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Techniques in the
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Introduction

The application of particle-hole theory and second quantization techniques to one-body composite systems, such as atoms or nuclei, is an old and well-established field of endeavor (refs. 1 to 4). The techniques are especially useful for calculating ground-state properties, such as nuclear binding energies, for which one assumes two-nucleon interactions among the A nucleons within the nucleus and calculates matrix elements for the Fermi vacuum or ground state (ref. 1). (A list of symbols used in this paper appears after the references.) However, when there are two composite bodies (e.g., nuclei composed of nucleons or hadrons composed of quarks) interacting via some scattering process, it is often desirable or necessary to simplify the collision description by restricting the two-body interactions to those *between* individual target and projectile constituents (refs. 5 and 6) while ignoring the constituent interactions *within* the target and projectile themselves. This assumption is usually made in high-energy, heavy-ion collisions for which solution of the full many-body problem is intractable and often not required, since the incident-ion kinetic energy is large enough to permit nuclear binding energies to be safely ignored.

Although somewhat straightforward, the generalization of particle-hole theory and second quantization techniques from one composite body to the interaction of two composite bodies is full of potential pitfalls for the unwary. It is often easy to develop, for two-body potentials and matrix elements, expressions which appear physically reasonable when applied to slightly different or more general problems but which, in fact, lead to wrong results. The present paper avoids this through development of expressions for two-body interaction potentials in both first- and second-quantized form. The expressions are rigorously generalized for use in composite two-body scattering. Although the techniques presented are valid for any system of interacting composite bodies, the actual discussions herein are limited either to the properties of a single nucleus containing A nucleons or to an interacting nucleus-nucleus system containing A_P (projectile) and A_T (target) nucleons. In particular, these techniques are used to evaluate matrix elements for nucleus-nucleus elastic (ref. 6) and inelastic (ref. 5) scattering. For the latter, the inelastic excitations are treated as particle-hole states through use of a Tamm-Dancoff approximation (ref. 7). The present paper deals only with interactions between nonidentical composite bodies, since this simplifies the treatment of overall symmetry considerations between the projectile and target nuclei (refs. 8 to 11). Interactions between identical composites will be treated in a future paper.

First Quantization

For a single nucleus composed of A nucleons, the total nuclear potential, written in terms of the constituent two-nucleon interactions, is

$$V = \frac{1}{2} \sum_{\substack{i \\ i \neq j}}^A \sum_j^A v_{ij} \quad (1)$$

where we assume

$$v_{ij} = v_{ji} \quad (2)$$

Equation (1) can be alternatively and equivalently expressed as

$$V = \sum_{\substack{i \\ i < j}}^A \sum_j^A v_{ij} \quad (3)$$

Note that double counting (for terms where $v_{ij} = v_{ji}$) is included in equation (1) and is corrected for by the multiplicative factor $1/2$. In addition, the restriction $i \neq j$ excludes terms such as v_{ii} , which denotes self-interaction. In equation (2) the use of the restriction $i < j$ automatically excludes both double-counting and self-interaction terms.

We now need to generalize or extend these methods from the case of a single nucleus to one in which separate projectile and target nuclei interact. As is customary in nucleus-nucleus collision theory, we include only those two-nucleon interactions for which individual constituents in the projectile nucleus interact with individual constituents in the target nucleus. For convenience, the i th and j th particles, which belong to the projectile and target nuclei, respectively, are labeled as i_P and j_T . The resulting expression for the nucleus-nucleus interaction potential is

$$V = \sum_{i_P}^{A_P} \sum_{j_T}^{A_T} v_{i_P j_T} \quad (4)$$

Note that equation (4) is *not* obtained, as might be assumed (especially when second quantization is used), by merely dropping the multiplicative factor $1/2$ in the generalization of equation (1). The appropriate generalization of equation (1) to the nucleus-nucleus interaction problem is

$$V = \frac{1}{2} \left(\sum_{i_P}^{A_P} \sum_{j_T}^{A_T} v_{i_P j_T} + \sum_{i_T}^{A_T} \sum_{j_P}^{A_P} v_{i_T j_P} \right) \quad (5)$$

whereas equation (4) is actually the generalization of equation (3). Although these questions concerning the correct generalizations appear to be somewhat trivial in first quantization, they become much more significant in second quantization methodology, in which one labels states instead of particles and the proper ordering of operators is crucial to obtaining correct descriptions of the physical processes involved.

Second Quantization

In second-quantized form, the composite two-body potential is (eq. C-23 of ref. 1 with notation of current paper)

$$V = \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} \sum_{\substack{kl \\ k \neq l}} \langle ij|v|kl \rangle a_i^+ a_j^+ a_l a_k \quad (6)$$

where a is an annihilation operator, a^+ is a creation operator, and the summations over i , j , k , and l are not yet restricted to sums over individual particles but indicate sums over the complete set of states (refs. 1, 3, and 6). In coordinate space, the matrix element is

$$\langle ij|v|kl \rangle = \iint \phi_i^*(\mathbf{r}_1) \phi_j^*(\mathbf{r}_2) v(\mathbf{r}_1, \mathbf{r}_2) \phi_k(\mathbf{r}_1) \phi_l(\mathbf{r}_2) d^3\mathbf{r}_1 d^3\mathbf{r}_2 \quad (7)$$

If the constituent particles are indistinguishable, then exchange is possible. If the individual matrix element in equation (7) represents a direct term, then the corresponding exchange term (see ref. 6) is written

$$\langle ji|v|kl \rangle = \iint \phi_j^*(\mathbf{r}_1) \phi_i^*(\mathbf{r}_2) v(\mathbf{r}_1, \mathbf{r}_2) \phi_k(\mathbf{r}_1) \phi_l(\mathbf{r}_2) d^3\mathbf{r}_1 d^3\mathbf{r}_2 \quad (8)$$

where the exchange occurs symbolically in the bra vector rather than in the ket, as was done in references 1, 3, and 6 (the two conventions yield equivalent results). In equation (8) the wave functions associated with quantum numbers i and j are now associated with position coordinates \mathbf{r}_2 and \mathbf{r}_1 , respectively, rather than with \mathbf{r}_1 and \mathbf{r}_2 as in the direct term (eq. (7)). In terms of a physical description, we say that the particles with position coordinates \mathbf{r}_1 and \mathbf{r}_2 have exchanged quantum numbers or, alternatively, that the particles with quantum numbers i and j have exchanged position coordinates (i.e., the particles are exchanged).

Since \mathbf{r}_1 and \mathbf{r}_2 are dummy variables, equations (7) and (8) could also be written as

$$\langle ij|v|kl \rangle = \iint \phi_i^*(\mathbf{r}_2) \phi_j^*(\mathbf{r}_1) v(\mathbf{r}_2, \mathbf{r}_1) \phi_k(\mathbf{r}_2) \phi_l(\mathbf{r}_1) d^3\mathbf{r}_2 d^3\mathbf{r}_1 \quad (9)$$

for the direct term and as

$$\langle ji|v|kl \rangle = \iint \phi_j^*(\mathbf{r}_2) \phi_i^*(\mathbf{r}_1) v(\mathbf{r}_2, \mathbf{r}_1) \phi_k(\mathbf{r}_2) \phi_l(\mathbf{r}_1) d^3\mathbf{r}_2 d^3\mathbf{r}_1 \quad (10)$$

for the exchange term. In subsequent discussions it is assumed that

$$v(\mathbf{r}_1, \mathbf{r}_2) = v(\mathbf{r}_2, \mathbf{r}_1) \quad (11)$$

If the generalization to nucleus-nucleus interactions is made, again with the two-nucleon interactions restricted to those between projectile nucleons and target nucleons, then the direct-term matrix element is

$$\langle ij|v|kl \rangle = \iint \phi_i^*(\xi_P) \phi_j^*(\xi_T) v(\xi_P, \xi_T) \phi_k(\xi_P) \phi_l(\xi_T) d^3\xi_P d^3\xi_T \quad (12)$$

The exchange-term matrix element is thus given by

$$\langle ji|v|kl \rangle = \iint \phi_j^*(\xi_P) \phi_i^*(\xi_T) v(\xi_P, \xi_T) \phi_k(\xi_P) \phi_l(\xi_T) d^3\xi_P d^3\xi_T \quad (13)$$

The position coordinates for the projectile and target nucleons are denoted by ξ_P and ξ_T , respectively. Equations (12) and (13) may also be written as

$$\langle ij|v|kl \rangle = \langle i(P) j(T)|v|k(P) l(T) \rangle \quad (14)$$

and

$$\langle ji|v|kl \rangle = \langle j(P) i(T)|v|k(P) l(T) \rangle \quad (15)$$

to explicitly indicate their actual projectile P and target T dependences. Noting that equations (12) and (13) can be obtained from equations (7) and (8) by substituting ξ_P for \mathbf{r}_1 and ξ_T for \mathbf{r}_2 and that integration is over ξ_P and ξ_T , we can rewrite equations (12) and (13) in their alternative form by substituting ξ_P and ξ_T into equations (9) and (10) to yield

$$\langle ij|v|kl \rangle = \iint \phi_i^*(\xi_T) \phi_j^*(\xi_P) v(\xi_T, \xi_P) \phi_k(\xi_T) \phi_l(\xi_P) d^3\xi_P d^3\xi_T \quad (16)$$

and

$$\langle ji|v|kl \rangle = \iint \phi_j^*(\xi_T) \phi_i^*(\xi_P) v(\xi_T, \xi_P) \phi_k(\xi_T) \phi_l(\xi_P) d^3\xi_P d^3\xi_T \quad (17)$$

Symbolically these equations ((16) and (17)) can be written as

$$\langle ij|v|kl \rangle = \langle i(T) j(P)|v|k(T) l(P) \rangle \quad (18)$$

and

$$\langle ji|v|kl \rangle = \langle j(T) i(P)|v|k(T) l(P) \rangle \quad (19)$$

Having considered the generalization to nucleus-nucleus interactions of the matrix element part of equation (6), we now focus attention on the use and labeling of the creation and annihilation operators a_i^+ , a_j^+ , a_l , and a_k . Based upon physical arguments, it is possible a priori to label the operators as belonging to either the projectile or the target nucleus to obtain a correct result. This was previously

done in references 5 and 6, for example. This procedure, however, could lead to errors if the correct labeling is not initially assumed. In fact, the procedure is unnecessary, since the correct application of Wick's theorem (ref. 1) will *automatically* yield the correct form for the nucleus-nucleus potential when equation (6) is evaluated for the particular nucleus-nucleus collision of interest.

Evaluation of Second-Quantized Matrix Elements

For a single nucleus, the matrix elements of equation (6) have been extensively studied (refs. 1 to 4 and 7). Recall from the previous discussion of first quantization that the generalization from the single-nucleus to the nucleus-nucleus case produces different expressions for the interaction potential. As discussed in the previous section, this does not occur in second quantization, since equation (6) remains intact. Differences in second quantization occur in the evaluation of matrix elements.

In the ensuing discussions, hole creation and annihilation operators are denoted by h^+ and h with the corresponding particle operators denoted by p^+ and p . From reference 1 (p. 657), the only nonzero contractions of these operators are

$$\underline{p_i p_j^+} = \delta_{ij} \quad (20)$$

and

$$\underline{h_i h_j^+} = \delta_{ij} \quad (21)$$

All other possible contractions are identically zero. We also assume that the total Fock state for *nonidentical* nuclei can be expanded as a simple product of projectile and target states as follows:

$$|PT\rangle = |P\rangle |T\rangle \quad (22)$$

for both the initial and final states of the system. Equation (22) is *not valid* when there are *identical* nuclei in the initial or final states.

Elastic Scattering of Target and Projectile Nuclei

In this section the evaluation of the matrix elements of equation (6) for nucleus-nucleus elastic scattering is illustrated by our rederiving equation (34) of reference 6 with the notation of this paper. Thus, we want to evaluate

$$\langle P_0 T_0 | V | P_0 T_0 \rangle = \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} \sum_{\substack{kl \\ k \neq l}} \langle ij | v | kl \rangle \langle P_0 T_0 | a_i^+ a_j^+ a_l a_k | P_0 T_0 \rangle \quad (23)$$

Noting that the annihilation operators act below the Fermi sea, we have two distinct possibilities for the nucleus-nucleus interaction problem:

$$\text{if } l \leq A_P \text{ then } k \leq A_T \quad (24a)$$

or

$$\text{if } l \leq A_T \text{ then } k \leq A_P \quad (24b)$$

We exclude the possibility that k and l both act upon either the target space or the projectile space since that would, by orthogonality, require the creation operators (a_i^+ and a_j^+) to both act upon the same projectile or target space. This possibility is not desired because we are limiting the two-body interactions to those only occurring between individual projectile and target nucleons. Through use of the above restrictions, the annihilation operators can be labeled so that equation (23) becomes

$$\begin{aligned}
\langle P_0 T_0 | V | P_0 T_0 \rangle &= \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} \left(\sum_{k_T l_P}^{A_T A_P} \langle ij | v | k_T l_P \rangle \langle P_0 T_0 | a_i^\dagger a_j^\dagger a_{l_P} a_{k_T} | P_0 T_0 \rangle \right. \\
&\quad \left. + \sum_{k_P l_T}^{A_P A_T} \langle ij | v | k_P l_T \rangle \langle P_0 T_0 | a_i^\dagger a_j^\dagger a_{l_T} a_{k_P} | P_0 T_0 \rangle \right) \quad (25)
\end{aligned}$$

Utilizing the following relationship between particle operators and hole operators

$$\left. \begin{aligned} a^+ &= h \\ a &= h^+ \end{aligned} \right\} \quad (26)$$

gives, in equation (24),

$$\begin{aligned}
\langle P_0 T_0 | V | P_0 T_0 \rangle &= \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} \left(\sum_{k_T l_P}^{A_T A_P} \langle ij | v | k_T l_P \rangle \langle P_0 T_0 | h_i h_j h_{l_P}^+ h_{k_T}^+ | P_0 T_0 \rangle \right. \\
&\quad \left. + \sum_{k_P l_T}^{A_P A_T} \langle ij | v | k_P l_T \rangle \langle P_0 T_0 | h_i h_j h_{l_T}^+ h_{k_P}^+ | P_0 T_0 \rangle \right) \quad (27)
\end{aligned}$$

where all operators are written as hole operators since they act below the Fermi sea. Note that restrictions (24a) and (24b) have resulted in two distinct matrix element labelings: $\langle ij | v | k_T l_P \rangle$ and $\langle ij | v | k_P l_T \rangle$. Since ξ_P and ξ_T are dummy variables in equations (12), (13), (16), and (17), we can label these matrix elements as either

$$\langle i(T) j(P) | v | k_T(T) l_P(P) \rangle \quad \text{and} \quad \langle i(T) j(P) | v | k_P(T) l_T(P) \rangle$$

or

$$\langle i(P) j(T) | v | k_T(P) l_P(T) \rangle \quad \text{and} \quad \langle i(P) j(T) | v | k_P(P) l_T(T) \rangle$$

If we choose the obvious convention of associating k_T with ξ_T and k_P with ξ_P , then equation (27) becomes

$$\begin{aligned}
\langle P_0 T_0 | V | P_0 T_0 \rangle &= \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} \left[\sum_{k_T l_P}^{A_T A_P} \langle i(T) j(P) | v | k_T(T) l_P(P) \rangle \langle P_0 T_0 | h_i h_j h_{l_P}^+ h_{k_T}^+ | P_0 T_0 \rangle \right. \\
&\quad \left. + \sum_{k_P l_T}^{A_P A_T} \langle i(P) j(T) | v | k_P(P) l_T(T) \rangle \langle P_0 T_0 | h_i h_j h_{l_T}^+ h_{k_P}^+ | P_0 T_0 \rangle \right] \quad (28)
\end{aligned}$$

At this point, we must be careful in interpreting and evaluating the operators in equation (28), since writing $\langle i(T) j(P) |$, for example, does not imply that $h_i h_j$ is to be interpreted as $h_{i_T} h_{j_P}$. When Wick's theorem is used, only contractions in the same projectile or target space are allowed. Therefore, if we let $h_i h_j \rightarrow h_{i_T} h_{j_P}$, then the only permissible nonzero operator contractions in equation (28) would be

$$\underbrace{h_{i_T} h_{j_P} h_{l_P}^+ h_{k_T}^+} \quad \text{and} \quad \underbrace{h_{i_T} h_{j_P} h_{l_T}^+ h_{k_P}^+}$$

since

$$\underbrace{h_{i_T} h_{j_P} h_{l_P}^+ h_{k_T}^+}_{\text{and}} \quad \underbrace{h_{i_T} h_{j_P} h_{l_T}^+ h_{k_P}^+}$$

are identically zero. Thus, there would be no exchange terms. Since the double summation \sum_{ij} in equation (28) is over the complete set, the labeling of h_i and h_j as either $h_{i_T} h_{j_P}$ or $h_{i_P} h_{j_T}$ is unjustified and incorrect. The possibility of exchange occurring is accounted for by contracting h_i and h_j with both the projectile and target spaces. Clearly, then, the generalization of nucleus-nucleus interaction in equation (6) is *not*

$$V = \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} \sum_{\substack{kl \\ k \neq l}} \langle ij|v|kl \rangle a_{i_T}^+ a_{j_P}^+ a_{l_T} a_{k_P} \quad (29)$$

Evaluation of $\langle P_0 T_0 | h_i h_j h_{l_P}^+ h_{k_T}^+ | P_0 T_0 \rangle$ is accomplished through the use of Wick's theorem (refs. 1 and 2), which states that the value of the matrix element is given by the sum of all possible allowed contractions. Therefore,

$$\langle P_0 T_0 | h_i h_j h_{l_P}^+ h_{k_T}^+ | P_0 T_0 \rangle = \underbrace{h_i h_j h_{l_P}^+ h_{k_T}^+}_{\text{}} + \underbrace{h_i h_j h_{l_P}^+ h_{k_T}^+}_{\text{}} = \underbrace{h_i h_{k_T}^+}_{\text{}} \underbrace{h_j h_{l_P}^+}_{\text{}} - \underbrace{h_i h_{l_P}^+}_{\text{}} \underbrace{h_j h_{k_T}^+}_{\text{}} \quad (30)$$

where we have used the anticommutation properties of these operators. Inserting equation (21) into (30) yields

$$\langle P_0 T_0 | h_i h_j h_{l_P}^+ h_{k_T}^+ | P_0 T_0 \rangle = \delta_{i k_T} \delta_{j l_P} - \delta_{i l_P} \delta_{j k_T} \quad (31)$$

In a similar manner, $\langle P_0 T_0 | h_i h_j h_{l_T}^+ h_{k_P}^+ | P_0 T_0 \rangle$ in equation (28) is evaluated as

$$\langle P_0 T_0 | h_i h_j h_{l_T}^+ h_{k_P}^+ | P_0 T_0 \rangle = \delta_{i k_P} \delta_{j l_T} - \delta_{i l_T} \delta_{j k_P} \quad (32)$$

Substituting equations (31) and (32) into (28) yields

$$\begin{aligned} \langle P_0 T_0 | V | P_0 T_0 \rangle = & \frac{1}{2} \left\{ \sum_{k_T} \sum_{l_P}^{A_T A_P} [\langle k_T(T) l_P(P) | v | k_T(T) l_P(P) \rangle - \langle l_P(T) k_T(P) | v | k_T(T) l_P(P) \rangle] \right. \\ & \left. + \sum_{k_P} \sum_{l_T}^{A_P A_T} [\langle k_P(P) l_T(T) | v | k_P(P) l_T(T) \rangle - \langle l_T(P) k_P(T) | v | k_P(P) l_T(T) \rangle] \right\} \quad (33) \end{aligned}$$

From appendix A,

$$\sum_{k_T} \sum_{l_P}^{A_T A_P} \langle k_T(T) l_P(P) | v | k_T(T) l_P(P) \rangle = \sum_{k_P} \sum_{l_T}^{A_P A_T} \langle k_P(P) l_T(T) | v | k_P(P) l_T(T) \rangle \quad (34)$$

and

$$\sum_{k_T} \sum_{l_P}^{A_T A_P} \langle l_P(T) k_T(P) | v | k_T(T) l_P(P) \rangle = \sum_{k_P} \sum_{l_T}^{A_P A_T} \langle l_T(P) k_P(T) | v | k_P(P) l_T(T) \rangle \quad (35)$$

allow equation (33) to be written as

$$\langle P_0 T_0 | V | P_0 T_0 \rangle = \sum_{k_P} \sum_{l_T}^{A_P A_T} [\langle k_P(P) l_T(T) | v | k_P(P) l_T(T) \rangle - \langle l_T(P) k_P(T) | v | k_P(P) l_T(T) \rangle] \quad (36)$$

which includes the exchange term.

Inelastic Scattering of Target and Projectile Nuclei to One-Particle–One-Hole States

The mutual excitation of projectile and target nuclei to one-particle–one-hole states is described in the Tamm-Dancoff approximation (ref. 1) by

$$|ph\rangle = a_{\nu_T}^+ a_{\lambda_T} a_{\gamma_P}^+ a_{\alpha_P} |P_0 T_0\rangle \quad (37)$$

where the order of the target and projectile quantum-number labels is discussed in appendix B. Since a particle-hole state represents a nucleus with a hole below the Fermi sea and a particle above it, equation (37) can be written in terms of particle-hole operators as

$$|ph\rangle = p_{\nu_T}^+ h_{\lambda_T}^+ p_{\gamma_P}^+ h_{\alpha_P}^+ |P_0 T_0\rangle \quad (38)$$

The matrix element which describes inelastic scattering of target and projectile nuclei to one-particle–one-hole states is

$$\langle ph | v | P_0 T_0 \rangle = \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} \sum_{\substack{kl \\ k \neq l}} \langle ij | v | kl \rangle \langle ph | a_i^+ a_j^+ a_l a_k | P_0 T_0 \rangle \quad (39)$$

Writing these creation and annihilation operators as particle and hole operators and inserting equation (38) yields

$$\begin{aligned} \langle ph | V | P_0 T_0 \rangle = & \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} \left[\sum_{k_T}^{A_T} \sum_{l_P}^{A_P} \langle i(T) j(P) | v | k_T(T) l_P(P) \rangle \langle P_0 T_0 | h_{\alpha_P} p_{\gamma_P} h_{\lambda_T} p_{\nu_T} p_i^+ p_j^+ h_{l_P}^+ h_{k_T}^+ | P_0 T_0 \rangle \right. \\ & \left. + \sum_{k_P}^{A_P} \sum_{l_T}^{A_T} \langle i(P) j(T) | v | k_P(P) l_T(T) \rangle \langle P_0 T_0 | h_{\alpha_P} p_{\gamma_P} h_{\lambda_T} p_{\nu_T} p_i^+ p_j^+ h_{l_T}^+ h_{k_P}^+ | P_0 T_0 \rangle \right] \quad (40) \end{aligned}$$

where we have rewritten $\langle ij | v | kl \rangle$ in a manner similar to that used in equation (28). As was done in the previous section, p_i^+ and p_j^+ are left unlabeled with respect to the projectile or target space. Evaluating the operator expressions with Wick's theorem gives

$$\begin{aligned} \langle P_0 T_0 | h_{\alpha_P} p_{\gamma_P} h_{\lambda_T} p_{\nu_T} p_i^+ p_j^+ h_{l_P}^+ h_{k_T}^+ | P_0 T_0 \rangle &= h_{\alpha_P} p_{\gamma_P} h_{\lambda_T} p_{\nu_T} p_i^+ p_j^+ h_{l_P}^+ h_{k_T}^+ \\ &+ h_{\alpha_P} p_{\gamma_P} h_{\lambda_T} p_{\nu_T} p_i^+ p_j^+ h_{l_P}^+ h_{k_T}^+ \\ &= \delta_{\alpha_P l_P} \delta_{\gamma_P j} \delta_{\lambda_T k_T} \delta_{\nu_T i} \\ &- \delta_{\alpha_P l_P} \delta_{\gamma_P i} \delta_{\lambda_T k_T} \delta_{\nu_T j} \quad (41) \end{aligned}$$

and

$$\begin{aligned}
\langle P_0 T_0 | h_{\alpha_P} p_{\gamma_P} h_{\lambda_T} p_{\nu_T} p_i^+ p_j^+ h_{l_T}^+ h_{k_P}^+ | P_0 T_0 \rangle &= h_{\alpha_P} p_{\gamma_P} h_{\lambda_T} p_{\nu_T} p_i^+ p_j^+ h_{l_T}^+ h_{k_P}^+ \\
&\quad + h_{\alpha_P} p_{\gamma_P} h_{\lambda_T} p_{\nu_T} p_i^+ p_j^+ h_{l_T}^+ h_{k_P}^+ \\
&= \delta_{\alpha_P k_P} \delta_{\gamma_P i} \delta_{\lambda_T l_T} \delta_{\nu_T j} \\
&\quad - \delta_{\alpha_P k_P} \delta_{\gamma_P j} \delta_{\lambda_T l_T} \delta_{\nu_T i}
\end{aligned} \tag{42}$$

Therefore, substituting equations (41) and (42) into (40) yields the inelastic scattering matrix element

$$\begin{aligned}
\langle ph | \dot{V} | P_0 T_0 \rangle &= \frac{1}{2} [\langle \nu_T(T) \gamma_P(P) | v | \lambda_T(T) \alpha_P(P) \rangle - \langle \gamma_P(T) \nu_T(P) | v | \lambda_T(T) \alpha_P(P) \rangle \\
&\quad + \langle \gamma_P(P) \nu_T(T) | v | \alpha_P(P) \lambda_T(T) \rangle - \langle \nu_T(P) \gamma_P(T) | v | \alpha_P(P) \lambda_T(T) \rangle]
\end{aligned} \tag{43}$$

As suggested by equations (A4) and (A6) of appendix A,

$$\langle \gamma_P(T) \nu_T(P) | v | \lambda_T(T) \alpha_P(P) \rangle = \langle \nu_T(P) \gamma_P(T) | v | \alpha_P(P) \lambda_T(T) \rangle \tag{44}$$

and

$$\langle \nu_T(T) \gamma_P(P) | v | \lambda_T(T) \alpha_P(P) \rangle = \langle \gamma_P(P) \nu_T(T) | v | \alpha_P(P) \lambda_T(T) \rangle \tag{45}$$

so that equation (43) becomes

$$\langle ph | V | P_0 T_0 \rangle = \langle \nu_T(T) \gamma_P(P) | v | \lambda_T(T) \alpha_P(P) \rangle - \langle \gamma_P(T) \nu_T(P) | v | \lambda_T(T) \alpha_P(P) \rangle \tag{46}$$

This result is analogous to equation (36), which was obtained for the elastic scattering case. Note that a comparison of equation (46) with equation (36) indicates that

$$\langle P_0 T_0 | V | P_0 T_0 \rangle \approx A_P A_T \langle ph | V | P_0 T_0 \rangle \tag{47}$$

if the summations in equation (36) simply imply multiplicative factors of A_P and A_T .

Inelastic Scattering of the Projectile to a One-Particle–One-Hole State With the Target Remaining in the Ground State

The process whereby one nucleus (e.g., the projectile) is excited to a one-particle–one-hole state while the other nucleus (e.g., the target) remains in the ground state may sometimes be forbidden by angular momentum conservation or by some similar selection rule. Nevertheless, the process is physically possible and provides a worthwhile illustration of second-quantized techniques. In the ensuing discussion we assume that the projectile nucleus is excited while the target nucleus remains in its ground state. The techniques displayed, however, can be applied to the situation in which the target is excited and the projectile remains in the ground state by merely swapping the projectile for the target and vice versa in the development.

The combined target ground state plus projectile nucleus particle-hole state is written as

$$|ph\rangle \equiv a_{\gamma_P}^+ a_{\alpha_P} |P_0 T_0\rangle = p_{\gamma_P}^+ h_{\alpha_P}^+ |P_0 T_0\rangle \tag{48}$$

We now evaluate the collision matrix element as

$$\langle ph | V | P_0 T_0 \rangle = \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} \sum_{\substack{kl \\ k \neq l}} \langle ij | v | kl \rangle \langle ph | a_i^+ a_j^+ a_l a_k | P_0 T_0 \rangle \tag{49}$$

Rewriting $\langle ij|v|kl \rangle$ as before (in eqs. (28) and (40)) and substituting particle and hole operators for the creation and annihilation operators yields

$$\begin{aligned} \langle ph|V|P_0T_0 \rangle = & \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} \left[\sum_{k_T}^{A_T} \sum_{l_P}^{A_P} \langle i(T) j(P)|v|k_T(T) l_P(P) \rangle \left(\langle P_0T_0|h_{\alpha_P} p_{\gamma_P} p_i^+ h_j h_{l_P}^+ h_{k_T}^+ |P_0T_0 \rangle \right. \right. \\ & + \langle P_0T_0|h_{\alpha_P} p_{\gamma_P} h_i p_j^+ h_{l_P}^+ h_{k_T}^+ |P_0T_0 \rangle \left. \right) + \sum_{k_P}^{A_P} \sum_{l_T}^{A_T} \langle i(P) j(T)|v|k_P(P) l_T(T) \rangle \\ & \left. \times \left(\langle P_0T_0|h_{\alpha_P} p_{\gamma_P} p_i^+ h_j h_{l_T}^+ h_{k_P}^+ |P_0T_0 \rangle + \langle P_0T_0|h_{\alpha_P} p_{\gamma_P} h_i p_j^+ h_{l_T}^+ h_{k_P}^+ |P_0T_0 \rangle \right) \right] \quad (50) \end{aligned}$$

where the additional terms result from the fact that a_i^+ , a_j^+ , a_k , and a_i are free to act on both projectile and target spaces. Evaluating the operators through the use of Wick's theorem yields

$$\langle P_0T_0|h_{\alpha_P} p_{\gamma_P} p_i^+ h_j h_{l_P}^+ h_{k_T}^+ |P_0T_0 \rangle = \underbrace{h_{\alpha_P} p_{\gamma_P} p_i^+ h_j h_{l_P}^+ h_{k_T}^+}_{\delta_{\alpha_P l_P} \delta_{\gamma_P i} \delta_{j k_T}} = -\delta_{\alpha_P l_P} \delta_{\gamma_P i} \delta_{j k_T} \quad (51)$$

$$\langle P_0T_0|h_{\alpha_P} p_{\gamma_P} h_i p_j^+ h_{l_P}^+ h_{k_T}^+ |P_0T_0 \rangle = \underbrace{h_{\alpha_P} p_{\gamma_P} h_i p_j^+ h_{l_P}^+ h_{k_T}^+}_{\delta_{\alpha_P l_P} \delta_{\gamma_P j} \delta_{i k_T}} = \delta_{\alpha_P l_P} \delta_{\gamma_P j} \delta_{i k_T} \quad (52)$$

$$\langle P_0T_0|h_{\alpha_P} p_{\gamma_P} p_i^+ h_j h_{l_T}^+ h_{k_P}^+ |P_0T_0 \rangle = \underbrace{h_{\alpha_P} p_{\gamma_P} p_i^+ h_j h_{l_T}^+ h_{k_P}^+}_{\delta_{\alpha_P k_P} \delta_{\gamma_P i} \delta_{j l_T}} = \delta_{\alpha_P k_P} \delta_{\gamma_P i} \delta_{j l_T} \quad (53)$$

and

$$\langle P_0T_0|h_{\alpha_P} p_{\gamma_P} h_i p_j^+ h_{l_T}^+ h_{k_P}^+ |P_0T_0 \rangle = \underbrace{h_{\alpha_P} p_{\gamma_P} h_i p_j^+ h_{l_T}^+ h_{k_P}^+}_{-\delta_{\alpha_P k_P} \delta_{\gamma_P j} \delta_{i l_T}} = -\delta_{\alpha_P k_P} \delta_{\gamma_P j} \delta_{i l_T} \quad (54)$$

Inserting equations (51) to (54) into (50) yields the following collision matrix element:

$$\begin{aligned} \langle ph|V|P_0T_0 \rangle = & \frac{1}{2} \left\{ \sum_{k_T}^{A_T} \left[\langle k_T(T) \gamma_P(P)|v|k_T(T) \alpha_P(P) \rangle - \langle \gamma_P(T) k_T(P)|v|k_T(T) \alpha_P(P) \rangle \right] \right. \\ & \left. + \sum_{l_T}^{A_T} \left[\langle \gamma_P(P) l_T(T)|v|\alpha_P(P) l_T(T) \rangle - \langle l_T(P) \gamma_P(T)|v|\alpha_P(P) l_T(T) \rangle \right] \right\} \quad (55) \end{aligned}$$

Again as suggested in appendix A,

$$\sum_{k_T}^{A_T} \langle k_T(T) \gamma_P(P)|v|k_T(T) \alpha_P(P) \rangle = \sum_{l_T}^{A_T} \langle \gamma_P(P) l_T(T)|v|\alpha_P(P) l_T(T) \rangle \quad (56)$$

and

$$\sum_{k_T}^{A_T} \langle \gamma_P(T) k_T(P)|v|k_T(T) \alpha_P(P) \rangle = \sum_{l_T}^{A_T} \langle l_T(P) \gamma_P(T)|v|\alpha_P(P) l_T(T) \rangle \quad (57)$$

which are inserted into equation (55) to give our final result:

$$\langle ph|V|P_0T_0 \rangle = \sum_{l_T}^{A_T} \left[\langle \gamma_P(P) l_T(T)|v|\alpha_P(P) l_T(T) \rangle - \langle l_T(P) \gamma_P(T)|v|\alpha_P(P) l_T(T) \rangle \right] \quad (58)$$

Comparing equation (58) with the elastic scattering result (eq. (36)) and with the particle-hole mutual-excitation result (eq. (46)), we note that the summations in the final matrix element remain only for those nuclei which remain in their ground states. The summations for those nuclei which undergo particle-hole excitations reduce to a single term since the operators for the final particle-hole state represent a particular single-particle configuration for that excited nucleus.

Summary

The methods of second quantization and the particle-hole model in the Tamm-Dancoff approximation have been generalized from their usual application to one composite body (such as a nucleus) to the study of interactions between two nonidentical composite bodies. The calculation techniques were illustrated through their application to the following: (1) elastic scattering of projectile and target nuclei; (2) projectile-nucleus-target-nucleus inelastic scattering with both nuclei excited to one-particle-one-hole states; and (3) inelastic scattering in which the projectile nucleus is excited to a one-particle-one-hole state while the target nucleus remains in its ground state.

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Appendix A

Derivation of Equations (34), (35), (44), (45), (56), and (57)

Recall from equations (12) and (14) for the direct term that

$$\langle i(P) j(T) | v | k(P) l(T) \rangle = \iint \phi_i^*(\xi_P) \phi_j^*(\xi_T) v(\xi_P, \xi_T) \phi_k(\xi_P) \phi_l(\xi_T) d^3 \xi_P d^3 \xi_T \quad (\text{A1})$$

whereas the exchange term (from eqs. (13) and (15)) is

$$\langle j(P) i(T) | v | k(P) l(T) \rangle = \iint \phi_j^*(\xi_P) \phi_i^*(\xi_T) v(\xi_P, \xi_T) \phi_k(\xi_P) \phi_l(\xi_T) d^3 \xi_P d^3 \xi_T \quad (\text{A2})$$

Since the order of the wave functions in equations (A1) and (A2) is unimportant, equation (A1) yields

$$\begin{aligned} & \iint \phi_i^*(\xi_P) \phi_j^*(\xi_T) v(\xi_P, \xi_T) \phi_k(\xi_P) \phi_l(\xi_T) d^3 \xi_P d^3 \xi_T \\ &= \iint \phi_j^*(\xi_T) \phi_i^*(\xi_P) v(\xi_P, \xi_T) \phi_l(\xi_T) \phi_k(\xi_P) d^3 \xi_P d^3 \xi_T \end{aligned} \quad (\text{A3})$$

or

$$\langle i(P) j(T) | v | k(P) l(T) \rangle = \langle j(T) i(P) | v | l(T) k(P) \rangle \quad (\text{A4})$$

which validates equation (45). Similarly, from equation (A2),

$$\begin{aligned} & \iint \phi_j^*(\xi_P) \phi_i^*(\xi_T) v(\xi_P, \xi_T) \phi_k(\xi_P) \phi_l(\xi_T) d^3 \xi_P d^3 \xi_T \\ &= \iint \phi_i^*(\xi_T) \phi_j^*(\xi_P) v(\xi_P, \xi_T) \phi_l(\xi_T) \phi_k(\xi_P) d^3 \xi_P d^3 \xi_T \end{aligned} \quad (\text{A5})$$

or

$$\langle j(P) i(T) | v | k(P) l(T) \rangle = \langle i(T) j(P) | v | l(T) k(P) \rangle \quad (\text{A6})$$

which validates equation (44).

Relabeling equation (A4), we have

$$\langle i_P(P) j_T(T) | v | k_P(P) l_T(T) \rangle = \langle j_T(T) i_P(P) | v | l_T(T) k_P(P) \rangle \quad (\text{A7})$$

which, for elastic scattering ($i_P = k_P$ and $j_T = l_T$), becomes

$$\langle k_P(P) l_T(T) | v | k_P(P) l_T(T) \rangle = \langle l_T(T) k_P(P) | v | l_T(T) k_P(P) \rangle \quad (\text{A8})$$

Summing equation (A8) yields

$$\sum_{l_T} \sum_{k_P}^{A_T A_P} \langle k_P(P) l_T(T) | v | k_P(P) l_T(T) \rangle = \sum_{l_T} \sum_{k_P}^{A_T A_P} \langle l_T(T) k_P(P) | v | l_T(T) k_P(P) \rangle \quad (\text{A9})$$

Since l_T and k_P are summed over A_T and A_P , we can relabel the right-hand side of equation (A9) by interchanging l and k to yield

$$\sum_{l_T} \sum_{k_P}^{A_T A_P} \langle k_P(P) l_T(T) | v | k_P(P) l_T(T) \rangle = \sum_{k_T} \sum_{l_P}^{A_T A_P} \langle k_T(T) l_P(P) | v | k_T(T) l_P(P) \rangle \quad (\text{A10})$$

which validates equation (34). In a similar manner, the elastic scattering exchange term from equations (A5), (A6), and (A9) becomes

$$\sum_{l_T}^{A_T} \sum_{k_P}^{A_P} \langle l_T(P) k_P(T) | v | k_P(P) l_T(T) \rangle = \sum_{l_T}^{A_T} \sum_{k_P}^{A_P} \langle k_P(T) l_T(P) | v | l_T(T) k_P(P) \rangle \quad (\text{A11})$$

Again, we can relabel the summation indices on the right-hand side by interchanging l and k . Hence, equation (A11) becomes

$$\sum_{l_T}^{A_T} \sum_{k_P}^{A_P} \langle l_T(P) k_P(T) | v | k_P(P) l_T(T) \rangle = \sum_{k_T}^{A_T} \sum_{l_P}^{A_P} \langle l_P(T) k_T(P) | v | k_T(T) l_P(P) \rangle \quad (\text{A12})$$

which validates equation (35).

Finally, the validity of equations (56) and (57) is demonstrated. Relabeling equations (A4) and (A6) yields a direct term

$$\langle i_P(P) j_T(T) | v | \alpha_P(P) l_T(T) \rangle = \langle j_T(T) i_P(P) | v | l_T(T) \alpha_P(P) \rangle \quad (\text{A13})$$

and an exchange term

$$\langle j_T(P) i_P(T) | v | \alpha_P(P) l_T(T) \rangle = \langle i_P(T) j_T(P) | v | l_T(T) \alpha_P(P) \rangle \quad (\text{A14})$$

Since the target remains in its ground state, $j_T = l_T$ and a summation over the target indices gives

$$\sum_{l_T}^{A_T} \langle i_P(P) l_T(T) | v | \alpha_P(P) l_T(T) \rangle = \sum_{l_T}^{A_T} \langle l_T(T) i_P(P) | v | l_T(T) \alpha_P(P) \rangle \quad (\text{A15})$$

for the direct term and

$$\sum_{l_T}^{A_T} \langle l_T(P) i_P(T) | v | \alpha_P(P) l_T(T) \rangle = \sum_{l_T}^{A_T} \langle i_P(T) l_T(P) | v | l_T(T) \alpha_P(P) \rangle \quad (\text{A16})$$

for the exchange term. Since l_T is summed over A_T , we can relabel the right-hand side of equations (A15) and (A16) by interchanging l and k to yield

$$\sum_{l_T}^{A_T} \langle i_P(P) l_T(T) | v | \alpha_P(P) l_T(T) \rangle = \sum_{k_T}^{A_T} \langle k_T(T) i_P(P) | v | k_T(T) \alpha_P(P) \rangle \quad (\text{A17})$$

and

$$\sum_{l_T}^{A_T} \langle l_T(P) i_P(T) | v | \alpha_P(P) l_T(T) \rangle = \sum_{k_T}^{A_T} \langle i_P(T) k_T(P) | v | k_T(T) \alpha_P(P) \rangle \quad (\text{A18})$$

which validate equations (56) and (57), respectively.

Appendix B

Nucleus-Nucleus Tamm-Dancoff State

In this appendix, the validity of the projectile-nucleus–target-nucleus Tamm-Dancoff state (eq. (37))

$$|ph\rangle = a_{\nu_T}^+ a_{\lambda_T} a_{\gamma_P}^+ a_{\alpha_P} |P_0 T_0\rangle \quad (\text{B1})$$

is discussed. If we start with the assumption that a two-particle–two-hole state in a *single* nucleus of A nucleons (ref. 1, p. 455; ref. 2, p. 163)

$$|2p-2h\rangle = a_i^+ a_j^+ a_l a_k |A\rangle \quad (\text{B2})$$

is analogous to a one-particle–one-hole state in each nucleus of a pair of nuclei, then the expression comparable to equation (B1) would be

$$|(ph)'\rangle = a_{\nu_T}^+ a_{\gamma_P}^+ a_{\lambda_T} a_{\alpha_P} |P_0 T_0\rangle \quad (\text{B3})$$

The correctness of equation (B1) *rather* than (B3) for the nucleus-nucleus case is supported by the fact that the use of equation (B3) leads to a negative sign for the direct term and to a positive sign for the exchange term, a result which is contrary to the usual convention. This correctness is because the order of the operators $a_{\gamma_P}^+$ and a_{λ_T} in equation (B3) is opposite to their order in (B1). In addition, there is no a priori reason to require mutual one-particle–one-hole excitations in colliding nuclei to be exactly analogous to a two-particle–two-hole excitation in a single nucleus. Finally, the single-nucleus one-particle–one-hole state

$$|ph\rangle = a_{\gamma_P}^+ a_{\alpha_P} |P_0 T_0\rangle \quad (\text{B4})$$

used in equation (48) follows naturally from equation (B1) rather than from (B3).

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Symbols

A	mass number of an arbitrary nucleus
A_P	mass number of projectile nucleus
A_T	mass number of target nucleus
a	annihilation operator
a^+	creation operator
$\langle ij v kl \rangle$	nucleon-nucleon two-body matrix element
h	hole annihilation operator
h^+	hole creation operator
i, j, k, l	nucleon quantum numbers
P	projectile
p	particle annihilation operator
p^+	particle creation operator
$ ph \rangle$	particle-hole state vector
$ PT \rangle$	total projectile-target state vector
$ P_0T_0 \rangle$	total projectile-target initial state vector
\mathbf{r}	position vector, fm
T	target
V	nucleus-nucleus interaction potential, MeV
v	nucleon-nucleon interaction potential, MeV
$\alpha, \lambda, \gamma, \nu$	nucleon quantum numbers used in particle-hole state vectors
δ_{ij}	Kronecker delta
ξ_P	projectile-nucleon position vector
ξ_T	target-nucleon position vector
ϕ	single-particle wave function
\sqcup	contraction symbol used in Wick's theorem

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