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FOR SYSTEMS OF CONSERVATION LAWS. I.**

Eitan Tadmor

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**THE NUMERICAL VISCOSITY OF ENTROPY STABLE SCHEMES
FOR SYSTEMS OF CONSERVATION LAWS. I.**

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ABSTRACT

Discrete approximations to hyperbolic systems of conservation laws are studied. We quantify the amount of numerical viscosity present in such schemes, and relate it to their entropy stability by means of comparison. To this end, conservative schemes which are also entropy conservative are constructed. These entropy conservative schemes enjoy second-order accuracy; moreover, they admit a particular interpretation within the finite-element framework, and hence can be formulated on various mesh configurations. We then show that conservative schemes are entropy stable if and only if they contain more viscosity than the mentioned above entropy conservative ones.

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1. INTRODUCTION

The systems of conservation laws referred to in the title are of the form

$$(1.1) \quad \frac{\partial}{\partial t} \mathbf{u} + \sum_{k=1}^d \frac{\partial}{\partial x_k} [\mathbf{f}^{(k)}] = 0, \quad (\bar{x}, t) \in \mathbb{R}^d \times [0, \infty);$$

here $\mathbf{f}^{(k)} \equiv \mathbf{f}^{(k)}(\mathbf{u}) = (f_1^{(k)}, \dots, f_N^{(k)})^T$ are smooth nonlinear flux mappings of the conservative variables $\mathbf{u} \equiv \mathbf{u}(\bar{x}, t) = (u_1, \dots, u_N)^T$. Owing to the nonlinearity of the fluxes $\mathbf{f}^{(k)}$, solutions of (1.1) may develop singularities at a finite time after which one must admit weak solutions, i.e., those derived directly from the underlying integral conservative relations.

Yet, such weak solutions of the conservative equations are not unique. Additional criteria are required in order to single out a unique physically relevant weak solution, the latter being identified as, roughly speaking, a stable limit of a vanishing viscosity mechanism. Entropy stability is then sought as the usual criterion to identify such vanishing viscosity solutions. Lax [10] has shown that entropy stability is in fact equivalent to a vanishing viscosity mechanism, at least in the small -- in the large for scalar problems, e.g., [12], [8].

We study entropy stable approximations to such systems of conservation laws. Entropy stability manifests itself here in terms of a conservative cell entropy inequality. We note in passing that, if holding for large enough class of entropy functions, such cell entropy inequality is intimately related to both, the question of convergence toward a limit solution as well as the question of this limit solution being the unique physically relevant one, e.g., [1], [4], [16] and the references therein.

Starting with Von Neumann and Richtmyer [13] it has long been a common practice to ensure the entropy stability of conservative schemes by tuning their numerical viscosity. In this paper we provide a framework for designing entropy stable schemes by quantifying precisely how much numerical viscosity is to be added. As in [18] this is accomplished by means of comparison: we show that entropy stability is achieved if and only if there is more numerical viscosity than that present in certain entropy conservative schemes. To this end we proceed as follows.

In Section 2 we begin by discussing the entropy variables associated with the conservative systems in question. As observed by Mock [12], [5], such systems are symmetrized w.r.t. these variables. Such symmetrization provides us with a natural order which then fits our goal to compare between the numerical viscosities of different schemes. Expressed in terms of these entropy variables, we then turn to construct the mentioned above entropy conservative schemes. The entropy conservative schemes -- discussed in Section 3 and 4, are second-order accurate. Moreover, they are particularly attractive in view of their interpretation within the finite element framework which enables possible generalizations to various mesh configurations. In Section 5 -- the main one in this paper -- we compare between the viscosity coefficients of different conservative schemes given in their appropriate viscosity form. It is shown that conservative schemes containing more viscosity than that of an entropy stable scheme are also entropy stable. In particular, when compared with the previously discussed entropy conservative schemes, we arrive at a necessary and sufficient criterion for entropy stability. Since our entropy conservative schemes are second-order accurate, such a comparison can still entertain the construction of entropy stable

schemes which are second-order accurate ones. Finally, in order to simplify the notations, we keep our presentation within the one-dimensional case; the multidimensional extension can be worked out dimension by dimension.

2. THE ENTROPY VARIABLES

We begin our discussion with the one-dimensional model

$$(2.1) \quad \frac{\partial}{\partial t} \mathbf{u} + \frac{\partial}{\partial x} [\mathbf{f}(\mathbf{u})] = 0.$$

We assume that the system (2.1) is equipped with a generalized

Entropy Function: a convex mapping $U = U(\mathbf{u})$ augmented with entropy flux mapping $F = F(\mathbf{u})$ such that the following compatibility relation holds

$$(2.2a) \quad U_{\mathbf{u}}^T A = F_{\mathbf{u}}^T.$$

Here $A = A(\mathbf{u})$ is the Jacobian matrix

$$(2.2b) \quad A(\mathbf{u}) = \mathbf{f}_{\mathbf{u}}.$$

We note that the entropy functions, $U(\mathbf{u})$, are exactly those whose positive Hessians $U_{\mathbf{u}\mathbf{u}} > 0$ symmetrize the system (2.1) upon multiplication on the left

[2]

$$(2.3) \quad U_{\mathbf{u}\mathbf{u}} A = [U_{\mathbf{u}\mathbf{u}} A]^T.$$

Mock [12], [5] has pointed out a more fundamental symmetrization of system (2.1), preserving both the strong as well as the weak solutions of the system. To this end one makes use of the entropy variables

$$(2.4) \quad \mathbf{v} \equiv \mathbf{v}(\mathbf{u}) = \frac{\partial U}{\partial \mathbf{u}}(\mathbf{u}).$$

Thanks to the convexity of $U(\mathbf{u})$ the mapping $\mathbf{u} \rightarrow \mathbf{v}$ is one-to-one. Hence we can make the change of variables $\mathbf{u} = \mathbf{u}(\mathbf{v})$ which puts the system (2.1) in its equivalent symmetric form

$$(2.5a) \quad \frac{\partial}{\partial t} [\mathbf{u}(\mathbf{v})] + \frac{\partial}{\partial x} [\mathbf{g}(\mathbf{v})] = 0, \quad \mathbf{g}(\mathbf{v}) \equiv \mathbf{f}(\mathbf{u}(\mathbf{v})).$$

The system (2.5a) is symmetric in the sense that the Jacobians of its temporal and spatial fluxes are

$$(2.5b) \quad \mathbf{H} \equiv \mathbf{H}(\mathbf{v}) = \frac{\partial}{\partial \mathbf{v}} [\mathbf{u}(\mathbf{v})] > 0, \quad \mathbf{B} \equiv \mathbf{B}(\mathbf{v}) = \frac{\partial}{\partial \mathbf{v}} [\mathbf{g}(\mathbf{v})].$$

This follows from the compatibility relation (2.2) equivalently expressed as

$$(2.6) \quad \mathbf{v}^T \mathbf{B}(\mathbf{v}) = \mathbf{G}_{\mathbf{v}}^T, \quad \mathbf{G}(\mathbf{v}) \equiv \mathbf{F}(\mathbf{u}(\mathbf{v})),$$

which in turn implies

$$(2.7) \quad \mathbf{u}(\mathbf{v}) = \frac{\partial}{\partial \mathbf{v}} [\mathbf{v}^T \mathbf{u}(\mathbf{v}) - U(\mathbf{u}(\mathbf{v}))], \quad \mathbf{g}(\mathbf{v}) = \frac{\partial}{\partial \mathbf{v}} [\mathbf{v}^T \mathbf{g}(\mathbf{v}) - G(\mathbf{v})].$$

Indeed, the Jacobians $H(\mathbf{v})$ and $B(\mathbf{v})$ in (2.5b) are the symmetric Hessians of the corresponding expressions inside the brackets in the right of (2.7).

Finally, we note that in contrast to the symmetrization on the left quoted in (2.3), the use of the entropy variables symmetrize the system (2.1) on the right [17], i.e., the original Jacobian $A = \mathbf{f}_{\mathbf{u}}$ is replaced here by the symmetric one $B = \mathbf{g}_{\mathbf{v}}$

$$(2.8) \quad B = AU_{\mathbf{uu}}^{-1} = [AU_{\mathbf{uu}}^{-1}]^T.$$

3. ENTROPY STABLE SCHEMES

We consider conservative discretizations of the form⁽¹⁾

$$(3.1) \quad \frac{d}{dt} \mathbf{u}_v(t) = - \frac{1}{\Delta x} [\mathbf{f}_{v+1/2} - \mathbf{f}_{v-1/2}]$$

serving as consistent approximations to the system of conservation laws

$$(3.2) \quad \frac{\partial}{\partial t} \mathbf{u} + \frac{\partial}{\partial x} [\mathbf{f}(\mathbf{u})] = 0.$$

Here $\mathbf{u}_v(t)$ denotes the approximation value along the gridline ($x_v \equiv v\Delta x, t$), Δx being the spatial mesh size, and

$$(3.3a) \quad \mathbf{f}_{v+1/2} = \phi_{\mathbf{f}}(\mathbf{u}_{v-p+1}, \dots, \mathbf{u}_{v+p})$$

(1) Both the differential and the discrete formulations will employ the same notations. The distinction between the two is made by the use of Greek indices in the discrete formulation.

is the Lipschitz continuous numerical flux consistent with the differential one

$$(3.3b) \quad \phi_f(\mathbf{w}, \mathbf{w}, \dots, \mathbf{w}) = f(\mathbf{w}).$$

To discuss entropy stability, we let (U, F) be any entropy pair associated with the system (3.2). Multiplying by $U_{\mathbf{u}}^T$ and employing (2.2) we conclude that under the smooth regime we have the additional conservation of entropy

$$(3.4) \quad \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0.$$

Taking into account the nonsmooth regime as well, following Lax [10] and Kružíkov [8] we postulate as an admissibility criterion an entropy stability requirement, expressed in terms of the following

Entropy Inequality: We have, in the sense of distributions

$$(3.5) \quad \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} \leq 0.$$

Similarly, for the scheme (3.1) to be entropy stable, a discrete cell entropy inequality is sought [4]

$$(3.6a) \quad \frac{d}{dt} U(\mathbf{u}_v(t)) + \frac{1}{\Delta x} [F_{v+1/2} - F_{v-1/2}] \leq 0;$$

here $F_{v+1/2}$ is the numerical entropy flux

$$(3.6b) \quad F_{v+1/2} = \phi_F(\mathbf{u}_{v-q+1}, \dots, \mathbf{u}_{v+q})$$

consistent with the differential one

$$(3.6c) \quad \phi_F(\mathbf{w}, \mathbf{w}, \dots, \mathbf{w}) = F(\mathbf{w}).$$

An approximate entropy equality of the above type was derived by Osher in [14, Section 3]. A slightly different version of such equality proved in the Appendix employs the consistent numerical flux.

$$(3.7a) \quad F_{v+1/2}^* = 1/2 [F(\mathbf{u}_v) + F(\mathbf{u}_{v+1})] + 1/2 [U_{\mathbf{u}}(\mathbf{u}_v) + U_{\mathbf{u}}(\mathbf{u}_{v+1})]^T \mathbf{f}_{v+1/2} \\ - 1/2 [U_{\mathbf{u}}^T(\mathbf{u}_v) \mathbf{f}(\mathbf{u}_v) + U_{\mathbf{u}}^T(\mathbf{u}_{v+1}) \mathbf{f}(\mathbf{u}_{v+1})].$$

The referred equality reads

Lemma 3.1: [Osher]

Let $\mathbf{u}_v(t)$ be the discrete solution of the conservative scheme (3.1) which is consistent with the system (3.2). Then, for any entropy pair (U, F) the following equality holds

$$(3.7b) \quad \frac{d}{dt} U(\mathbf{u}_v(t)) + \frac{1}{\Delta x} [F_{v+1/2}^* - F_{v-1/2}^*] = \\ = \frac{1}{2\Delta x} \left[\int_{\mathbf{u}_{v-1}}^{\mathbf{u}_v} d\mathbf{u}^T U_{\mathbf{uu}} [\mathbf{f}_{v-1/2} - \mathbf{f}(\mathbf{u})] + \int_{\mathbf{u}_v}^{\mathbf{u}_{v+1}} d\mathbf{u}^T U_{\mathbf{uu}} [\mathbf{f}_{v+1/2} - \mathbf{f}(\mathbf{u})] \right].$$

Hence the scheme (3.1) is entropy stable if the integrals on the right of (3.7b) are shown to be nonpositive. In order to examine the entropy stability question, we first study the case in which these integrals vanish, i.e., we study entropy conservative schemes.

4. ENTROPY CONSERVATIVE SCHEMES

In this section we identify particular schemes which satisfy a cell entropy equality. The numerical viscosity present in such entropy conservative schemes will then be used as the building block for the construction of entropy stable schemes.

In order to carry out the program above, it will prove useful to work with the entropy variables rather than the usual conservative ones. Thus, associated with an entropy function $U(\mathbf{u})$ are the entropy variables $\mathbf{v} = U_{\mathbf{u}}(\mathbf{u})$. Expressed in terms of the latter, the system of conservation laws considered is, see (2.5a)

$$(4.1) \quad \frac{\partial}{\partial t} [\mathbf{u}(\mathbf{v})] + \frac{\partial}{\partial x} [\mathbf{g}(\mathbf{v})] = 0, \quad \mathbf{g}(\mathbf{v}) = \mathbf{f}(\mathbf{u}(\mathbf{v})).$$

It is augmented by the corresponding entropy inequality

$$(4.2a) \quad \frac{\partial V}{\partial t} + \frac{\partial G}{\partial x} \leq 0$$

where (V, G) is the appropriate entropy pair

$$(4.2b) \quad V \equiv V(\mathbf{v}) = U(\mathbf{u}(\mathbf{v})), \quad G \equiv G(\mathbf{v}) = F(\mathbf{u}(\mathbf{v})).$$

In a similar manner, we interpret the conservative scheme (3.1) in terms of the appropriate entropy variables $\mathbf{v}_v = U_{\mathbf{u}}(\mathbf{u}_v)$. Thus, (4.1) is approximated by

$$(4.3a) \quad \frac{d}{dt} [\mathbf{u}(\mathbf{v}_v(t))] = - \frac{1}{\Delta x} [\mathbf{g}_{v+1/2} - \mathbf{g}_{v-1/2}]$$

where $g_{v+1/2}$ is the numerical flux

$$(4.3b) \quad g_{v+1/2} = \psi_g(\mathbf{v}_{v-p+1}, \dots, \mathbf{v}_{v+p}), \quad \psi_g(\dots \mathbf{v}_v, \dots) \equiv \phi_f(\dots u(\mathbf{v}_v) \dots)$$

consistent with the differential one

$$(4.3c) \quad \psi_g(\mathbf{w}, \mathbf{w}, \dots, \mathbf{w}) = g(\mathbf{w}).$$

The corresponding all entropy inequality takes the form

$$(4.4a) \quad \frac{d}{dt} v(\mathbf{v}_v(t)) + \frac{1}{\Delta x} [G_{v+1/2} - G_{v-1/2}] \leq 0.$$

Here $G_{v+1/2}$ is the numerical entropy flux

$$(4.4b) \quad G_{v+1/2} = \Psi_G(\mathbf{v}_{v-q+1}, \dots, \mathbf{v}_{v+q})$$

consistent with the differential one

$$(4.4c) \quad \Psi_G(\mathbf{w}, \mathbf{w}, \dots, \mathbf{w}) = G(\mathbf{w}).$$

Such a consistent numerical entropy flux suggested by (3.7a) is given by

$$(4.5a) \quad G_{v+1/2}^* = \frac{1}{2} [G(\mathbf{v}_v) + G(\mathbf{v}_{v+1})] + \frac{1}{2} [\mathbf{v}_v + \mathbf{v}_{v+1}]^T g_{v+1/2} - \frac{1}{2} [\mathbf{v}_v^T g(\mathbf{v}_v) + \mathbf{v}_{v+1}^T g(\mathbf{v}_{v+1})].$$

The equality (3.7) now simplifies into

Lemma 4.1: (An approximate cell entropy equality-revisited).

Let $\mathbf{v}_v(t)$ be the discrete solution of the conservative scheme (4.3).
then, for any entropy pair (V, G) the following equality holds

$$(4.5b) \quad \frac{d}{dt} V(\mathbf{v}_v(t)) + \frac{1}{\Delta x} [G_{v+1/2}^* - G_{v-1/2}^*] = \\ = \frac{1}{2\Delta x} \left[\int_{\mathbf{v}_{v-1}}^{\mathbf{v}_v} d\mathbf{v}^T [g_{v-1/2} - g(\mathbf{v})] + \int_{\mathbf{v}_v}^{\mathbf{v}_{v+1}} d\mathbf{v}^T [g_{v+1/2} - g(\mathbf{v})] \right].$$

Our entropy conservative schemes will be determined by setting the numerical flux $g_{v+1/2}$ to be

$$(4.6a) \quad g_{v+1/2} = g_{v+1/2}^* = \int_{\xi=0}^1 g(\mathbf{v}_{v+1/2}(\xi)) d\xi,$$

where $\mathbf{v}_{v+1/2}(\xi)$ denotes the segment connecting \mathbf{v}_v and \mathbf{v}_{v+1}

$$(4.6b) \quad \mathbf{v}_{v+1/2}(\xi) \equiv \mathbf{v}_v + \xi \Delta \mathbf{v}_{v+1/2}, \quad \Delta \mathbf{v}_{v+1/2} = \mathbf{v}_{v+1} - \mathbf{v}_v.$$

With this choice of numerical flux we have

$$(4.7) \quad \int_{\mathbf{v}_v}^{\mathbf{v}_{v+1}} d\mathbf{v}^T g_{v+1/2}^* \equiv \Delta \mathbf{v}_{v+1/2}^T g_{v+1/2}^* = \int_{\xi=0}^1 \Delta \mathbf{v}_{v+1/2}^T g(\mathbf{v}_v + \xi \Delta \mathbf{v}_{v+1/2}) d\xi.$$

It is here where we make use of the entropy variables formulation: thanks to the symmetry of g_v , the expression on the right equals the path independent integral, see (2.7),

$$(4.8) \quad \int_{\mathbf{v}_v}^{\mathbf{v}_{v+1}} d\mathbf{v}^T \mathbf{g}_{v+1/2}^* = \int_{\xi=0}^1 \Delta \mathbf{v}_{v+1/2}^T \mathbf{g}(\mathbf{v}_v + \xi \Delta \mathbf{v}_{v+1/2}) d\xi = \int_{\mathbf{v}_v}^{\mathbf{v}_{v+1}} d\mathbf{v}^T \mathbf{g}(\mathbf{v})$$

and therefore each of the integrals on the RHS of (4.5b) vanishes in this case. Let us summarize what we have shown in

Theorem 4.2: (Entropy conservative schemes).

The conservative scheme

$$(4.9) \quad \frac{d}{dt} [\mathbf{u}(\mathbf{v}_v(t))] = \frac{1}{\Delta x} [\mathbf{g}_{v+1/2}^* - \mathbf{g}_{v-1/2}^*], \quad \mathbf{g}_{v+1/2}^* = \int_{\xi=0}^1 \mathbf{g}(\mathbf{v}_{v+1/2}(\xi)) d\xi$$

is also entropy conservative, i.e., it satisfies the following cell entropy equality

$$(4.10) \quad \frac{d}{dt} v(\mathbf{v}_v(t)) + \frac{1}{\Delta x} [G_{v+1/2}^* - G_{v-1/2}^*] = 0.$$

We close this section by noting that the scheme (4.9) besides being entropy conservative is also a second-order accurate one. Both the second-order accuracy and in particular the entropy conservation of the scheme (4.9) can be directly verified within the framework of a finite-element formulation advocated by, e.g., Mock [11] and in particular by Hughes and his collaborators, [7] and the references therein. To this end one considers the weak formulation of (4.1)

$$(4.11) \quad \int_{\Omega} \left[\frac{\partial \omega}{\partial t} \mathbf{u}(\mathbf{v}) + \frac{\partial \omega}{\partial x} \mathbf{g}(\mathbf{v}) \right] dx dt = 0.$$

If now the trial solutions $\mathbf{v} \rightarrow \hat{\mathbf{v}}(x,t)$ and the weighting test functions $\omega \rightarrow \hat{\omega}(x,t)$ are chosen out of the typical finite-element set spanned by the C^0 "hat functions"

$$\hat{H}_v(x) = \begin{cases} \frac{x - x_{v-1}}{x_v - x_{v-1}} & x_{v-1} \leq x \leq x_v \\ \frac{x_{v+1} - x}{x_{v+1} - x_v} & x_v \leq x \leq x_{v+1} \end{cases}$$

we are led to

$$\frac{1}{2} [x_{v+1} - x_{v-1}] \frac{\partial}{\partial t} [\mathbf{u}(\mathbf{v}_v(t))] = - \int_{x_{v-1}}^{x_{v+1}} \frac{\partial \hat{H}_v(x)}{\partial x} \mathbf{g}(\hat{\mathbf{v}}(x, t) \equiv \sum_v \mathbf{v}_v(t) \hat{H}_v(x)) dx,$$

or, after changing variables on the right

$$(4.12) \quad \frac{\partial}{\partial t} [\mathbf{u}(\mathbf{v}_v(t))] = \frac{-2}{x_{v+1} - x_{v-1}} \left[\int_{\xi=0}^1 \mathbf{g}(\mathbf{v}_{v+1/2}(\xi)) d\xi - \int_{\xi=0}^1 \mathbf{g}(\mathbf{v}_{v-1/2}(\xi)) d\xi \right].$$

Observe that (4.12) also suggests the natural extension of the entropy conservative schemes (4.9) to non-uniformly spaced meshes. The entropy conservation follows, at once, by selecting the test function $\hat{\omega}$ to coincide with the trial function $\hat{\mathbf{v}}$, so that in view of (2.6), (4.11) yields

$$0 = \int_{\Omega} \left[\frac{\partial \hat{\mathbf{v}}^T}{\partial t} \mathbf{u}(\hat{\mathbf{v}}) + \frac{\partial \hat{\mathbf{v}}^T}{\partial x} \mathbf{g}(\hat{\mathbf{v}}) \right] dx dt \equiv \int_{\Omega} \left[\frac{\partial}{\partial t} V(\hat{\mathbf{v}}) + \frac{\partial}{\partial x} G(\hat{\mathbf{v}}) \right] dx dt.$$

5. NUMERICAL VISCOSITY AND ENTROPY STABILITY

The essential role played by the numerical viscosity has long been recognized starting with Von Neumann and Richtmyer [13]. In this section we quantify the amount of numerical viscosity required in order to guarantee entropy stability. As in [18], this will be done by means of comparison.

To start with, we consider the entropy conservative scheme (4.9). Integration by parts of its numerical flux formula (4.6a) yields

$$(5.1) \quad \mathbf{g}_{\nu+1/2}^* = \int_{\xi=0}^1 \mathbf{g}(\mathbf{v}_{\nu+1/2}(\xi)) d\xi = \left(\xi - \frac{1}{2} \right) \mathbf{g}(\mathbf{v}_{\nu+1/2}(\xi)) \Big|_{\xi=0}^{\xi=1} - \int_{\xi=0}^1 \left(\xi - \frac{1}{2} \right) \frac{d}{d\xi} \mathbf{g}(\mathbf{v}_{\nu+1/2}(\xi)) d\xi.$$

Recalling the notation for the Jacobian $\mathbf{g}_{\mathbf{v}}$, see (2.5b),

$$(5.2a) \quad \mathbf{B}(\mathbf{v}) = \mathbf{g}_{\mathbf{v}},$$

the RHS of (5.1) can be rewritten as

$$(5.2b) \quad \mathbf{g}_{\nu+1/2}^* = \frac{1}{2} [\mathbf{g}(\mathbf{v}_{\nu}) + \mathbf{g}(\mathbf{v}_{\nu+1})] - \frac{1}{2} \int_{\xi=0}^1 (2\xi - 1) \mathbf{B}(\mathbf{v}_{\nu+1/2}(\xi)) d\xi \Delta \mathbf{v}_{\nu+1/2}.$$

Inserting this into (4.9), our entropy conservative scheme assumes the viscosity form

$$(5.3a) \quad \frac{d}{dt} [\mathbf{u}(\mathbf{v}_{\nu+1}(t))] = - \frac{1}{2\Delta x} [\mathbf{g}(\mathbf{v}_{\nu+1}) - \mathbf{g}(\mathbf{v}_{\nu})] + \frac{1}{2\Delta x} [Q_{\nu+1/2}^* \Delta \mathbf{v}_{\nu+1/2} - Q_{\nu-1/2}^* \Delta \mathbf{v}_{\nu-1/2}].$$

Here $Q_{\nu+1/2}^*$ is the numerical viscosity coefficient matrix given by

$$(5.3b) \quad Q_{\nu+1/2}^* = \int_{\xi=0}^1 (2\xi - 1) \mathbf{B}(\mathbf{v}_{\nu+1/2}(\xi)) d\xi.$$

We note that the second brackets on the right of (5.3a) mimic a diffusive-like term $\Delta x (Q_{\mathbf{v}_x}^*)_{\mathbf{x}}$. Yet, though the viscosity matrix $Q_{\nu+1/2}^*$ is symmetric, it is not necessarily a positive definite one; rather, it is determined so as to

counterbalance the dispersive flux central differencing inside the first brackets on the right of (5.3a).

Motivated by the discussion above, we would like to consider schemes given in a similar viscosity form [18]

$$(5.4) \quad \frac{d}{dt} [u(\mathbf{v}_v(t))] = -\frac{1}{2\Delta x} [g(\mathbf{v}_{v+1}) - g(\mathbf{v}_{v-1})] + \\ + \frac{1}{2\Delta x} [Q_{v+1/2} \Delta \mathbf{v}_{v+1/2} - Q_{v-1/2} \Delta \mathbf{v}_{v-1/2}].$$

The matrix $Q_{v+1/2}$ on the right will be referred to as the numerical viscosity coefficient matrix.

Remark: We observe that the above definition of the numerical viscosity coefficient depends on the specific entropy function under consideration $U(\mathbf{u})$. The special choice $U(\mathbf{u}) = u^2$ corresponds to our earlier viscosity definition [18] in the scalar case.

What schemes admit a viscosity form-like (5.4)? To answer this question we observe that the numerical flux determined by (5.4)

$$(5.5) \quad \mathbf{g}_{v+1/2} = \psi_g(\mathbf{v}_{v-p+1}, \dots, \mathbf{v}_{v+p}) = \frac{1}{2} [g(\mathbf{v}_v) + g(\mathbf{v}_{v+1})] \\ - \frac{1}{2} Q_{v+1/2} [\mathbf{v}_{v+1} - \mathbf{v}_v]$$

satisfies the consistency relation

$$(5.6) \quad \psi_g(\mathbf{v}_{v-p+1}, \dots, \mathbf{v}_{v-1}, \mathbf{w}, \mathbf{w}, \mathbf{v}_{v+2}, \dots, \mathbf{v}_{v+p}) = g(\mathbf{w}).$$

The consistency relation (5.6) is slightly more stringent than the usual one (4.3c). It characterizes the class of essentially three-point schemes, a class which includes, beside the standard three-point schemes, most of the recently constructed second-order accurate TVD schemes, e.g., [3], [15], [6]. In the Appendix we show that the converse of the above implication holds, namely we have

Lemma 5.1: The consistent conservative scheme (4.3) can be rewritten in the viscous form (5.4) if and only if it is an essentially three-point scheme.

Granted the viscosity form (5.4), we now turn to discuss the question of entropy stability. We first note that there is only one degree of freedom in setting up the viscosity form (5.4), that is, setting up the viscosity coefficient $Q_{v+1/2}$. Motivated by Lemma 4.1, we shall examine the quantity in light of the integrals on the RHS of (4.5b). Thus, using (5.5) we find

$$\int_{\mathbf{v}_v}^{\mathbf{v}_{v+1}} d\mathbf{v}^T [\mathbf{g}_{v+1/2} - \mathbf{g}(\mathbf{v})] = \frac{1}{2} \Delta \mathbf{v}_{v+1/2}^T [\mathbf{g}(\mathbf{v}_v) + \mathbf{g}(\mathbf{v}_{v+1})] \\ - \int_{\mathbf{v}_v}^{\mathbf{v}_{v+1}} d\mathbf{v}^T \mathbf{g}(\mathbf{v}) - \frac{1}{2} \Delta \mathbf{v}_{v+1/2}^T Q_{v+1/2} \Delta \mathbf{v}_{v+1/2}.$$

The first two terms on the right are fixed by the differential flux; abbreviate their difference by $K_{v+1/2}$. Then, Lemma 4.1 tells us that

$$\frac{d}{dt} V(\mathbf{v}_v(t)) + \frac{1}{\Delta x} [G_{v+1/2}^* - G_{v-1/2}^*] = K_{v-1/2} + K_{v+1/2} \\ (5.7) \\ - \frac{1}{2} \Delta \mathbf{v}_{v-1/2}^T Q_{v-1/2} \Delta \mathbf{v}_{v-1/2} - \frac{1}{2} \Delta \mathbf{v}_{v+1/2}^T Q_{v+1/2} \Delta \mathbf{v}_{v+1/2}.$$

Hence we conclude that the cell expression on the left decreases whenever the quadratic forms $\Delta \mathbf{v}_{v \pm 1/2}^T Q_{v \pm 1/2} \Delta \mathbf{v}_{v \pm 1/2}$ increase. This suggests to compare between different schemes in terms of their numerical viscosity coefficients used as our scale.

We shall say one scheme contains more viscosity than another scheme if the viscosity coefficient of the first scheme, say $Q_{v+1/2}^{(1)}$, dominates that of the second one, $Q_{v+1/2}^{(2)}$, i.e., if we have

$$(5.8a) \quad \Delta \mathbf{v}_{v+1/2}^T Q_{v+1/2}^{(2)} \Delta \mathbf{v}_{v+1/2} \leq \Delta \mathbf{v}_{v+1/2}^T Q_{v+1/2}^{(1)} \Delta \mathbf{v}_{v+1/2} .$$

An strengthened formulation of this, in terms of the natural order among symmetric matrices, reads

$$(5.8b) \quad \text{Re} Q_{v+1/2}^{(2)} \leq \text{Re} Q_{v+1/2}^{(1)} .$$

In case the entropy variables were used to begin with, they would lead to symmetric viscosity coefficients very much the same way they led to symmetric Jacoians, e.g., (5.3b), and we would arrive at the natural hierarchy

$$(5.8c) \quad Q_{v+1/2}^{(2)} \leq Q_{v+1/2}^{(1)} .$$

The last three inequalities mean the same when dealing with the scalar case.

Equipped with the above terminology we now turn to a particular comparison with the entropy conservative scheme (5.3). Thus, we consider a conservative scheme in its viscous form (5.4) and we let $D_{v+1/2}$

$$(5.9) \quad D_{v+1/2} = Q_{v+1/2} - Q_{v+1/2}^*$$

denotes its viscosity deviation from the entropy conservative one (5.3). Then we can write the scheme under conservation (5.4)

$$\begin{aligned} \frac{d}{dt} [\mathbf{u}(\mathbf{v}_v(t))] &= \frac{1}{2\Delta x} [g(\mathbf{v}_{v+1}) - g(\mathbf{v}_{v-1})] + \frac{1}{2\Delta x} [Q_{v+1/2}^* \Delta \mathbf{v}_{v+1/2} - Q_{v-1/2}^* \Delta \mathbf{v}_{v-1/2}] \\ &\quad + \frac{1}{2\Delta x} [D_{v+1/2} \Delta \mathbf{v}_{v+1/2} - D_{v-1/2} \Delta \mathbf{v}_{v-1/2}] \\ &= -\frac{1}{\Delta x} [g_{v+1/2}^* - g_{v-1/2}^*] + \frac{1}{2\Delta x} [D_{v+1/2} \Delta \mathbf{v}_{v+1/2} - D_{v-1/2} \Delta \mathbf{v}_{v-1/2}]. \end{aligned}$$

Multiplying the last equality by $\mathbf{u}_v^T(\mathbf{u}_v(t)) = \mathbf{v}_v^T(t)$ on the left yields

$$\begin{aligned} \frac{d}{dt} v(\mathbf{v}_v(t)) + \frac{1}{\Delta x} \mathbf{v}_v^T [g_{v+1/2}^* - g_{v-1/2}^*] \\ (5.10) \quad &= \frac{1}{2\Delta x} \mathbf{v}_v^T [D_{v+1/2} \Delta \mathbf{v}_{v+1/2} - D_{v-1/2} \Delta \mathbf{v}_{v-1/2}]. \end{aligned}$$

By Theorem 4.2 the second expression on the RHS of (5.10) equals the conservative difference

$$\begin{aligned} \frac{1}{\Delta x} \mathbf{v}_v^T [g_{v+1/2}^* - g_{v-1/2}^*] &= -\mathbf{v}_v^T(t) \frac{d}{dt} [\mathbf{u}(\mathbf{v}_v(t))] \\ (5.11) \quad &= -\frac{d}{dt} v(\mathbf{v}_v(t)) = \frac{1}{\Delta x} [G_{v+1/2}^* - G_{v-1/2}^*]. \end{aligned}$$

Regarding the RHS of (5.10), the following identity puts it as the sum of familiar quadratic terms plus a conservative difference

$$\begin{aligned}
 & \frac{1}{2\Delta x} \mathbf{v}_v^T [D_{v+1/2} \Delta \mathbf{v}_{v+1/2} - D_{v-1/2} \Delta \mathbf{v}_{v-1/2}] \\
 (5.12) \quad & \equiv - \frac{1}{2\Delta x} \left[\frac{1}{2} \Delta \mathbf{v}_{v+1/2}^T D_{v+1/2} \Delta \mathbf{v}_{v+1/2} + \frac{1}{2} \Delta \mathbf{v}_{v-1/2}^T D_{v-1/2} \Delta \mathbf{v}_{v-1/2} \right] \\
 & + \frac{1}{2\Delta x} \left[\frac{1}{2} [\mathbf{v}_v + \mathbf{v}_{v+1}]^T D_{v+1/2} \Delta \mathbf{v}_{v+1/2} - \frac{1}{2} [\mathbf{v}_{v-1} + \mathbf{v}_v]^T D_{v-1/2} \Delta \mathbf{v}_{v-1/2} \right].
 \end{aligned}$$

Using the last equality we conclude the main result of this section, asserting

Theorem 5.2: (Entropy stability).

A conservative scheme is entropy stable if and only if it contains more numerical viscosity than the entropy conservative scheme. Moreover, the entropy dissipates in this case at rate governed by the cell-estimate

$$\begin{aligned}
 & \frac{d}{dt} V(\mathbf{v}_v(t)) + \frac{1}{\Delta x} [G_{v+1/2} - G_{v-1/2}] \\
 (5.13a) \quad & = - \frac{1}{4\Delta x} \left[\Delta \mathbf{v}_{v+1/2}^T D_{v+1/2} \Delta \mathbf{v}_{v+1/2} + \Delta \mathbf{v}_{v-1/2}^T D_{v-1/2} \Delta \mathbf{v}_{v-1/2} \right] \leq 0.
 \end{aligned}$$

Here $G_{v+1/2}$ is the consistent numerical entropy flux

$$\begin{aligned}
 (5.13b) \quad G_{v+1/2} & = \frac{1}{2} [G(\mathbf{v}_v) + G(\mathbf{v}_{v+1})] + \frac{1}{2} [\mathbf{v}_v + \mathbf{v}_{v+1}]^T \left[\mathbf{g}_{v+1/2} - \frac{1}{2} D_{v+1/2} \Delta \mathbf{v}_{v+1/2} \right] \\
 & - \frac{1}{2} \left[\mathbf{v}_v^T \mathbf{g}(\mathbf{v}_v) + \mathbf{v}_{v+1}^T \mathbf{g}(\mathbf{v}_{v+1}) \right].
 \end{aligned}$$

Proof: Inserting (5.11) and (5.12) into (5.10) we obtain

$$\begin{aligned}
 (5.14) \quad & \frac{d}{dt} V(\mathbf{v}_v(t)) + \frac{1}{\Delta x} [G_{v+1/2} - G_{v-1/2}] \\
 & = - \frac{1}{4\Delta x} [\Delta \mathbf{v}_{v+1/2}^T D_{v+1/2} \Delta \mathbf{v}_{v+1/2} + \Delta \mathbf{v}_{v-1/2}^T D_{v-1/2} \Delta \mathbf{v}_{v-1/2}]
 \end{aligned}$$

where $G_{v+1/2}$ is the consistent numerical entropy flux given in (5.13b). Now, if the scheme considered contains more viscosity than the entropy conservative scheme, then by (5.9) the RHS of (5.14) is nonpositive and entropy stability follows. Conversely, assume that our scheme is entropy stable, i.e., that it satisfies a consistent all entropy inequality of the form

$$\frac{d}{dt} V(\mathbf{v}_v(t)) + \frac{1}{\Delta x} [\tilde{G}_{v+1/2} - \tilde{G}_{v-1/2}] \leq 0.$$

Subtracting this from (5.14) we get after multiplication by Δx

$$(5.15a) \quad H_{v+1/2} - H_{v-1/2} \geq - \frac{1}{4} [\Delta \mathbf{v}_{v+1/2}^T D_{v+1/2} \Delta \mathbf{v}_{v+1/2} + \Delta \mathbf{v}_{v-1/2}^T D_{v-1/2} \Delta \mathbf{v}_{v-1/2}].$$

Here $H_{v+1/2} = H(\mathbf{v}_{v-q+1}, \dots, \mathbf{v}_{v+q})$ stands for the entropy flux difference $G_{v+1/2} - \tilde{G}_{v+1/2}$; by Lemma 5.1 the entropy flux $G_{v+1/2}$ is essentially a three-point one and a similar form of $\tilde{G}_{v+1/2}$ implies that $H_{v+1/2}$ satisfies the essentially three-point consistency relation, see (5.6),

$$(5.15b) \quad H(\mathbf{v}_{v-q+1}, \dots, \mathbf{v}_{v-1}, \mathbf{w}, \mathbf{w}, \mathbf{v}_{v+2}, \dots, \mathbf{v}_{v+q}) = 0.$$

Choosing $\mathbf{v}_{v-1} = \mathbf{v}_v$ in (5.15a) we obtain, in view of (5.15b)

$$(5.16)_v \quad H_{v+1/2} \geq - \frac{1}{4} \Delta \mathbf{v}_{v+1/2}^T D_{v+1/2} \Delta \mathbf{v}_{v+1/2}.$$

Similarly, taking $\mathbf{v}_v = \mathbf{v}_{v+1}$ in (5.15a) yields, in view of (5.15b)

$$(5.17)_v \quad -H_{v-1/2} \geq -\frac{1}{4} \Delta \mathbf{v}_{v-1/2}^T D_{v-1/2} \Delta \mathbf{v}_{v-1/2}.$$

Adding (5.16)_v together with (5.17)_{v+1} implies

$$0 = H_{v+1/2} - H_{v+1/2} \geq -\frac{1}{2} \Delta \mathbf{v}_{v+1/2}^T D_{v+1/2} \Delta \mathbf{v}_{v+1/2}$$

or, according to (5.9),

$$(5.18) \quad \Delta \mathbf{v}_{v+1/2}^T Q_{v+1/2}^* \Delta \mathbf{v}_{v+1/2} \leq \Delta \mathbf{v}_{v+1/2}^T Q_{v+1/2} \Delta \mathbf{v}_{v+1/2}.$$

Thus the scheme considered, (5.4), contains more viscosity than the entropy conservative one, (5.3), as asserted.

We recall that the entropy conservative scheme (4.9) is a second-order accurate one. Hence, Theorem 5.2 allows us, in particular, to tune additional numerical viscosity so that we retain both the entropy stability and second-order accuracy [15], [16].

Finally, using arguments similar to those employed in the proof of the last theorem, we conclude with the following extension of [18, Theorem 6.1], dealing with systems of conservation laws.

Theorem 5.3: (Entropy stability by comparison).

Conservative schemes containing more viscosity than an entropy stable scheme are also entropy stable.

The last theorem provides us with a framework for designing entropy stable schemes. One could start with any scheme which is already known to be entropy stable and then tune in additional numerical viscosity according to the above guidelines so as to obtain a better performance in terms of, e.g, simplicity, controlled variation, avoiding entropy "glitches" etc. Virtually all the first-order entropy stable schemes fall within that framework when compared to Godunov scheme — these are the E schemes discussed in [14], [18].

APPENDIX

A. AN APPROXIMATE CELL ENTROPY EQUALITY

We consider the scheme

$$(a.1) \quad \frac{d}{dt} \mathbf{u}_v(t) = - \frac{1}{\Delta x} [\mathbf{f}_{v+1/2} - \mathbf{f}_{v-1/2}].$$

Multiplication by $U_{\mathbf{u}}^T(\mathbf{u}_v(t))$ on the left gives us

$$(a.2) \quad \frac{d}{dt} U(\mathbf{u}_v(t)) = - \frac{1}{\Delta x} [U_{\mathbf{u}}^T(\mathbf{u}_v) \mathbf{f}_{v+1/2} - U_{\mathbf{u}}^T(\mathbf{u}_v) \mathbf{f}_{v-1/2}].$$

Adding the term $\frac{1}{\Delta x} [F_{v+1/2}^* - F_{v-1/2}^*]$ to both sides, we find on account of

(3.7a)

(a.3)

$$\begin{aligned} \frac{d}{dt} U(\mathbf{u}_v(t)) + \frac{1}{\Delta x} [F_{v+1/2}^* - F_{v-1/2}^*] &= \frac{1}{2\Delta x} [(F(\mathbf{u}_{v+1}) - F(\mathbf{u}_{v-1})) \\ &- (U_{\mathbf{u}}^T(\mathbf{u}_{v+1}) \mathbf{f}(\mathbf{u}_{v+1}) - U_{\mathbf{u}}^T(\mathbf{u}_{v-1}) \mathbf{f}(\mathbf{u}_{v-1}))] \\ &+ \frac{1}{2\Delta x} \left[[U_{\mathbf{u}}(\mathbf{u}_{v+1}) - U_{\mathbf{u}}(\mathbf{u}_v)]^T \mathbf{f}_{v+1/2} + [U_{\mathbf{u}}(\mathbf{u}_v) - U_{\mathbf{u}}(\mathbf{u}_{v-1})]^T \mathbf{f}_{v-1/2} \right]. \end{aligned}$$

Integration by parts of the compatibility relation (2.2) implies that the first brackets on the right of (a.3) equal

$$\begin{aligned}
 (a.4) \quad F(\mathbf{u}) - U_{\mathbf{u}}^T(\mathbf{u})f(\mathbf{u}) \Big|_{\mathbf{u}_{v-1}}^{\mathbf{u}_{v+1}} &= \int_{\mathbf{u}_{v-1}}^{\mathbf{u}_{v+1}} F_{\mathbf{u}}^T d\mathbf{u} - U_{\mathbf{u}}^T(\mathbf{u})f(\mathbf{u}) \Big|_{\mathbf{u}_{v-1}}^{\mathbf{u}_{v+1}} \\
 &= \int_{\mathbf{u}_{v-1}}^{\mathbf{u}_{v+1}} U_{\mathbf{u}}^T f_{\mathbf{u}} d\mathbf{u} - U_{\mathbf{u}}^T f(\mathbf{u}) \Big|_{\mathbf{u}_{v-1}}^{\mathbf{u}_{v+1}} = - \int_{\mathbf{u}_{v-1}}^{\mathbf{u}_{v+1}} d\mathbf{u}^T U_{\mathbf{u}\mathbf{u}} f(\mathbf{u}).
 \end{aligned}$$

The second brackets on the right of (a.3) can be rewritten as

$$(a.5) \quad U_{\mathbf{u}}^T(\mathbf{u}) \Big|_{\mathbf{u}_v}^{\mathbf{u}_{v+1}} f_{v+1/2} - U_{\mathbf{u}}^T(\mathbf{u}) \Big|_{\mathbf{u}_{v-1}}^{\mathbf{u}_v} f_{v-1/2} = \int_{\mathbf{u}_v}^{\mathbf{u}_{v+1}} d\mathbf{u}^T U_{\mathbf{u}\mathbf{u}} f_{v+1/2} - \int_{\mathbf{u}_{v-1}}^{\mathbf{u}_v} d\mathbf{u}^T U_{\mathbf{u}\mathbf{u}} f_{v-1/2}.$$

Inserting (a.4) and (a.5) into (a.3) yields the desired approximate cell entropy equality, see Lemma 3.1,

$$\begin{aligned}
 \frac{d}{dt} U(\mathbf{u}_v(t)) + \frac{1}{\Delta x} [F_{v+1/2}^* - F_{v-1/2}^*] &= \\
 &= \frac{1}{2\Delta x} \left[\int_{\mathbf{u}_{v-1}}^{\mathbf{u}_v} d\mathbf{u}^T U_{\mathbf{u}\mathbf{u}} [f_{v-1/2} - f(\mathbf{u})] + \int_{\mathbf{u}_v}^{\mathbf{u}_{v+1}} d\mathbf{u}^T U_{\mathbf{u}\mathbf{u}} [f_{v+1/2} - f(\mathbf{u})] \right].
 \end{aligned}$$

The above proof essentially follows that of Osher in [16, Section 3]. It differs, however, in that it employs a numerical entropy flux, $F_{v+1/2}^*$, centered at mesh midpoints.

B. THE VISCOSITY FORM OF ESSENTIALLY THREE-POINT SCHEMES

We consider conservative schemes in the viscous form

$$\begin{aligned}
 (b.1) \quad \frac{d}{dt} [\mathbf{u}(\mathbf{v}_v(t))] &= \frac{1}{\Delta x} [g_{v+1/2} - g_{v-1/2}] = \frac{1}{2\Delta x} [g(\mathbf{v}_{v+1}) - g(\mathbf{v}_{v-1})] \\
 &+ \frac{1}{2\Delta x} [Q_{v+1/2} \Delta \mathbf{v}_{v+1/2} - Q_{v-1/2} \Delta \mathbf{v}_{v-1/2}].
 \end{aligned}$$

The numerical flux associated with scheme (b.1)

$$\begin{aligned}
 (b.2) \quad g_{v+1/2} &= \psi_g(\mathbf{v}_{v-p+1}, \dots, \mathbf{v}_{v+p}) \\
 &= \frac{1}{2} [g(\mathbf{v}_v) + g(\mathbf{v}_{v+1})] - \frac{1}{2} Q_{v+1/2} [\mathbf{v}_{v+1} - \mathbf{v}_v].
 \end{aligned}$$

satisfies the essentially three-point consistency relation

$$(b.3) \quad \psi_g(\mathbf{v}_{v-p+1}, \dots, \mathbf{v}_{v-1}, \mathbf{w}, \mathbf{w}, \mathbf{v}_{v+2}, \dots, \mathbf{v}_{v+p}) = g(\mathbf{w}).$$

Conversely, consider a conservative scheme

$$(b.4) \quad \frac{d}{dt} [u(\mathbf{v}_v(t))] = - \frac{1}{\Delta x} [\psi_g(\mathbf{v}_{v-p+1}, \dots, \mathbf{v}_{v+p}) - \psi_g(\mathbf{v}_{v-p}, \dots, \mathbf{v}_{v+p-1})].$$

Subject to the essentially three-point consistency relation

$$(b.5) \quad \psi_g(\mathbf{v}_{v-p+1}, \dots, \mathbf{v}_{v-1}, \mathbf{w}, \mathbf{w}, \mathbf{v}_{v+2}, \dots, \mathbf{v}_{v+p}) = g(\mathbf{w})$$

the scheme (b.4) can be put into the viscosity form (b.1) provided a numerical viscosity coefficient, $Q_{v+1/2}$, can be found such that

$$(b.6) \quad Q_{v+1/2} \Delta \mathbf{v}_{v+1/2} = g(\mathbf{v}_v) + g(\mathbf{v}_{v+1}) - 2\psi_g(\mathbf{v}_{v-p+1}, \dots, \mathbf{v}_{v+p}).$$

Using the consistency relation (b.5) we can rewrite the RHS of (b.6) as

$$\begin{aligned}
 & [\psi_{\mathbf{g}}(\dots, \mathbf{v}_\nu, \mathbf{v}_\nu, \dots) - \psi_{\mathbf{g}}(\dots, \mathbf{v}_\nu, \mathbf{v}_{\nu+1}, \dots)] \\
 & + [\psi_{\mathbf{g}}(\dots, \mathbf{v}_{\nu+1}, \mathbf{v}_{\nu+1}, \dots) - \psi_{\mathbf{g}}(\dots, \mathbf{v}_\nu, \mathbf{v}_{\nu+1}, \dots)]
 \end{aligned}$$

and equality (b.6) is then fulfilled by setting the numerical viscosity coefficient to be

$$\begin{aligned}
 (b.7) \quad Q_{\nu+1/2} &= - \int_{\xi=0}^1 \frac{\partial}{\partial \mathbf{v}_{\nu+1}} [\psi_{\mathbf{g}}(\dots, \mathbf{v}_\nu, \mathbf{v}_{\nu+1/2}(\xi), \dots)] d\xi \\
 &+ \int_{\xi=0}^1 \frac{\partial}{\partial \mathbf{v}_\nu} [\psi_{\mathbf{g}}(\dots, \mathbf{v}_{\nu+1/2}(\xi), \mathbf{v}_{\nu+1}, \dots)] d\xi.
 \end{aligned}$$

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