

NASA
Reference
Publication
1160

1986

Comparative Accuracy of
the Albedo, Transmission
and Absorption for
Selected Radiative
Transfer Approximations

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PREFACE

In recent years much attention has been devoted to the development of simple and computationally fast analytical approximations to the radiative transfer equation. This has largely been the result of the need to parameterize the radiative properties of clouds and aerosols in general circulation climate models. In these and other climate model applications, it becomes necessary to rapidly calculate the plane albedo, total transmission and fractional absorption as a function of optical thickness and solar zenith angle for a wide range of atmospheric conditions.

This report contains a comparison of the absolute and relative accuracy of eight different radiative transfer approximations as a function of optical thickness ($0.1 \leq \tau_t \leq 100$) and cosine of the solar zenith angle ($0 \leq \mu_0 \leq 1$). Contour plots of the approximate plane albedo, total transmission and fractional absorption are presented for each model, as well as contour plots for the relative and absolute errors in each model. These results have been obtained for four values of the single scattering albedo (*viz.*, 1.0, 0.99, 0.9 and 0.8) and for a cloud phase function having an asymmetry factor $g = 0.843$. The radiative transfer approximations considered in this report are asymptotic theory for thick layers and the following widely used two-stream approximations: Coakley-Chýlek's models 1 and 2, Meador-Weaver, Eddington, delta-Eddington, PIFM and delta-discrete ordinates. The baseline computations for these comparisons were obtained using the doubling method.

The authors are grateful to Howard G. Meyer for aid in performing the computations and in generating the numerous graphical results. The research was in part supported by NASA Grant NAG 5-309.

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COMPARATIVE ACCURACY OF THE ALBEDO, TRANSMISSION AND ABSORPTION FOR SELECTED RADIATIVE TRANSFER APPROXIMATIONS

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SUMMARY

In the present study the plane albedo, total transmission and fractional absorption predicted by eight different radiative transfer approximations are compared with doubling computations as a function of optical thickness, solar zenith angle and single scattering albedo. The phase function used in these computations is intended to be representative of a fair weather cumulus cloud having an asymmetry factor $g = 0.843$. These results show that specific regions can be identified where one radiative transfer approximation is more accurate than another. For climate model applications in which a single radiative transfer approximation is required to accurately model reflection, transmission and absorption at all optical depths, solar zenith angles and single scattering albedos, our results show that this requirement is not satisfied by any of the approximate methods. However, for remote sensing applications involving flux measurements of either reflected or transmitted radiation, it is generally possible to use these results as a guide in selecting the most accurate approximation to use.

Of the models we have examined, asymptotic theory (Sobolev, 1975, van de Hulst, 1980), although seldom used, is the most accurate approximation for optical thicknesses greater than about 6, where errors less than 5% are obtained for reflection, transmission and absorption for all solar zenith angles and all single scattering albedos. Among the two-stream methods, Coakley-Chýlek's model 1 is accurate to within 5% for optically thin atmospheres having optical thicknesses less than about 0.2 for most values of the solar zenith angle. Though the accuracies of the delta-Eddington and Meador-Weaver approximations are less easily summarized, it can generally be concluded that the delta-Eddington approximation is the most accurate two-stream method for conservative scattering in optically thick atmospheres, whereas the Meador-Weaver approximation is the most accurate two-stream method for nonconservative scattering ($\omega_0 \leq 0.9$). In general, the PIFM and delta-discrete ordinates methods are quite similar to, but slightly worse than, the delta-Eddington method over the entire range of variables. The other two-stream models presented in this report are the Eddington approximation and Coakley-Chýlek's model 2. In both of these methods it is always possible to find one of the other six models for which more accurate estimates of the plane albedo, total transmission or fractional absorption are obtained. This precludes the use of either of these models in any radiative transfer problem involving large asymmetry factors, such as those examined in this report.

INTRODUCTION

In many remote sensing and climate model applications, it is necessary to calculate the plane albedo, total transmission and fractional absorption of atmospheric layers as a function of optical thickness and solar zenith angle. Among the simplest and most widely used approximations to the radiative transfer equation are the two-stream approximations. These approximations

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have been discussed and analyzed by Irvine (1968), Kawata and Irvine (1970), Shettle and Weinman (1970), Liou (1973, 1974), Coakley and Chýlek (1975), Joseph *et al.* (1976) and Schaller (1979). Recently, Meador and Weaver (1980) and Zdunkowski *et al.* (1980) have shown that a whole class of approximate two-stream solutions can be reduced to a standard form with only a few coefficients. These coefficients depend on the cosine of the solar zenith angle (μ_0), single scattering albedo (ω_0) and one or more moments of the single scattering phase function, while the general equations for the plane albedo and total transmission depend in addition on the total optical thickness of the layer (τ_t). In spite of this long history of development, no generally agreed-upon variable ranges exist within which one can use a given approximation with assurance that accurate reflected, transmitted and absorbed flux densities will be obtained.

Once a layer is sufficiently thick, an asymptotic regime is established within the layer which permits the extension of the plane albedo, total transmission and fractional absorption to comparable values for a semi-infinite layer (van de Hulst, 1968a, 1980; Sobolev, 1975). In the present study, this asymptotic method for thick layers is compared in accuracy with selected two-stream approximations for $0 \leq \mu_0 \leq 1$ and $0.1 \leq \tau_t \leq 100$. Results have been obtained for four values of the single scattering albedo (*viz.*, 1.0, 0.99, 0.9 and 0.8) and for a cloud phase function having an asymmetry factor $g = 0.843$. Following the suggestion of Wiscombe and Joseph (1977), who considered the accuracy of the Eddington approximation for $g \leq 0.5$, both absolute and relative errors in the plane albedo, total transmission and fractional absorption are presented in this report. We concentrate our comparisons on asymptotic theory for thick layers and Coakley-Chýlek's models 1 and 2, Meador-Weaver, Eddington, delta-Eddington, PIFM and delta-discrete ordinates approximations.

MULTIPLE SCATTERING COMPUTATIONS

To provide a baseline for assessing the accuracy of various radiative transfer approximations, numerical computations were performed for a model atmosphere composed of cloud particles. Fig. 1 illustrates the phase function employed in these calculations, which is based on Mie theory for a wavelength $\lambda = 0.754 \mu\text{m}$, refractive index $m = 1.332$, and a size distribution of particles of a given radius proportional to $r^6 \exp(-1.6187r)$, where r is the particle radius in μm . This distribution of particles is a gamma distribution with an effective radius of $5.56 \mu\text{m}$ and an effective variance of 0.111, and is considered typical of fair weather cumulus (FWC) clouds (Hansen, 1971). This distribution is similar to Deirmendjian's (1963) cloud C. 1 model, except that the effective radius in Deirmendjian's model is $6.0 \mu\text{m}$.

In performing our radiative transfer calculations, we have followed the common practice of expressing the product of the single scattering albedo ω_0 and phase function $\Phi(\cos \Theta)$, as a finite expansion of Legendre polynomials of the form

$$\omega_0 \Phi(\cos \Theta) = \sum_{l=0}^L \omega_l P_l(\cos \Theta), \quad (1)$$

where Θ is the scattering angle and $P_l(\cos \Theta)$ a Legendre polynomial of order l . With this definition, the phase function obeys the normalization condition

$$\frac{1}{2} \int_{-1}^1 \Phi(\cos \Theta) d(\cos \Theta) = 1, \quad (2)$$

with the asymmetry factor g related to the Legendre coefficient ω_1 by

$$\frac{1}{2} \int_{-1}^1 \Phi(\cos \Theta) \cos \Theta d(\cos \Theta) = g = \omega_1 / (3 \omega_0). \quad (3)$$

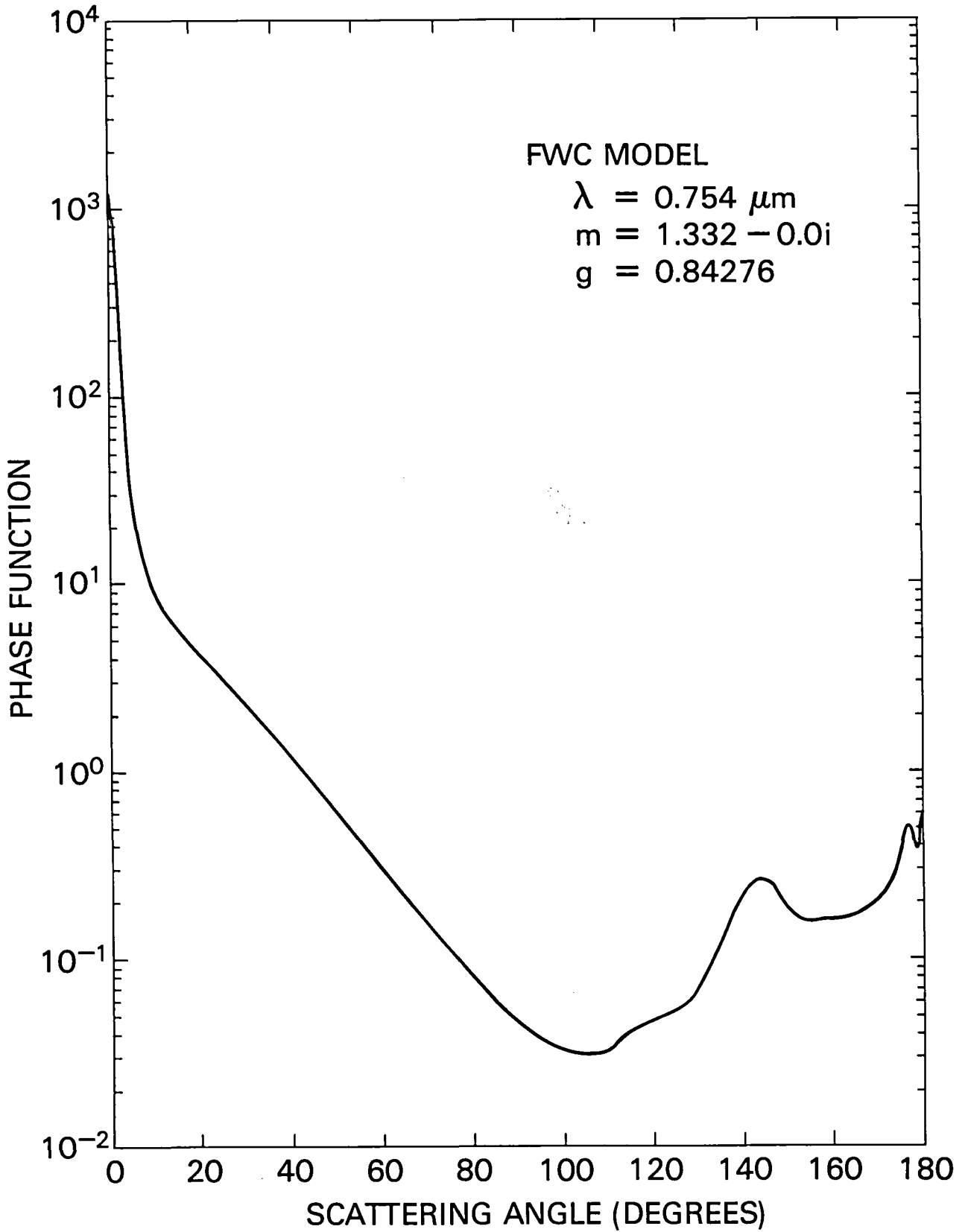


Figure 1. Phase function as a function of scattering angle for a fair weather cumulus (FWC) size distribution given by $n(r) \propto r^6 \exp(-1.6187r)$, where $\lambda = 0.754 \mu\text{m}$ and $m = 1.332 - 0.0i$.

The asymmetry factor for the FWC cloud model is $g = 0.843$.

Multiple scattering calculations were performed for the azimuth-independent term of the reflection and transmission functions using the doubling method described by Hansen and Travis (1974), together with the invariant imbedding initialization described by King (1983). In terms of these functions, the azimuthally-averaged reflected $I^0(0, -\mu)$ and transmitted $I^0(\tau_t, \mu)$ intensities from a horizontally homogeneous atmosphere illuminated from above by a parallel beam of radiation of incident flux density F_0 may be expressed as

$$I^0(0, -\mu) = (\mu_0 F_0 / \pi) R^0(\tau_t; \mu, \mu_0), \quad (4)$$

$$I^0(\tau_t, \mu) = (\mu_0 F_0 / \pi) T^0(\tau_t; \mu, \mu_0), \quad (5)$$

where τ_t is the total optical thickness of the atmosphere, μ_0 the cosine of the solar zenith angle and μ the cosine of the zenith angle with respect to the outward normal ($0 \leq \mu, \mu_0 \leq 1$).

In terms of the azimuth-independent reflection function $R^0(\tau_t; \mu, \mu_0)$ and transmission function $T^0(\tau_t; \mu, \mu_0)$, the plane albedo $[r(\tau_t, \mu_0)]$, total transmission $[t(\tau_t, \mu_0)]$ and fractional absorption $[a(\tau_t, \mu_0)]$ of the layer are given by

$$r(\tau_t, \mu_0) = 2 \int_0^1 R^0(\tau_t; \mu, \mu_0) \mu d\mu, \quad (6)$$

$$t(\tau_t, \mu_0) = 2 \int_0^1 T^0(\tau_t; \mu, \mu_0) \mu d\mu + \exp(-\tau_t / \mu_0), \quad (7)$$

$$a(\tau_t, \mu_0) = 1 - r(\tau_t, \mu_0) - t(\tau_t, \mu_0). \quad (8)$$

Due to the use of a highly anisotropic phase function and the requirement that accurate computations of the plane albedo, total transmission and fractional absorption be obtained, we have subdivided the μ angular interval using a Gaussian quadrature of order 80 (King, 1983).

Since the major purpose of this study is to examine the accuracy of various radiative transfer approximations over a wide range of optical depths, solar zenith angles and single scattering albedos, we have ignored the effects of surface reflection. As a result, the reflection and transmission functions appearing in (4) - (7) apply to those of an isolated cloud layer only.

RADIATIVE TRANSFER APPROXIMATIONS

Asymptotic Theory

When the optical thickness is sufficiently large, an asymptotic regime is established within the layer such that the reflection and transmission functions can be expressed in terms of functions applicable to a semi-infinite layer (van de Hulst, 1968a, 1980). From these expressions, coupled with the definitions of plane albedo and total transmission given previously, it can be shown that the asymptotic theory approximations for the plane albedo $[\hat{r}(\tau_t, \mu_0)]$, total transmission $[\hat{t}(\tau_t, \mu_0)]$ and fractional absorption $[\hat{a}(\tau_t, \mu_0)]$ of the layer are given by

$$\hat{r}(\tau_t, \mu_0) = r_\infty(\mu_0) - \frac{m n l}{1 - l^2} \frac{K(\mu_0) e^{-2k\tau_t}}{e^{-2k\tau_t}}, \quad (9)$$

$$\hat{r}(\tau_t, \mu_0) = \frac{m n}{1 - l 2 e^{-2k\tau_t}} K(\mu_0) e^{-k\tau_t} , \quad (10)$$

$$\hat{a}(\tau_t, \mu_0) = 1 - \hat{r}(\tau_t, \mu_0) - \hat{l}(\tau_t, \mu_0) . \quad (11)$$

In these expressions $r_\infty(\mu_0)$ is the plane albedo of a semi-infinite atmosphere, $K(\mu_0)$ the escape function, k the diffusion exponent describing the attenuation of radiation in the diffusion domain, and m , n and l constants which depend primarily on the single scattering albedo and asymmetry factor (King, 1981).

The escape function and diffusion exponent, as well as other asymptotic functions and constants appearing in (9) and (10), can be obtained by applying the asymptotic fitting method of van de Hulst (1968b). In this method, computational results from the doubling method are fit to asymptotic expressions for the plane albedo, diffuse transmission and internal intensity field as a function of the optical depth. The functions $K(\mu_0)$ and $r_\infty(\mu_0)$ thus obtained are illustrated in King and Harshvardhan (1986) as a function of μ_0 for the FWC phase function and for four values of ω_0 . In addition to these functions, the plane albedo and total transmission of thick layers depend on the constants m , n , l and k [cf. Eqs. (9) and (10)]. Each of these constants is strongly ω_0 dependent with a somewhat weaker dependence on g . King (1981) has shown that m , n and l can be well described by a function of a similarity parameter s , defined by

$$s = \left(\frac{1 - \omega_0}{1 - \omega_0 g} \right)^{1/2} , \quad (12)$$

where s reduces to $(1 - \omega_0)^{1/2}$ for isotropic scattering and spans the range 0 ($\omega_0 = 1$) to 1 ($\omega_0 = 0$).

Although the diffusion exponent k does not obey such a similarity relationship, the function $k/(1 - \omega_0 g)$ does. Fig. 2 illustrates $k/(1 - \omega_0 g)$ as a function of s for both the FWC ($\omega_0 = 0.99, 0.9$ and 0.8) and Henyey-Greenstein phase functions for varying values of ω_0 (0.9999, 0.999, 0.996, 0.99, 0.96, 0.9, 0.8 and 0.6) and g (0.8, 0.85 and 0.9). The computational results presented in Fig. 2 were fit to $k/(1 - \omega_0 g)$ as a function of s . The formulas for m , n , l and $k/(1 - \omega_0 g)$ are summarized in Table 1. The formula for m is identical to that obtained by King (1981), whereas the coefficients in the formulas for n and l differ slightly in order to give a better fit for small values of s .

For conservative scattering, when $r_\infty(\mu_0) = n = l = 1$ and $m = k = 0$, the asymptotic expressions for the plane albedo and total transmission given in (9) and (10) are indeterminate. Expanding n , l , m and k to first order in s , it can be shown that (9) and (10) can be rewritten as (King, 1981)

$$\hat{r}(\tau_t, \mu_0) = 1 - \frac{4K(\mu_0)}{3(1 - g)(\tau_t + 2q_0)} , \quad (13)$$

$$\hat{l}(\tau_t, \mu_0) = \frac{4K(\mu_0)}{3(1 - g)(\tau_t + 2q_0)} , \quad (14)$$

where q_0 is the extrapolation length. The reduced extrapolation length $q' = (1 - g)q_0$ is known to range between 0.709 and 0.715 for all possible phase functions (van de Hulst, 1980), and can be well approximated by 0.714 for anisotropic cloud phase functions (King, 1981). Thus it is seen that the plane albedo and total transmission in optically thick, conservatively scattering layers are a function of $(1 - g)\tau_t$, with all of the solar zenith angle dependence contained in $K(\mu_0)$.

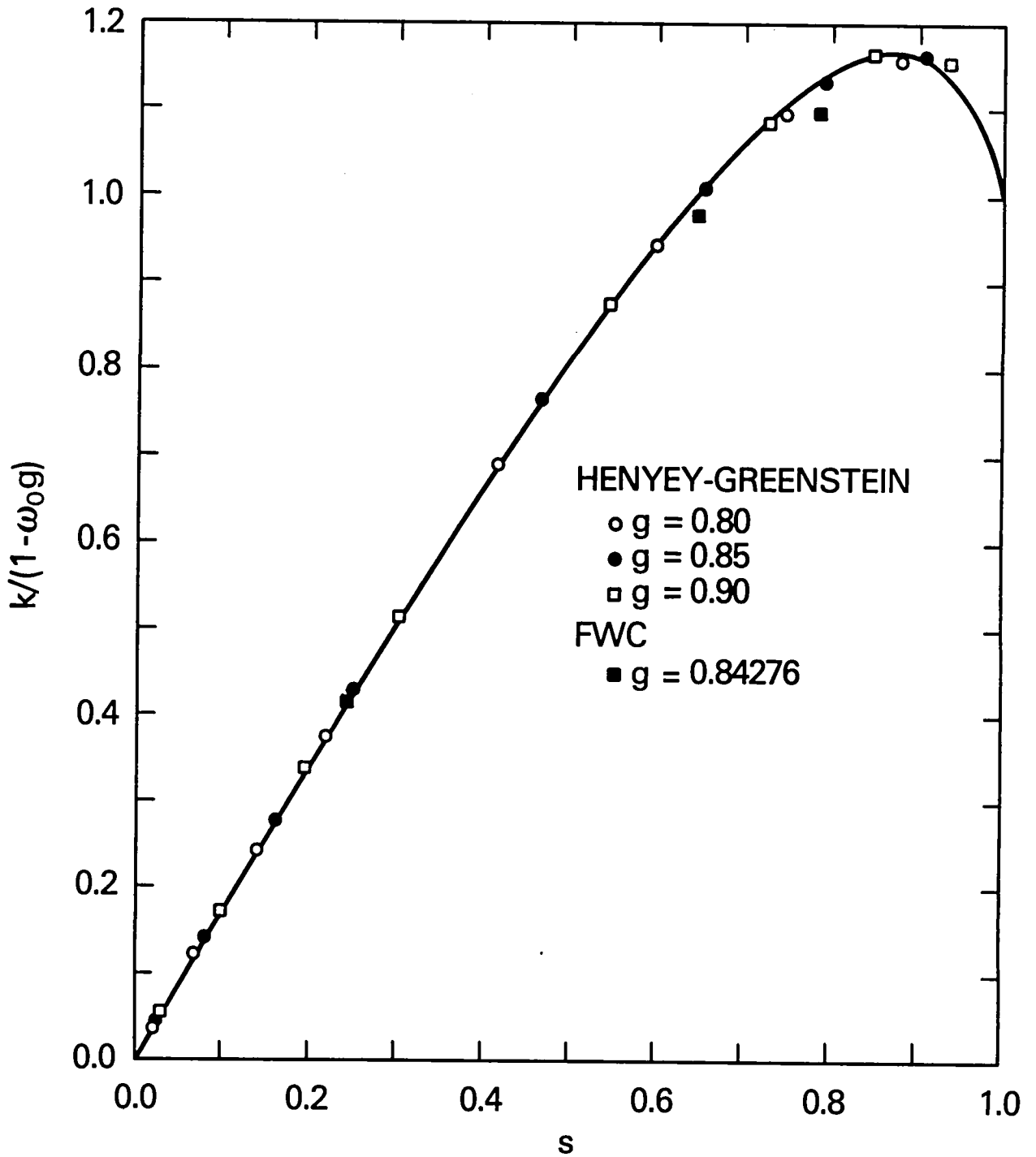


Figure 2. $k/(1 - \omega_0 g)$ as a function of similarity parameter $s = [(1 - \omega_0)/(1 - \omega_0 g)]^{1/2}$, where k is the diffusion exponent. The symbols represent values obtained by numerical computation for FWC and Henyey-Greenstein phase functions, and the curve the result of a least-squares fit to an analytic equation.

TABLE 1

Similarity relations satisfied by the constants which arise in asymptotic expressions for the plane albedo, total transmission and fractional absorption of thick layers.

$$l = \frac{(1 - 0.681s)(1 - s)}{(1 + 0.792s)}$$

$$n = \left[\frac{(1 + 0.414s)(1 - s)}{(1 + 1.888s)} \right]^{\frac{1}{2}}$$

$$m = (1 + 1.537s) \ln \left[\frac{1 + 1.800s - 7.087s^2 + 4.740s^3}{(1 - 0.819s)(1 - s)^2} \right]$$

$$k/(1 - \omega_0 g) = \sqrt{3} s - \frac{(0.985 - 0.253s)s^2}{(6.464 - 5.464s)}$$

Two-Stream Approximations

The two-stream approximations in radiative transfer are based on assuming various analytic forms for the upward and downward intensity fields within and at the boundaries of a plane-parallel medium. Substituting the assumed angular distribution into the integro-differential form of the equation of transfer results in a set of differential equations for the upward $[F^-(\tau, \mu_0)]$ and downward $[F^+(\tau, \mu_0)]$ flux densities (Meador and Weaver, 1980; Zdunkowski *et al.*, 1980):

$$\frac{dF^-(\tau, \mu_0)}{d\tau} = \gamma_1 F^-(\tau, \mu_0) - \gamma_2 F^+(\tau, \mu_0) - F_0 \omega_0 \gamma_3 e^{-\tau/\mu_0} \quad (15)$$

$$\frac{dF^+(\tau, \mu_0)}{d\tau} = \gamma_2 F^-(\tau, \mu_0) - \gamma_1 F^+(\tau, \mu_0) + F_0 \omega_0 \gamma_4 e^{-\tau/\mu_0} \quad (16)$$

where

$$F^\pm(\tau, \mu_0) = 2\pi \int_0^1 I^0(\tau, \pm\mu) \mu d\mu \quad (17)$$

In order to obtain the forms given in (15) and (16), it is often necessary to approximate the scattering phase function in order to integrate the source function analytically. Expressions for the plane albedo, total transmission and fractional absorption are obtained by solving (15) and (16), subject to the boundary conditions $F^-(\tau_1, \mu_0) = F^+(0, \mu_0) = 0$. The results may be obtained in the form (Meador and Weaver, 1980)

$$\hat{\gamma}(\tau_t, \mu_0) = \frac{\omega_0}{(1 - k^2 \mu_0^2) [(k + \gamma_1) e^{k\tau_t} + (k - \gamma_1) e^{-k\tau_t}]} \times [(1 - k\mu_0)(a_2 + k\gamma_3) e^{k\tau_t} - (1 + k\mu_0)(a_2 - k\gamma_3) e^{-k\tau_t} - 2k(\gamma_3 - a_2\mu_0) e^{-\tau_t/\mu_0}], \quad (18)$$

$$\hat{i}(\tau_t, \mu_0) = e^{-\tau_t/\mu_0} \times \left\{ 1 - \frac{\omega_0}{(1 - k^2 \mu_0^2) [(k + \gamma_1) e^{k\tau_t} + (k - \gamma_1) e^{-k\tau_t}]} \times [(1 + k\mu_0)(a_1 + k\gamma_4) e^{k\tau_t} - (1 - k\mu_0)(a_1 - k\gamma_4) e^{-k\tau_t} - 2k(\gamma_4 + a_1\mu_0) e^{\tau_t/\mu_0}] \right\}, \quad (19)$$

where

$$a_1 = \gamma_1 \gamma_4 + \gamma_2 \gamma_3, \quad (20)$$

$$a_2 = \gamma_1 \gamma_3 + \gamma_2 \gamma_4, \quad (21)$$

$$k = (\gamma_1^2 - \gamma_2^2)^{1/2}, \quad (22)$$

$$\gamma_4 = 1 - \gamma_3, \quad (23)$$

and the fractional absorption $\hat{a}(\tau_t, \mu_0)$ is given by (11).

The γ_1 , γ_2 and γ_3 coefficients in (15) and (16) for various two-stream approximations, along with references to their original description in the literature, are given in Table 2. Several of these methods employ delta scaling (Joseph *et al.*, 1976) in which a fraction f of the scattered energy is considered to be in the forward peak, approximated as a Dirac delta function. For each of these methods, which include the delta-Eddington, Practical Improved Flux Method (PIFM) and delta-discrete ordinates methods, (18) and (19) can still be used as long as the following transformations are made in the coefficients and solutions:

$$\tau_t \rightarrow \tau_t' = (1 - \omega_0 f) \tau_t, \quad (24)$$

$$\omega_0 \rightarrow \omega_0' = (1 - f) \omega_0 / (1 - \omega_0 f), \quad (25)$$

$$g \rightarrow g' = (g - f) / (1 - f). \quad (26)$$

The primed quantities in (24) - (26), when substituted into the expressions for γ_1 , γ_2 and γ_3 (cf. Table 2) as well as into (18) and (19), yield the relevant expressions for $\hat{\gamma}(\tau_t, \mu_0)$, $\hat{i}(\tau_t, \mu_0)$ and $\hat{a}(\tau_t, \mu_0)$ for the delta-scaled approximations. Though various choices of f are possible, the most frequently used choice, and the one used in all computational results to be presented below, is $f = g^2$.

TABLE 2. Summary of γ_i coefficients in selected two-stream approximations.

Method	Reference	γ_1	γ_2	γ_3
Eddington	Kawata and Irvine (1970)	$1/4[7 - \omega_0(4 + 3g)]$	$-1/4[1 - \omega_0(4 - 3g)]$	$1/4(2 - 3g\mu_0)$
delta-Eddington	Joseph <i>et al.</i> (1976)	$1/4[7 - \omega_0'(4 + 3g')]$	$-1/4[1 - \omega_0'(4 - 3g')]$	$1/4(2 - 3g'\mu_0)$
PIFM	Zdunkowski <i>et al.</i> (1980)	$1/4[8 - \omega_0'(5 + 3g')]$	$3/4[\omega_0'(1 - g')]$	$1/4(2 - 3g'\mu_0)$
discrete ordinates	Liou (1973, 1974)	$\sqrt{3}/2[2 - \omega_0(1 + g)]$	$\sqrt{3}/2[\omega_0(1 - g)]$	$1/2(1 - \sqrt{3}g\mu_0)$
delta-discrete ordinates	Schaller (1979)	$\sqrt{3}/2[2 - \omega_0'(1 + g')]$	$\sqrt{3}/2[\omega_0'(1 - g')]$	$1/2(1 - \sqrt{3}g'\mu_0)$
Coakley-Chýlek (I)	Coakley and Chýlek (1975)	$\{1 - \omega_0[1 - \beta(\mu_0)]\} / \mu_0$	$\omega_0\beta(\mu_0) / \mu_0$	$\beta(\mu_0)$
Coakley-Chýlek (II)	Coakley and Chýlek (1975)	$2[1 - \omega_0(1 - \bar{\beta})]$	$2\omega_0\bar{\beta}$	$\beta(\mu_0)$
Meador-Weaver	Meador and Weaver (1980)	$\frac{7 - 3g^2 - \omega_0(4 + 3g) + \omega_0g^2[4\beta(\mu_0) + 3g]}{4[1 - g^2(1 - \mu_0)]}$	$\frac{-1 + g^2 + \omega_0(4 - 3g) + \omega_0g^2[4\beta(\mu_0) + 3g - 4]}{4[1 - g^2(1 - \mu_0)]}$	$\beta(\mu_0)$

In the Meador-Weaver and Coakley-Chýlek approximations, γ_3 is set equal to the backscatter fraction $\beta(\mu_0)$, defined as

$$\beta(\mu_0) = \frac{1}{2\omega_0} \int_0^1 h^0(-\mu, \mu_0) d\mu, \quad (27)$$

where $h^0(-\mu, \mu_0)$ is the azimuthal average of $\omega_0 \Phi(\cos \Theta)$ for incident solar radiation in the direction μ_0 and reflected in the direction $-\mu$. The backscatter fraction $\beta(\mu_0)$ is illustrated in King and Harshvardhan (1986) for the FWC phase function used in the present investigation.

The Coakley and Chýlek (1975) model 2, which we will hereafter refer to as Coakley-Chýlek (II), uses the average backscatter fraction $\bar{\beta}$ in the expression for γ_1 and γ_2 . This constant is defined as

$$\bar{\beta} = \int_0^1 \beta(\mu_0) d\mu_0. \quad (28)$$

For the phase function used in the present investigation, $\bar{\beta} = 0.1772$.

For conservative scattering, for which $\gamma_1 = \gamma_2$ and $k = 0$, Eqs. (18) and (19) reduce to

$$\hat{r}(\tau_t, \mu_0) = 1 - \hat{t}(\tau_t, \mu_0) = \frac{1}{1 + \gamma_1 \tau_t} [\gamma_1 \tau_t + (\gamma_3 - \gamma_1 \mu_0) (1 - e^{-\tau_t/\mu_0})]. \quad (29)$$

For Coakley and Chýlek's (1975) model 1, hereafter referred to as Coakley-Chýlek (I), $\gamma_3 = \gamma_1 \mu_0$ and thus (29) reduces to an especially simple form (see Table 2).

Since k exceeds unity for strongly absorbing atmospheres in all two-stream approximations (cf. Table 2), conditions can easily exist for which $\mu_0 = k^{-1}$, especially in the water vapor bands. Though this condition can lead to a numerical singularity in (18) and (19) the singularity is removable, and when $\mu_0 = k^{-1}$ it is rather straightforward to show that (18) and (19) reduce to

$$\hat{r}(\tau_t, \mu_0) = \frac{\omega_0}{2[(1 + \gamma_1 \mu_0) e^{\tau_t/\mu_0} + (1 - \gamma_1 \mu_0) e^{-\tau_t/\mu_0}]} \times \left\{ (a_2 \mu_0 + \gamma_3) e^{\tau_t/\mu_0} - [(a_2 \mu_0 + \gamma_3) + 2(a_2 \mu_0 - \gamma_3) \tau_t/\mu_0] e^{-\tau_t/\mu_0} \right\}, \quad (30)$$

$$\hat{t}(\tau_t, \mu_0) = e^{-\tau_t/\mu_0} \times \left\{ 1 - \frac{\omega_0}{2[(1 + \gamma_1 \mu_0) e^{\tau_t/\mu_0} + (1 - \gamma_1 \mu_0) e^{-\tau_t/\mu_0}]} \times \left\{ [(a_1 \mu_0 - \gamma_4) - 2(a_1 \mu_0 + \gamma_4) \tau_t/\mu_0] e^{\tau_t/\mu_0} - (a_1 \mu_0 - \gamma_4) e^{-\tau_t/\mu_0} \right\} \right\}. \quad (31)$$

This case may be avoided by either applying these formulae when $\mu_0 = k^{-1}$, or by displacing μ_0 by a very small increment and applying (18) and (19), as suggested by Zdunkowski *et al.* (1980).

In computing fluxes for multi-layer systems overlying a reflecting surface, it is also necessary to compute the albedo and transmission of layers for diffuse radiation. For parameterization purposes, it is usual to compute these

quantities for an isotropic incident source. Under this situation, the global (spherical) albedo and global transmission can be obtained by integrating the corresponding plane albedo and total transmission solutions as a function of μ_0 :

$$\bar{r}(\tau_t) = 2 \int_0^1 \hat{r}(\tau_t, \mu_0) \mu_0 d\mu_0, \quad (32)$$

$$\bar{t}(\tau_t) = 2 \int_0^1 \hat{t}(\tau_t, \mu_0) \mu_0 d\mu_0. \quad (33)$$

Inspection of (18) and (19), or even the simpler (29) for conservative scattering, shows that a *general* closed form solution for (32) and (33) does not exist. Although these integrations can be carried out numerically for each specific two-stream model, this is not practical for most modeling applications. For those two-stream models for which $\beta(\mu_0)$ does not explicitly appear in any of the γ_i coefficients, it is in principle possible to analytically integrate (18) and (19) to obtain expressions for $\bar{r}(\tau_t)$ and $\bar{t}(\tau_t)$ for specific models. In asymptotic theory it is possible to obtain simple analytic formulae for $\bar{r}(\tau_t)$ and $\bar{t}(\tau_t)$. These expressions can be found in King (1981).

Coakley and Chýlek (1975) suggest that (32) and (33) may be avoided by using a second set of globally-averaged two-stream equations in which the incident isotropic radiation is treated as an upper boundary condition. In general, however, the results obtained by this approach yield different results from those obtained using (32) and (33) for the same model. In the results to be presented below, we have restricted our comparisons of the radiative properties of various radiative transfer models to those for an isolated layer composed of cloud particles. The complicating effects of ground albedo have therefore not been considered, but a comparable intercomparison of results for $\bar{r}(\tau_t)$ and $\bar{t}(\tau_t)$ as a function of τ_t and ω_0 remains an important study to be pursued in the future.

RESULTS

We have examined both the absolute and relative accuracies of the plane albedo, total transmission and fractional absorption (where applicable) as a function of τ_t and μ_0 for four values of the single scattering albedo ($\omega_0 = 1.0, 0.99, 0.9$ and 0.8) and for all radiative transfer approximations discussed in the previous section. Fig. 3 illustrates $r(\tau_t, \mu_0) = 1 - t(\tau_t, \mu_0)$ as a function of τ_t ($0.1 \leq \tau_t \leq 100$) and μ_0 ($0 \leq \mu_0 \leq 1$) for conservative scattering ($\omega_0 = 1$), and Fig. 4 illustrates corresponding results for $\hat{r}(\tau_t, \mu_0)$ for each of eight different radiative transfer approximations. The shaded regions in the asymptotic theory and Eddington approximation results of Fig. 4 delineate regions for which nonphysical (negative) plane albedos are obtained in the approximations. In both of these models, nonphysical regions occur for optically thin atmospheres and for small solar zenith angles (large values of μ_0).

Fig. 5 illustrates a 4×3 plot composite of errors for conservative scattering and for four radiative transfer models, where the first row applies to asymptotic theory and succeeding rows to the delta-Eddington, Meador-Weaver and Coakley-Chýlek (1) approximations. Individual plots in the first column of Fig. 5 represent absolute errors in the plane albedo, defined as

$$\Delta r(\tau_t, \mu_0) = \hat{r}(\tau_t, \mu_0) - r(\tau_t, \mu_0). \quad (34)$$

With this definition, positive (negative) errors indicate that the radiative transfer approximation overestimates (underestimates) the exact albedo, taken as the computational results presented in Fig. 3. Similar definitions apply to errors in the total transmission $[\Delta t(\tau_t, \mu_0)]$ and fractional absorption $[\Delta a(\tau_t, \mu_0)]$. The relative errors in the plane albedo $[\Delta r(\tau_t, \mu_0)/r(\tau_t, \mu_0)]$ and total transmission $[\Delta t(\tau_t, \mu_0)/t(\tau_t, \mu_0)]$ are presented in succeeding columns of Fig. 5, and are given in per cent. Relative errors with magnitudes greater than 20% and absolute errors with magnitudes greater than 0.2 are not plotted in Fig. 5 or in subsequent figures.

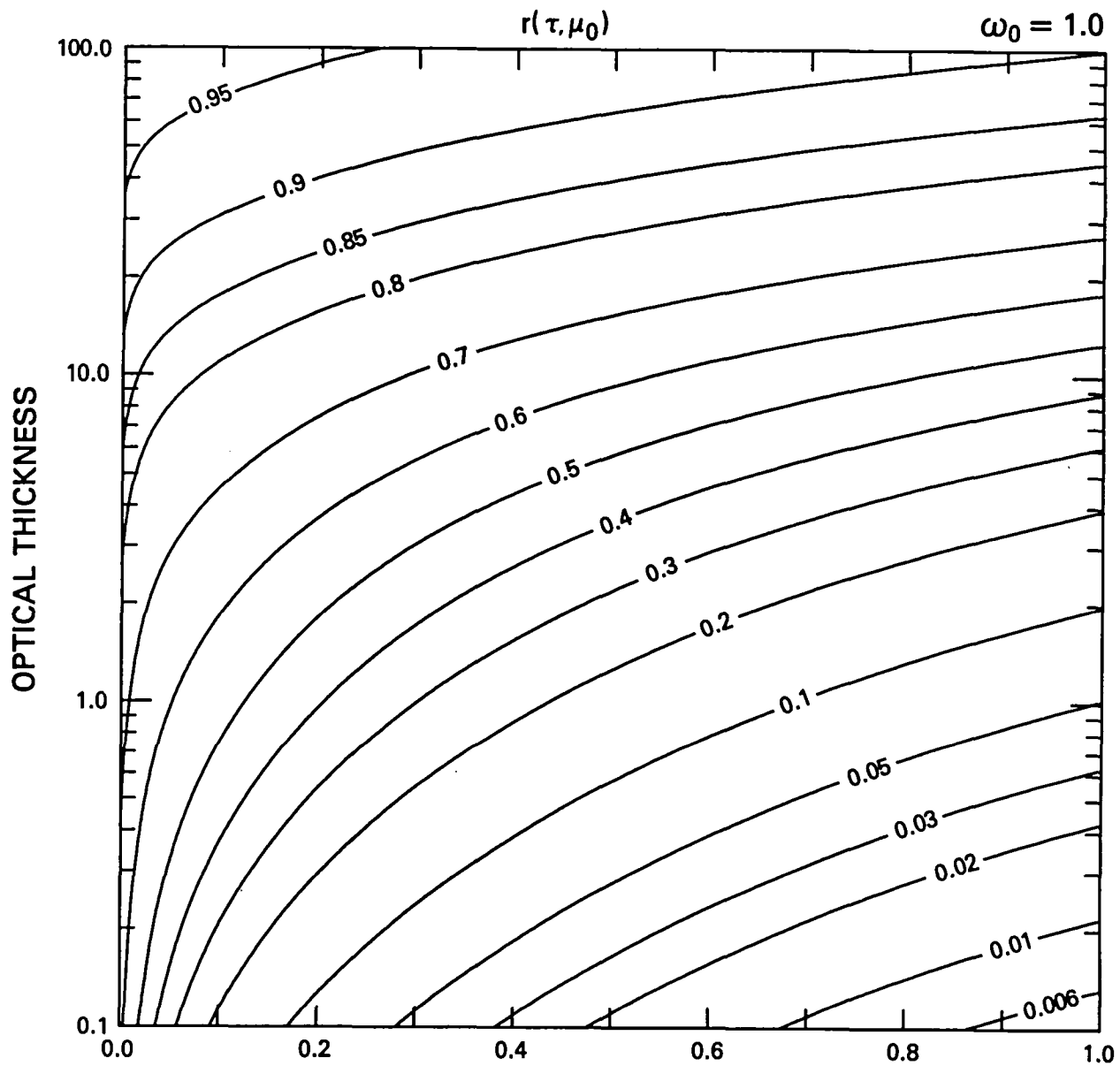


Figure 3. Doubling computations of the plane albedo as a function of optical thickness and cosine of the solar zenith angle for a FWC phase function with conservative scattering ($\omega_0 = 1.0$).

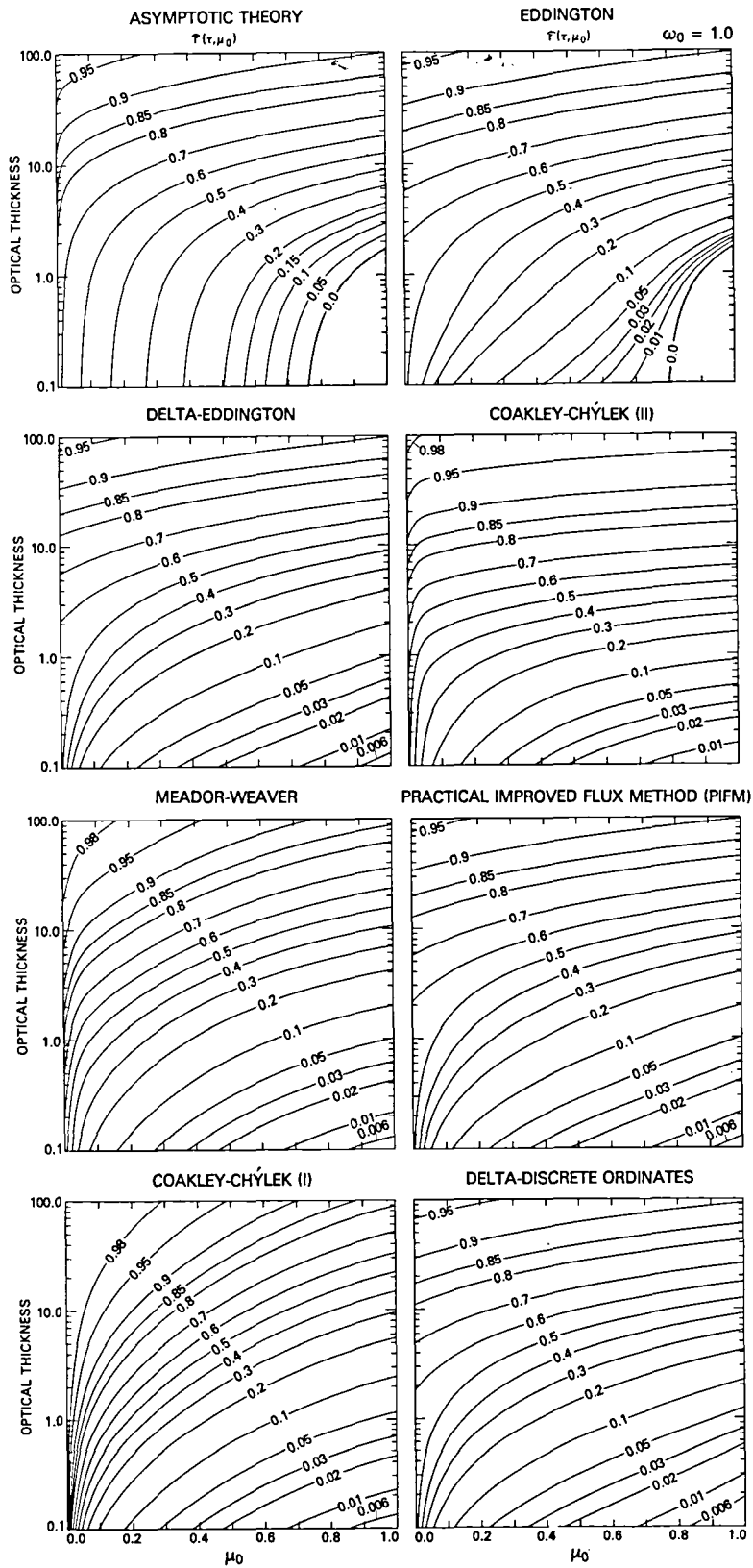


Figure 4. Computations of the plane albedo as a function of optical thickness and cosine of the solar zenith angle for eight different radiative transfer approximations identified in the figure.

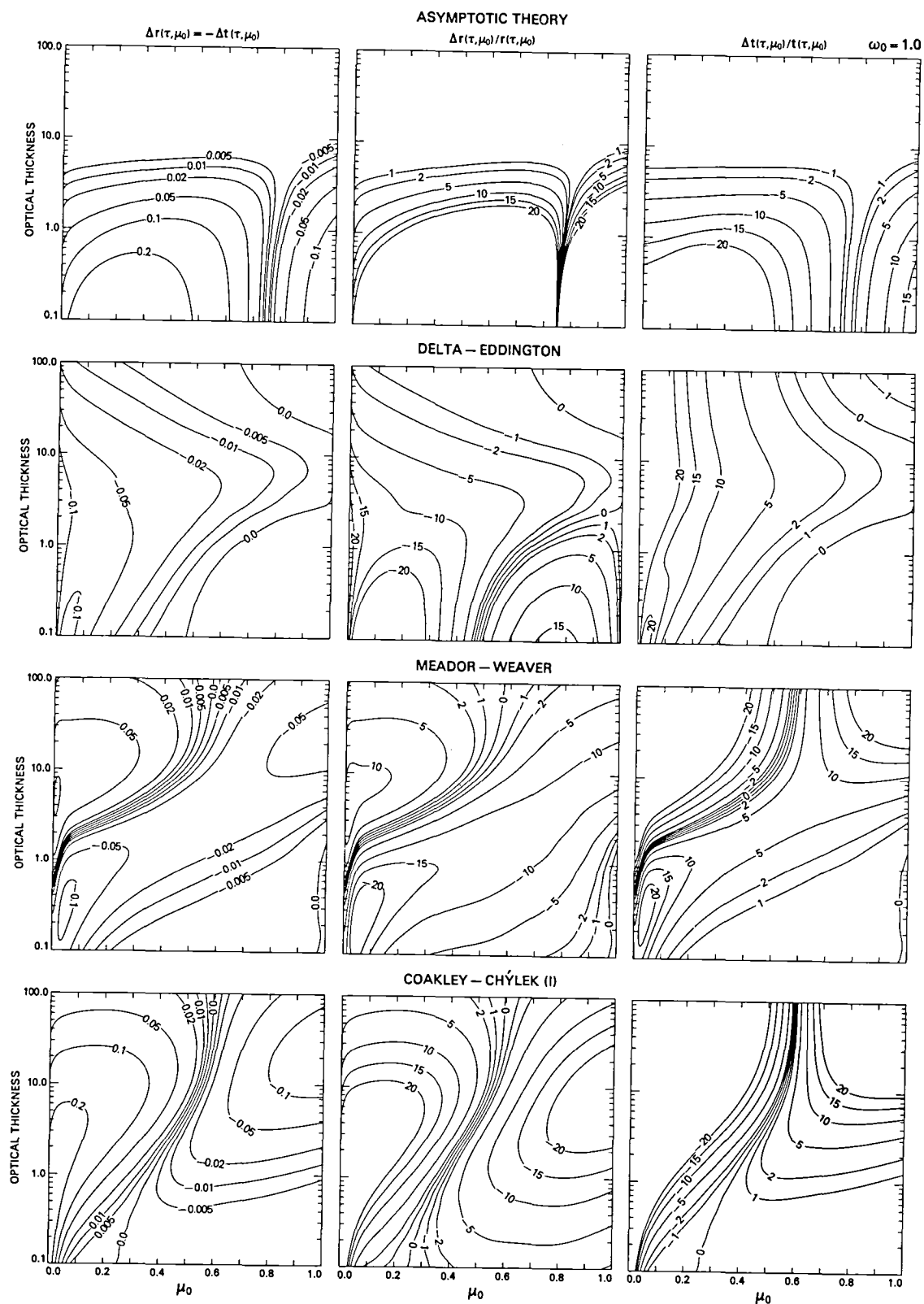


Figure 5. Absolute and relative accuracy of asymptotic theory, delta-Eddington, Meador-Weaver and Coakley-Chylek (I) approximations to the plane albedo and total transmission as a function of optical thickness and cosine of the solar zenith angle for conservative scattering ($\omega_0 = 1.0$). All relative accuracy values are in per cent. The FWC phase function is assumed throughout.

Fig. 5 shows that asymptotic theory is accurate to within 5% in both reflection and transmission for $\tau_t \gtrsim 3$ when $\mu_0 \lesssim 0.9$. Furthermore, asymptotic theory is accurate to within 1% for all solar zenith angles when $\tau_t \gtrsim 8$. The Coakley-Chýlek (I) approximation, on the other hand, is accurate to within 5% for all $\tau_t \lesssim 0.2$ when $\mu_0 > 0.1$. Since $r(\tau_t, \mu_0)$ is small when $\tau_t/\mu_0 \lesssim 0.5$ (cf. Fig. 3), a relative error of 5% is too stringent a criterion to use for accepting a model in this range. Adopting instead the absolute error criterion $|\Delta r(\tau_t, \mu_0)| \lesssim 0.005$, we see that the range of validity of the Coakley-Chýlek (I) approximation can be extended to $\tau_t \lesssim 0.5$ for a wide range of solar zenith angles. An advantage of both of these models is that their absolute and relative accuracies show little sensitivity to μ_0 in their respective ranges of validity.

In contrast to these models, the delta-Eddington approximation tends to have its greatest accuracy when $\mu_0 \gtrsim 0.5$, regardless of optical thickness. The large relative albedo errors which occur when τ_t/μ_0 is small are not critical, since the absolute errors are small in this range. Similarly, when $\tau_t \gtrsim 10$ and $\mu_0 \lesssim 0.5$, the small values of $t(\tau_t, \mu_0)$ allow one to extend the range of validity of the delta-Eddington model to values of μ_0 somewhat lower than 0.5.

The Meador-Weaver approximation, which was developed as a composite of the Eddington and Coakley-Chýlek (I) methods, has most of the characteristics of the latter for conservative scattering, especially for thick atmospheres. In optically thin atmospheres, on the other hand, it has a much greater μ_0 sensitivity than the Coakley-Chýlek (I) method. This makes the Meador-Weaver approximation less suitable than alternative methods for conservative scattering over the entire range of variables, at least for the high values of asymmetry factor considered in the present investigation.

Detailed results for conservative scattering analogous to Fig. 5 are presented in Fig. 6 for four other radiative transfer approximations, *viz.*, the Eddington, Coakley-Chýlek (II), PIFM and delta-discrete ordinates methods. On examination of Table 2, one can readily show that the PIFM method of Zdunkowski *et al.* (1980) is identical to the delta-Eddington method for conservative scattering, which accounts for the identical appearance of the corresponding panels in Figs. 5 and 6 for these two models. The difference Zdunkowski *et al.* report between the delta-Eddington and PIFM methods when $\omega_0 = 1$ is likely a result of their using different values of f in the scaling formulae (24) - (26) for each method.

A careful examination of the results in Fig. 6 shows that the Eddington, PIFM and delta-discrete ordinates methods are quite accurate for optically thick, conservative atmospheres, although they differ substantially in their accuracy at small and intermediate optical depths. From a comparison of the results in Figs. 5 and 6, it follows that the Coakley-Chýlek (II) approximation is everywhere less accurate than the Coakley-Chýlek (I) approximation. In general, it can be concluded that when $\omega_0 = 1$ the majority of models considered in this report are inaccurate in their approximation of the plane albedo when $\mu_0 \ll 1$ and $1 \lesssim \tau_t \lesssim 5$, with asymptotic theory being the best suited in this difficult range of variables.

Fig. 7 illustrates doubling computations of the plane albedo, total transmission and fractional absorption as a function of τ_t and μ_0 for nonconservative scattering ($\omega_0 = 0.99$), where we have used the same phase function illustrated in Fig. 1, but simply scaled the Legendre coefficients by ω_0 . Figs. 8 and 9 show corresponding figures for each of the eight radiative transfer approximations considered in this report. Again we have used shading to identify all nonphysical regions where either the plane albedo or fractional absorption is negative or the total transmission exceeds unity.

Fig. 10 illustrates a 4 x 3 plot composite showing absolute errors in the plane albedo, total transmission and fractional absorption as a function of τ_t and μ_0 for $\omega_0 = 0.99$ and for each of the four models presented in Fig. 8. A similar plot composite for the remaining four models is presented in Fig. 11. Corresponding results for relative errors for all eight models are presented in Figs. 12 and 13. In all of these figures, individual plots in the first column represent errors in the plane albedo, while plots in succeeding columns represent errors in the total transmission and fractional absorption, respectively. Due to the small amount of absorption when $\omega_0 = 0.99$ and $\tau_t \lesssim 5$, the relative and absolute errors for all eight models are nearly identical to corresponding errors in the conservative case. Only in the case of high sun and large optical depths do appreciable changes occur in the accuracy of different models. Further discussion of these differences is reserved for the following treatment of the $\omega_0 = 0.9$ case.

Fig. 14 illustrates doubling computations of the plane albedo, total transmission and fractional absorption for $\omega_0 = 0.9$, while Figs. 15 and 16 show corresponding figures for all eight radiative transfer approximations considered in this report. Figs. 17 - 20 illustrate absolute and relative errors obtained for $\omega_0 = 0.9$ in the same format

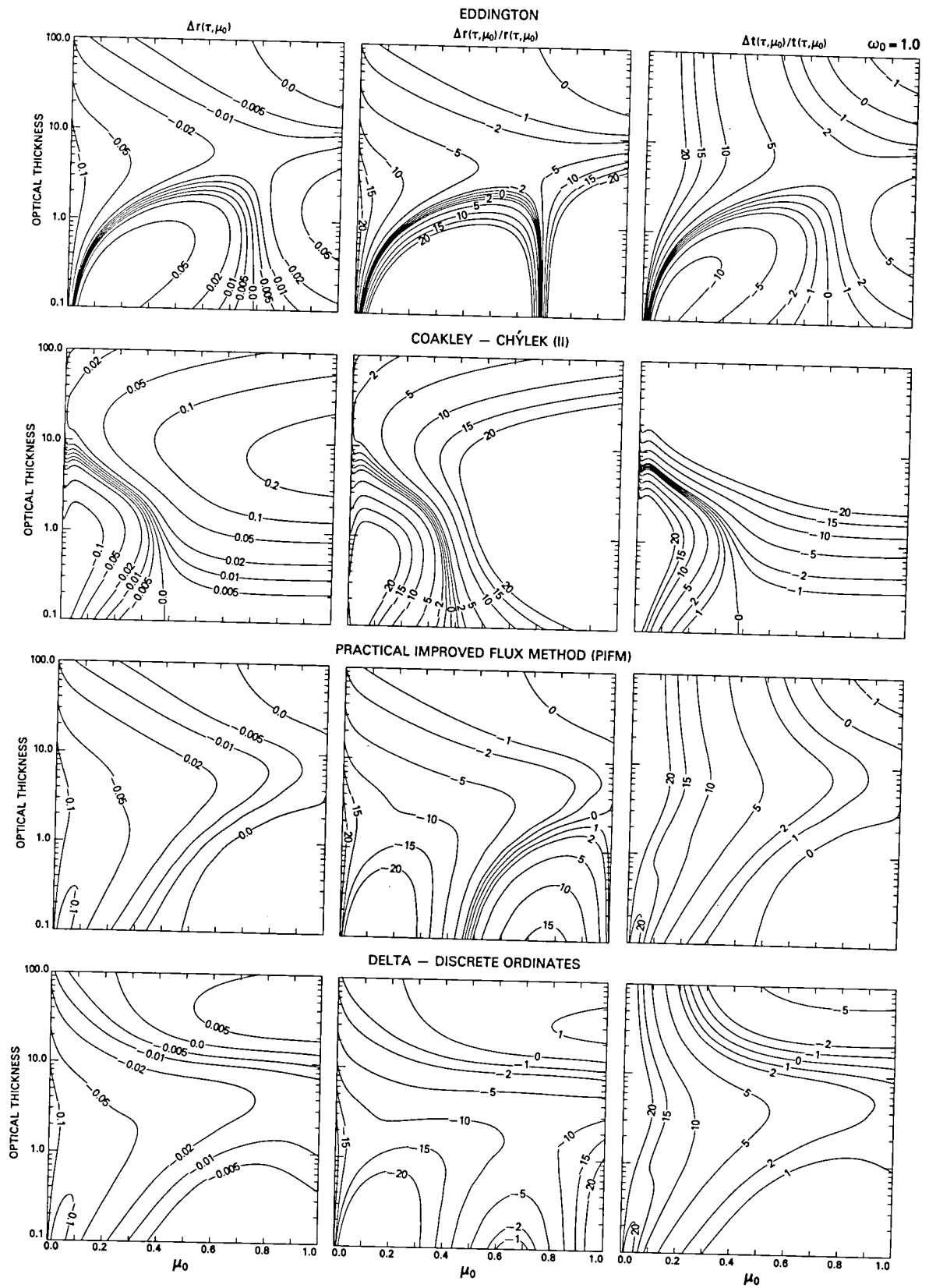


Figure 6. As in Fig. 5 except for the Eddington, Coakley-Chýlek(II), PIFM and delta-discrete ordinates approximations.

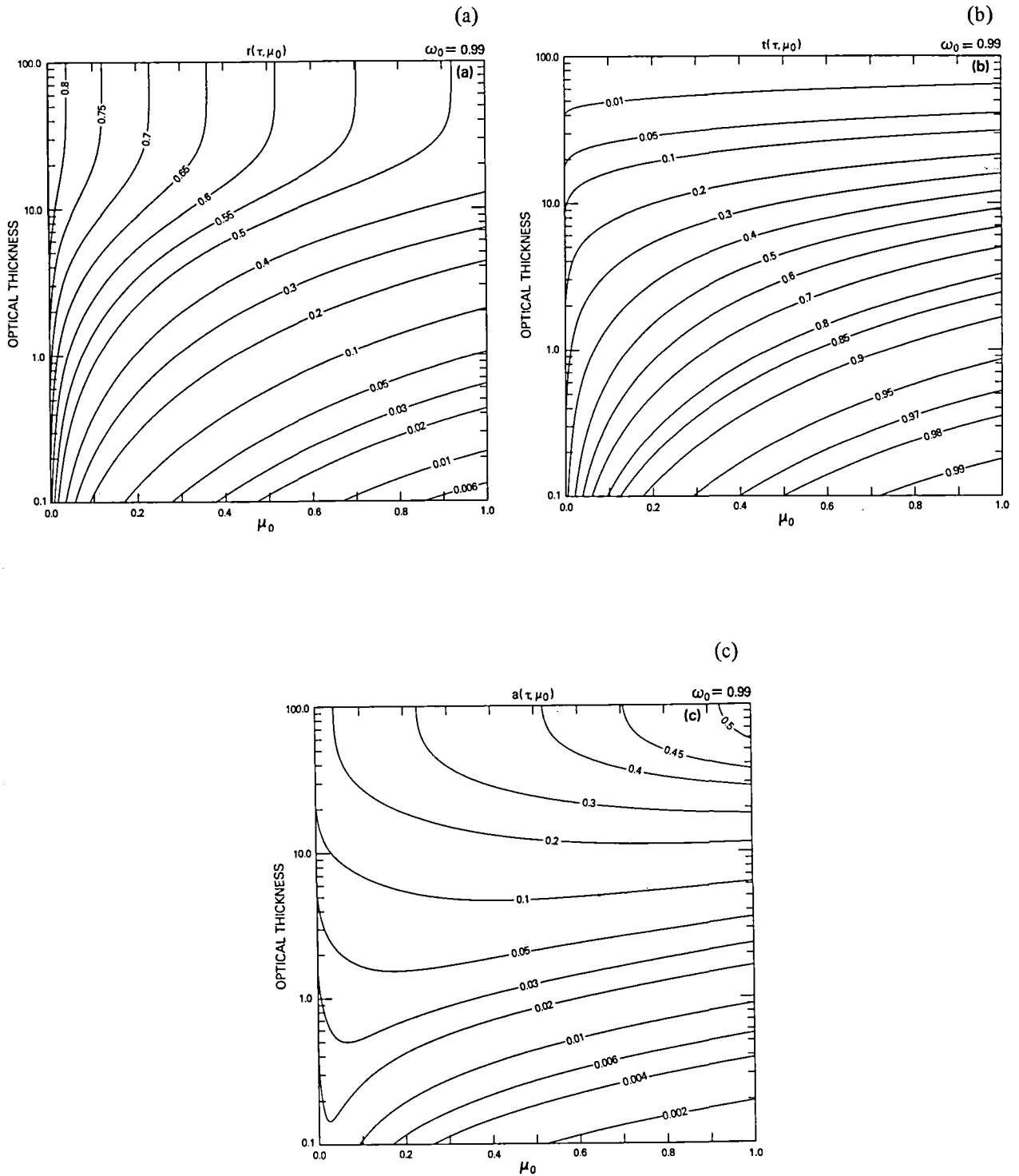


Figure 7. Doubling computations of the (a) plane albedo, (b) total transmission and (c) fractional absorption as a function of optical thickness and cosine of the solar zenith angle for a FWC phase function with nonconservative scattering ($\omega_0 = 0.99$).

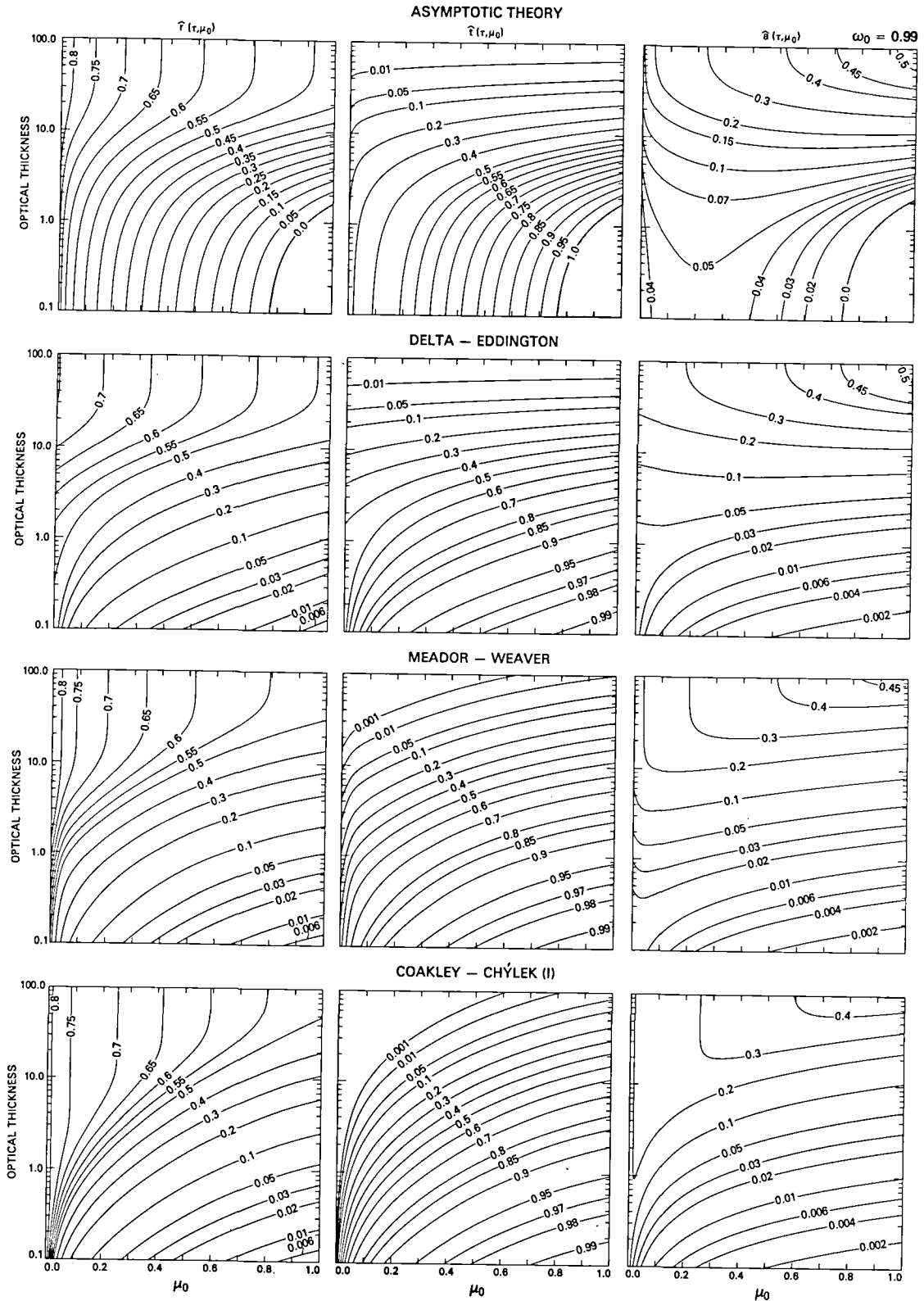


Figure 8. Computations of the plane albedo, total transmission and fractional absorption as a function of optical thickness and cosine of the solar zenith angle for asymptotic theory, delta-Eddington, Meador-Weaver and Coakley-Chýlek (I) approximations when $\omega_0 = 0.99$.

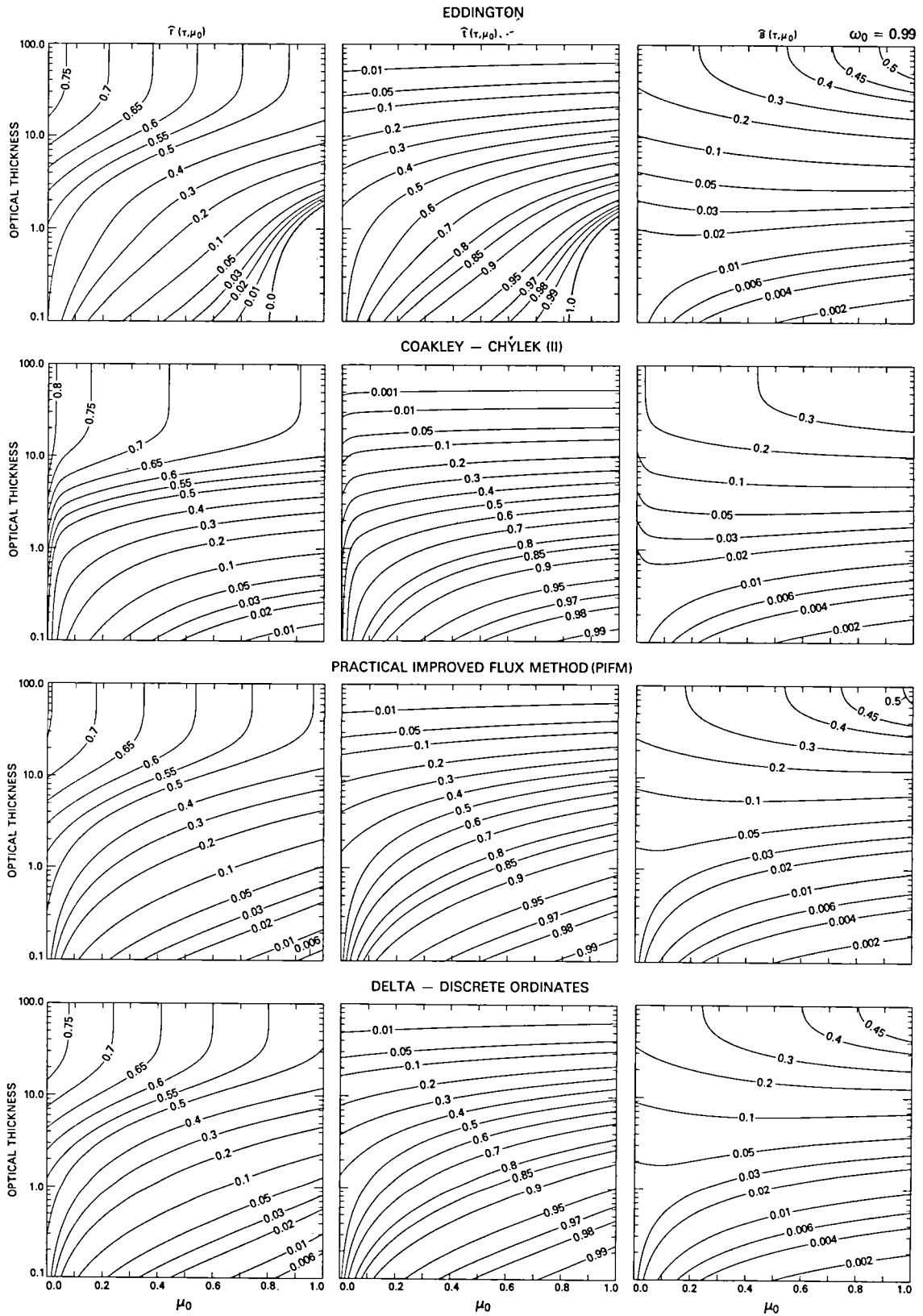


Figure 9. As in Fig. 8 except for the Eddington, Coakley-Chylek (II), PIFM and delta-discrete ordinates approximations.

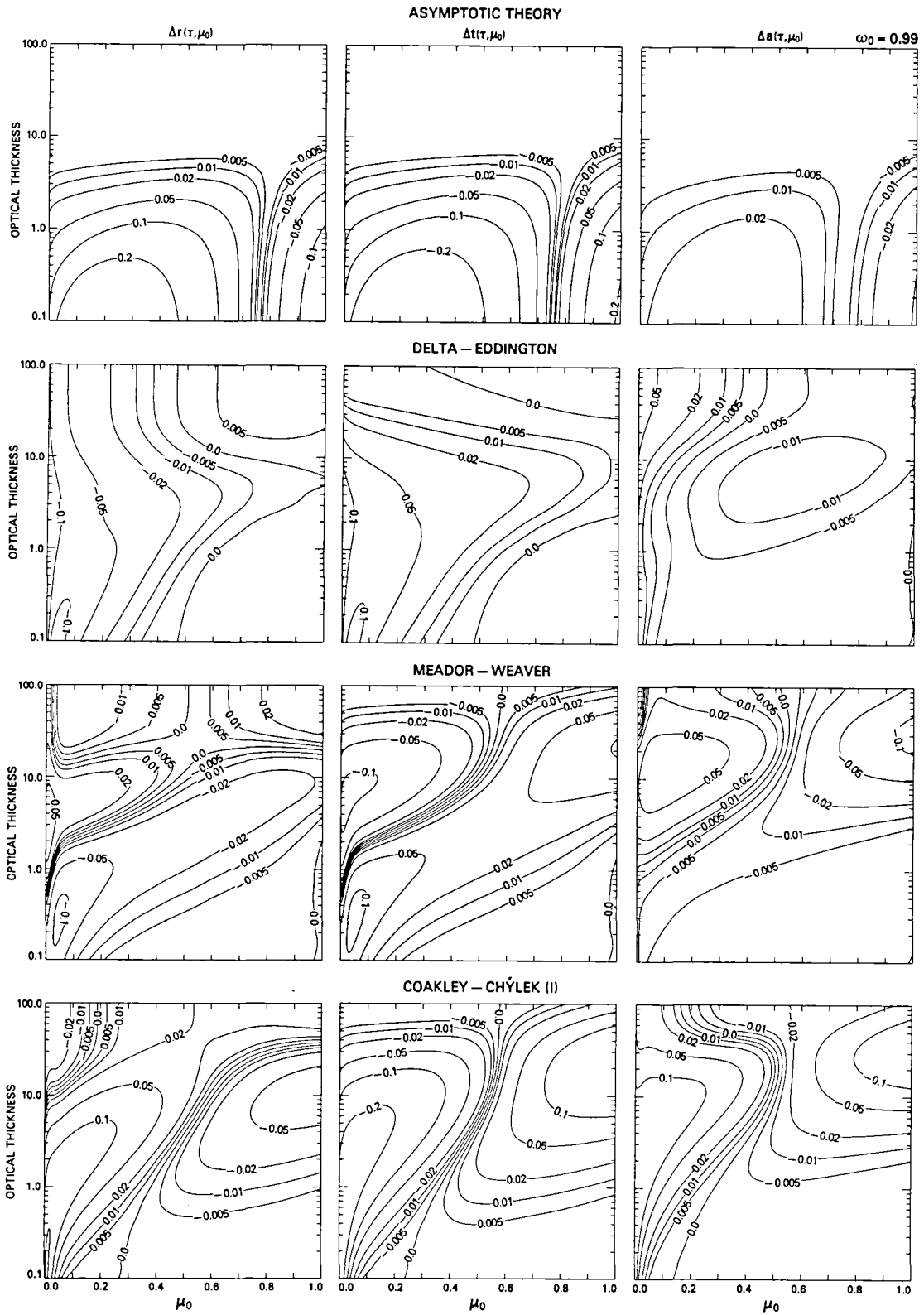


Figure 10. Absolute accuracy of asymptotic theory, delta-Eddington, Meador-Weaver and Coakley-Chýlek (I) approximations to the plane albedo, total transmission and fractional absorption as a function of optical thickness and cosine of the solar zenith angle for nonconservative scattering ($\omega_0 = 0.99$).

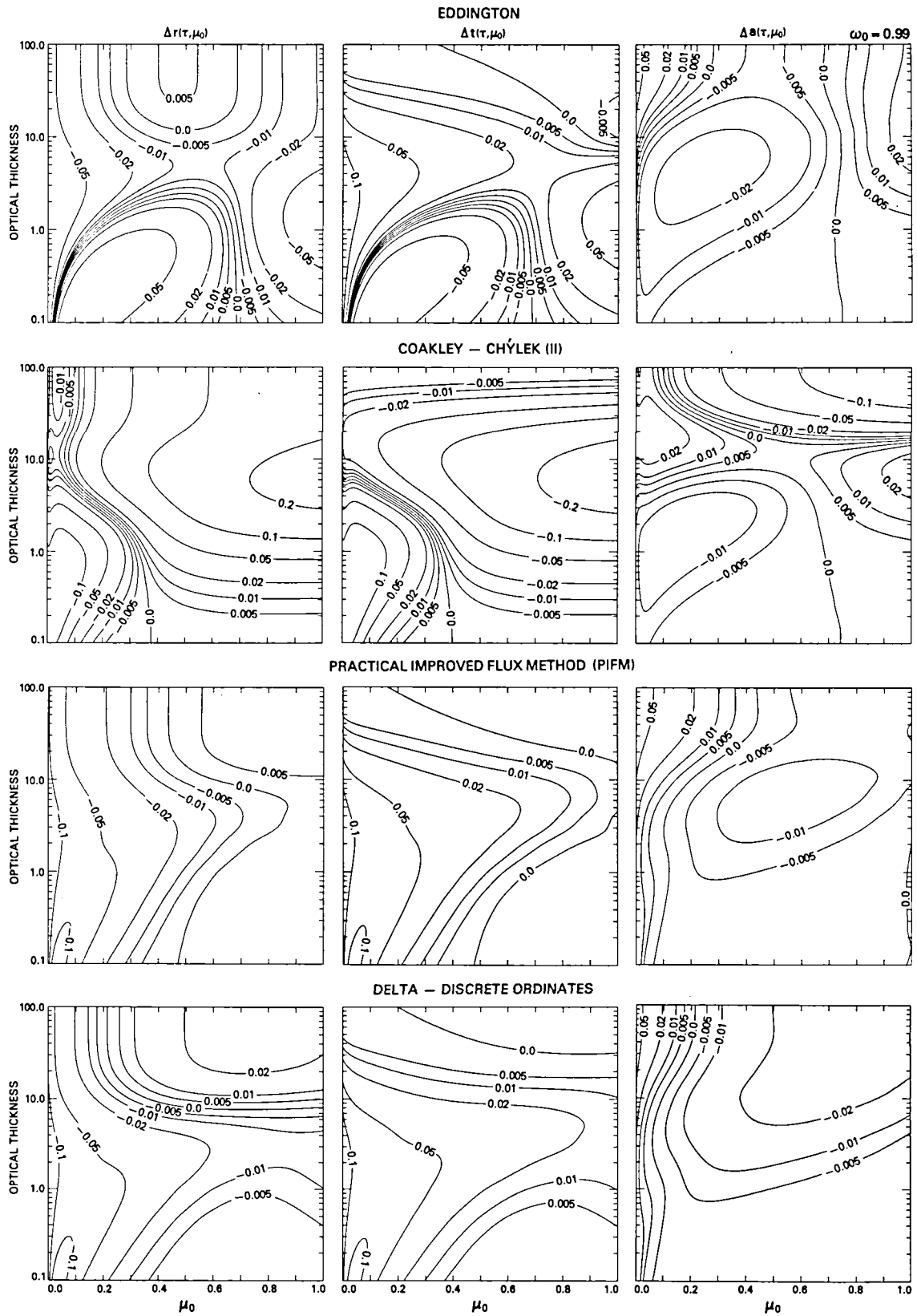


Figure 11. As in Fig. 10 except for the Eddington, Coakley-Chýlek (II), PIFM and delta-discrete ordinates approximations.

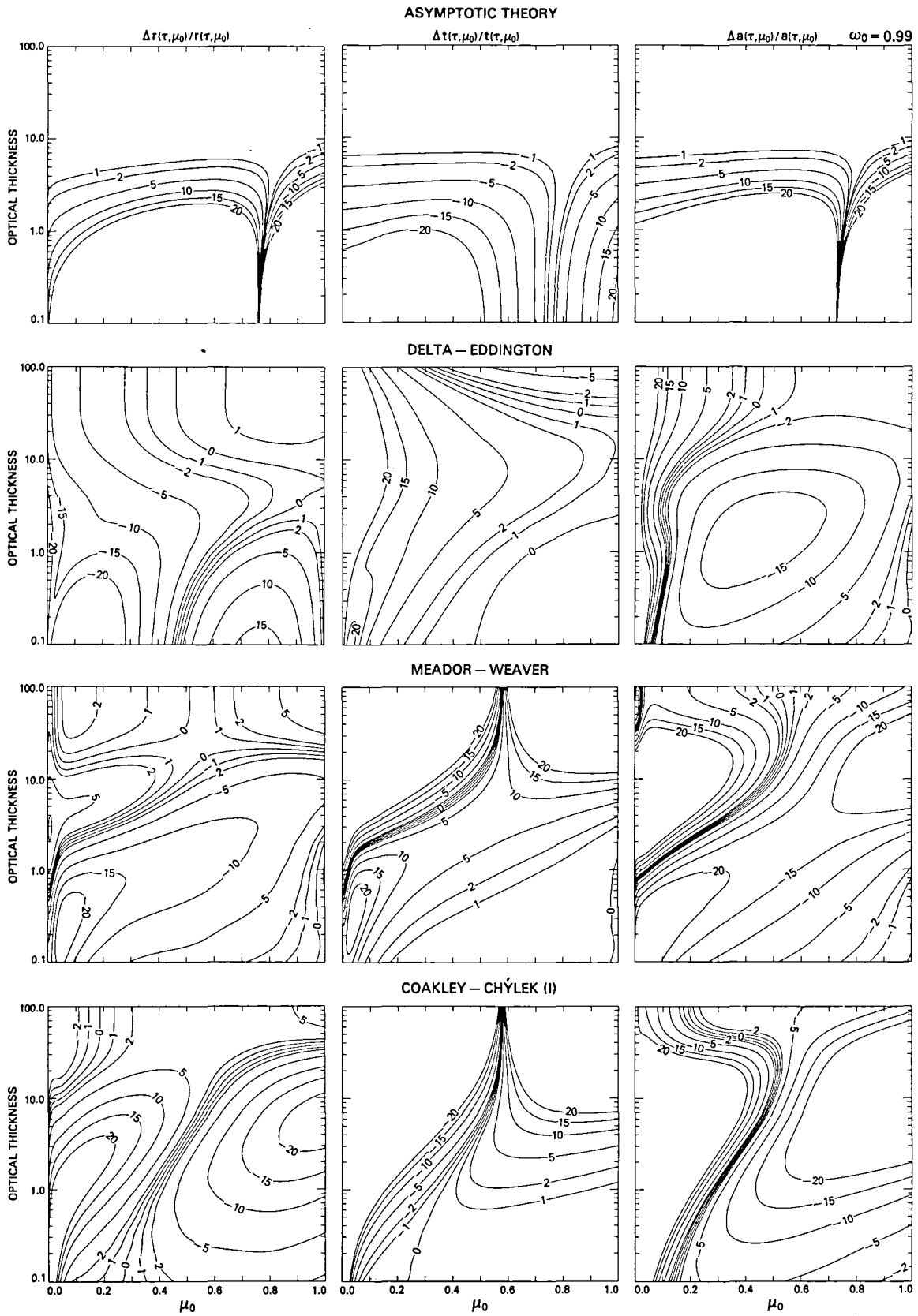


Figure 12. As is Fig. 10 except for relative accuracies (in per cent).

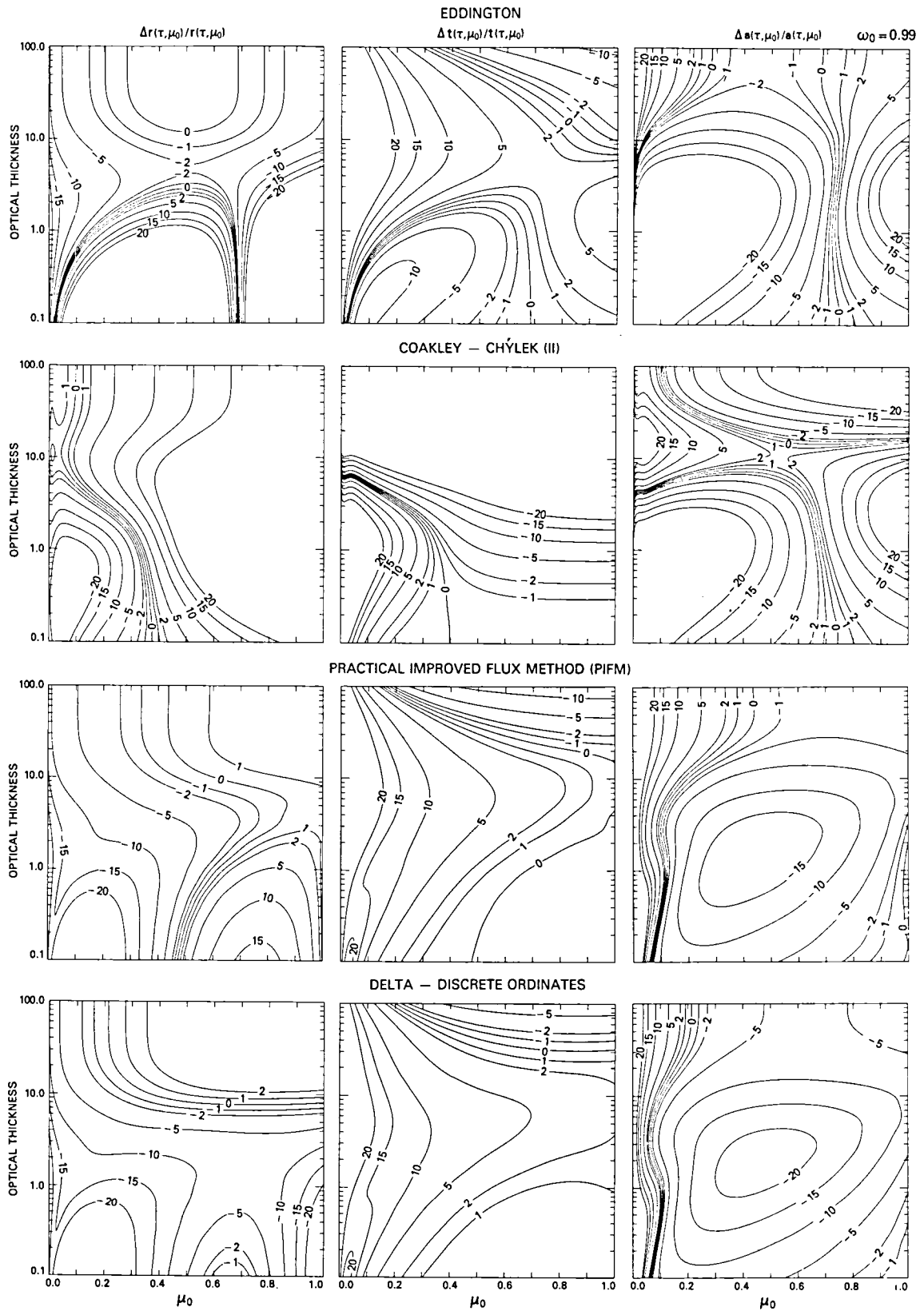


Figure 13. As is Fig. 11 except for relative accuracies (in per cent).

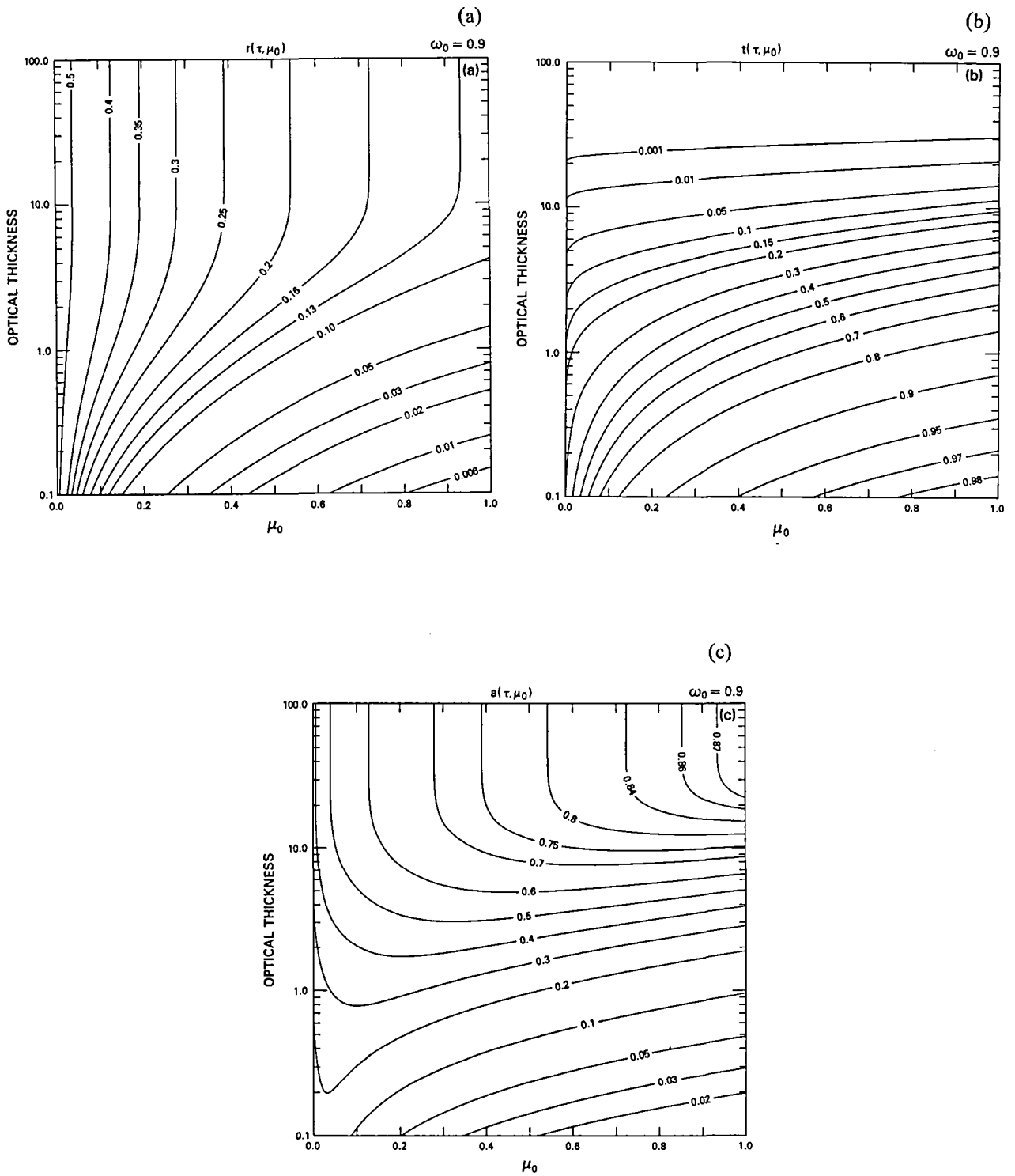


Figure 14. Doubling computations of the (a) plane albedo, (b) total transmission and (c) fractional absorption as a function of optical thickness and cosine of the solar zenith angle for a FWC phase function with nonconservative scattering ($\omega_0 = 0.9$).

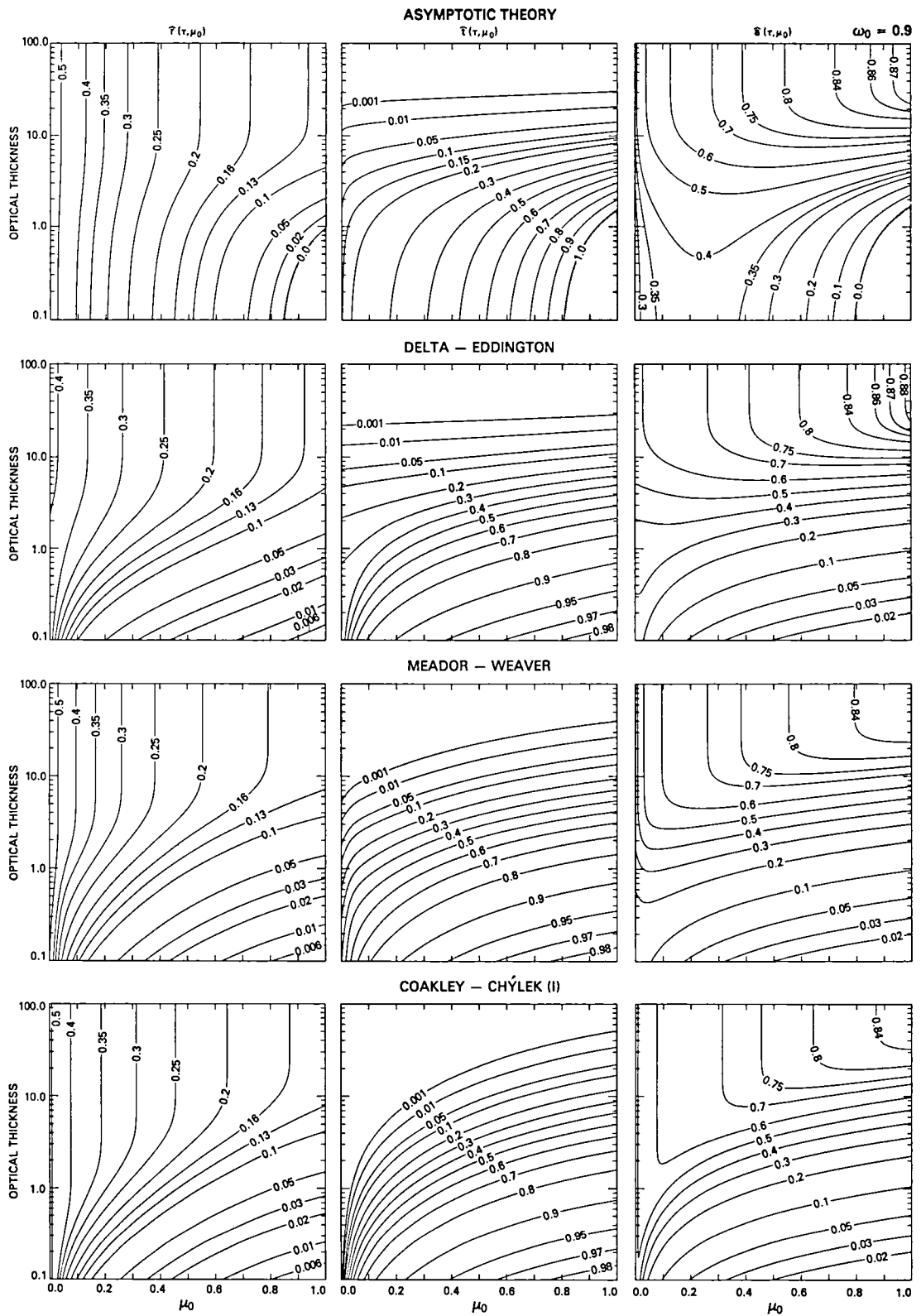


Figure 15. Computations of the plane albedo, total transmission and fractional absorption as a function of optical thickness and cosine of the solar zenith angle for asymptotic theory, delta-Eddington, Meador-Weaver and Coakley-Chýlek (I) approximations when $\omega_0 = 0.9$.

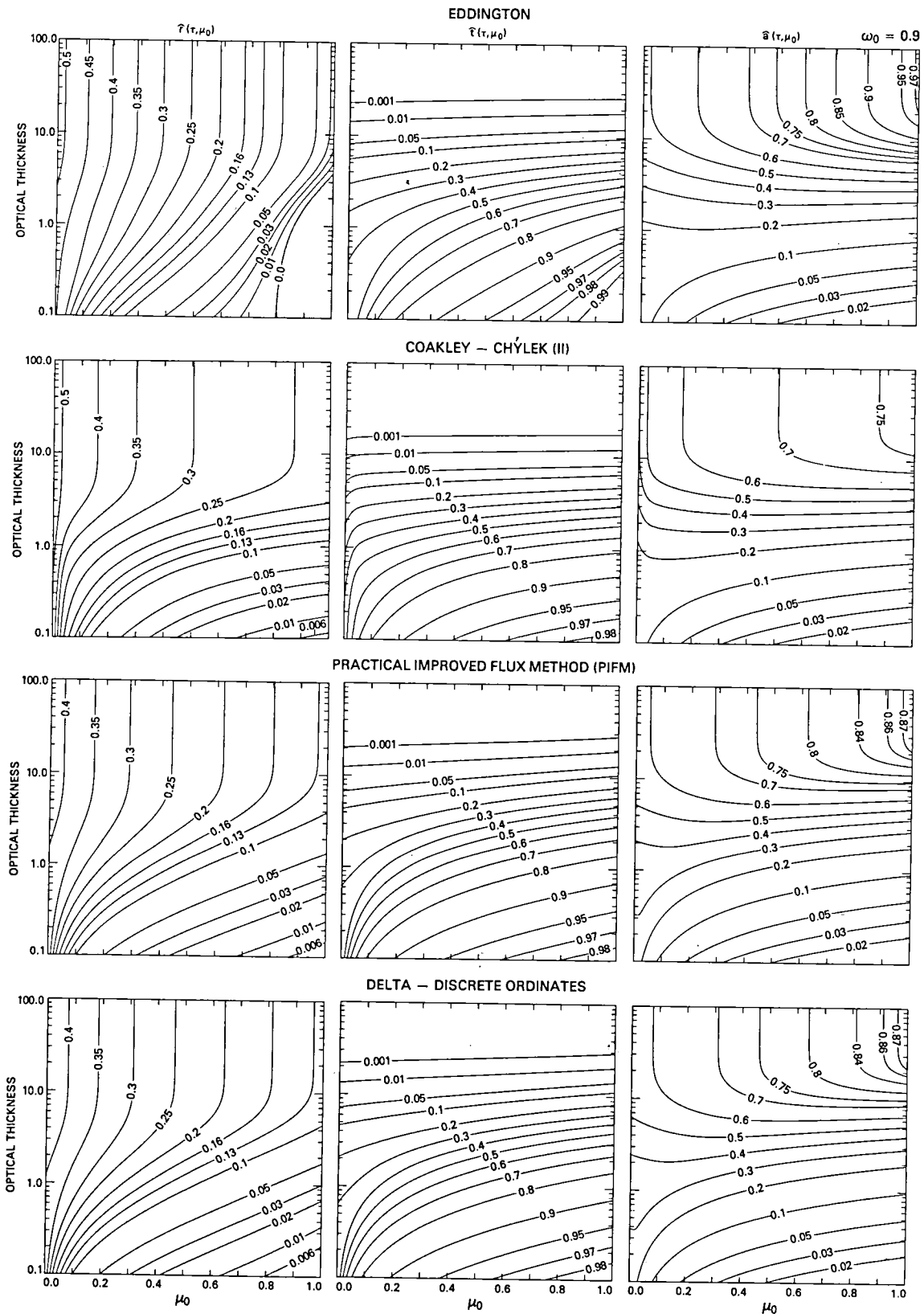


Figure 16. As in Fig. 15 except for the Eddington, Coakley-Chýlek (II), PIFM and delta-discrete ordinates approximations.

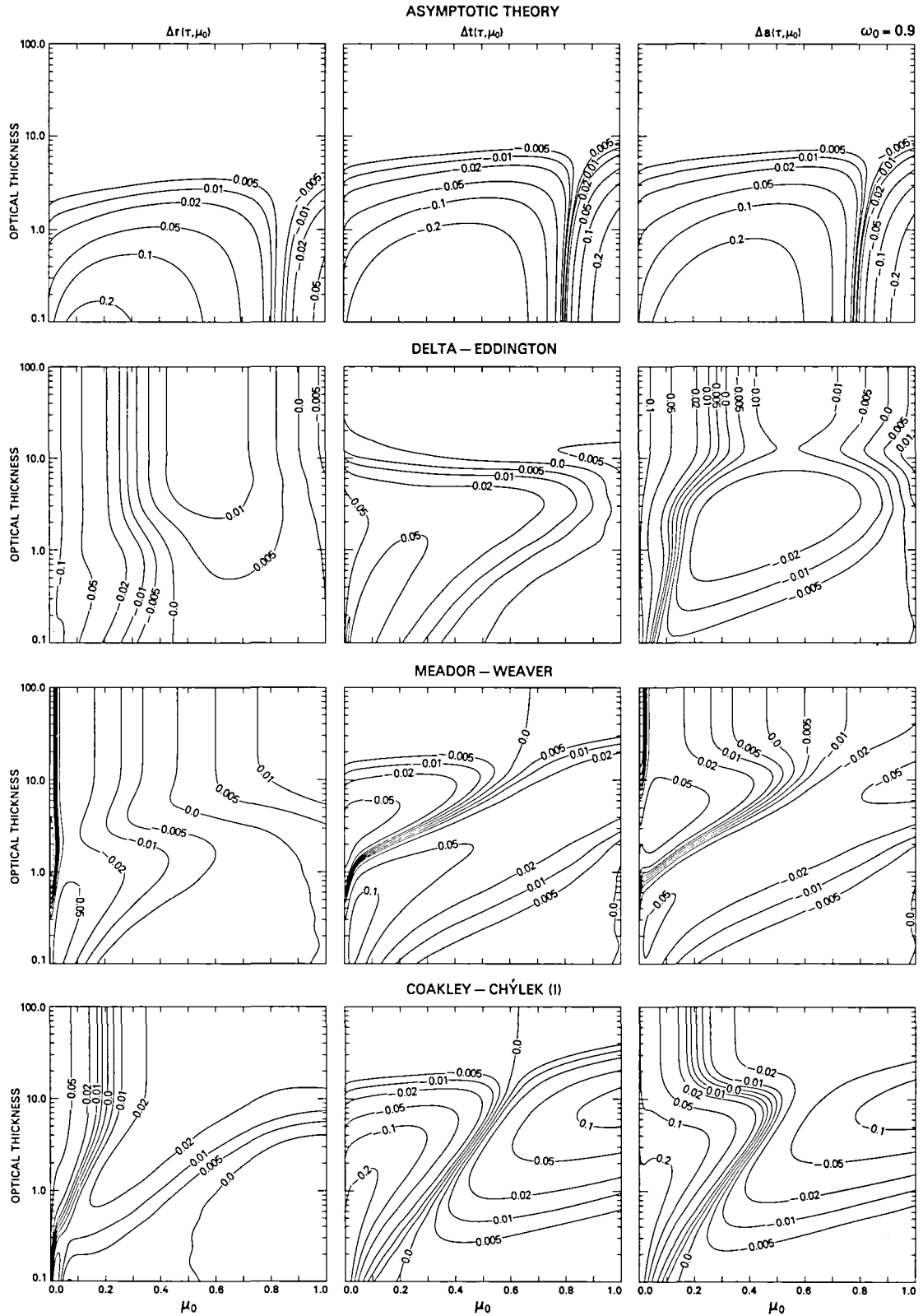


Figure 17. Absolute accuracy of asymptotic theory, delta-Eddington, Meador-Weaver and Coakley-Chýlek (I) approximations to the plane albedo, total transmission and fractional absorption as a function of optical thickness and cosine of the solar zenith angle for nonconservative scattering ($\omega_0 = 0.9$).

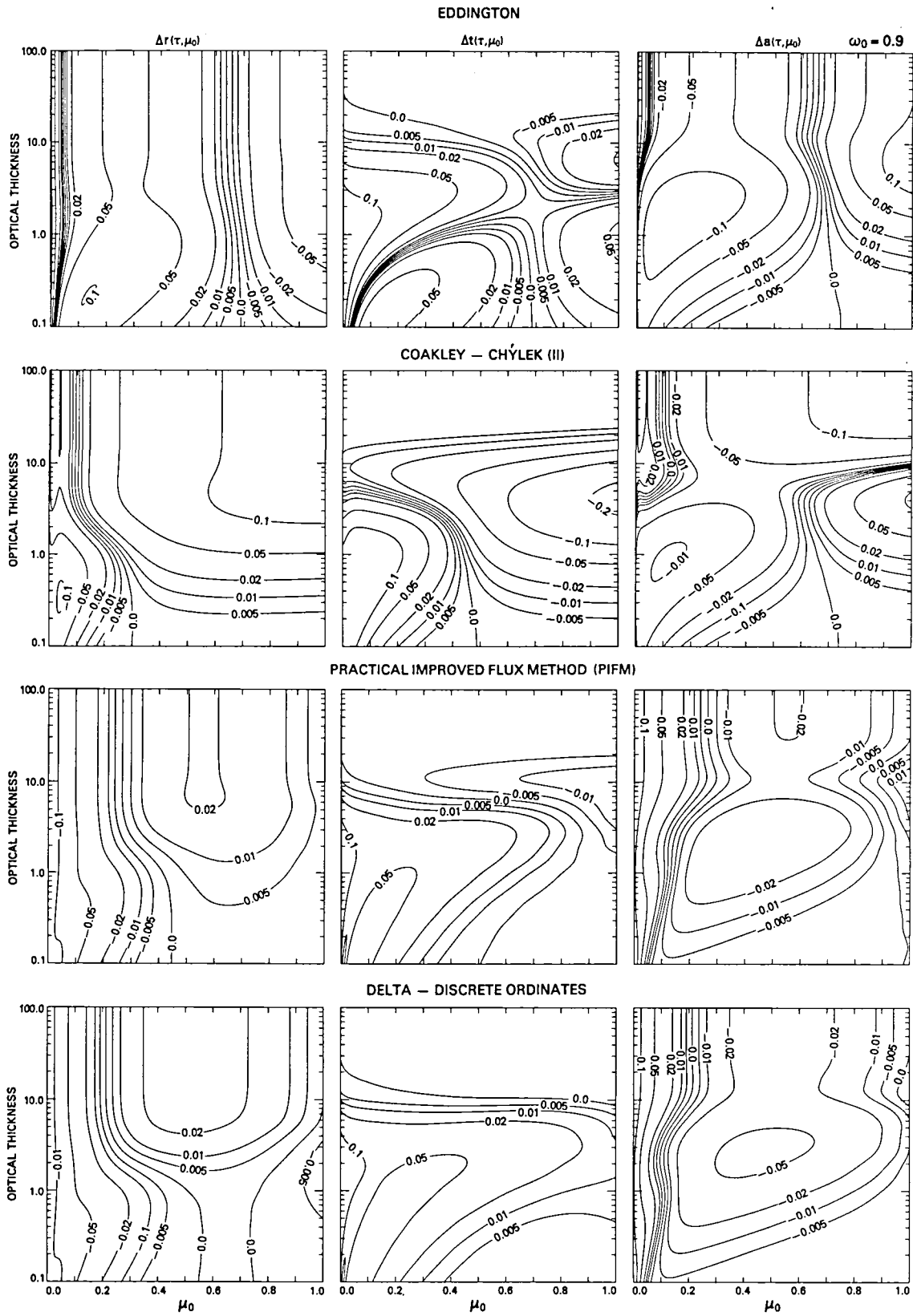


Figure 18. As in Fig. 17 except for the Eddington, Coakley-Chýlek (II), PIFM and delta-discrete ordinates approximations.

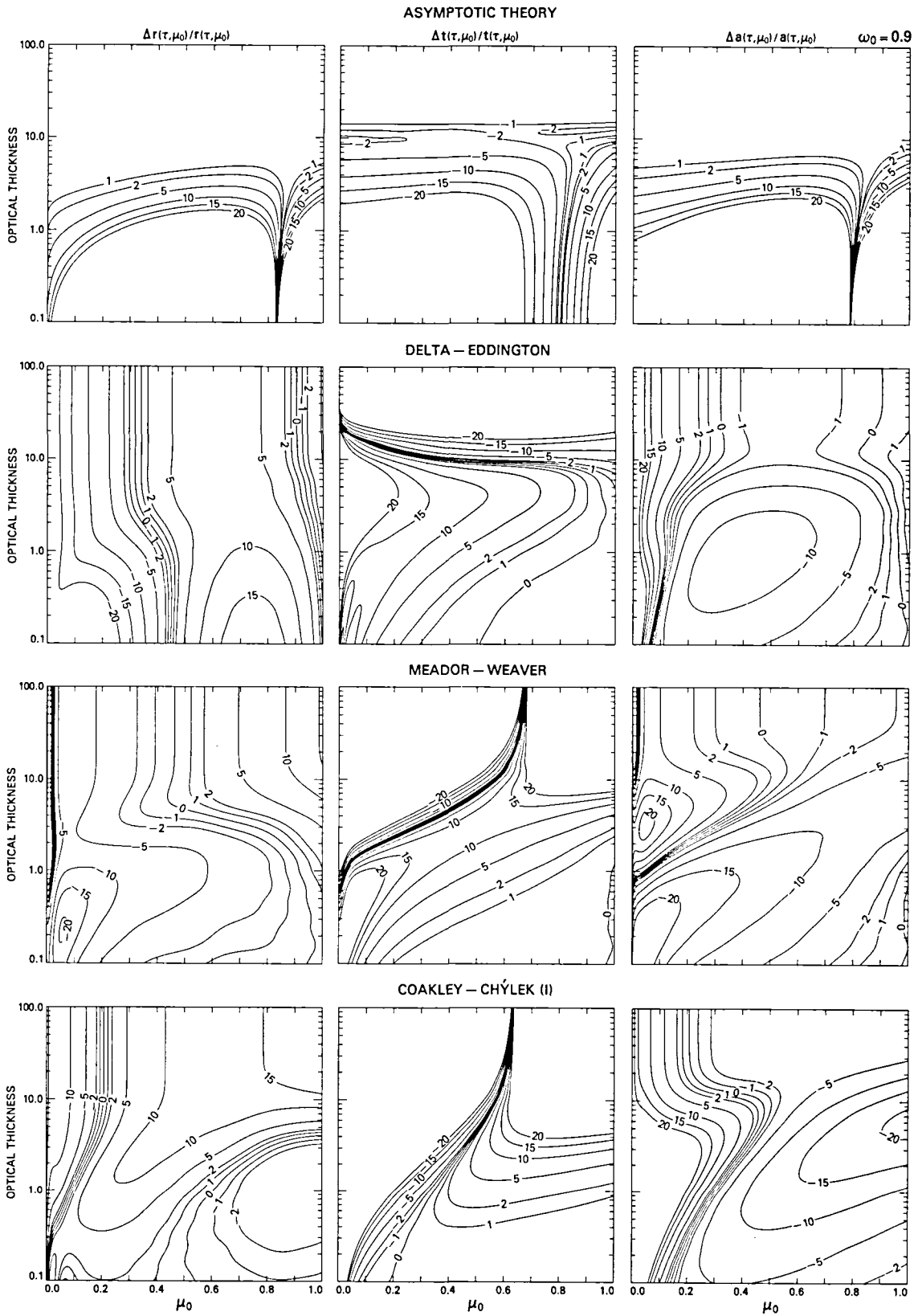


Figure 19. As is Fig. 17 except for relative accuracies (in per cent).

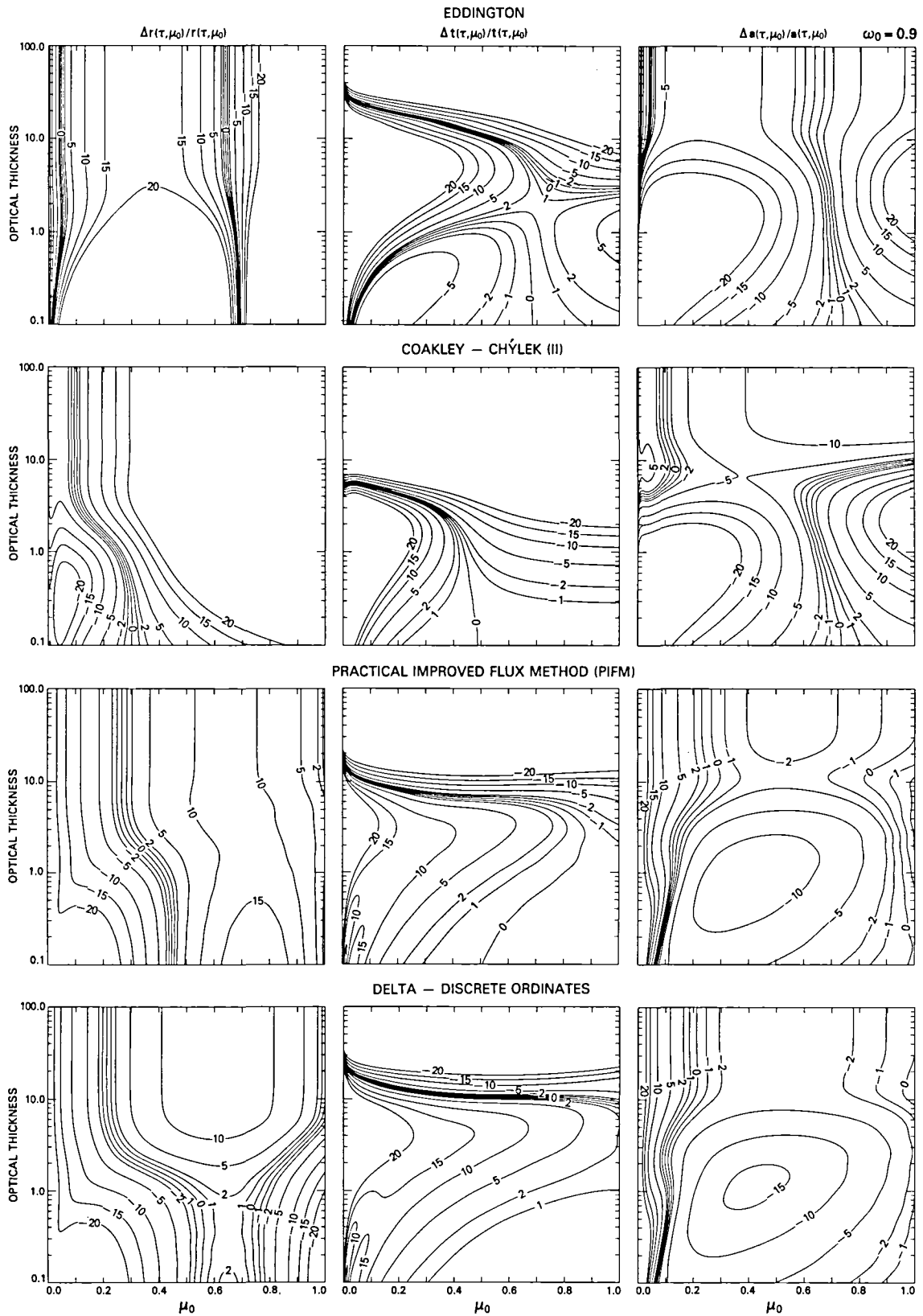


Figure 20. As is Fig. 18 except for relative accuracies (in per cent).

as used previously in Figs. 10 - 13. Figs. 17 and 19 show that asymptotic theory is equally as valid an approximation for optically thick, nonconservative atmospheres as it is for optically thick, conservative atmospheres (cf. Fig. 5). Relative errors of 5% or less are achieved in asymptotic theory for reflection, transmission and absorption when $\tau_t \gtrsim 6$, regardless of solar zenith angle. For cases in which reflection is the most important, the results presented in Fig. 19 show that asymptotic theory can be applied to optical depths as low as 4 with an accuracy of better than 5%.

As in the case of conservative scattering, the Coakley-Chýlek (I) method is the most accurate approximation for optically thin atmospheres, with a tendency to be somewhat more accurate for small solar zenith angles (large values of μ_0). In order to have an accuracy of better than 5% in reflection, transmission and absorption, Fig. 19 suggests that it is necessary for $\tau_t \lesssim 0.1$ and $\mu_0 \gtrsim 0.1$. However, since $a(\tau_t, \mu_0) \ll 1$ when $\tau_t/\mu_0 \lesssim 0.5$ (cf. Fig. 14c), it is more meaningful to use the absolute error criterion $|\Delta a(\tau_t, \mu_0)| \lesssim 0.005$. Thus we conclude that the range of validity of the Coakley-Chýlek (I) approximation can generally be extended to include all optical depths less than some maximum in the range $0.2 \lesssim \tau_t \lesssim 0.7$, depending on solar zenith angle.

For the delta-Eddington approximation, comparison of Figs. 5 and 19 show that relative errors in the plane albedo and total transmission degrade somewhat in accuracy as absorption increases, especially for optically thick atmospheres. This is a consequence of the fact that the $\hat{r}_\infty(\mu_0)$ computed in the delta-Eddington approximation is nearly linear in μ_0 for all single scattering albedos, whereas the true $r_\infty(\mu_0)$ has increasing curvature as absorption increases (cf. King and Harshvardhan, 1986). Both the absolute and relative errors in the other delta-scaled approximations (*viz.*, the PIFM and delta-discrete ordinates methods) are very similar to, but slightly worse than, the delta-Eddington method. A feature of all of these methods is the isolated region at intermediate values of τ_t and μ_0 where absorption errors in excess of 10% occur (cf. Figs. 19 and 20).

Although the Meador-Weaver approximation was previously shown to be an inferior model for conservative scattering, it is clear from Figs. 17 and 19 that its accuracy improves dramatically as absorption increases, especially for reflection. This is true for both optically thin and thick atmospheres. Moreover, it is the only two-stream model which has an albedo accuracy of better than 5% over a wide range of solar zenith angles when $\tau_t \gtrsim 2$, although the absorption error is sometimes as large as 10% in this range of variables. The explanation for the exceptional accuracy of the Meador-Weaver approximation in optically thick, nonconservative atmospheres is that $\hat{r}_\infty(\mu_0)$ exhibits significant curvature in μ_0 for all single scattering albedos, as does the true $r_\infty(\mu_0)$ (cf. King and Harshvardhan, 1986). This feature is unique to the Meador-Weaver method among two-stream approximations.

Although the Eddington approximation was previously shown to be quite accurate for optically thick, nearly conservative atmospheres, the results presented in Figs. 18 and 20 show that the Eddington approximation has no useful regime where albedo errors less than 5% occur when $\omega_0 = 0.9$. From a similar examination of results for the Coakley-Chýlek (II) method, we conclude that this model is also a poor approximation for nonconservative atmospheres involving collimated radiation.

Detailed results analogous to Figs. 14 - 20 are presented in Figs. 21 - 27 for $\omega_0 = 0.8$. From a comparison of the relative errors in Figs. 5, 12, 19 and 26, one can see that there is a clear tendency for the Meador-Weaver approximation to improve in accuracy as ω_0 decreases, at least to $\omega_0 = 0.8$, at which point albedo errors of less than 7.5% occur for all optical depths when $\mu_0 \gtrsim 0.2$. In addition, these figures show that asymptotic theory continues to be a valid approximation for optically thick, nonconservative atmospheres. Although the Coakley-Chýlek (I) model can still be used for optically thin atmospheres, the other two-stream methods generally degrade in accuracy as absorption increases. Again it is clear from Figs. 26 and 27 that all of the delta-scaled two-stream methods have an isolated region at intermediate values of τ_t and μ_0 where absorption errors in excess of 10% occur. In general, it can be concluded that all models with the exception of asymptotic theory and the Meador-Weaver approximation have difficulty in optically thick, nonconservative atmospheres.

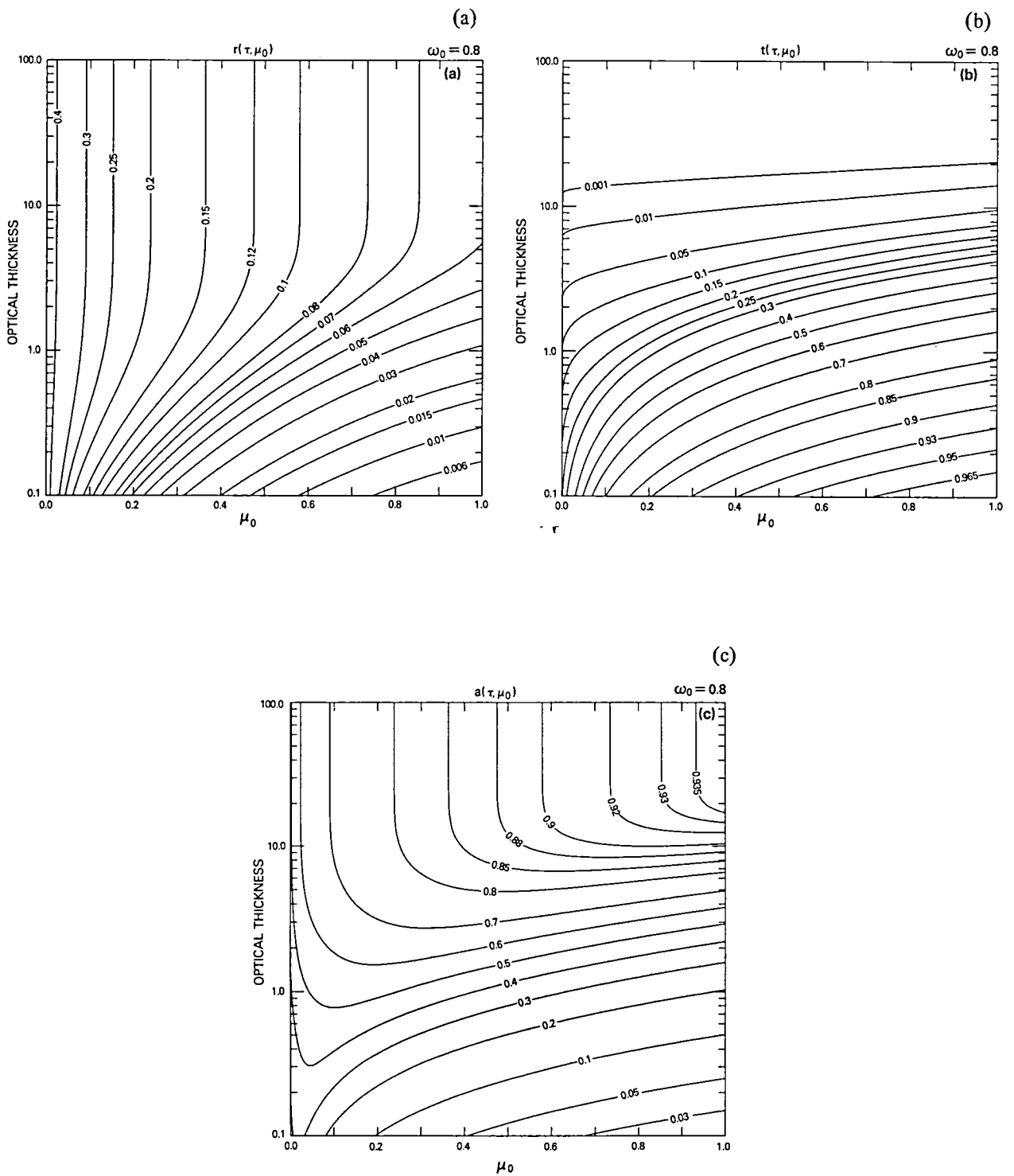


Figure 21. Doubling computations of the (a) plane albedo, (b) total transmission and (c) fractional absorption as a function of optical thickness and cosine of the solar zenith angle for a FWC phase function with nonconservative scattering ($\omega_0 = 0.8$).

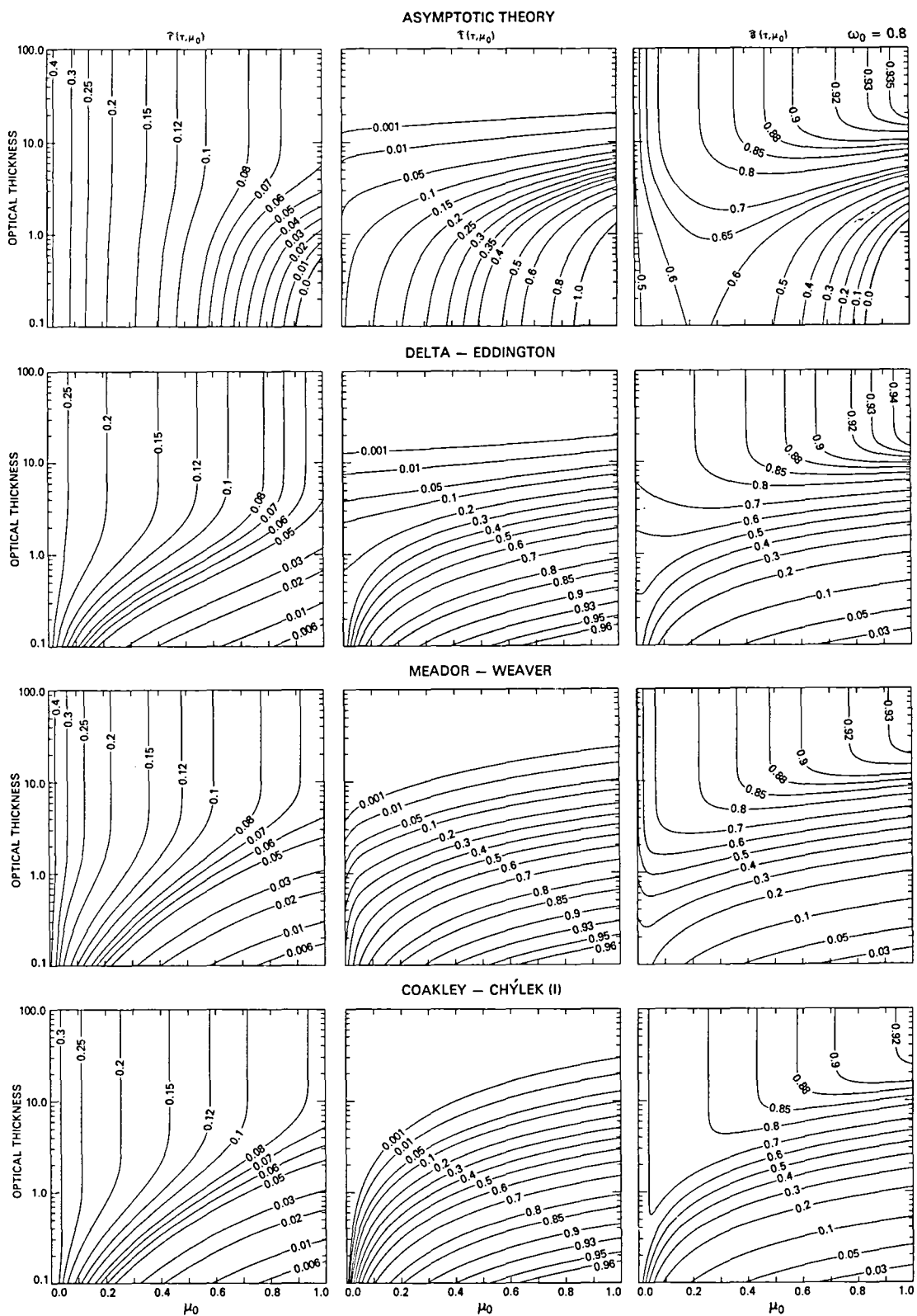


Figure 22. Computations of the plane albedo, total transmission and fractional absorption as a function of optical thickness and cosine of the solar zenith angle for asymptotic theory, delta-Eddington, Meador-Weaver and Coakley-Chýlek (I) approximations when $\omega_0 = 0.8$.

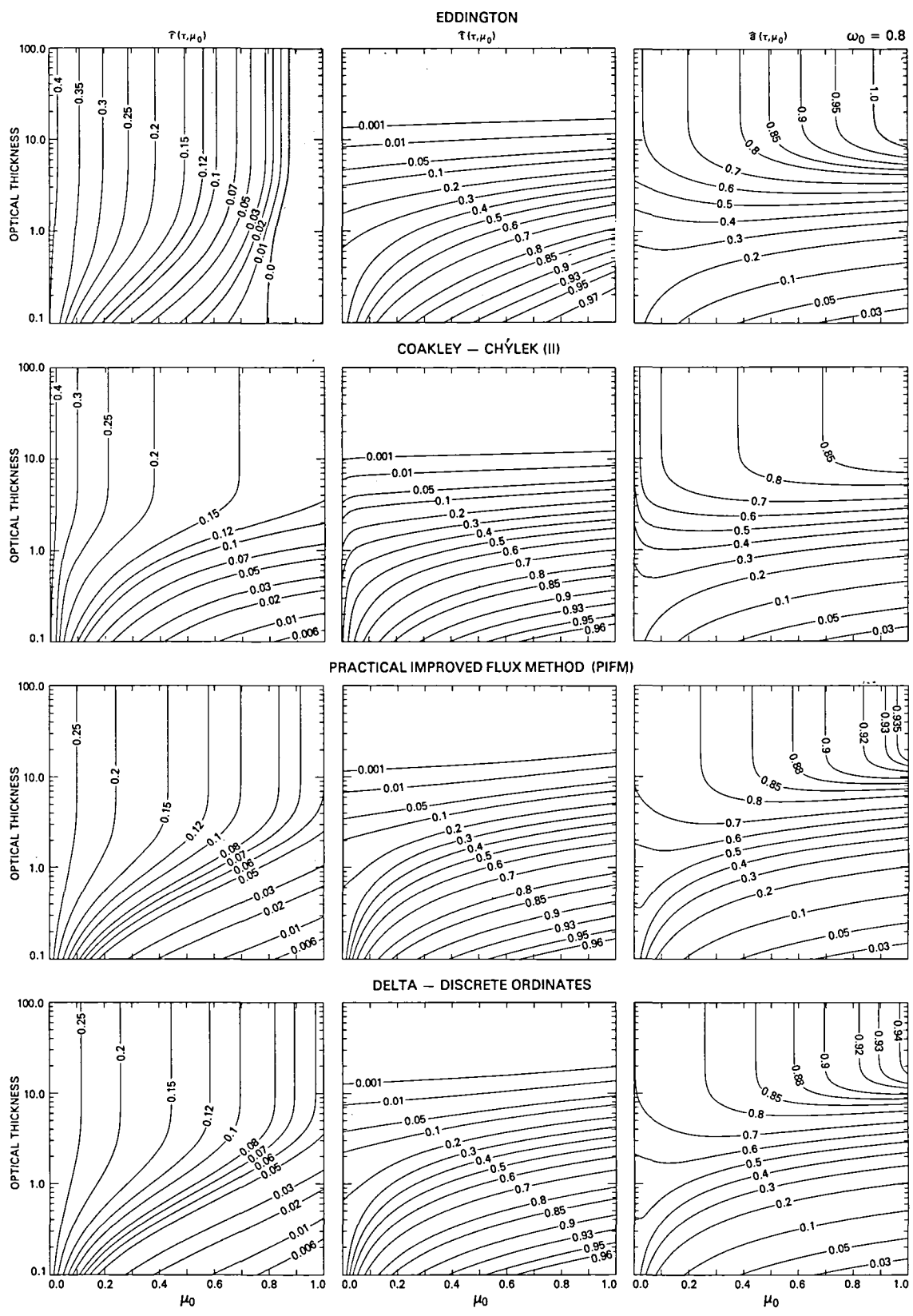


Figure 23. As in Fig. 22 except for the Eddington, Coakley-Chylek (II), PIFM and delta-discrete ordinates approximations.

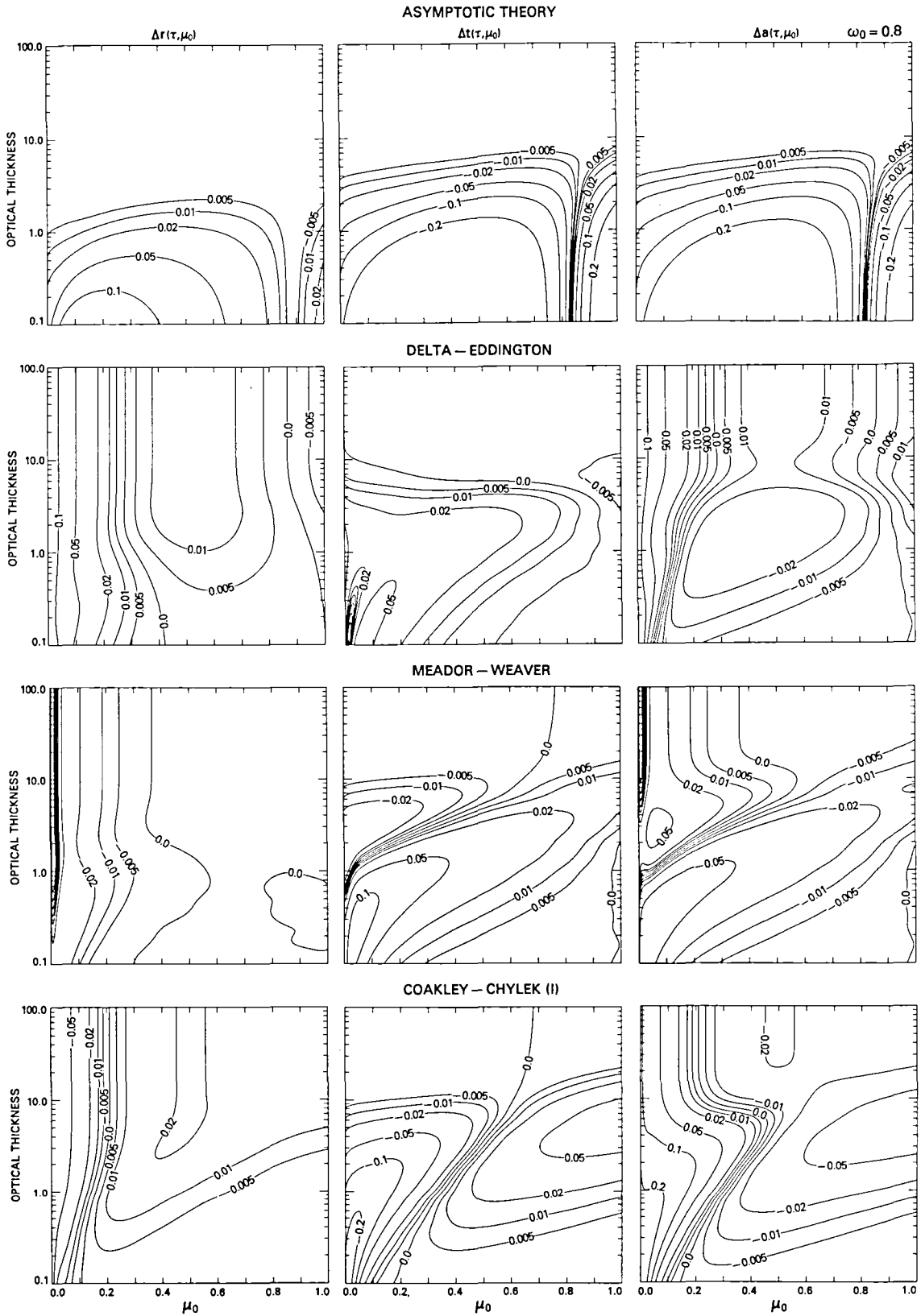


Figure 24. Absolute accuracy of asymptotic theory, delta-Eddington, Meador-Weaver and Coakley-Chylek (I) approximations to the plane albedo, total transmission and fractional absorption as a function of optical thickness and cosine of the solar zenith angle for nonconservative scattering ($\omega_0 = 0.8$).

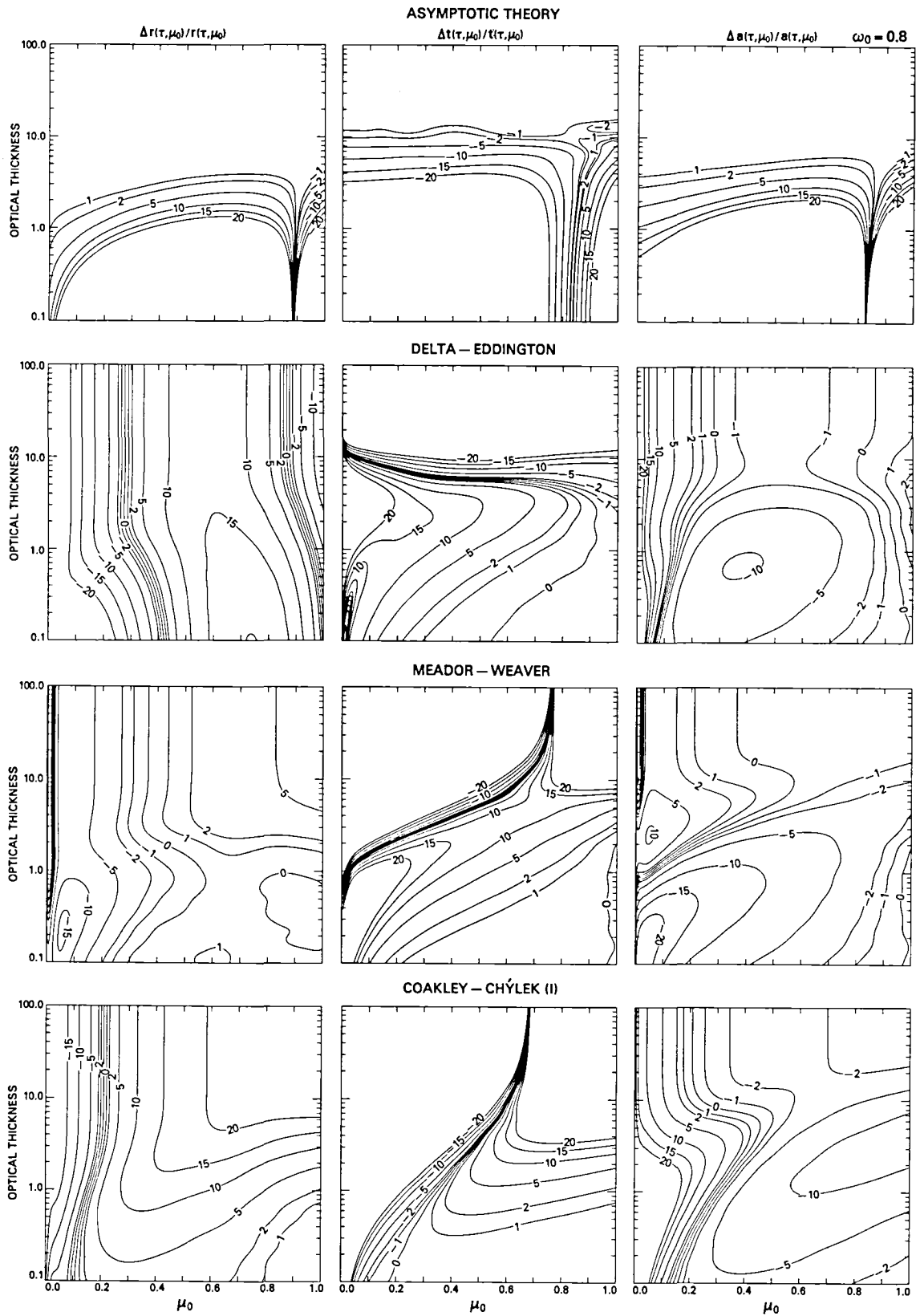


Figure 26. As is Fig. 24 except for relative accuracies (in per cent).

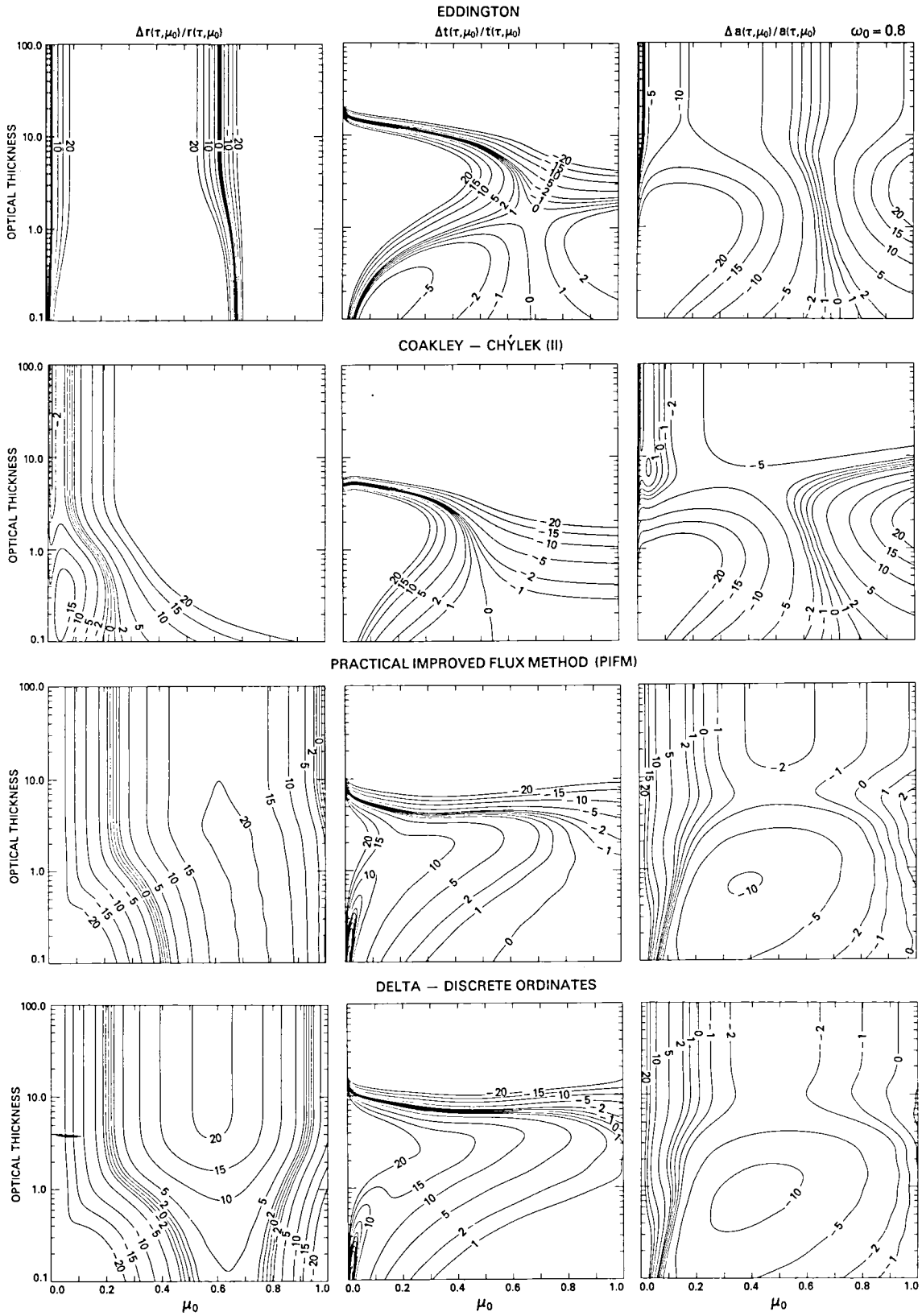


Figure 27. As is Fig. 25 except for relative accuracies (in per cent).

CONCLUDING REMARKS

Most previous intercomparisons of radiative transfer approximations have concentrated on presenting results for the plane albedo as a function of cosine of the solar zenith angle for selected values of the optical depth. On some occasions, intercomparisons have been further restricted to selected values of μ_0 . Although plane albedo errors in the delta-Eddington approximation are less than 5% for all optical depths when $\omega_0 = 0.9$ and $\mu_0 = 0.4$ (cf. Fig. 19), a generalized conclusion on its overall accuracy based on this restricted intercomparison would be highly misleading. We therefore feel it is important to examine the accuracy of the plane albedo, total transmission and fractional absorption as a function of optical depth and solar zenith angle before drawing conclusions about the overall accuracy of a given approximation.

After examining a wide variety of radiative transfer approximations over a large range of optical depths, solar zenith angles and single scattering albedos, it has become evident why some approximations succeed while others fail in specific regimes. For example, a straight-forward comparison of (13) and (29) shows that when $\omega_0 = 1$ and $\tau_t/\mu_0 \gg 1$, the plane albedo obtained from asymptotic theory and two-stream approximations are equivalent, provided the two-stream coefficients γ_1 and γ_3 satisfy the following criteria:

$$\gamma_1 = \frac{1}{2q'}(1 - g), \quad (35)$$

$$\gamma_3 = 1 + \gamma_1 \mu_0 - \frac{2K(\mu_0)}{3q'}. \quad (36)$$

For the delta-scaled approximations, the only difference in these criteria is the substitution $g \rightarrow g'$ in (35). Since $q' \simeq 0.714$ for all possible phase functions, (35) implies that $\gamma_1 \simeq 0.7(1 - g)$ for unscaled approximations and $0.7(1 - g')$ for scaled approximations. Table 2 shows that the Eddington and delta-Eddington methods satisfy these requirements the most closely. Furthermore, in order for a two-stream method to perform well for optically thick, conservative atmospheres, it is necessary for γ_3 to be a linear function of μ_0 over most of the range of solar zenith angles. This readily follows from a comparison of (36) with the escape function $K(\mu_0)$ presented in King and Harshvardhan (1986, Fig. 4). The poor performance of the Meador-Weaver and Coakley-Chylek (I) methods under these conditions is at least in part a result of their choice of $\gamma_3 = \beta(\mu_0)$, a function which is highly nonlinear in μ_0 . The Eddington and delta-Eddington methods, on the other hand, very nearly satisfy (36). Note that although γ_3 can be negative for high sun in the Eddington method, this is in accord with the requirement given by (36).

Due to the importance of developing a radiative transfer approximation which is accurate for all solar zenith angles and over a wide range of optical depths, our results suggest that a hybrid two-stream model that reduces to asymptotic theory for thick atmospheres but extends the range of validity of asymptotic theory to thinner atmospheres would be extremely valuable. The development of such a model remains a challenge for further study. Finally, we would like to note that none of the conclusions drawn in the present investigation are affected by our choice of a Mie theory phase function. Limited intercomparisons with doubling computations using the Henyey-Greenstein phase function with the same asymmetry factor as in the fair weather cumulus model ($g = 0.843$) yield error plots with virtually the same appearance as those of the FWC phase function.

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September 6, 1985

REFERENCES

- Coakley, J.A., Jr., and P. Chýlek, 1975: The two-stream approximation in radiative transfer: Including the angle of the incident radiation. *J. Atmos. Sci.*, **32**, 409-418.
- Deirmendjian, D., 1963: Scattering and polarization properties of polydispersed suspensions with partial absorption. In *Electromagnetic Scattering*, M. Kerker, Ed., Pergamon Press, 171-189.
- Hansen, J.E., 1971: Multiple scattering of polarized light in planetary atmospheres. Part I. The doubling method. *J. Atmos. Sci.*, **36**, 508-518.
- _____, and L.D. Travis, 1974: Light scattering in planetary atmospheres. *Space Sci. Rev.*, **16**, 527-610.
- Irvine, W.M., 1968: Multiple-scattering by large particles. II. Optically thick layers. *Astrophys. J.*, **152**, 823-834.
- Joseph, J.H., W.J. Wiscombe and J.A. Weinman, 1976: The delta-Eddington approximation for radiative flux transfer. *J. Atmos. Sci.* **33**, 2452-2459.
- Kawata, Y., and W. M. Irvine, 1970: The Eddington approximation for planetary atmospheres. *Astrophys. J.*, **160**, 787-790.
- King, M.D., 1981: A method for determining the single scattering albedo of clouds through observation of the internal scattered radiation field. *J. Atmos. Sci.*, **38**, 2031-2044.
- _____, 1983: Number of terms required in the Fourier expansion of the reflection function for optically thick atmospheres. *J. Quant. Spectrosc. Radiat. Transfer*, **30**, 143-161.
- _____, and Harshvardhan, 1986: Comparative accuracy of selected multiple scattering approximations. *J. Atmos. Sci.*, **43**, in press.
- Liou, K.N., 1973: A numerical experiment on Chandrasekhar's discrete-ordinate method for radiative transfer: Applications to cloudy and hazy atmospheres. *J. Atmos. Sci.*, **30**, 1303-1326.
- _____, 1974: Analytic two-stream and four-stream solutions for radiative transfer. *J. Atmos. Sci.*, **31**, 1473-1475.
- Meador, W.E., and W.R. Weaver, 1980: Two-stream approximations to radiative transfer in planetary atmospheres: A unified description of existing methods and a new improvement. *J. Atmos. Sci.*, **37**, 630-643.
- Schaller, E., 1979: A delta-two-stream approximation in radiative flux calculations. *Contrib. Atmos. Phys.*, **52**, 17-26.
- Shettle, E.P., and J.A. Weinman, 1970: The transfer of solar irradiance through inhomogeneous turbid atmospheres evaluated by Eddington's approximation. *J. Atmos. Sci.*, **27**, 1048-1055.
- Sobolev, V.V., 1975: *Light Scattering in Planetary Atmospheres* (Transl. by W.M. Irvine). Pergamon Press, 256 pp.
- van de Hulst, H.C., 1968a: Radiative transfer in thick atmospheres with an arbitrary scattering function. *Bull. Astron. Inst. Netherlands*, **20**, 77-86.
- _____, 1968b: Asymptotic fitting, a method for solving anisotropic transfer problems in thick layers. *J. Comput. Phys.* **3**, 291-306.
- _____, 1980: *Multiple Light Scattering. Tables, Formulas, and Applications*, Vols. 1 and 2. Academic Press, 739 pp.

Wiscombe, W.J., and J.H. Joseph, 1977: The range of validity of the Eddington approximation. *Icarus*, 32, 362-377.

Zdunkowski, W.G., R.M. Welch and G. Korb, 1980: An investigation of the structure of typical two-stream-methods for the calculation of solar fluxes and heating rates in clouds. *Contrib. Atmos. Phys.*, 53, 147-166.

1. Report No. NASA RP-1160		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle Comparative Accuracy of the Albedo, Transmission and Absorption for Selected Radiative Transfer Approximations				5. Report Date January 1986	
				6. Performing Organization Code 613	
7. Author(s) Michael D. King and Harshvardhan				8. Performing Organization Report No. 86B0034	
9. Performing Organization Name and Address NASA Goddard Space Flight Center Greenbelt, MD 20771				10. Work Unit No.	
				11. Contract or Grant No.	
				13. Type of Report and Period Covered Reference Publication	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546				14. Sponsoring Agency Code	
15. Supplementary Notes					
16. Abstract					
<p>This report contains illustrations of both the relative and absolute accuracy of eight different radiative transfer approximations as a function of optical thickness, solar zenith angle and single scattering albedo. Computational results for the plane albedo, total transmission and fractional absorption have been obtained for plane-parallel atmospheres composed of cloud particles. These computations, which were obtained using the doubling method, are compared with comparable results obtained using selected radiative transfer approximations. We concentrate our comparisons on asymptotic theory for thick layers and the following widely used two-stream approximations: Coakley-Chylek's models 1 and 2, Meador-Weaver, Eddington, delta-Eddington, PIFM and delta-discrete ordinates.</p>					
17. Key Words (Selected by Author(s)) Radiative transfer approximations Clouds Albedo Absorption Transmission			18. Distribution Statement Unclassified-Unlimited Subject Category 46		
19. Security Classif. (of this report) Unclassified		20. Security Classif. (of this page) Unclassified		21. No. of Pages 46	22. Price A03

For sale by the National Technical Information Service, Springfield, Virginia 22161.

**National Aeronautics and
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