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**Symmetry Considerations
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Identical Composite Bodies**

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Introduction

The interaction between two nonidentical composite bodies in terms of second quantization and particle-hole theory has been investigated previously (ref. 1). In that study, the initial projectile and target composites were assumed to be nonidentical as were the final scattered composites. This assumption greatly simplifies the symmetry considerations for the (initial or final) total projectile-target Fock state $|PT\rangle$, since it decomposes as

$$|PT\rangle = |P\rangle|T\rangle \quad (1)$$

which is *unsymmetrized* with respect to interchange of projectile and target. (Note also that $|P\rangle|T\rangle$ is not necessarily equal to $|T\rangle|P\rangle$.) Equation (1) is a fundamental result for *all* cases of the scattering of nonidentical particles (whether single particles or composite systems). The simple decomposition in equation (1) makes the evaluation of matrix elements such as $\langle P_0T_0|V|P_0T_0\rangle$ or $\langle PT|V|P_0T_0\rangle$ particularly simple. (Here V refers to the total interaction potential between the composites, and $|P_0T_0\rangle$ refers to the total ground-state wave function.) The evaluation of these matrix elements, with the use of equation (1), was the subject considered in reference 1. For the scattering of identical particles, however, the simple separation in equation (1) does *not* occur. This statement is true not only for identical initial states with identical final states, but also for nonidentical initial states with identical final states and identical initial states with nonidentical final states. (Eq. (1) is, of course, still appropriate for any nonidentical states.)

The aim of the present work is to consider the matrix elements of the total interaction potential, V , for the case of identical, as well as nonidentical, interacting composite systems. Important references are pages 362–378 of reference 2, pages 80–81 of reference 3, and pages 332–348 of reference 4. A commonly used abbreviation for a wave function such as the projectile wave function $\Phi(\mathcal{E}_P)$ is $\Phi(P)$.

Symmetries of Scattering Amplitudes and Matrix Elements

The “exchange” term of a scattering amplitude $f(\theta, \phi)$ can be written as $f(\pi-\theta, \phi+\pi)$ (ref. 2). In the following development, $f(\theta, \phi)$ is abbreviated by $f(\theta)$ and $f(\pi-\theta, \phi+\pi)$ by $f(\pi-\theta)$. For elastic scattering of identical aggregates, the cross section is written in terms of the scattering amplitude and the potential matrix elements. Results are also obtained for inelastic scattering.

First recall that for the elastic scattering of identical particles (such as ^{12}C incident upon ^{12}C), the total elastic differential cross section is given by

$$\frac{d\sigma_{\text{el}}}{d\Omega} = |f(\theta)|^2 \quad (2)$$

where $f(\theta)$ is the elastic scattering amplitude. Merzbacher (ref. 5) shows that the scattering amplitude is directly proportional to the T-matrix, which, in the Born approximation, means that $f(\theta)$ is proportional to the matrix element of the potential $\langle K|V|I\rangle$, where $|I\rangle$ and $|K\rangle$ are the initial and final states, respectively.

Thus, in the Born approximation,

$$\frac{d\sigma_{\text{el}}}{d\Omega} \propto |\langle K|V|I\rangle|^2 \quad (3)$$

where

$$|I\rangle \equiv |P_0T_0\rangle \quad (4)$$

and

$$|K\rangle \equiv |PT\rangle \quad (5)$$

and in the case of elastic scattering,

$$|PT\rangle = |P_0T_0\rangle \quad (6)$$

Recall the unsymmetrized decomposition of equation (1) for nonidentical composites. Equation (1) can also be written as

$$\left. \begin{aligned} \Psi &= \Phi_i(P) \Phi_j(T) \\ &\equiv |P_i \rangle |T_j \rangle \end{aligned} \right\} \quad (7)$$

where i and j refer to an appropriate collection of spatial quantum numbers and various types of “spin” quantum numbers.

Identical Composites

The total wave function Ψ for two identical composites (see appendix A) is

$$\left. \begin{aligned} \Psi &= \frac{1}{\sqrt{2}} [\Phi_i(P) \Phi_j(T) \pm \Phi_j(P) \Phi_i(T)] \\ &\equiv \frac{1}{\sqrt{2}} (|P_i \rangle |T_j \rangle \pm |P_j \rangle |T_i \rangle) \\ &\equiv \frac{1}{\sqrt{2}} (|i \rangle |j \rangle \pm |j \rangle |i \rangle) \end{aligned} \right\} \quad (8)$$

In equations (8), the plus sign is appropriate for a boson aggregate, in which the total wave function is symmetric, and the minus sign is for a fermion aggregate, which must have an antisymmetric total wave function. Let us be very clear about the meaning of equations (8). When the identical projectile and targets are interchanged, this means that the projectile now has the spatial and spin quantum numbers of the target and vice versa. This aspect is emphasized in the first and second expressions of equations (8). The third expression is what one would write from the Young Tableau (ref. 6), with the unnormalized symmetric state being

$$\boxed{1 \ 2} |i \rangle |j \rangle = |i \rangle |j \rangle + |j \rangle |i \rangle \quad (9)$$

and the unnormalized antisymmetric state given by

$$\boxed{\frac{1}{2}} |i \rangle |j \rangle = |i \rangle |j \rangle - |j \rangle |i \rangle \quad (10)$$

With the Young Tableau approach in equations (9) and (10), we must remember what is meant with respect to the projectile and target as in equations (8).

If the aggregates themselves consist of a collection of fermions, then the aggregate will be a boson if the number of fermions is even, and it will be a fermion if it consists of an odd number of fermions (ref. 7). Hadrons are made up of quarks with spin equal to $1/2$ (i.e., fermions), and thus it is understood why baryons are fermions and mesons are bosons.)

For the case of elastic scattering of identified composites, equation (2) becomes (ref. 2)

$$\frac{d\sigma_{el}}{d\Omega} = |f(\theta) \pm f(\pi-\theta)|^2 \quad (11)$$

Elastic Scattering

Up to now the wave functions and scattering amplitudes have been presented for the case of two nonidentical and two identical aggregates. Equations (8) and (11) reflect the fact of the identity of the aggregates. We wish now to show that the use of either of these equations will increase the cross sections by the same amount over the cross section for nonidentical aggregate scattering. That the cross section be increased by the same amount when one uses $f(\theta) \pm f(\pi-\theta)$ in equation (11) or $\langle \Psi | V | \Psi \rangle$ from equations (8) is an essential consistency check.

First we deal with the scattering amplitude, which is appropriate only in the case of elastic scattering. Equation (11) becomes (ref. 2)

$$\frac{d\sigma_{\text{el}}}{d\Omega} = |f(\theta)|^2 + |f(\pi-\theta)|^2 \pm 2\text{Re}[f(\theta) f^*(\pi-\theta)] \quad (12)$$

which is of similar form to

$$|\Psi|^2 = \frac{1}{2}\{|\Phi_i(P) \Phi_j(T)|^2 + |\Phi_j(P) \Phi_i(T)|^2 \pm 2\text{Re}[\Phi_i(P) \Phi_j(T) \Phi_j^*(P) \Phi_i^*(T)]\} \quad (13)$$

It is misleading to seek analogy between equation (12) and the square of the wave function in equation (13) because the scattering amplitude corresponds to $\langle \Psi|V|\Psi \rangle$ (ref. 5) and *not* to $|\Psi|^2$. Thus

$$f(\theta) \pm f(\pi-\theta) \propto \langle \Psi|V|\Psi \rangle \quad (14)$$

We now introduce the notation

$$u(PT) \equiv \Phi_i(P) \Phi_j(T) = |P_i \rangle |T_j \rangle \quad (15)$$

and

$$u(TP) \equiv \Phi_j(P) \Phi_i(T) = |P_j \rangle |T_i \rangle \quad (16)$$

with $u(PT)$ and $u(TP)$ being analogous to $u(12)$ and $u(21)$ used by Schiff (ref. 2, p. 367). Thus from equations (8) and (14)

$$\left. \begin{aligned} f(\theta) \pm f(\pi-\theta) &\propto \frac{1}{2}\{ \langle [u(PT) \pm u(TP)]|V|[u(PT) \pm u(TP)] \rangle \} \\ &= \frac{1}{2}\{ \langle u(PT)|V|u(PT) \rangle + \langle u(TP)|V|u(TP) \rangle \\ &\quad \pm \langle u(TP)|V|u(PT) \rangle \pm \langle u(PT)|V|u(TP) \rangle \} \end{aligned} \right\} \quad (17)$$

Now, although $u(PT)$ is not equal to $u(TP)$, it is true that (see appendix B)

$$\langle u(PT)|V|u(PT) \rangle = \langle u(TP)|V|u(TP) \rangle \quad (18)$$

and

$$\langle u(TP)|V|u(PT) \rangle = \langle u(PT)|V|u(TP) \rangle \quad (19)$$

so that equations (17) become

$$f(\theta) \pm f(\pi-\theta) \propto \langle u(PT)|V|u(PT) \rangle \pm \langle u(TP)|V|u(PT) \rangle \quad (20)$$

Clearly then

$$\left. \begin{aligned} \frac{d\sigma_{\text{el}}}{d\Omega} &= |f(\theta)|^2 + |f(\pi-\theta)|^2 \pm 2\text{Re}[f(\theta) f^*(\pi-\theta)] \\ &\propto \{ \langle u(PT)|V|u(PT) \rangle^2 + \langle u(TP)|V|u(PT) \rangle^2 \\ &\quad \pm 2\text{Re}[\langle u(PT)|V|u(PT) \rangle \langle u(TP)|V|u(PT) \rangle^*] \} \end{aligned} \right\} \quad (21)$$

(See appendix C for the inclusion of spin.) Just because particles are identical does *not* require that they are indistinguishable. In the laboratory we always distinguish the ^{12}C beam from the ^{12}C target, in which case we have a situation of distinguishable identical particles. A case of indistinguishable identical particles would occur, for example, in a molecule consisting of two ^{12}C atoms. Schiff (ref. 2) points out that "it is to be expected that the result of an experiment is independent of the symmetry character of the

wave function if the coordinates of the particles do not overlap," which corresponds to a situation in which the particles can be distinguished by their position vectors or some other quantum number such as spin. This means that $u(PT)$ is nonzero only when the projectile coordinate in some region, \mathcal{E}_P , and the target coordinate in some region, \mathcal{E}_T , do not overlap. Consequently, the interference term in equation (40.8) of Schiff (ref. 2) is zero everywhere.

Schiff (ref. 2, p. 370) points out that for distinguishable identical particles, the interference term in equations (21) drops out, so that

$$\frac{d\sigma_{el}}{d\Omega} = |f(\theta)|^2 + |f(\pi-\theta)|^2 \quad (22)$$

and

$$\left. \begin{aligned} \frac{d\sigma_{el}}{d\Omega} &\propto |\langle u(PT)|V|u(PT)\rangle|^2 + |\langle u(TP)|V|u(PT)\rangle|^2 \\ &= |\langle P_i|\langle T_j|V|P_i\rangle|T_j\rangle|^2 + |\langle P_j|\langle T_i|V|P_i\rangle|T_j\rangle|^2 \end{aligned} \right\} \quad (23)$$

which is just the sum of the individual differential cross sections for observation of the incident particle $|f(\theta)|^2$ and of the struck particle $|f(\pi-\theta)|^2$. This is also the result obtained if the particles are *not* identical, but no attempt is made to distinguish between them in the experiment (ref. 2).

For the scattering of nonidentical particles, one obtains only the first terms in equations (22) and (23). Equations (22) and (23) represent the fulfillment of one of our initial aims, namely to show that the cross section increases (for the case of identical particle scattering over that of nonidentical scattering) by the same amount independent of whether our description is in terms of a scattering amplitude or a matrix element. Note that if

$$|f(\theta)|^2 = |f(\pi-\theta)|^2 \quad (24)$$

and

$$|\langle P_i|\langle T_j|V|P_i\rangle|T_j\rangle|^2 = |\langle P_j|\langle T_i|V|P_i\rangle|T_j\rangle|^2 \quad (25)$$

then

$$\left(\frac{d\sigma_{el}}{d\Omega}\right)_{\text{identical aggregates}} = 2 \left(\frac{d\sigma_{el}}{d\Omega}\right)_{\text{nonidentical aggregates}} \quad (26)$$

It is to be emphasized that this result depends on (1) the vanishing of the interference term and (2) the validity of equations (24) and (25).

We come now to a very important point. Suppose that we have developed a theory for calculating the cross section for production of a ^{12}C projectile elastically scattered from a ^{12}C target. Clearly an experimental detector cannot distinguish whether the detected ^{12}C nucleus is the original projectile or the scattered target. Equations (22) and (23) imply that the experimental cross section will be the *addition* of the cross sections of the elastically scattered ^{12}C projectile and the elastically scattered ^{12}C target. A similar result is demonstrated subsequently for the case of inelastic scattering.

Inelastic Scattering

For the case of elastic scattering, it is clear that $|P_i\rangle$, $|T_j\rangle$, and so forth are just the Fermi vacua ($|P_0\rangle$, $|T_0\rangle$, etc.) in both the initial and final channels. In inelastic scattering, the final states are different from the initial states and are generally written in terms of particle-hole operators acting on the initial states. For the inelastic scattering of two identical initial aggregates, we can either have identical or nonidentical aggregates in the final state.

Let us first consider the former case. The initial state is

$$|P_0T_0\rangle = \frac{1}{\sqrt{2}}(|P_i\rangle|T_j\rangle \pm |P_j\rangle|T_i\rangle) \quad (27)$$

and the final state, now different from the initial state, is

$$|PT\rangle_f = \frac{1}{\sqrt{2}}(|P_k\rangle|T_l\rangle \pm |P_l\rangle|T_k\rangle) \quad (28)$$

Thus

$$f \langle PT|V|P_0T_0 \rangle = \frac{1}{2}[\langle P_k | \langle T_l | V | P_i \rangle | T_j \rangle + \langle P_l | \langle T_k | V | P_j \rangle | T_i \rangle \pm \langle P_k | \langle T_l | V | P_j \rangle | T_i \rangle \pm \langle P_l | \langle T_k | V | P_i \rangle | T_j \rangle] \quad (29)$$

and again, the first and second matrix elements in equation (29) are equal, as are the third and fourth. (See appendix B.) Thus,

$$f \langle PT|V|P_0T_0 \rangle = \langle P_k | \langle T_l | V | P_i \rangle | T_j \rangle \pm \langle P_k | \langle T_l | V | P_j \rangle | T_i \rangle \quad (30)$$

is the result for identical initial and identical final aggregates analogous to equation (20).

For nonidentical final aggregates,

$$|PT \rangle_{f'} = |P_k \rangle |T_i \rangle \quad (31)$$

Thus

$$f' \langle PT|V|P_0T_0 \rangle = \frac{1}{\sqrt{2}}(\langle P_k | \langle T_l | V | P_i \rangle | T_j \rangle \pm \langle P_k | \langle T_l | V | P_j \rangle | T_i \rangle) \quad (32)$$

is the result for identical initial and nonidentical final aggregates.

Thus, if the matrix elements in equations (30) and (32) were equal (which perhaps is unlikely), then one would obtain

$$\left(\frac{d\sigma}{d\Omega}\right)_{\substack{\text{identical initial} \\ \text{identical final}}} = 2 \left(\frac{d\sigma}{d\Omega}\right)_{\substack{\text{identical initial} \\ \text{nonidentical final}}} \quad (33)$$

One cannot immediately relate the above inelastic cross sections to the elastic case because the final states are radically different.

Equation (32) has a very important application in the pion work of Norbury, Deutchman, and Townsend (ref. 8) when extended to a ^{12}C projectile inelastically scattering from a ^{12}C target. An isobar giant resonance (which decays via pion emission) is formed in one nucleus, and a magnetic giant dipole resonance (decaying via photon emission) is formed in the other nucleus. An experimental measurement of the pion alone or of the pion in coincidence with the photon cannot distinguish whether the pion comes from the projectile or the target. The theory (ref. 8), however, was originally developed for pions coming from the projectile. Equation (32) tells one how to extend the projectile pion theory to pions coming from the projectile or the target. Evidently equation (32) is analogous to equation (20). Thus, *if interference terms are neglected*, the cross section pertaining to equation (32) will be identical in form to equations (22) and (23), which are the cross sections pertaining to equation (20). Clearly, then, the experimental pion cross section will be the *addition* of the theoretical cross section for pions coming from the projectile and the theoretical cross section for pions coming from the target.

Finally, for nonidentical initial aggregates and identical final aggregates, such as the reaction $\pi^+\pi^- \rightarrow \pi^0\pi^0$, one has the initial state

$$|P_0T_0 \rangle' = |P_i \rangle |T_j \rangle \quad (34)$$

and the final state given by equation (28), so that

$$f \langle PT|V|P_0T_0 \rangle' = \frac{1}{\sqrt{2}}(\langle P_k | \langle T_l | V | P_i \rangle | T_j \rangle \pm \langle P_l | \langle T_k | V | P_i \rangle | T_j \rangle) \quad (35)$$

Equation (35) is to be compared with the result for the inelastic reaction of nonidentical initial and final states (e.g., $\pi^+\pi^- \rightarrow K^+K^-$) considered in equation (39) of reference 1, which would be written *without* the interference term as

$$f \langle PT|V|P_0T_0 \rangle = \langle P_k | \langle T_l | V | P_i \rangle | T_j \rangle \quad (36)$$

This is similar to the elastic reaction of nonidentical initial and final states ($\pi^+\pi^- \rightarrow \pi^+\pi^-$) considered in equation (23) of reference 1, which would be written as

$$\langle P_0 T_0 | V | P_0 T_0 \rangle = \langle P_i | \langle T_j | V | P_i \rangle | T_j \rangle \quad (37)$$

The results for all these matrix elements are listed in table I.

Explicit Evaluation of Matrix Elements

So far the actual form of the total composite wave functions in terms of the single-particle wave functions has not been considered, and thus the matrix elements have not been explicitly evaluated. This separate problem is not considered in the present work. However, we do provide the guidelines as to how to proceed.

The matrix elements of equations (36) and (37), for the case of nonidentical initial and final states, have been explicitly evaluated in reference 1. For elastic scattering, the initial and final states were taken to be simply $|P_0 \rangle |T_0 \rangle$ with the result that (ref. 1)

$$\left. \begin{aligned} \langle P_i | \langle T_j | V | P_i \rangle | T_j \rangle &= \langle P_0 | \langle T_0 | V | P_0 \rangle | T_0 \rangle \\ &= \sum_{k_P}^{A_P} \sum_{l_T}^{A_T} [\langle k_P(P) l_T(T) | v | k_P(P) l_T(T) \rangle \\ &\quad - \langle l_T(P) k_P(T) | v | k_P(P) l_T(T) \rangle] \end{aligned} \right\} \quad (38)$$

For the case of inelastic scattering, the excited (nonidentical) projectile-target final state was taken to be the particle-hole Tamm-Dancoff state

$$|P_k \rangle |T_l \rangle \equiv |ph \rangle = a_{\nu_T}^+ a_{\lambda_T} a_{\gamma_P}^+ a_{\alpha_P} |P_0 T_0 \rangle \quad (39)$$

which gives the result (ref. 1) that

$$\left. \begin{aligned} \langle P_k | \langle T_l | V | P_i \rangle | T_j \rangle &= \langle ph | V | P_0 \rangle | T_0 \rangle \\ &= \langle \nu_T(T) \gamma_P(P) | v | \lambda_T(T) \alpha_P(P) \rangle \\ &\quad - \langle \gamma_P(T) \nu_T(P) | v | \lambda_T(T) \alpha_P(P) \rangle \end{aligned} \right\} \quad (40)$$

Similar particle-hole and ground-state wave functions can be inserted for explicit evaluation of the other matrix elements.

Summary of Results

Previous studies of the interactions between composite particles have been extended to the case in which the composites are identical. The form of the total interaction potential matrix elements has been obtained, and guidelines for their explicit evaluation have been given. For the case of elastic scattering of identical composites, the matrix element approach has been shown to be equivalent to the scattering amplitude method.

Often in an experiment it is not possible to determine whether a detected particle has its origin in the projectile or the target. Justification for simply adding the projectile and target theoretical cross sections in order to compare theoretical and experimental results has been given in some instances. It is very important to bear in mind, however, that in this simple addition of cross sections, the interference terms are assumed to be negligible.

It was also shown that *if* the direct term is equal to the exchange term (which may be quite unlikely) and if the interference terms are negligible, then for certain reactions the cross section for identical aggregates is double that for nonidentical aggregates.

This doubling of the cross section depends on the very stringent condition of direct and exchange terms being equal, which probably rarely happens, and thus the result is of pedagogic interest only. However, the cross section addition alluded to above is an important and more useful result.

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TABLE I. MATRIX ELEMENTS WITH EXAMPLES OF REACTIONS

[The reactions are schematic only and do not necessarily represent the real physical
process in that there may be more particles in the final state than indicated.]

		Elastic scattering		Inelastic scattering		
		Nonidentical	Identical	Nonidentical	Identical	
Initial state aggregates	Final state aggregates					
Nonidentical	Nonidentical	$\langle P_i \langle T_j V P_i \rangle T_j \rangle$ equation (37) (explicitly evaluated in ref. 1) e.g., $\pi^+ \pi^- \rightarrow \pi^+ \pi^-$ $^{12}\text{C} + ^{16}\text{O} \rightarrow ^{12}\text{C} + ^{16}\text{O}$	Identical	$\langle P_k \langle T_l V P_i \rangle T_j \rangle$ equation (36) (explicitly evaluated in ref. 1) e.g., $\pi^+ \pi^- \rightarrow K^+ K^-$ $^{16}\text{O} + ^{12}\text{C} \rightarrow ^{16}\text{O}_{\Delta^0}^* + ^{12}\text{C}_{M1}$ <div style="display: flex; justify-content: center; gap: 20px; margin-top: 5px;"> <div style="text-align: center;"> \downarrow $^{16}\text{N} + \pi^0$ </div> <div style="text-align: center;"> \downarrow $^{12}\text{N} + \gamma$ </div> </div>	Identical	$\frac{1}{\sqrt{2}} (\langle P_k \langle T_l V P_i \rangle T_j \rangle$ $\pm \langle P_l \langle T_k V P_i \rangle T_j \rangle)$ equation (35) e.g., $\pi^+ \pi^- \rightarrow \pi^0 \pi^0$
Identical	Identical	$\langle P_i \langle T_j V P_i \rangle T_j \rangle$ $\pm \langle P_j \langle T_i V P_i \rangle T_j \rangle$ equations (20), (15), and (16) e.g., $\pi^0 \pi^0 \rightarrow \pi^0 \pi^0$ $^{12}\text{C} + ^{12}\text{C} \rightarrow ^{12}\text{C} + ^{12}\text{C}$	$\frac{1}{\sqrt{2}} (\langle P_k \langle T_l V P_i \rangle T_j \rangle$ $\pm \langle P_k \langle T_l V P_j \rangle T_i \rangle)$ equation (32) e.g., $\pi^0 \pi^0 \rightarrow \pi^+ \pi^-$ $^{12}\text{C} + ^{12}\text{C} \rightarrow ^{12}\text{C}_{\Delta}^* + ^{12}\text{C}^*$ <div style="display: flex; justify-content: center; gap: 20px; margin-top: 5px;"> <div style="text-align: center;"> \downarrow $^{12}\text{C} + \pi$ </div> <div style="text-align: center;"> \downarrow $^{12}\text{C} + \gamma$ </div> </div>	$\langle P_k \langle T_l V P_i \rangle T_j \rangle$ $\pm \langle P_k \langle T_l V P_j \rangle T_i \rangle$ equation (30) e.g., $\pi^0 \pi^0 \rightarrow K^0 K^0$ $^{12}\text{C} + ^{12}\text{C} \rightarrow ^{12}\text{C}_{\Delta}^* + ^{12}\text{C}_{\Delta}^*$ <div style="display: flex; justify-content: center; gap: 20px; margin-top: 5px;"> <div style="text-align: center;"> \downarrow $^{12}\text{C} + \pi$ </div> <div style="text-align: center;"> \downarrow $^{12}\text{C} + \pi$ </div> </div>		

Appendix A

Resonating Group Method

In this section, the overall symmetry of the two-composite body wave function is discussed in connection with the resonating group method (RGM) and the generator coordinate method (GCM) for the choice of the wave function as given in equations (8). Interest in the RGM and GCM arises because these methods imply a choice of the symmetry properties of the overall wave function, and this choice is *different* from that in the present work (eqs. (8)).

The generator coordinate method was originally proposed by Hill and Wheeler (ref. 9) and Griffin and Wheeler (ref. 10) to describe the phenomenon of nuclear fission. For many years, it was applied with considerable success to bound-state problems such as nuclear rotations both with (ref. 11) and without (ref. 12) axial symmetry and also to studies of nuclear shapes and collective motions in light nuclei (refs. 13 and 14). The resonating group method also dates back to Wheeler (refs. 15–17) in studies of clustering phenomena in nuclei. According to Harvey, Le Tourneux, and Lorazo (ref. 18) and Onsi and Le Tourneux (ref. 19), there is a complete physical equivalence between the RGM and the GCM for nuclear scattering and reactions. Because of this, we do not go into the details of the differences between these two methods.

As far as scattering problems are concerned, the RGM and GCM have been especially applied to the case of ^4He - ^4He scattering (refs. 19–23) and also to the more difficult problem of nucleus-nucleus scattering (refs. 24–28), which is of most relevance to the present work. Efforts in this area have involved deriving a nucleus-nucleus optical potential from the basic nucleon-nucleon interaction. Finally, the RGM and GCM have found substantial applications in nucleon-nucleon scattering in terms of the underlying quark degrees of freedom (refs. 18 and 29–37). The authors of references 36 and 37 have also introduced a modified potential to deal with the spurious van der Waals color force.

In the present work, the overall wave function for the scattering of two identical composites has been chosen by the form given in equations (8). Let us consider the scattering of two identical composites, each made of two particles. An example of this is deuteron-deuteron scattering with each deuteron composed of two identical nucleons (with their isospin difference ignored). (Note that meson-meson scattering is slightly different, as the mesons are composed of nonidentical particles, namely quark and antiquark.) If the constituent particles are labeled as 1, 2, 3, and 4, equations (8) become

$$\left. \begin{aligned} \Psi &= |P_i \rangle |T_j \rangle \pm |P_j \rangle |T_i \rangle \\ &= \Phi_P(12) \Phi_T(34) \pm \Phi_P(34) \Phi_T(12) \end{aligned} \right\} \quad (\text{A1})$$

where the normalization factor $1/\sqrt{2}$ has been suppressed, and the symmetric combination, for instance, is appropriate for the two-boson (deuteron) interaction. Here, for example

$$\left. \begin{aligned} \Phi_P(12) &= \begin{vmatrix} \phi_{P_\alpha}(1) & \phi_{P_\alpha}(2) \\ \phi_{P_\beta}(1) & \phi_{P_\beta}(2) \end{vmatrix} \\ &= \phi_{P_\alpha}(1) \phi_{P_\beta}(2) - \phi_{P_\alpha}(2) \phi_{P_\beta}(1) \\ &= a_{P_\alpha}^+ a_{P_\beta}^+ |0 \rangle \\ &= |P_i \rangle \end{aligned} \right\} \quad (\text{A2})$$

where $|0 \rangle$ represents the absolute vacuum and $|P_i \rangle$ represents the projectile Fermi vacuum. The RGM wave function, however, is (refs. 18, 36, 38, and 39)

$$\Psi = A[\Phi_T(\xi_T) \Phi_P(\xi_P) \chi(R_{PT})] \quad (\text{A3})$$

where ξ_T and ξ_P are internal coordinates, χ is the relative wave function in terms of the relative coordinate R_{PT} , and A is the antisymmetrizer operator between all constituents, so that Ψ actually contains a sum

term with all possible choices of constituents separated into antisymmetric cluster Φ . (Note that both in the present work and in ref. 1, the relative wave function has always been suppressed. If, for example, one is working in the projectile frame, then the target wave function can be written as the product of the relative wave function and the internal target wave function (ref. 8).) Thus for the deuteron problem, the RGM wave function would be

$$\begin{aligned} \Psi = & \Phi_T(12) \Phi_P(34) + \Phi_T(34) \Phi_P(12) - \Phi_P(32) \Phi_T(14) \\ & - \Phi_P(42) \Phi_T(31) - \Phi_P(13) \Phi_T(24) - \Phi_P(14) \Phi_T(32) \end{aligned} \quad (\text{A4})$$

Note that for meson-meson scattering (nonidentical quark-antiquark constituents), the fourth and fifth terms in equation (A4) would not appear (ref. 39). By comparison of the wave function used in the present work (eqs. (A1)) and the RGM wave function (eq. (A4)), one can see that the basic difference between the two is that the present work ignores the symmetry terms arising from exchange of constituents between *different* composites.

Another comment to be made is that potential models of hadrons lead to spurious van der Waals forces. Greenberg and Hietarinta (refs. 39 and 40) have introduced the idea of link operators (often called strings or flux tubes) to avoid these forces at the outset by modifying the form of the RGM wave function which contains quark-quark or antiquark-antiquark exchange between the two mesons. Hybrid models (refs. 38 and 41–43), which are somewhat simpler than the link operator formalism, have also been used to avoid van der Waals effects, and again these models modify the RGM wave function. The conclusion here is that a straight application of the RGM wave function in hadron potential models can produce these van der Waals problems. Finally, note that Harvey (ref. 13) has argued against the use of equation (A3) in favor of emphasizing the underlying quark symmetries within a given hadron.

Thus, in conclusion, there are four main reasons why the present work has made use of the overall wave function given in equations (8) rather than the RGM wave function. The first and foremost reason is that equations (8) lead to a much simpler analysis of scattering, as presented herein. This simpler form is certainly appropriate when looking at these questions for the first time. A more complete inclusion of symmetry can be incorporated later. Second, the use of the RGM wave function as applied to nucleus-nucleus scattering involved several model approximations in its own right (refs. 19–28). The third and fourth reasons involve applications of quark potential models of hadrons. The two problems here have been mentioned above.

Appendix B

Proof of Equations (18) and (19)

In this appendix we wish to prove equations (18) and (19) and the statement following equation (29). We begin with equation (18) where $u(PT)$ and $u(TP)$ are defined in equations (15) and (16), so that

$$\langle u(PT)|V|u(PT) \rangle = \iint \Phi_i^*(\mathcal{E}_P) \Phi_j^*(\mathcal{E}_T) V \Phi_i(\mathcal{E}_P) \Phi_j(\mathcal{E}_T) d\mathcal{E}_P d\mathcal{E}_T \quad (\text{B1})$$

and

$$\langle u(TP)|V|u(TP) \rangle = \iint \Phi_j^*(\mathcal{E}_P) \Phi_i^*(\mathcal{E}_T) V \Phi_j(\mathcal{E}_P) \Phi_i(\mathcal{E}_T) d\mathcal{E}_P d\mathcal{E}_T \quad (\text{B2})$$

Because \mathcal{E}_P and \mathcal{E}_T are dummy integration variables, we can interchange them in equation (B2) without altering the value of the integral and thus verify equation (18).

In equation (19) we have

$$\langle u(TP)|V|u(PT) \rangle = \iint \Phi_j^*(\mathcal{E}_P) \Phi_i^*(\mathcal{E}_T) V \Phi_i(\mathcal{E}_P) \Phi_j(\mathcal{E}_T) d\mathcal{E}_P d\mathcal{E}_T \quad (\text{B3})$$

and

$$\langle u(PT)|V|u(TP) \rangle = \iint \Phi_i^*(\mathcal{E}_P) \Phi_j^*(\mathcal{E}_T) V \Phi_j(\mathcal{E}_P) \Phi_i(\mathcal{E}_T) d\mathcal{E}_P d\mathcal{E}_T \quad (\text{B4})$$

Again upon interchange of the dummy integration variables, it is evident that equation (19) is correct.

We now wish to verify the statement following equation (29). Evidently, from equations (7)

$$\langle P_k | \langle T_l | V | P_i \rangle | T_j \rangle = \iint \Phi_k^*(\mathcal{E}_P) \Phi_l^*(\mathcal{E}_T) V \Phi_i(\mathcal{E}_P) \Phi_j(\mathcal{E}_T) d\mathcal{E}_P d\mathcal{E}_T \quad (\text{B5})$$

and

$$\langle P_l | \langle T_k | V | P_j \rangle | T_i \rangle = \iint \Phi_l^*(\mathcal{E}_P) \Phi_k^*(\mathcal{E}_T) V \Phi_j(\mathcal{E}_P) \Phi_i(\mathcal{E}_T) d\mathcal{E}_P d\mathcal{E}_T \quad (\text{B6})$$

which, again by interchanging integration variables, means that the first two terms on the right-hand side of equation (29) are equal. The same argument is easily applied to the second two terms also. Thus, the statement below equation (29) is verified.

Appendix C

Inclusion of Spin

The whole formalism presented herein is independent of spin. That is, the spin of the projectile and target has been ignored. If the projectile and target have *different* spin values, they are then *distinguishable*, and the interference term in equations (21) drops out (ref. 2, p. 374), so that

$$\frac{d\sigma_{\text{el}}}{d\Omega} = |f(\theta)|^2 + |f(\pi-\theta)|^2 \quad (\text{C1})$$

If the projectile and target have the *same* spin, S , and their interaction does not involve spin (ref. 2, p. 373), then equations (21) generalize to (ref. 2, p. 374)

$$\frac{d\sigma_{\text{el}}}{d\Omega} = |f(\theta)|^2 + |f(\pi-\theta)|^2 + \frac{(-1)^{2S}}{2S+1} 2 \operatorname{Re}[f(\theta) f^*(\pi-\theta)] \quad (\text{C2})$$

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Symbols

A_P	projectile nucleon number
A_T	target nucleon number
a	annihilation operator
a^+	creation operator
$f(\theta, \phi)$	scattering amplitude
$f(\theta)$	abbreviation for $f(\theta, \phi)$
i, j, k, l	collections of spatial and spin quantum numbers
$ I \rangle$	initial state vector
$\langle ij v kl \rangle$	nucleon-nucleon two-body matrix element, MeV
$ K \rangle$	final state vector
$\langle k V i \rangle$	nucleus-nucleus matrix element, MeV
P	projectile
$ P \rangle$	projectile state vector
$ PT \rangle$	total projectile-target state vector
$ P_0 \rangle$	projectile initial state vector
$ P_0 T_0 \rangle$	total projectile-target initial state vector
$ ph \rangle$	particle-hole state vector
T	target
$ T \rangle$	target state vector
$ T_0 \rangle$	target initial state vector
u	total projectile-target wave function with quantum numbers in a specific order
V	nucleus-nucleus interaction potential, MeV
v	nucleon-nucleon interaction potential, MeV
θ	angle, rad
Ξ	nuclear coordinate vector, fm
σ	cross section, mb
$\frac{d\sigma_{el}}{d\Omega}$	elastic differential cross section, mb/sr
Φ	single nucleus wave function
ϕ	angle, rad
Ψ	total projectile-target wave function
Ω	solid angle, sr
$\begin{array}{ c c } \hline & \\ \hline \end{array}$	symmetric two-particle Young Tableau
$\begin{array}{ c } \hline \\ \hline \end{array}$	antisymmetric two-particle Young Tableau

Subscripts:

f final state

P projectile

T target

$\alpha, \beta, \gamma, \lambda, \nu$ nucleon quantum numbers used in particle-hole state vectors

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