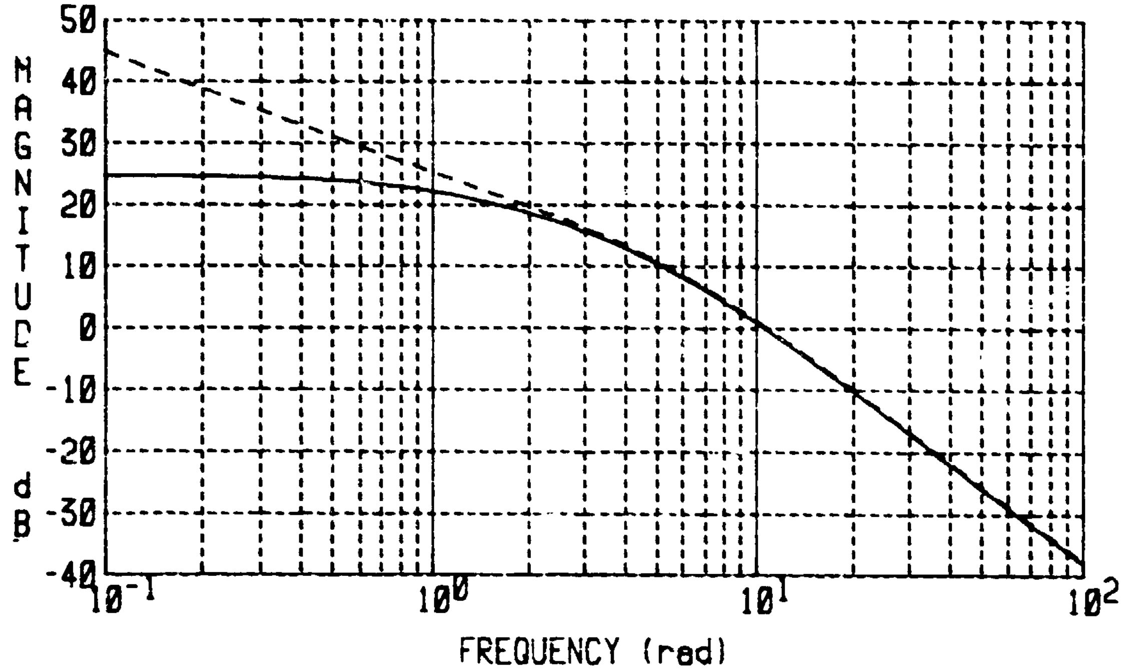
and (29)

$$[X_o, Z_o] = \text{diag}\ [1,1,1,1]$$

Frequency shaping resulted in the compensator transfer function

$$K(s) = 592.4(s+4.28)/(s+2.05)(s+50) \quad (30)$$

The phase margin was increased to 75 degrees. Bode diagrams of the loop transfer function G(s)K(s) for the first order compensator (25) and the second order compensator with frequency shaping (30) are compared in Figure 2.



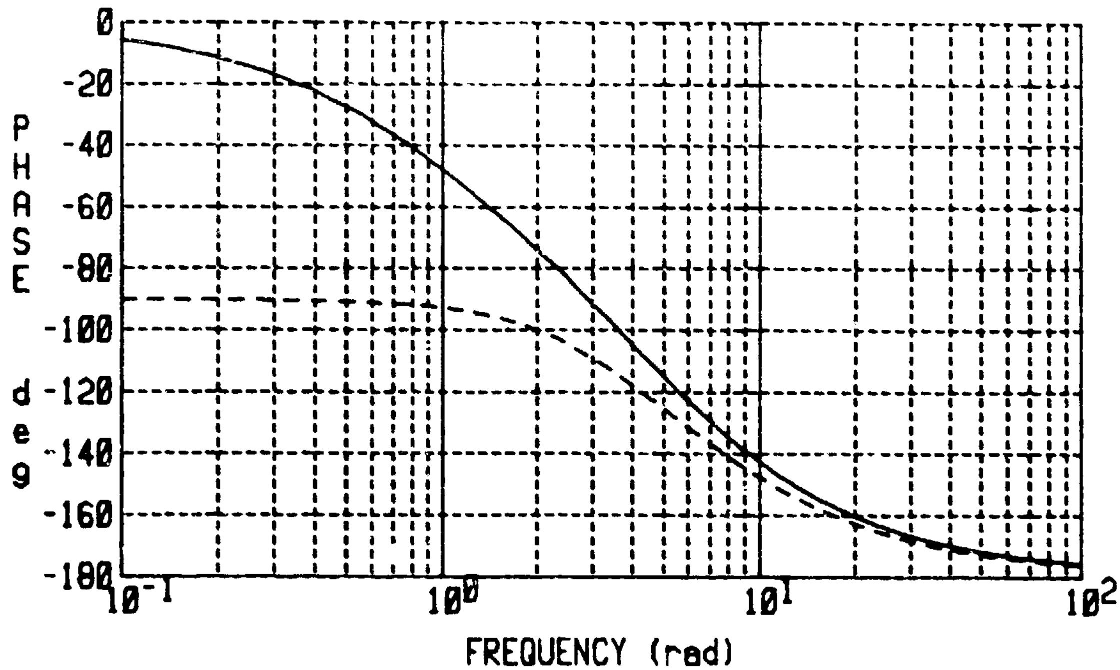


Figure 1. Frequency response of G(s)K(s) for the first order compensator (solid) and the type 1 second order compensator (dashed).

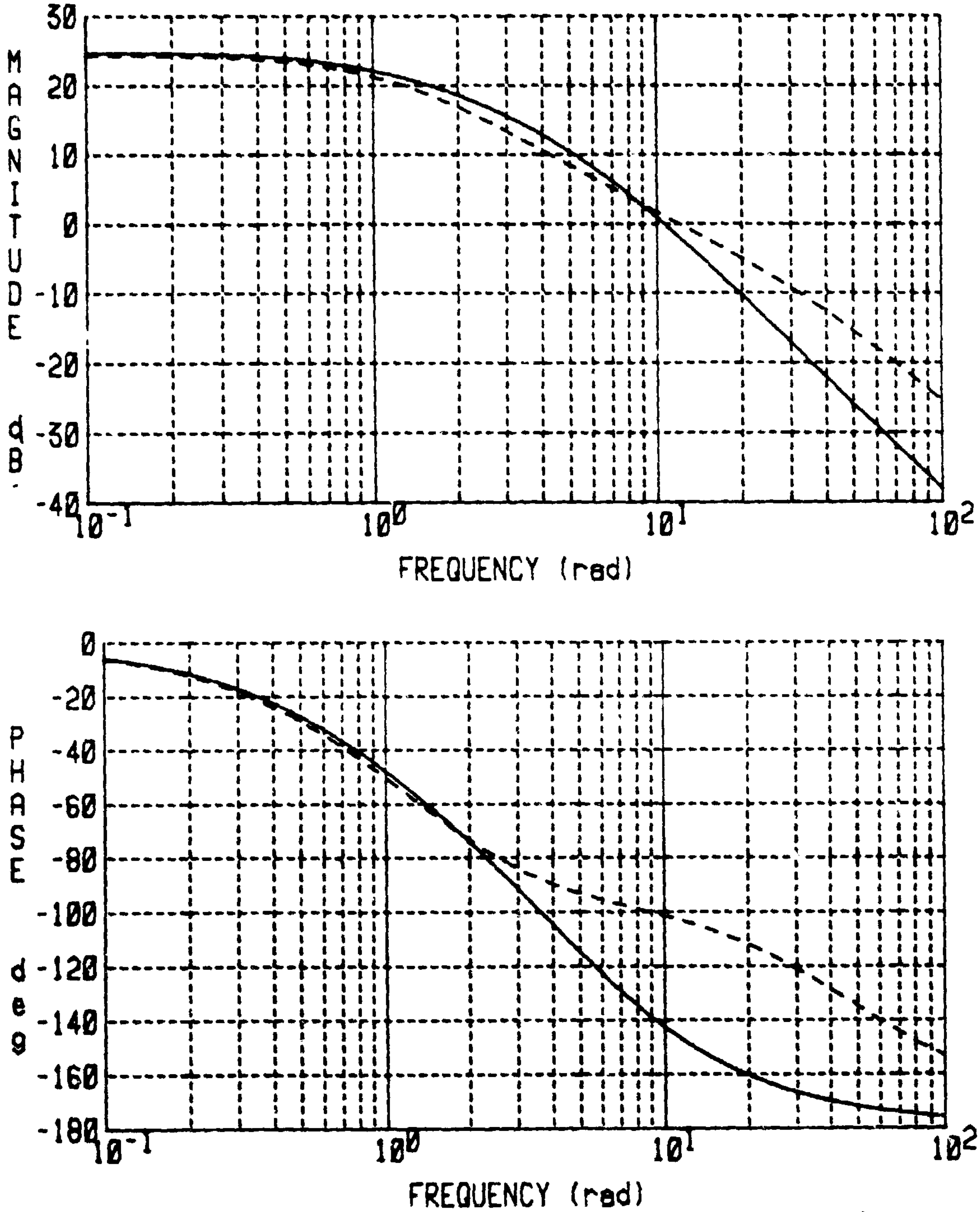


Figure 2. Frequency response of G(s)K(s) for the first order compensator (solid) and the second order compensator with frequency shaping (dashed).

5. Conclusions

A design methodology has been introduced for the LQ synthesis of fixed order dynamic compensators without output feedthrough for SISO systems. A controller canonical structure was imposed on the compensator description which minimized the number of free parameters and correlated with a unique transfer function realization. A frequency-dependent cost functional provided the means to enhance the robustness of the fixed order compensator design to improve the stability margins of the closed loop system. In particular, first order lead-lag type cost functionals were introduced to shape the input and output of the system and were shown to have a classical lead-lag counterpart when used with the dynamic compensator.

Appendix

Proof of Theorem

The necessary conditions satisfying the LQ optimal output feedback problem are given by

$$\hat{A}^t K + K\hat{A} + Q + \tilde{C}^t\tilde{G}^t R\tilde{G}\tilde{C} = 0 \tag{A1}$$

$$\hat{A}L + L\hat{A}^t + XX_o = 0 \tag{A2}$$

$$R\tilde{G}(\tilde{C}L\tilde{C}^t) = \tilde{B}^t KL\tilde{C}^t \tag{A3}$$

where $\hat{A} = \tilde{A} - \tilde{B}\tilde{G}\tilde{C}$ is asymptotically stable, the quadruple $(\tilde{A},\tilde{B},\tilde{C},\tilde{G})$ is defined in (6) and (7), $Q \geq 0$, $R = \text{diag}\,[\rho,\rho_c] > 0$ and $XX_o = \text{block diag}\,[X_o,Z_o] \geq 0$. In order to uniquely determine $\tilde{G}$ in (A3), it is necessary that $(\tilde{C}L\tilde{C}^t)^{-1}$ exist. For the form of $\tilde{C}$ in (7), this inverse can be reduced by first partitioning L into

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{12}^t & L_{22} \end{bmatrix} \tag{A4}$$

and expanding $\tilde{C}L\tilde{C}^t$. Thus

$$\tilde{C}L\tilde{C}^t = L_{22} \tag{A5}$$

and the existence of $(\tilde{C}L\tilde{C})^{-1}$ is solely dependent upon the existence of $(L_{22})^{-1}$.

Expanding equation (A2) results in

$$PL_{22} + L_{22}P^t + L_{12}^t\underline{cn}^t + \underline{nc}^t L_{12} + Z_o = 0 \tag{A6}$$

Referring to [8] and the references therein, the following remarks hold. Equation (A6) can be written as

$$(P + \underline{n}\underline{c}^t L_{12} L_{22}^{+}) L_{22} + L_{22}(P + \underline{n}\underline{c}^t L_{12} L_{22}^{+})^t + Z_o = 0 \qquad \text{(A7)}$$

where + denotes a pseudoinverse. Since $(P,\underline{n})$ is controllable, then so is $(P + \underline{n}\underline{c}^t L_{12} L_{22}^{+},\underline{n})$. Let $Z_o = \underline{n}\underline{n}^t$. Furthermore, since L_{22} is at least non-negative and $(P + \underline{n}\underline{c}^t L_{12} L_{22}^{+},\underline{n})$ is controllable, then $(P + \underline{n}\underline{c}^t L_{12} L_{22}^{+})$ is a stability matrix; and the solution L_{22} to (A7) is positive definite.

We can broaden the class of Z_o to that which makes $(P + \underline{n}\underline{c}^t L_{12} L_{22}^{+},D)$ controllable, where $DD^t = Z_o \supset \underline{n}\underline{n}^t$.

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