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CONVERGENCE TO STEADY STATE OF SOLUTIONS OF BURGERS' EQUATION

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CONVERGENCE TO STEADY STATE OF SOLUTIONS OF BURGERS' EQUATION

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Dedicated to Milton E. Rose
on occasion of his sixtieth birthday

Abstract

Consider the initial-boundary value problem for Burgers' equation. It is shown that its solutions converge, in time, to a unique steady state. The speed of the convergence depends on the boundary conditions and can be exponentially slow. Methods to speed up the rate of convergence are also discussed.

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1. Introduction. In many gasdynamical problems one tries to calculate the steady state solution by solving the corresponding time dependant problem. One hopes that for $t \to \infty$ the solution converges to a unique steady state. Recently, M. D. Salas, S. Abarbanel and D. Gottlieb [1]considered the initial-boundary value problem

$$\begin{aligned} u_t + \frac{1}{2}(u^2)_x &= f(x), \quad t \geq 0, \quad 0 \leq x \leq \pi, \\ u(x,0) &= g(x). \end{aligned} \tag{1.1}$$

They used

$$f(x) = \sin x \cos x, \quad g(x) = b \sin x, \quad 0 < b,$$

and showed that the solution $u(x,t)$ of the above problem converges to a steady state $v(x)$, as $t \to \infty$, but that $v(x)$ depends on the initial data.

In this paper we consider the viscous problem

$$u_t + \frac{1}{2}(u^2)_x = \varepsilon u_{xx} + f(x), \quad t \geq 0, \quad 0 \leq x \leq 1, \quad \varepsilon > 0, \tag{1.2a}$$

with initial and boundary conditions

$$\begin{aligned} u(x,0) &= g(x), \\ u(0,t) = a, \quad u(1,t) &= b, \end{aligned} \tag{1.2b}$$

and the corresponding steady state problem

$$\begin{aligned} \frac{1}{2}(y^2)_x &= \varepsilon y_{xx} + f(x), \quad 0 \leq x \leq 1, \quad \varepsilon > 0, \\ y(0) = a, \quad y(1) &= b. \end{aligned} \tag{1.3}$$

For simplicity we restrict ourselves to two cases:

1) $a > 0 \geq b$, $a \geq -b$, $f(x) \equiv 0$,
2) $a = b = 0$, f is such that there exists an α with $0 < \alpha < 1$ such that $f(x) > 0$ for $0 < x < \alpha$, $f(x) < 0$ for $\alpha < x < 1$, $f(0) = f(1) = 0$, $f_x(0) \geq f_0 > 0$ and $f_x(1) \geq f_0$.

We will show that (1.3) has a unique solution and discuss the properties of $y(x)$. We shall also show that in all cases we consider, the limit of $y(x)$ as $\varepsilon \to 0$ exists. Thus, if

$$\lim_{t \to \infty} u(x,t) = y(x)$$

exists, we obtain a unique steady state solution of the inviscid equation (1.1) if we first let $t \to \infty$ and then $\varepsilon \to 0$. This is in contrast to the procedure in [1], where the two limit procedures are taken in the reverse order.

We shall prove that the eigenvalues of the eigenvalue problem

$$\lambda\varphi = -(y\varphi)_x + \varepsilon\varphi_{xx}, \quad \varphi(0) = \varphi(1) = 0, \tag{1.4}$$

are all negative. Therefore, the solution of (1.2) converges to the solution of (1.3) provided $u(x,0) = g(x)$ is sufficiently close to $y(x)$. In another paper we shall prove that $u(x,t)$ converges to $y(x)$ as $t \to \infty$ for arbitrary initial data. The speed of convergence is determined by the eigenvalues, λ_j, of (1.4). We shall show that the eigenvalue distribution depends on $f(x)$ and on a, b in the following way:

There is a constant $c > 0$ which does not depend on ε such that

$$\begin{aligned}
&(1) \quad \text{if} \quad a > -b,\ f \equiv 0 \quad \text{then} \quad 0 > -c/\varepsilon > \lambda_1 > \lambda_2 > \cdots \\
&(2) \quad \text{if} \quad a = -b,\ f \equiv 0 \quad \text{then} \quad -\lambda_1 = O(e^{-1/\varepsilon}) > 0,\ -c/\varepsilon > \lambda_2 > \lambda_3 > \cdots \\
&(3) \quad \text{if} \quad a = b = 0,\ \int_0^1 f(x)dx \neq 0, \quad \text{then} \quad -c > \lambda_1 > \lambda_2 > \cdots \\
&(4) \quad \text{if} \quad a = b = 0,\ \int_0^1 f(x)dx = 0, \quad \text{then} \quad -\lambda_1 = O(e^{-1/\varepsilon}) > 0,\ -c > \lambda_2 > \lambda_3 > \cdots
\end{aligned} \tag{1.5}$$

We expect a reasonable speed of convergence in the first and third case, while in the second and fourth case the speed should be extremely slow due to the eigenvalue $-\lambda_1 = O(e^{-1/\varepsilon})$. This is confirmed by numerical experiments. We see that at first $u(x,t)$ quite rapidly approches the same limit as the inviscid equation (1.1), which consists of solutions of the stationary equation

$$\frac{1}{2}(u^2)_x = f(x)$$

connected by a shock. Once the viscous shock has been formed, the solution of (1.2) becomes quasi-stationary and the shock creeps extremely slowly to the "right" position. We can explain the behavior, because by linearizing around the quasistationary solution we find that the eigenvalues of the corresponding eigenvalue problem have a similar distribution as earlier.

If $-\lambda_1 = O(e^{-1/\varepsilon})$ then the speed of convergence is so slow that the above method to calculate the steady state is impractical, see figures (1) and (3). However, we can use the same technique as Hafez, Parlette and Salas in [2] to speed up the convergence. See figures (2) and (4).

Unfortunatly, not only the speed of convergence but also the condition number of the stationary problem deteriorates. We have to calculate with $O(e^{1/\varepsilon})$ decimals to obtain correct

results. To avoid an excessive number of decimals we have used a quite large ε in our numerical calculations.

The situation becomes much better in a two dimensional case, which we discuss in the last section. Now there is a whole sequence of eigenvalues

$$-\mu_{1j} = O(j^2\varepsilon), \qquad j = 1, 2, \ldots,$$

close to zero. However, they are only algebraically and not exponentially close to zero. We indicate how to modify the procedure to accellerate the speed of convergence.

We believe that the viscous model (1.2) better explains what happens in actual calculations than the inviscid equation (1.1). Practically all numerical methods have some viscosity built in. Also, from a physical point of view, the solution we are interested in is the limit of solutions of a viscous equation.

Finally we want to point out that the appearance of small eigenvalues has also been observed by D. Brown, W. Kath, H. O. Kreiss and W. Henshaw, M. Naughton (private communication).

2. Uniqueness,existence and properties of the steady state solution. We start with uniqueness, which can be proven by standard techniques.

Lemma 2.1. If the steady equation (1.3) has a solution, then it is unique.

Proof. Let u, v be two solutions. Then $w = u - v$ is the solution of

$$\frac{1}{2}(pw)_x = \varepsilon w_{xx}, \quad p = u + v, \quad w(0) = w(1) = 0. \tag{2.1}$$

If $w \not\equiv 0$ then the zeros of w are isolated. Let $\overline{x}$ with $0 < \overline{x} \leq 1$ be the first zero to the right of $x = 0$. Without restriction we can assume that $w > 0$ for $0 < x < \overline{x}$, i.e. $w_x(0) \geq 0$ and $w_x(\overline{x}) \leq 0$. Integration of (2.1) gives us

$$-\varepsilon(|w_x(\overline{x})| + |w_x(0)|) = \varepsilon[w_x]_0^{\overline{x}} = \frac{1}{2}[pw]_0^{\overline{x}} = 0.$$

Thus $w_x(0) = w_x(\overline{x}) = 0$. We can consider (2.1) as an initial value problem with initial data $w(0) = w_x(0) = 0$ whose solution is $w(x) \equiv 0$, and the lemma is proved.

We shall now discuss the properties of the solution. Let us start with the case $f(x) \equiv 0$, $a > 0 \geq b$, $a > -b$. Integrating (1.3) gives us

$$\begin{aligned} &\varepsilon y_x = \frac{1}{2}y^2 - c, \quad 0 \leq x \leq 1, \\ &y(0) = a. \end{aligned} \tag{2.2}$$

The constant c has to be determined so that $y(1) = b$. We necessarily have $c = d^2/2 > a^2/2$, because with $c \le a^2/2$, $y_x \ge 0$ for all x, and $y(1) = b$ cannot be satisfied. We can solve equation (2.2) explicitly. This is done by writing (2.2) in the form

$$2\varepsilon \int_a^{y(x)} \frac{d\tilde{y}}{\tilde{y}^2 - d^2} = \int_0^x d\tilde{x},$$

i.e.

$$(\frac{a+d}{a-d})(\frac{y(x)-d}{y(x)+d}) = e^{dx/\varepsilon}.$$

Therefore $y(1) = b$ implies $d = a + O(e^{-1/\varepsilon})$, and

$$y(x) = a\frac{1 - \tau e^{-a(1-x)/\varepsilon}}{1 + \tau e^{-a(1-x)/\varepsilon}}, \quad \text{with} \quad \tau = \frac{a-b}{a+b}. \tag{2.3}$$

Away from the boundary layer at x=1 we have $y(x) = a + O(e^{-a(1-x)/\varepsilon})$. Thus, for $\varepsilon \to 0$, $y(x)$ converges to a for $0 \le x < 1$.

If $a = -b$ we consider (2.2) on the interval $0 \le x \le \frac{1}{2}$, with boundary conditions $y(0) = a$, $y(\frac{1}{2}) = 0$ and obtain a solution $y_1(x)$ of the form (2.3). The solution on the whole interval is given by

$$y(x) = \begin{cases} y_1(x), & \text{if } 0 \le x \le \frac{1}{2}, \\ -y_1(1-x), & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

In figures (9) and (10) we have plotted $y(x)$ for two different sets of boundary values.

Consider case 2, where f only vanishes at $x = 0, \alpha, 1$ and $a = b = 0$. Without restrictions we can assume that

$$\int_0^1 f(x) \ge 0. \tag{2.4}$$

If this is not true, we transform the problem by introducing new variables,

$$\tilde{x} = 1 - x, \quad \tilde{f} = -f, \quad \tilde{y} = -y.$$

The new problem satisfies (2.4).

Lemma 2.2. Let $y(x)$ be the solution of (1.3), $F(x) = \int_0^x f(\xi)d\xi$ and $h(x) = \sqrt{2F(x)}$. Then

$$y_x(1) \le y_x(0) \le K_1, \quad K_1 = \max_{0 \le x \le 1} \{|h_x(x)|\} + |h_x(0)|.$$

Proof. Integration of (1.3) gives

$$\begin{aligned} \varepsilon(y_x - y_x(0)) &= \frac{1}{2}y^2 - F, \\ y(0) &= 0, \end{aligned} \tag{2.5}$$

where $y_x(0)$ is determined by $y(1) = 0$. If $u = y - h$, then u is the solution of

$$\begin{aligned} u_x &= y_x(0) - h_x + \varepsilon^{-1}uh + \frac{1}{2}\varepsilon^{-1}u^2, \\ u(0) &= 0. \end{aligned}$$

Assume that $y_x(0) > K_1$. It follows that $y_x(0) - h_x(x)$ is positive and thus u and u_x are positive for all $x > 0$. In particular $u(1) > 0$ and $y(1) = u(1) + h(1) > 0$, which contradicts $y(1) = 0$. Thus $y_x(0) \leq K_1$. Also

$$\varepsilon y_x(1) = \varepsilon y_x(0) - F(1) \leq \varepsilon y_x(0).$$

This proves the lemma.

Lemma 2.3. Let $y(x)$ be the solution of (1.3) and let ε be sufficiently small. If $F(1) > 0$ then $y(x) > 0$ for $0 < x < 1$ and $y(x)$ has exactly one maximum. If $F(1) = 0$ then there exists an $\overline{x}$ with $0 < \overline{x} < 1$ such that $y(x) > 0$ for $0 < x < \overline{x}$, and $y(x) < 0$ for $\overline{x} < x < 1$. Also $y(x)$ has exactly one minimum and one maximum. In both cases $|y(x)| \leq \max|F(x)|$.

Proof. At extrema $y_x = 0$ and

$$y_{xx} = -\varepsilon^{-1}f = \begin{cases} < 0 & \text{for } 0 < x < \alpha, \\ = 0 & \text{for } x = \alpha, \\ > 0 & \text{for } \alpha < x < 1. \end{cases} \tag{2.6}$$

Thus y cannot have a minimum to the left of a maximum. Since $y(0) = y(1) = 0$ there are only three possibilities, namely

$$y > 0 \quad \text{for} \quad 0 < x < 1, \quad \text{y has exactly one maximum,} \tag{2.7a}$$

$$y < 0 \quad \text{for} \quad 0 < x < 1, \quad \text{y has exactly one minimum,} \tag{2.7b}$$

$$\begin{aligned} &y > 0 \quad \text{for} \quad 0 < x < \overline{x}, \qquad 0 < \overline{x} < 1, \\ &y < 0 \quad \text{for} \quad \overline{x} < x < 1, \quad \text{y has exactly one maximum and one minimum.} \end{aligned} \tag{2.7c}$$

We shall prove that if $F(1) > 0$ then (b) and (c) are not possible, and that if $F(1) = 0$ then (a) and (b) are not possible.

Let $F(1) > 0$. Suppose (2.7b) holds. Then

$$y_x(0) \leq 0, \quad y_x(1) \geq 0.$$

By (2.5)

$$0 \leq \varepsilon(y_x(1) - y_x(0)) = -F(1) < 0. \tag{2.8}$$

This is a contradiction, so (2.7b) cannot hold. Now supposse (2.7c) is valid. Then $y_x(0) \geq 0$ and by (2.8)

$$y_x(0) \geq \varepsilon^{-1} F(1).$$

If ε is small enough this is impossible by lemma 2.2.

Let $F(1) = 0$. Assume that (2.7a) or (2.7b) are valid. By (2.8) $y_x(0) = y_x(1)$, which is only possible if $y_x(0) = y_x(1) = 0$. Differentiating (1.3) gives us

$$\varepsilon y_{xxx} = y y_{xx} + (y_x)^2 - f_x. \tag{2.9}$$

Thus

$$\begin{aligned} y(0) = y_x(0) = y_{xx}(0) = 0, \quad y_{xxx}(0) < 0, \\ y(1) = y_x(1) = y_{xx}(1) = 0, \quad y_{xxx}(1) < 0. \end{aligned}$$

This implies that y must change sign at least once, which contradicts the assumption, and therefore (2.7c) must hold.

It remains to show that $|y(x)|$ is bounded by $\max|F(x)|$. Since $y(0) = y(1) = 0$, the maximum absolute value of y is found at a local extrema, where $y_x = 0$. Thus, from (2.5) it follows that

$$|y(x)| \leq \max_{0 \leq x \leq 1} |F(x) - \varepsilon y_x(0)| \leq \max_{0 \leq x \leq 1} |F(x)|.$$

This finishes the proof.

We can use the usual singular perturbation methods to discuss the behavior of the solution in detail, see for ex. [3].

Theorem 2.1. Let $F(1) > 0$, assume that (1.3) has a solution and that ε is sufficiently small. Then $y(x)$ has a boundary layer at $x = 1$. For $1 - O(\varepsilon|\log(\varepsilon)|) \leq x \leq 1$, $y(x)$ is close to $w(x)$ which is the solution of

$$\varepsilon w_x = \frac{1}{2} w^2 - F(1), \quad -\infty < x \leq 1, \quad w(1) = 0. \tag{2.9}$$

In any interval $0 < x_0 \leq x \leq 1 - O(\varepsilon|\log(\varepsilon)|)$

$$y(x) = h(x) + \varepsilon u_1(x, \varepsilon), \quad h(x) = \sqrt{2F(x)} =: x g(x), \tag{2.10}$$

where u_1 and its derivatives are bounded independantly of ε. For $0 \le x \le x_0 < \alpha$ we have

$$y(x) = h(x) + \varepsilon u(\tilde{x}), \quad \tilde{x} = x/\sqrt{\varepsilon}, \tag{2.11}$$

where u and the derivatives $d^\nu u/d\tilde{x}^\nu$ are bounded independantly of ε. Thus, for $\varepsilon \to 0$, $y(x)$ converges to $h(x)$ for $0 \le x < 1$.

Proof. We indicate only the proof of (2.11). In the proof we shall use I_1, I_2 and I to denote the intervals $0 \le \tilde{x} \le 1$, $1 \le \tilde{x} \le x_0/\sqrt{\varepsilon}$ and $0 \le \tilde{x} \le x_0/\sqrt{\varepsilon}$, respectively. We shall also use

$$\|f\|_I := \max_{\tilde{x} \in I} |f(\tilde{x})|,$$

where I is an interval.

We introduce a new variable in (1.3),

$$y(x) = h(x) + \varepsilon u(x/\sqrt{\varepsilon}).$$

This gives us

$$u_{\tilde{x}\tilde{x}} - (\tilde{x}g(x) + \sqrt{\varepsilon}u)u_{\tilde{x}} - h_x u = -h_{xx}, \quad 0 \le \tilde{x} \le x_0/\sqrt{\varepsilon}, \quad u(0) = 0, \quad u(x_0/\sqrt{\varepsilon}) = u_0, \tag{2.12}$$

where $u_0 = u_1(x_0, \varepsilon)$ is bounded independantly of ε. From $x_0 < \alpha$ and the assumtion $f_x(0) \ge f_0 > 0$ it follows that $h_x(x) \ge h_0 > 0$ for $0 \le x \le x_0$. Therefore we can use the maximum principle. The maximum of u is found either on the boundary or at a local extrema, where $u_{\tilde{x}} = 0$. At local extrema

$$|u| \le |\frac{h_{xx}}{h_x}| \le \frac{1}{h_0}\|h_{xx}(x)\|_I =: \alpha.$$

Thus

$$\|u\|_I \le \max(u_0, \alpha). \tag{2.13}$$

Next we want to estimate $\|u_{\tilde{x}}\|_I$. First we consider the interval $I_1 = [0, 1]$. By (2.12) and (2.13) there are constants C_1 and C_2 such that

$$\|u_{\tilde{x}\tilde{x}}\|_{I_1} \le C_1\|u_{\tilde{x}}\|_{I_1} + C_2.$$

It is well known, see Landau [4], that one can estimate $\|u_{\tilde{x}}\|_{I_1}$ in terms of $\|u\|_{I_1}$, and $\|u_{\tilde{x}\tilde{x}}\|_{I_1}$, i.e. for every constant δ there is a constant $C(\delta)$ such that

$$\|u_{\tilde{x}}\|_{I_1} \le \delta\|u_{\tilde{x}\tilde{x}}\|_{I_1} + C(\delta)\|u\|_{I_1}.$$

Thus for $\delta = \frac{1}{2}(C_1)^{-1}$ we obtain a bound for $\|u_{\tilde{x}\tilde{x}}\|_{I_1}$, which gives us a bound for $\|u_{\tilde{x}}\|_{I_1}$. Especially, $|u_{\tilde{x}}(1)|$ is bounded.

In the remaining interval $I_2 = [1, x_0/\sqrt{\varepsilon}]$, we have

$$F \ge F(\sqrt{\varepsilon}) = \varepsilon f_x(0)(1 + O(\sqrt{\varepsilon})).$$

Thus

$$\tilde{x}g + \sqrt{\varepsilon}u = \sqrt{2F}/\sqrt{\varepsilon} + \sqrt{\varepsilon} \ge \sqrt{f_x(0)} + O(\sqrt{\varepsilon}),$$

i.e. for sufficiently small $\sqrt{\varepsilon}$

$$\tilde{x}g + \sqrt{\varepsilon}u \geq \frac{1}{2}\sqrt{f_x(0)}\,.$$

At local extrema of $u_{\tilde{x}}$, $u_{\tilde{x}\tilde{x}} = 0$ and we have, by (2.12),

$$|u_{\tilde{x}}| \leq \frac{2}{\sqrt{f_x(0)}}|h_{xx} - h_x u| \leq \frac{2}{\sqrt{f_x(0)}}(\|h_{xx}\|_{I_2} + \|h_x\|_{I_2}\|u\|_{I_2}) =: \beta.$$

Thus

$$\|u_{\tilde{x}}\|_{I_2} \leq \max(|u_{\tilde{x}}(1)|, |u_{\tilde{x}}(\frac{x_0}{\sqrt{\varepsilon}})|, \beta),$$

and $u_{\tilde{x}}$ is bounded independantly of ε in the whole interval. By differentiating (2.12) bounds for higher derivatives of u can be obtained.

It is also clear that as $\varepsilon \to 0$, $y(x)$ converges to $h(x)$. This finishes the proof.

If $F(1) = 0$ then the solution switches at $\overline{x}$ from $\sqrt{2F} + O(\varepsilon)$ to $-\sqrt{2F} + O(\varepsilon)$. In each subinterval $0 \leq x < \overline{x}$ and $\overline{x} \leq x \leq 1$ the local behavior of the solution is of the same type as in the first case. As $\varepsilon \to 0$, $y(x)$ converges to $h(x)$ for $0 \leq x < \overline{x}$ and to $-h(x)$ for $\overline{x} < x \leq 1$. In general, the position of $\overline{x}$ can only be obtained by detailed calculation. However, if $f(x)$ is antisymmetric around $x = \frac{1}{2}$ then $\overline{x} = \frac{1}{2}$. This is the only case we consider.

We shall now discuss the existence of a solution. For this we need two lemmata.

Lemma 2.4. For sufficiently large ε the steady state equation (1.2) has a solution.

Proof. By integrating (1.3) twice, we can write the equation in the form

$$y(x) = \frac{1}{2}\eta \int_0^x y^2(\xi)d\xi - \eta \int_0^x F(\xi)d\xi + \eta x c_0, \quad \eta = 1/\varepsilon,$$

$$\frac{1}{2}\int_0^1 y^2(\xi)d\xi - \int_0^1 F(\xi)d\xi + c_0 = 0,$$

or after the change of variable $y = \eta\tilde{y}$

$$\tilde{y}(x) = \frac{1}{2}\eta^2 \int_0^x \tilde{y}^2(\xi)d\xi - \int_0^x F(\xi)d\xi + x c_0,$$

$$\frac{1}{2}\eta^2 \int_0^1 \tilde{y}^2(\xi)d\xi - \int_0^1 F(\xi)d\xi + c_0 = 0.$$

For $\eta = 0$ the above equations have a unique solution. Therefore the same is true for all sufficiently small η. This proves the lemma.

Lemma 2.5. Let $p(x)$ be a smooth function. Consider the eigenvalue problem

$$\lambda\varphi = -(p\varphi)_x + \varepsilon\varphi_{xx}, \quad \varphi(0) = \varphi(1) = 0. \tag{2.14}$$

The eigenvalues are real and negative.

Proof. We introduce a new variable $\psi(x)$ by

$$\varphi(x) = e^{\frac{1}{2}\varepsilon^{-1}\int_1^x p(\xi)d\xi}\psi(x),$$

and obtain

$$\begin{gathered}\lambda\psi = \varepsilon\psi_{xx} - c\psi =: L\psi, \quad c(x) = \frac{1}{2}p_x(x) + \frac{1}{4\varepsilon}(p(x))^2, \\ \psi(0) = \psi(1) = 0.\end{gathered} \tag{2.15}$$

(2.15) is selfadjoint and therefore the eigenvalues are real. Let $\varphi \not\equiv 0$, λ be a solution of (2.14), and let $\tilde{x}$ be the first zero of φ to the right of $x = 0$. We can assume that $\varphi > 0$ for $0 < x < \tilde{x}$. Thus $\varphi_x(0) \geq 0$ and $\varphi_x(\tilde{x}) \leq 0$, and integration of (2.14) gives us

$$\lambda\int_0^{\tilde{x}} \varphi(x)dx = \varepsilon[\varphi_x]_0^{\tilde{x}} \leq 0.$$

It follows that $\lambda \leq 0$. If $\lambda = 0$, the only possible solution of (2.14) would be $\varphi(x) \equiv 0$. Thus $\lambda < 0$, which proves the lemma.

Now we can prove

Theorem 2.2. The equation (1.3) has a unique solution for all $\varepsilon > 0$.

Proof. We have already shown that (1.3) has a solution for sufficiently large ε. We will now employ continuation in ε to prove existance for all $\varepsilon > 0$. Assume we have shown existance for $\varepsilon > \bar{\varepsilon}$. We want to show that there is a solution for $\varepsilon = \bar{\varepsilon}$. By lemma 2.3 the solutions of (1.3) are uniformly bounded for $\bar{\varepsilon} < \varepsilon \leq \bar{\varepsilon} + 1$. Therefore the same is true for the first three derivatives. Thus we can select a sequence of solutions

$$y(x, \varepsilon_\nu), \quad \nu = 1, 2, \ldots, \quad \lim_{\nu\to\infty} \varepsilon_\nu = \bar{\varepsilon},$$

such that

$$\lim_{\nu\to\infty} \frac{d^j}{dx^j} y(x, \varepsilon_\nu) = \frac{d^j}{dx^j} y(x, \bar{\varepsilon}), \quad j = 0, 1, 2 \quad ,$$

and $y(x,\bar{\varepsilon})$ is the desired solution.Linearizing the equation around $y(x,\bar{\varepsilon})$ gives us

$$(y(x,\bar{\varepsilon})\delta y)_x = \varepsilon(\delta y)_{xx} + (\varepsilon - \bar{\varepsilon})y(x,\bar{\varepsilon}), \quad \delta y(0) = \delta y(1) = 0.$$

By the previous lemma $\lambda = 0$ is not an eigenvalue of the above equation and therefore we can solve (1.3) for all sufficiently small $\varepsilon - \bar{\varepsilon}$. This proves the theorem.

3. Speed of convergence. In this section we want to discuss the speed of convergence to steady state. We assume that the initial data $g(x)$ of (1.3) are sufficiently close to the solution of the steady problem, so that we only have to discuss the behavior of the solutions of the linearized equation

$$\begin{aligned} &w_t + (yw)_x = \varepsilon w_{xx}, \quad 0 \le x \le 1, \quad t \ge 0, \\ &w(x,0) = \tilde{g}(x), \\ &w(0,t) = w(1,t) = 0. \end{aligned} \tag{3.1}$$

To determine the speed of convergence we study the distribution of eigenvalues of

$$\lambda\varphi + (y\varphi)_x = \varepsilon\varphi_{xx}, \quad \varphi(0) = \varphi(1) = 0. \tag{3.2}$$

Theorem 3.1 . The eigenvalues of (3.2) are real and negative and their distribution is given by (1.5).

Proof . Lemma 2.5 tells us that the eigenvalues are real and negative. First we consider the case $f \equiv 0, \quad a > -b$. We write (3.2) in the selfadjoint form (2.15) with $p = y$. Let $\lambda = \lambda_1$ be the largest eigenvalue. The corresponding eigenfunction ψ_1 does not change sign, and we can assume that $\psi_1 > 0$ for $0 < x < 1$ and that $\max|\psi_1(x)| = 1$. We assume that $\lambda_1 > -a^2/8\varepsilon$. Then there is a constant K such that $c(x) + \lambda_1 > 0$ for $0 \ge x \ge 1 - K\varepsilon$. Thus ψ_1 is monotone in the interval $0 \le x \le 1 - K\varepsilon$, and therefore ψ_1 must have its maximum in the remaining interval, $1 - K\varepsilon \le x < 1$. By assumption $\max\psi_1(x) = 1$ and therefore there must be a constant $\delta > 0$ such that $\psi_{1x}(1) \le -\delta/\varepsilon$. Now consider the corresponding eigenfunction

$$\varphi_1(x) = e^{\frac{1}{2}\varepsilon^{-1}\int_1^x y(\xi)d\xi}\psi_1(x), \quad \varphi_{1x}(1) = \psi_{1x}(1), \quad 0 \le \varphi_1(x) \le \psi_1(x).$$

Integrating (3.2) gives us

$$-\delta \ge \varepsilon(\varphi_{1x}(1) - \varphi_{1x}(0)) = \lambda_1 \int_0^1 \varphi_1 dx \ge$$

$$\ge \lambda_1 \int_0^1 e^{\frac{1}{2}\varepsilon^{-1}\int_1^x y d\xi} dx = \lambda_1 \varepsilon d.$$

Thus

$$\lambda_1 < -\min(\frac{a^2}{8\varepsilon}, \frac{\delta}{d\varepsilon}),$$

and the theorem is proven for this case.

When $f \not\equiv 0$, $a = b = 0$, and $\int_0^1 f(x)dx > 0$ the corresponding estimate follows in the same way, since by theorem 2.1 there are constants $C_0 > 0$ and K such that

$$c(x) = \frac{1}{2}(h_x(x) + O(\sqrt{\varepsilon}) + \frac{1}{4}\varepsilon^{-1}(h(x) + O(\sqrt{\varepsilon}))^2) \geq C_0 > 0 \quad \text{for} \quad 0 \leq x \leq 1 - K\varepsilon.$$

We now consider the antisymmetric case when $a = -b$, $f \equiv 0$ or $a = b = 0$ and $f(x)$ is antisymmetric around $x = \frac{1}{2}$. We want to show that

$$-\lambda_1 = O(\varepsilon^{-1}e^{-1/\varepsilon}).$$

We shall use the fact that for our selfadjoint eigenvalue problem (2.15) the eigenvalue with the smallest absolute value, λ_1, satisfies

$$|\lambda_1| \leq \frac{\|L\phi\|_2}{\|\phi\|_2},$$

for any smooth function $\phi \not\equiv 0$ satisfying the boundary conditions. We chose

$$\phi(x) = e^{\frac{1}{2}\varepsilon^{-1}\int_{1/2}^{x} y(\xi)d\xi} - e^{-\frac{1}{2}\varepsilon^{-1}\int_0^{1/2} y(\xi)d\xi}$$

as trial function. $y(x)$ is antisymmetric around $x = \frac{1}{2}$, and $\phi(0) = \phi(1) = 0$. Also

$$L\phi = (\frac{y^2}{4\varepsilon} + \frac{y_x}{2})e^{-\frac{1}{2}\varepsilon^{-1}\int_0^{1/2} y(\xi)d\xi}.$$

Both ϕ^2 and $(\frac{1}{4}\varepsilon^{-1}y^2 + \frac{1}{2}y_x)^2$ are symmetric around $x = \frac{1}{2}$. Therefore

$$\|L\phi\|^2 = 2\int_0^{1/2} (\frac{y^2}{4\varepsilon} + \frac{y_x}{2})^2 e^{-\varepsilon^{-1}\int_0^{1/2} y d\xi} dx,$$

$$\|\phi\|^2 = 2\int_0^{1/2} e^{-\varepsilon^{-1}\int_0^{1/2} y d\xi}(e^{\frac{1}{2}\varepsilon^{-1}\int_0^x y d\xi} - 1)^2 dx,$$

and by (2.5) and theorem 2.1

$$\lambda_1^2 \le \frac{\|L\phi\|^2}{\|\phi\|^2} = \frac{\int_0^{1/2} (\frac{y^2}{4\varepsilon} + \frac{y_x}{2})^2 dx}{\int_0^{1/2} (e^{\frac{1}{2}\varepsilon^{-1}\int_0^x y d\xi} - 1)^2 dx} \le C^2 \varepsilon^{-2} e^{-2D/\varepsilon},$$

where $C > 0$, $D > 0$ are constants which do not depend on ε.

We shall now estimate the size of the second eigenvalue for the case with an interior boundary layer at $x = \frac{1}{2}$. By assumption $y(x)$ is antisymmetric around $x = \frac{1}{2}$. Consider the eigenvalue problem (3.2) on half the interval, $0 \le x \le \frac{1}{2}$, and denote its solutions by

$$\tilde{\varphi}_i(x), \quad \tilde{\lambda}_i, \quad i = 1, 2 \ldots .$$

We know that $\tilde{\varphi}_i$ has $i - 1$ sign changes, and we have already shown how the $\tilde{\lambda}_i$'s are bounded away from zero. The function

$$\varphi_{2i}(x) = \begin{cases} \tilde{\varphi}_i(x) & \text{for } 0 \le x \le \frac{1}{2} \\ -\tilde{\varphi}_i(x - \frac{1}{2}) & \text{for } \frac{1}{2} < x \le 1 \end{cases}, \quad i = 1, 2 \ldots ,$$

will satisfy (3.2) on the full interval, $0 \le x \le 1$ with $\lambda = \lambda_{2i} = \tilde{\lambda}_i$. Also φ_{2i} changes sign $2(i-1)+1$ times. Thus φ_{2i} is the $2i^{th}$ eigenfunction and λ_{2i} is the $2i^{th}$ eigenvalue. Therefore λ_2 is bounded away from zero. This finishes the proof.

4. Numerical results. We shall discuss difference approximations for the time dependant problem (1.2) and the eigenvalue problem (3.2). We introduce gridpoints

$$(x_i = ih, t_j = jk), \quad i = 0, 1, \ldots \quad j = 0, 1, \ldots, N, \quad h = \frac{1}{N},$$

where N is a natural number and $k > 0$ is the time step. We also introduce gridfunctions

$$v_i^j = v(x_i, t_j).$$

We approximate (1.2) by the usual implicit method

$$(I - \varepsilon k D_+ D_-) v_i^{j+1} + \frac{1}{2} k D_0 (v_i^{j+1})^2 = v_i^j + k f_i, \quad i = 1, 2, \ldots, N-1 \qquad (4.1)$$

with initial and boundary conditions

$$u_i^0 = g_i, \quad i = 1, 2, \ldots, N-1,$$

$$u_0^j = a, \quad u_N^j = b, \quad j = 1, 2, \ldots \quad .$$

Here

$$h^2 D_+ D_- u_i = u_{i+1} - 2u_i + u_{i-1} \quad \text{and} \quad 2hD_0(u_i)^2 = (u_{i+1})^2 - (u_{i-1})^2$$

denote the usual centered difference operators. At every time step one has to solve a nonlinear system to determine u_i^{j+1} . This is done by the iteration

$$(I - \varepsilon k D_+ D_-) u_i^{(l+1)} = -\frac{1}{2} k D_0 (u_i^{(l)})^2 + u_i^j + k f_i, \quad l = 0, 1 \ldots \quad , \tag{4.2}$$

where $u^{(0)}$ is choosen by a predictor process.

In all our experiments the solution of (4.1) converges to a steady state solution. However, the speed of convergence depends on the location of the shock. If the shock is located at the boundary, corresponding to the first and third case of (1.5), then the convergence to steady state is quite rapid. See figure (5). If on the other hand the shock is located in the interior, corresponding to the other cases of (1.5), the convergence is, in general, very slow. When the shock is formed at an early stage it is in general in the "wrong" place, depending on the initial data. From then on, the the shock moves slowly to the correct position. See figures (1),(3). This process can be considered quasi-stationary, which makes it possible to use the same convergence acceleration as in [2].

Formally we can write our iteration (4.1) as

$$H(u^{n+1}) = u^{n+1} - u^n := r^n. \tag{4.3}$$

We can linearize the realation and obtain

$$(I - L) r^{n+1} = r^n. \tag{4.4}$$

In our case

$$L r_i = \varepsilon k D_+ D_- r_i - k D_0 (u_i^{n+1} r_i). \tag{4.5}$$

This is a discretization of the right hand side of the eigenvalue problem (2.14), with $p = u^n$. If the process is quasi-stationary we can consider L to be independant of n. Then we have

$$r^{n+j} = (I - L)^{-j} r^n$$

and

$$u^{n+p} = u^n + \sum_{j=0}^{p-1} (I - L)^{-j} r^n.$$

If the eigenvalues λ_i, of L are negative the eigenvalues κ_i, of $(I - L)^{-1}$ satisfy $|\kappa_i| < 1$ and

$$\lim_{p \to \infty} u^{n+p} = u^n + (I - (I - L)^{-1})^{-1} r^n = u^n + (I - L^{-1}) r^n. \tag{4.6}$$

Instead of taking a large number of time steps we can take one large step, which we call an extrapolation step. We put

$$u = u^n + \beta e, \tag{4.7}$$

where e is the solution of the equation

$$Le = (L - I)r^n, \tag{4.8}$$

and β is a stabilizing parameter. We choose β in such a way that $H(u^n + \beta e)$ has no component in the direction of e, i.e.

$$\langle H(u^n + \beta e), e\rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. There are other possible choices, for example choose β such that

$$\|H(u^n + \beta e)\| = \min_{\beta} \|H(u^n + \beta e)\|.$$

Of course (4.7) is not the steady solution we are seeking. We use the new u to restart the time iteration, and make a new extrapolation step once a new quasi-stationary state is reached. In our experiments we use an a priori fixed number of time steps between the extrapolation steps. Better strategies are under development.

We have calculated the first eigenvalues and eigenvectors of the discrete linearized operator (4.5), provided u_i^{n+1} is the discrete steady state solution. The calculations show that the eigenvalues are negative and their distribution is of the same type as for the corresponding continous case. See table (1). In figures (6),(7) the first few eigenvectors are plotted. Note that in the case of an interior shock the first eigenvector is exponentially small away from the shock region. Also, we have no doubt, and it is confirmed by the calculations, that the position of the shock does not change the nature of the eigenvalue distribution. In fact, in the proof of theorem 3.1, y can be replaced by any function of the same structure.

In our case, when the shock is located in the interior, $(I - L)^{-1}$ has only one eigenvalue, κ_1, close to zero. All other eigenvalues are small. Therefore, when we have reached the quasi-stationary state, r^n is in the direction of the eigenvector corresponding to κ_1. See figure (8). Therefore we do not need to solve (4.8), and instead of (4.7) we use

$$u = u^n + \beta r^n. \tag{4.9}$$

In figures (2),(4) we have plotted u at different time stages to show how the convergence is accelerated.

5. A twodimensional case. Consider the following problem

$$u_t + (\frac{1}{2}u^2)_x = \varepsilon(u_{xx} + u_{yy}), \quad 0 \le x \le 1, \quad 0 \le y \le 1, \quad t \ge 0,$$

$$\begin{gathered} u(0,y,t) = a, \quad u(1,y,t) = -a, \quad a > 0, \\ u(x,0,t) = u(x,1,t) = w(x), \\ u(x,y,0) = g(x,y), \end{gathered} \tag{5.1}$$

where $W(x)$ is the solution of the one dimensional problem (1.3) with $b = -a$, and $f(x) \equiv 0$. See (2.3). A steady solution of (5.1) is $u(x,y) = w(x)$.

The speed of convergence can be studied by analyzing the corresponding eigenvalue problem

$$\mu\varphi + (w\varphi)_x = \varepsilon(\varphi_{xx} + \varphi_{yy}), \quad \varphi = 0 \text{ on the boundary.} \tag{5.2}$$

We can solve (5.2) by separation of variables. Let $\varphi(x,y) = X(x)Y(y)$. Then

$$(wX)' - \varepsilon X'' = \lambda X, \quad X(0) = X(1) = 0, \tag{5.3a}$$

$$Y'' = -qY, \quad Y(0) = Y(1) = 0, \tag{5.3b}$$

with $\mu = \lambda - \varepsilon q$. We recognize (5.3a) as (3.2). Therefore $-\lambda_1 = O(e^{-1/\varepsilon})$ and $-\lambda_j > O(1/\varepsilon)$, $j = 2,3,\ldots$. We can solve (5.3b). The solution is

$$Y_j(y) = \sin(j\pi y), \quad q_j = (j\pi)^2, \quad j = 1,2\ldots.$$

There is a whole sequence of eigenvalues, μ_{1j}, of order $O(\varepsilon)$. The eigenfunctions corresponding to this sequence, φ_{1j}, will be exponentially small away from the shock. All other eigenvalues will be of order $O(1/\varepsilon)$.

We expect that the time iteration will again lead to a quasi-stationary state, and that the residual will be composed of eigenfunctions corresponding to the eigenvalues of order $O(\varepsilon)$. Therefore e in (4.8) will be of the same form, and we can replace all components of e away from the shock by zero, thus obtaining a linear system of equations of order N instead of N^2. More details will be given in another paper.

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Table 1.

Eigenvalues of the eigenvalueproblem (3.2), y is the solution of (1.3). Three different cases were treated. The discretization is done according to (4.5), with $N = 100$ gridpoints. The eigenvalues were found using inverse iteration. Eigenvectors corresponding to case (1) are plotted in figure (6a,b).

	λ_1	λ_2	λ_3
$f(x) = \sin(2\pi x)/2$ $a = b = 0$ $\varepsilon = 0.04$	$-8.64 \cdot 10^{-3}$	-4.34	-5.32
$f(x) = \sin(2\pi x)/2$ $a = b = 0$ $\varepsilon = 0.02$	$-4.62 \cdot 10^{-6}$	-5.617	-5.622
$f(x) \equiv 0$ $a = 1,\ b = -1$ $\varepsilon = 0.02$	$-1.24 \cdot 10^{-9}$	-12.8	-13.5

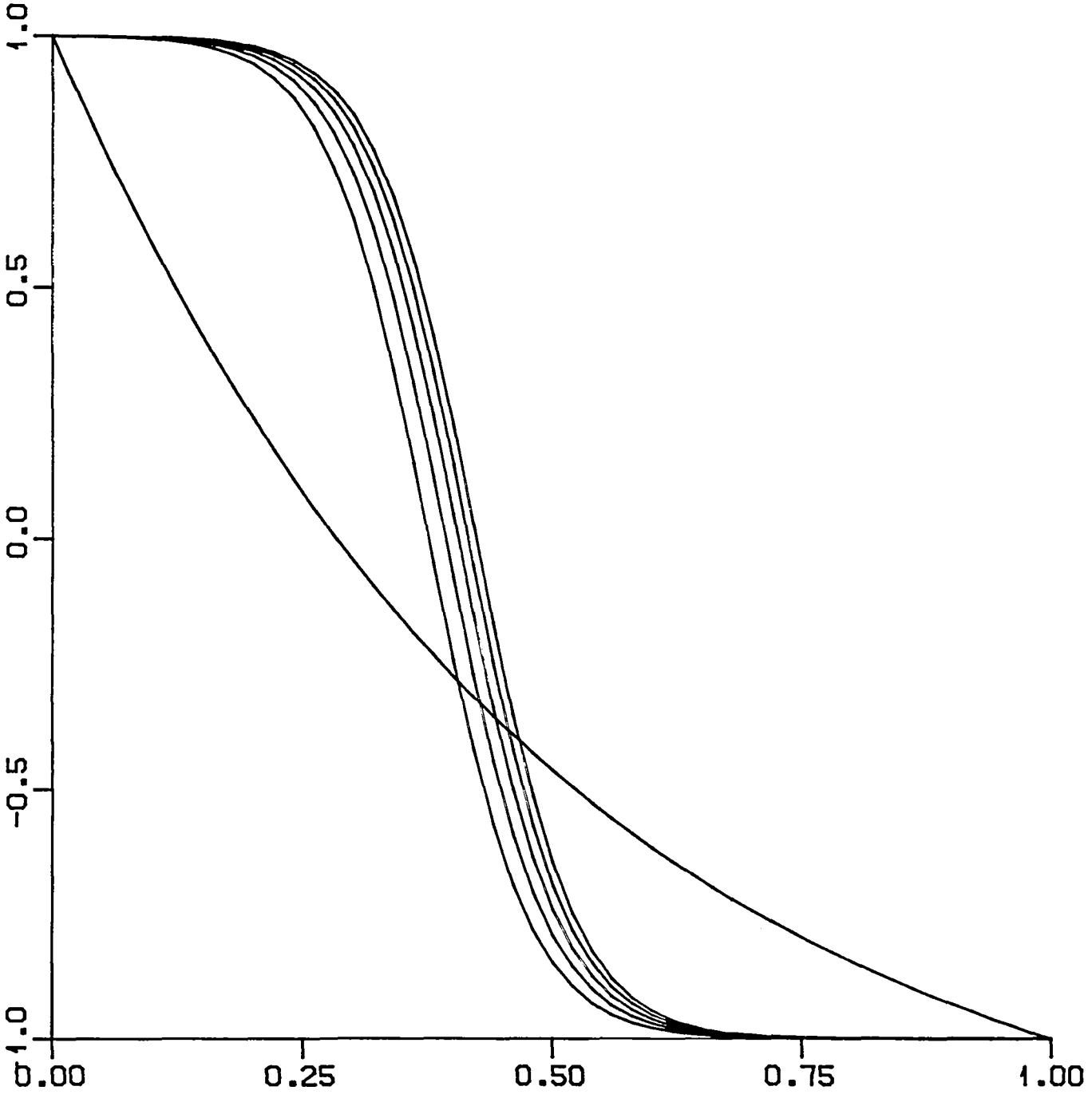

Figure 1. Convergence in time without convergence acceleration. Numerical solutions at different time stages for the case $\varepsilon = 0.05$, $f \equiv 0$, $a = 1$, $b = -1$, $u(x,0) = 1 + 2(e^{-2x} - 1)/(1 - e^{-2})$. Between each curve there are 200 time steps = 40 time units. The calculation is made with time step k = 0.2 and N=50 grid points.

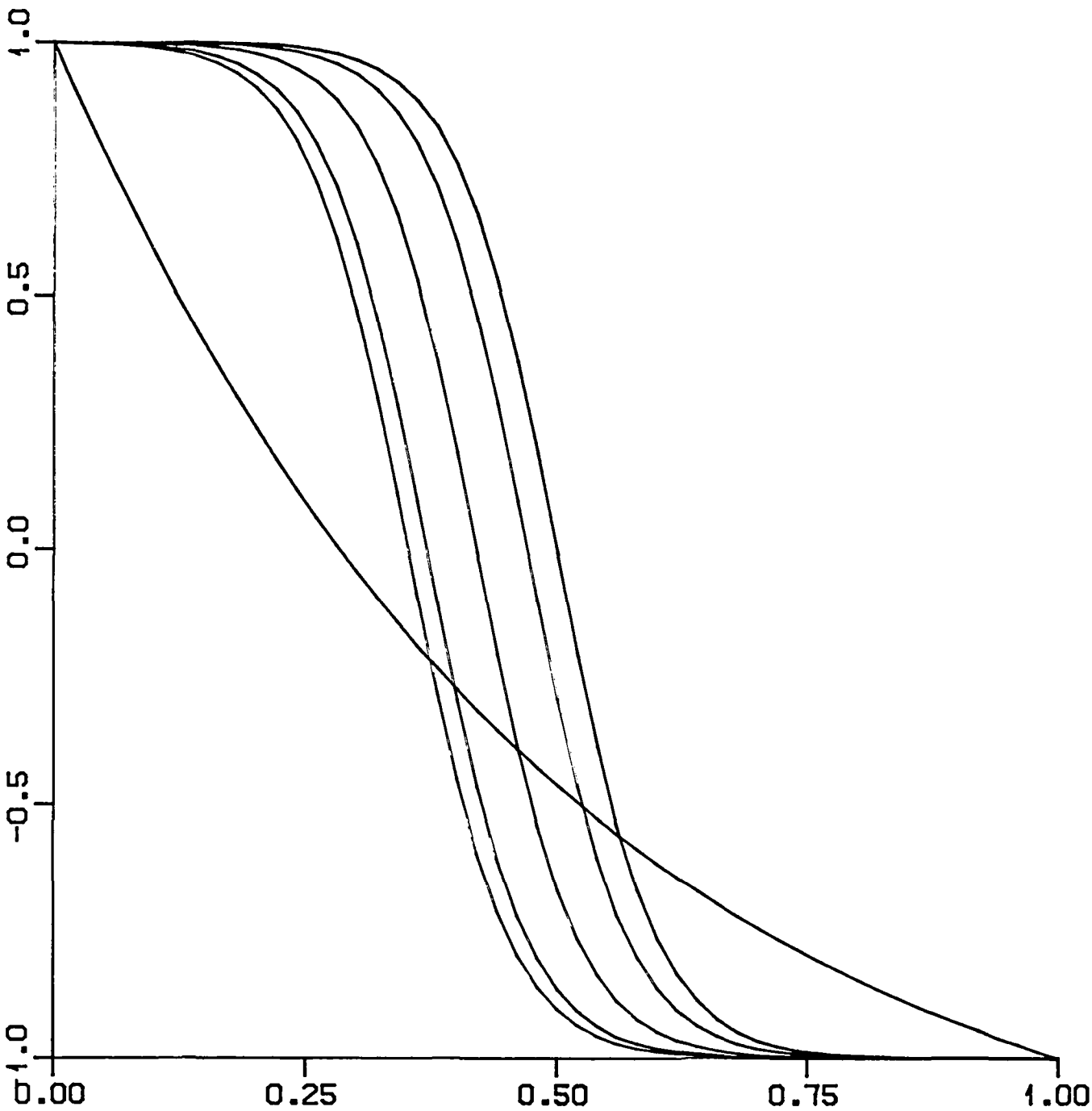

Figure 2. Convergence in time with convergence acceleration. Numerical solutions at different time stages for the same case as in figure 1. Between each curve there are 15 time steps and one extrapolation step. The same time step, k=0.2, and number of grid points , N=50, are used.

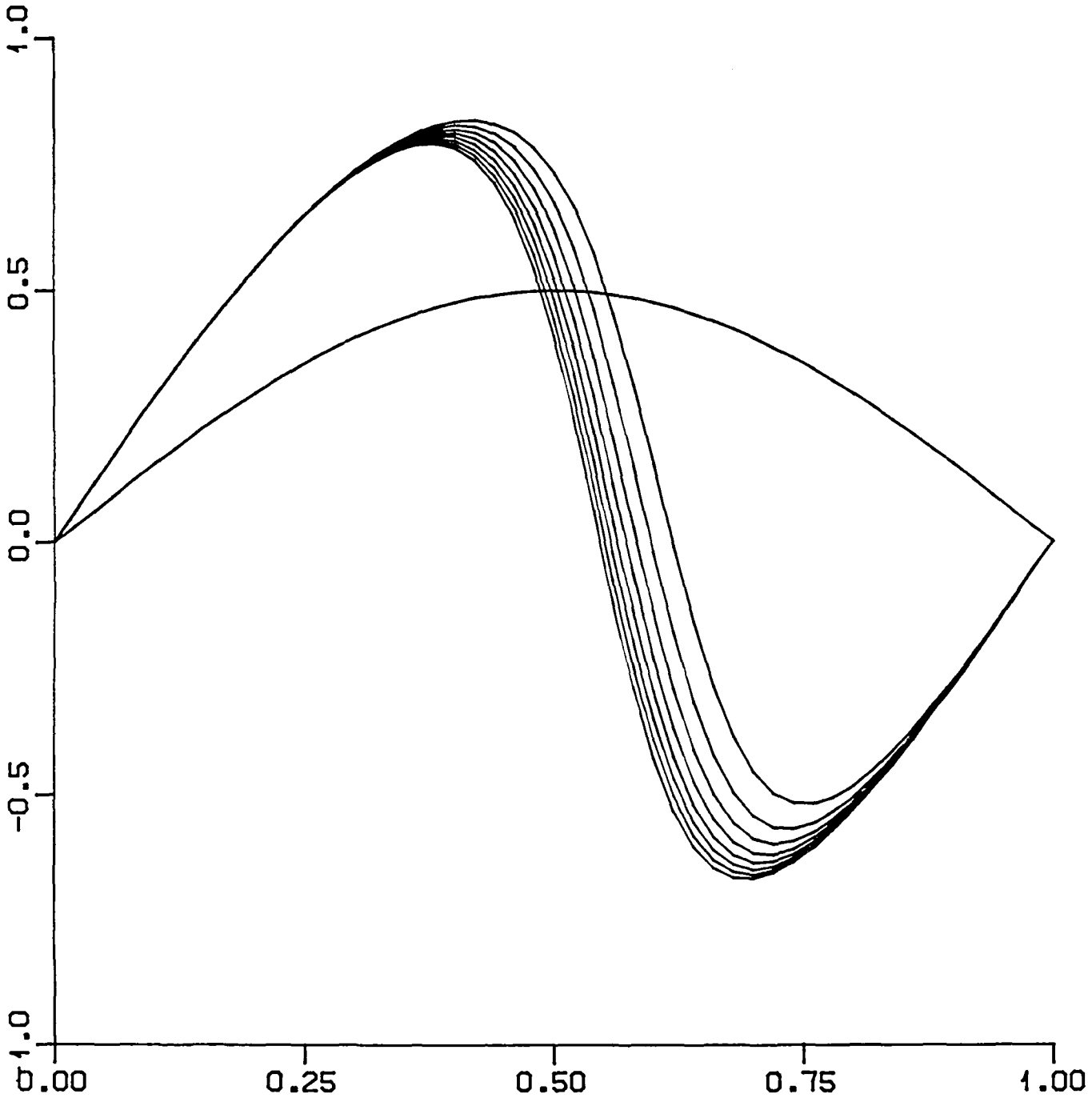

Figure 3. Convergence in time without convergence acceleration. Numerical solutions at different time stages for the case $\varepsilon = 0.04$, $f = \frac{\pi}{4}\sin(\pi x)\cos(\pi x)$, $a = b = 0$, $u(x,0) = \frac{1}{2}\sin(\pi x)$. Between each curve there are 100 time steps . The calculation is made with time step k = 0.1 and N=50 grid points.

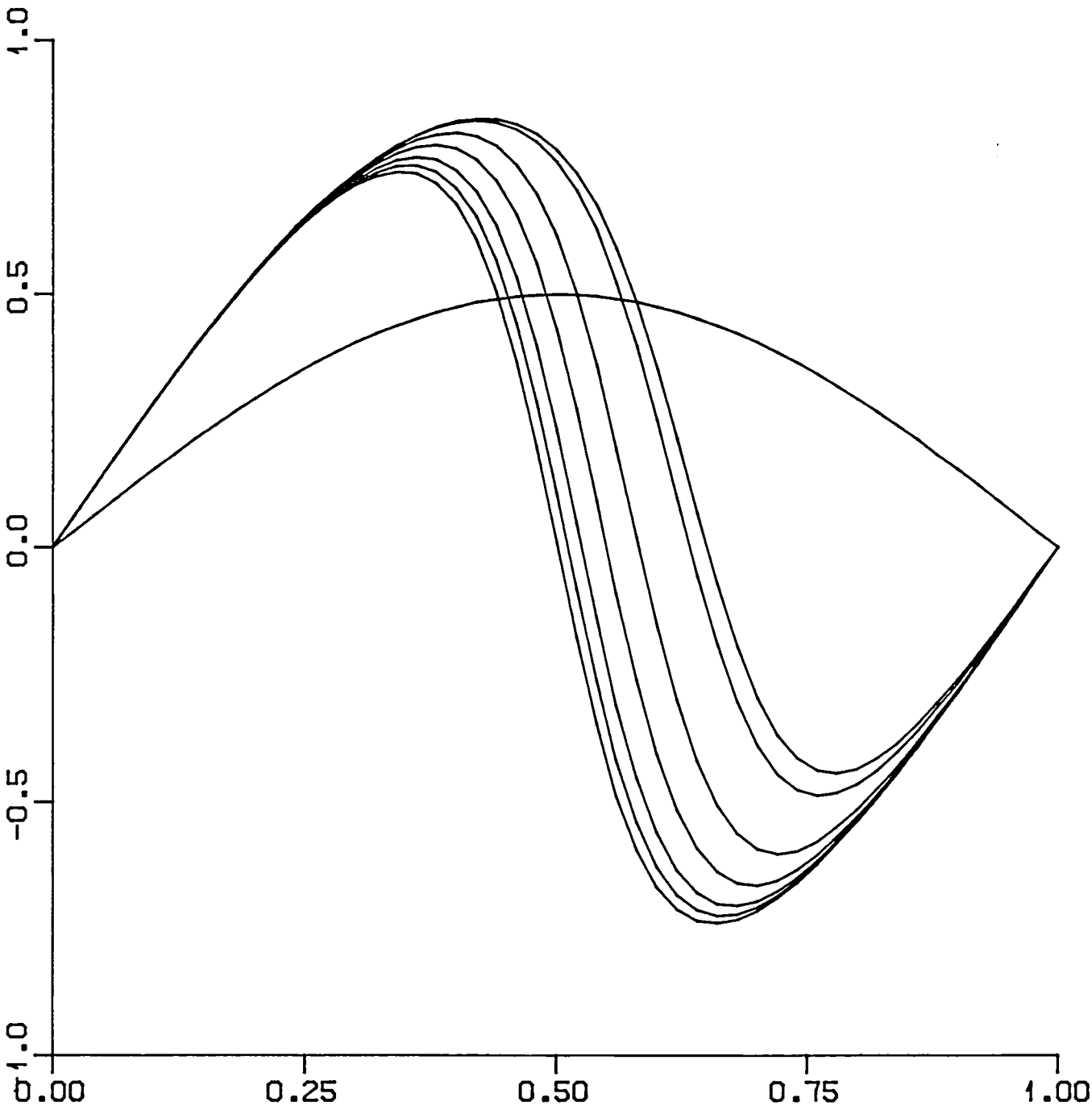

Figure 4. Convergence in time with convergence acceleration. Numerical solutions at different time stages for the same case as in figure 3. Between each curve there are 20 time steps and one extrapolation step. The same time step, k=0.1, and number of grid points , N=50, are used.

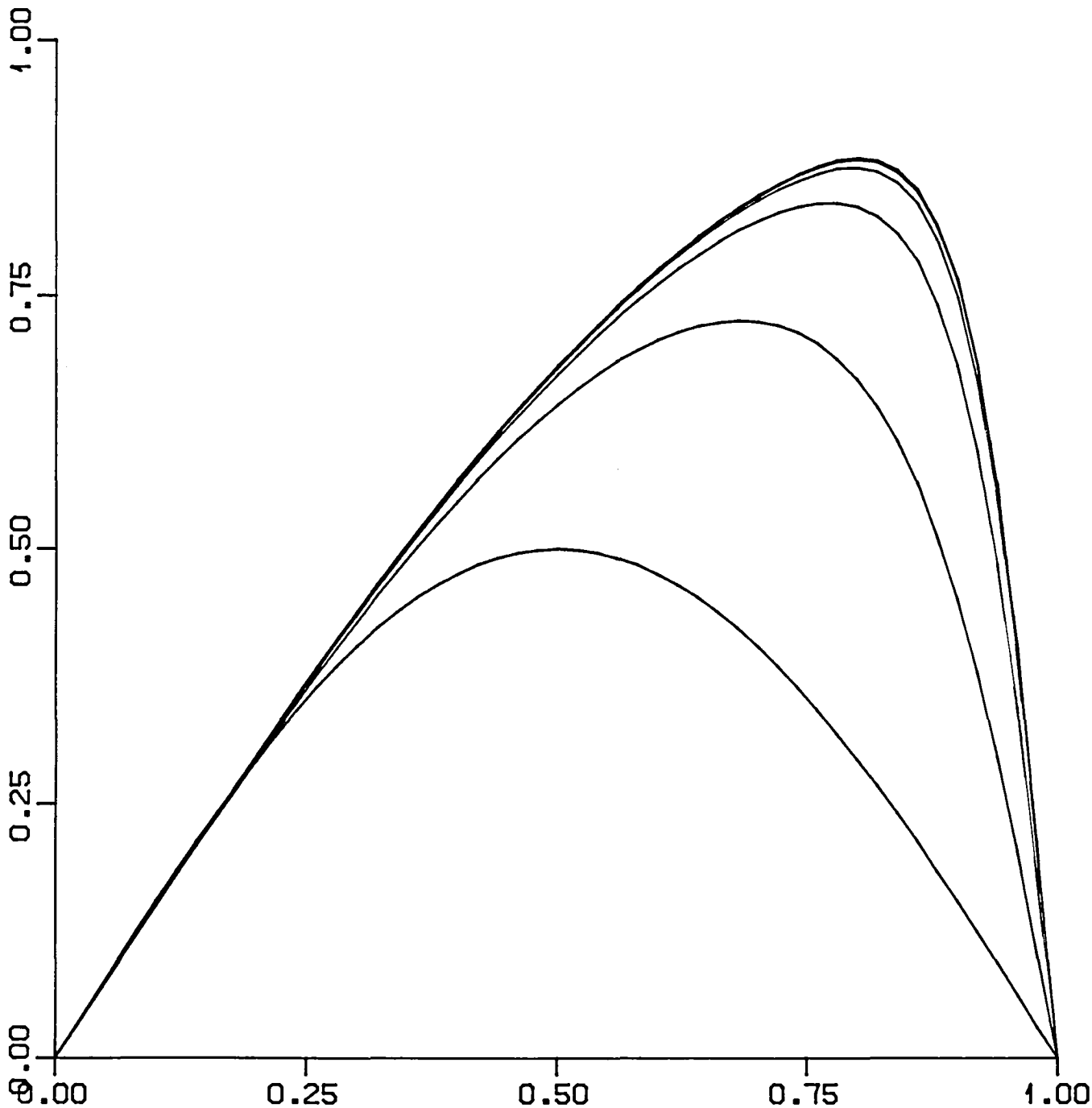

Figure 5. Convergence when the shock is located at the boundary. Here $\varepsilon = 0.04$, $f(x) = \frac{\pi}{4}\sin(\pi x)$, $u(x,0) = \frac{1}{2}\sin(\pi x)$, $N = 50, k = 0.1$. Between each curve there are 5 time steps.

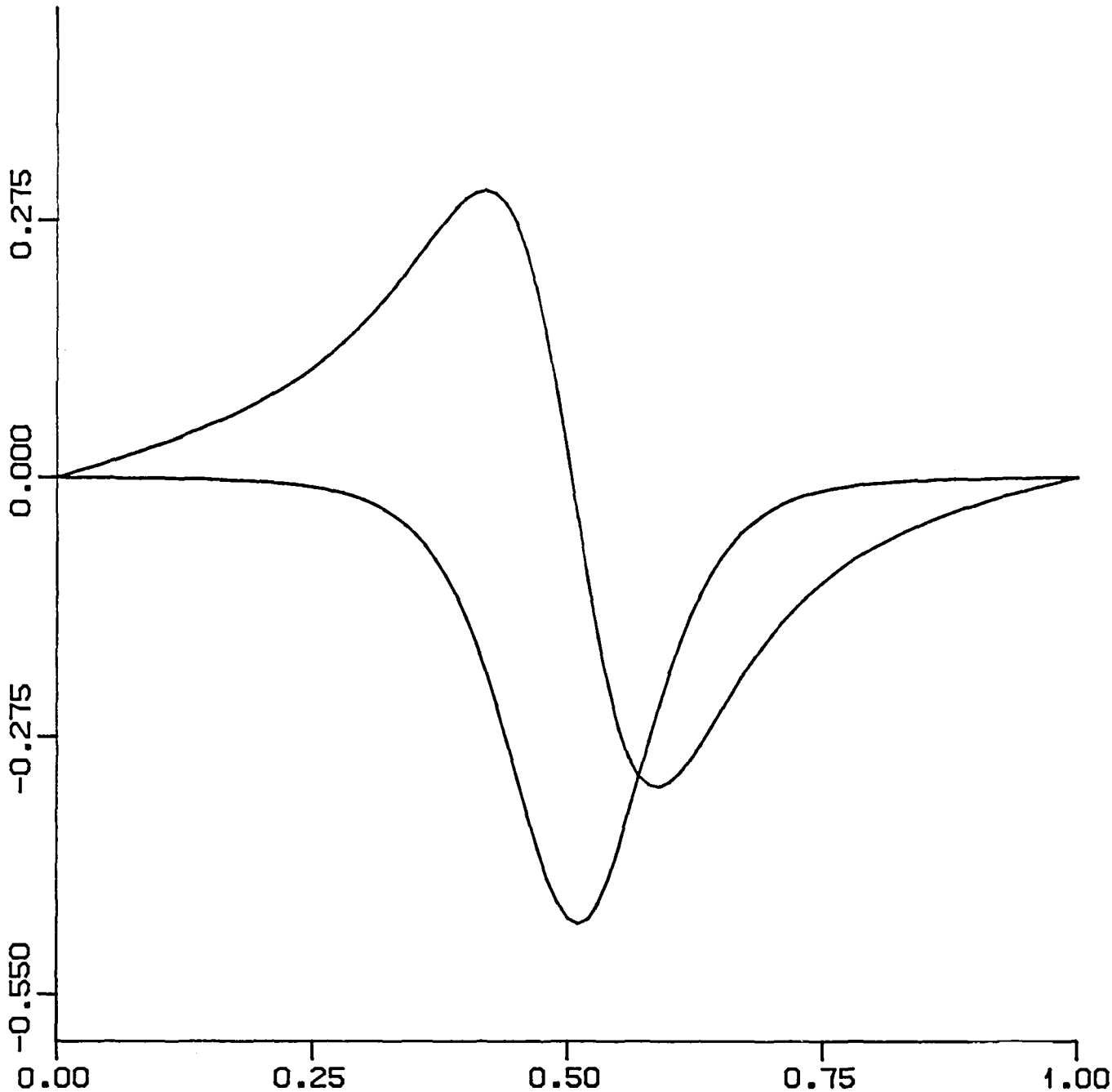

Figure 6a. Eigenvectors. The first two eigenfunctions of problem (3.2), when y, the solution of (1.3), has a shock in the interior. In this case $\varepsilon = 0.04$, $f(x) = \frac{\pi}{4}\sin(\pi x)\cos(\pi x)$, $a = b = 0$, $N = 100$.

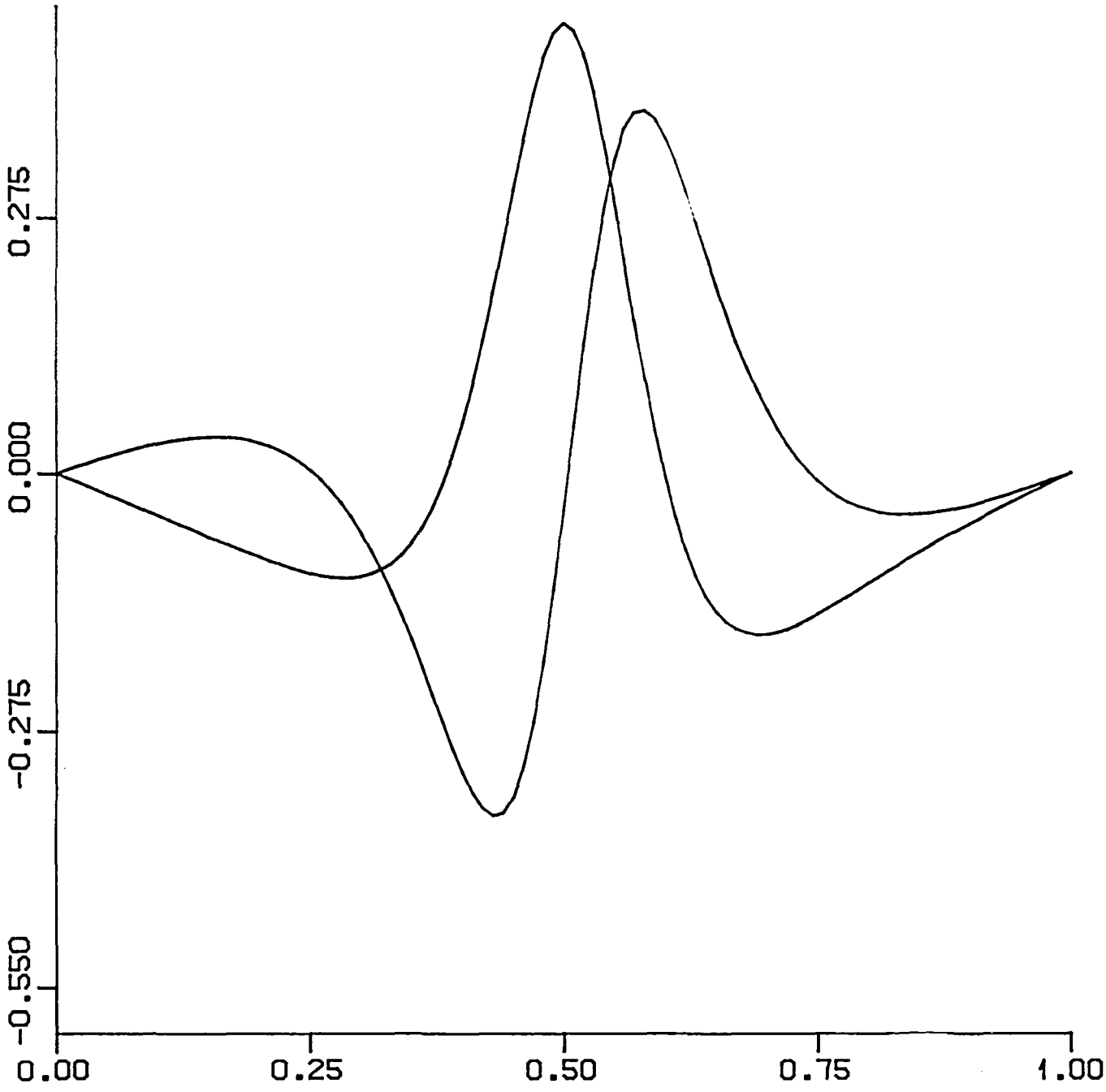

Figure 6b. Eigenvectors. The third and fourth eigenfunctions of problem (3.2), when y, the solution of (1.3), has a shock in the interior. In this case $\varepsilon = 0.04$, $f(x) = \frac{\pi}{4}\sin(\pi x)\cos(\pi x)$, $a = b = 0$, $N = 100$.

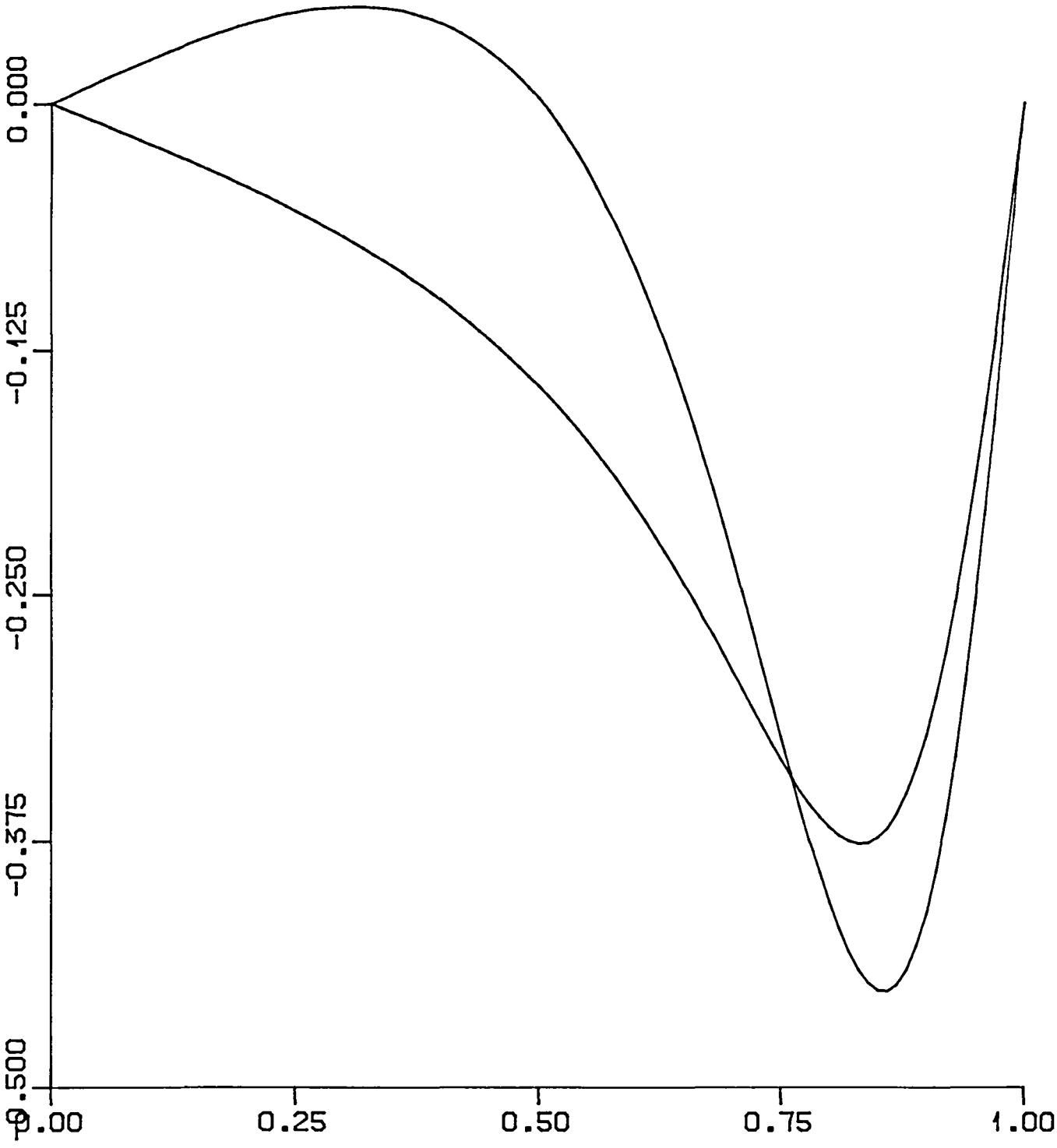

Figure 7. Eigenvectors. The first two eigenvectors, φ_1 and φ_2, of problem (3.2), when y, the solution of (1.3), has a shock $x = 1$. In this case $\varepsilon = 0.08$, $f(x) = \frac{\pi}{4}\sin(\pi x)$, $a = b = 0$, $N = 100$.

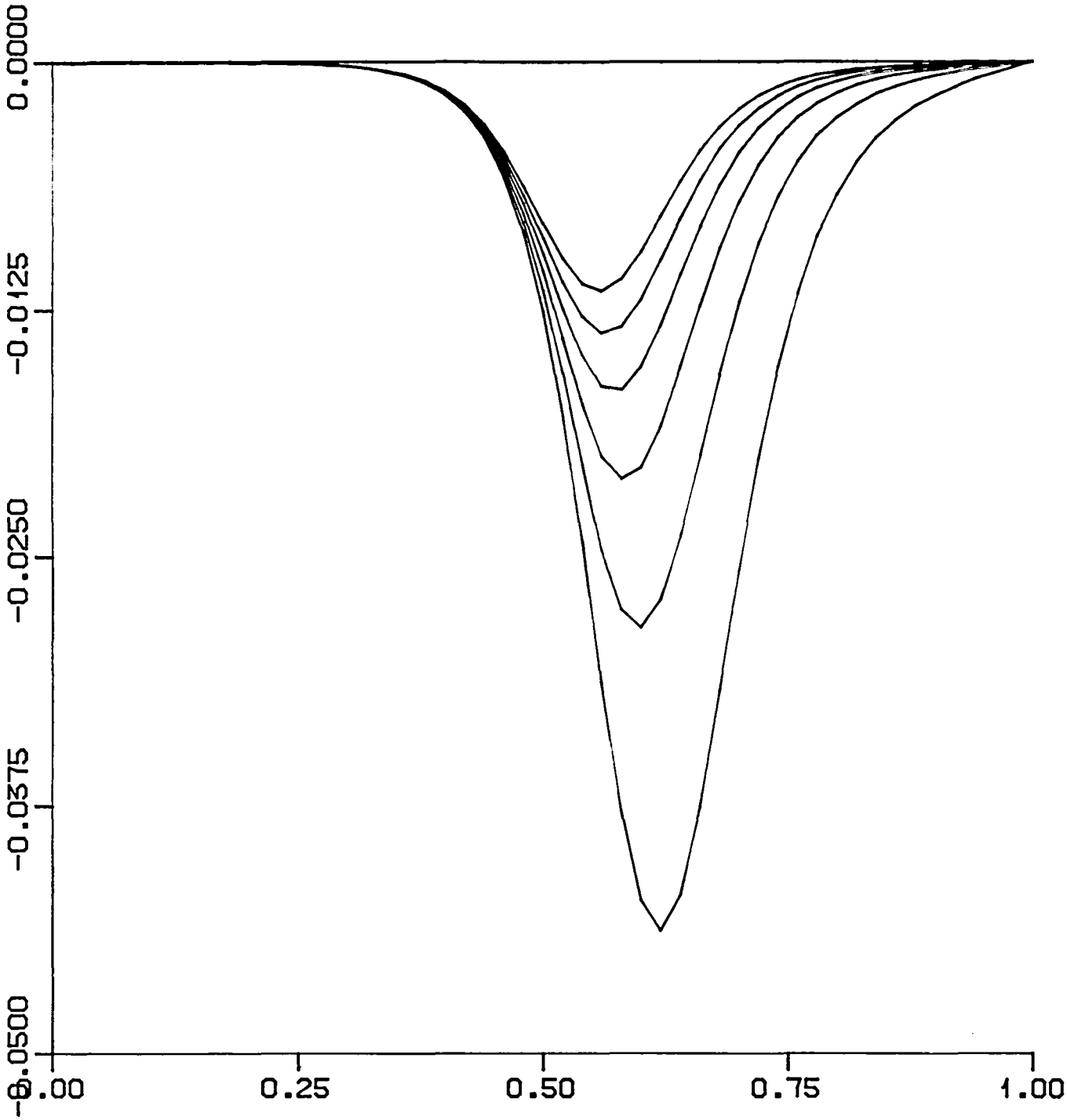

Figure 8. Differences between consecutive solutions at different time stages, when $\varepsilon = 0.04$, $f = \frac{\pi}{4}\sin(\pi x)\cos(\pi x)$, $a = b = 0$, $u(x,0) = \frac{1}{2}\sin(\pi x)$. Between each curve there are 100 time steps . The calculation is made with time step k = 0.1 and N=50 grid points.

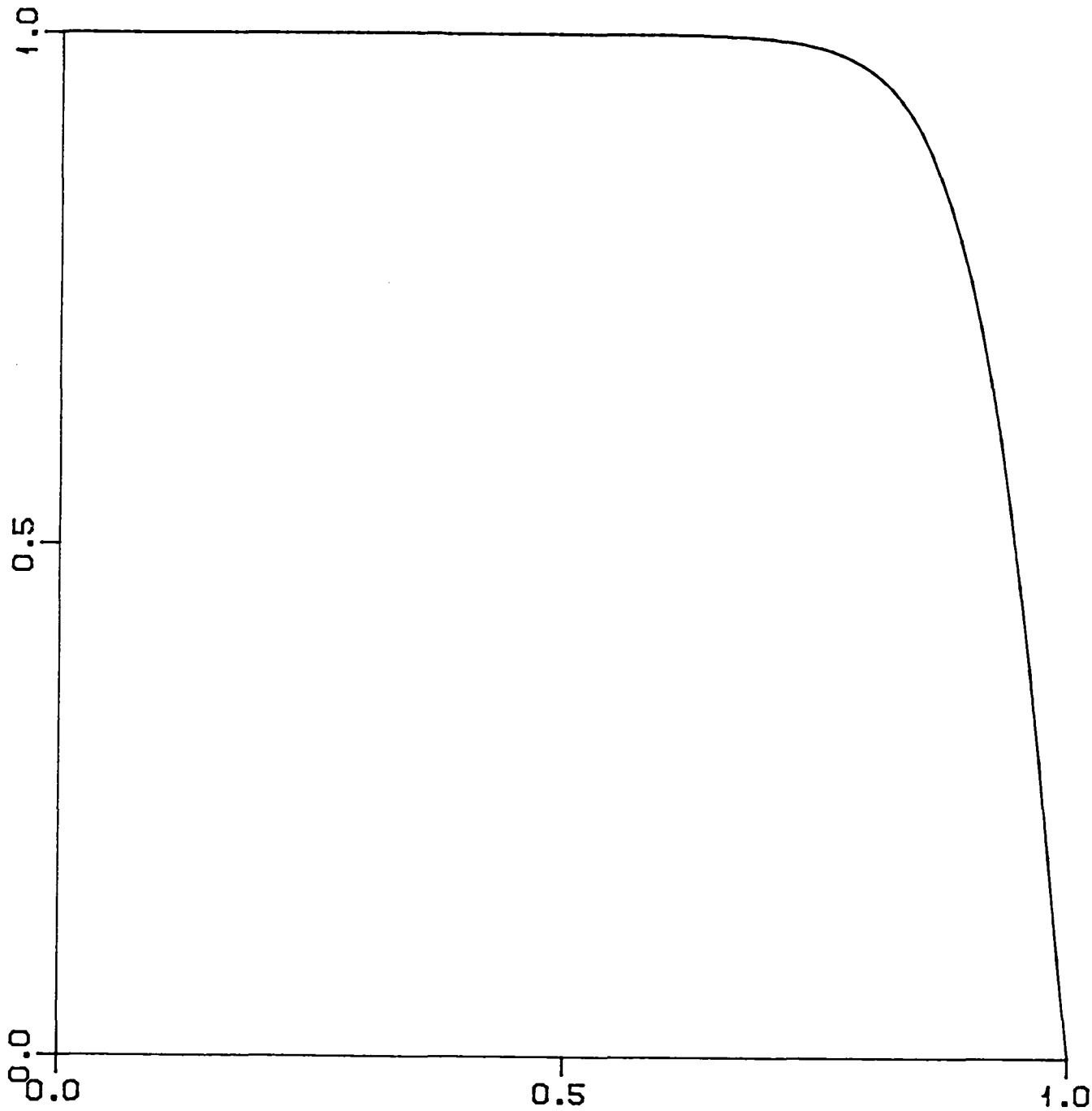

Figure 9. The solution of (1.2) when $f \equiv 0$, $a = 1$, $b = 0$ and $\varepsilon = 0.05$.

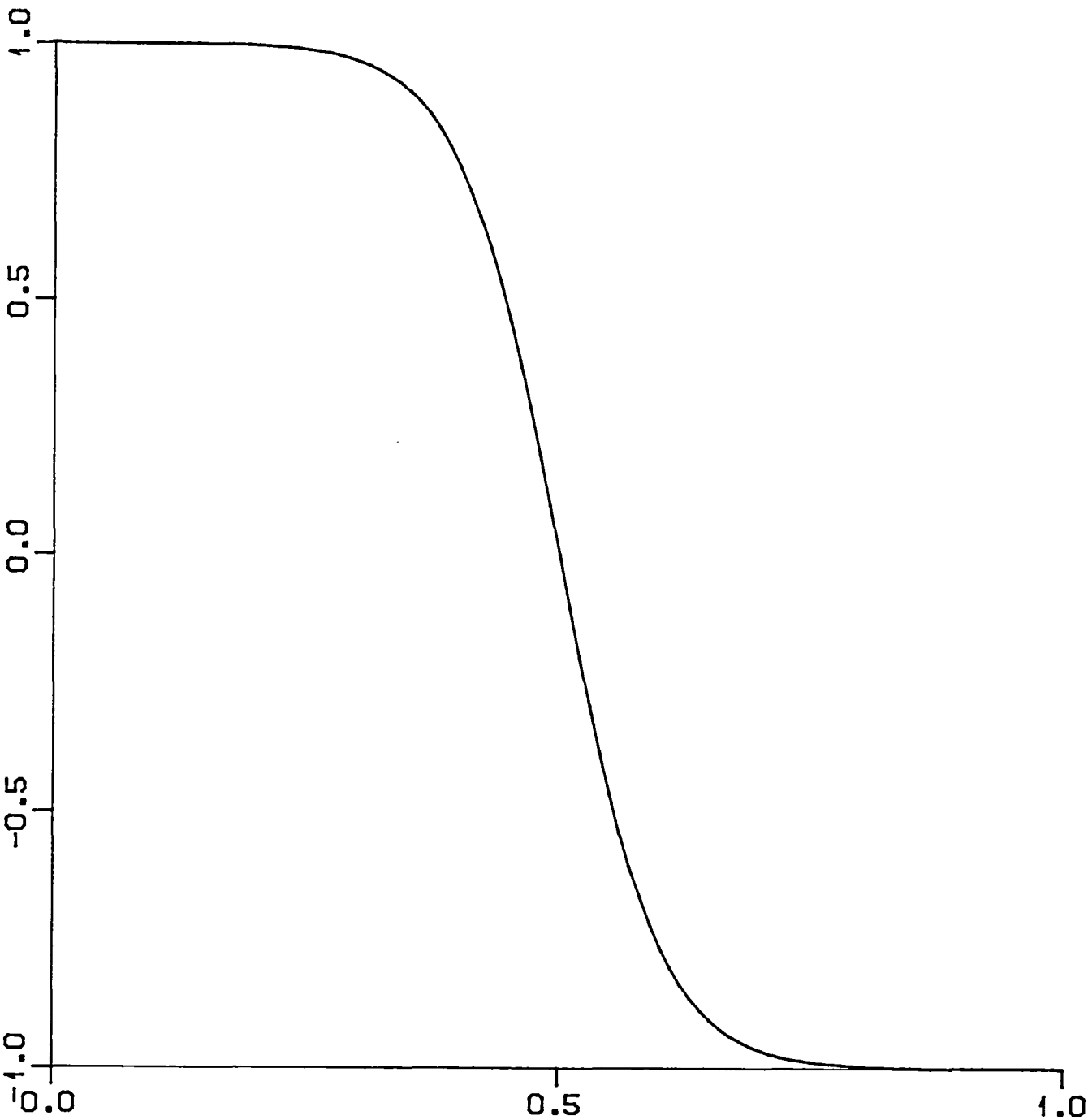

Figure 10. The solution of (1.2) when $f \equiv 0$, $a = 1$, $b = -1$ and $\varepsilon = 0.05$.

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16. Abstract

Consider the initial-boundary value problem for Burgers' equation. It is shown that its solutions converge, in time, to a unique steady state. The speed of the convergence depends on the boundary conditions and can be exponentially slow. Methods to speed up the rate of convergence are also discussed.

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