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# MATHEMATICAL CORRELATION OF MODAL PARAMETER IDENTIFICATION METHODS VIA SYSTEM REALIZATION THEORY

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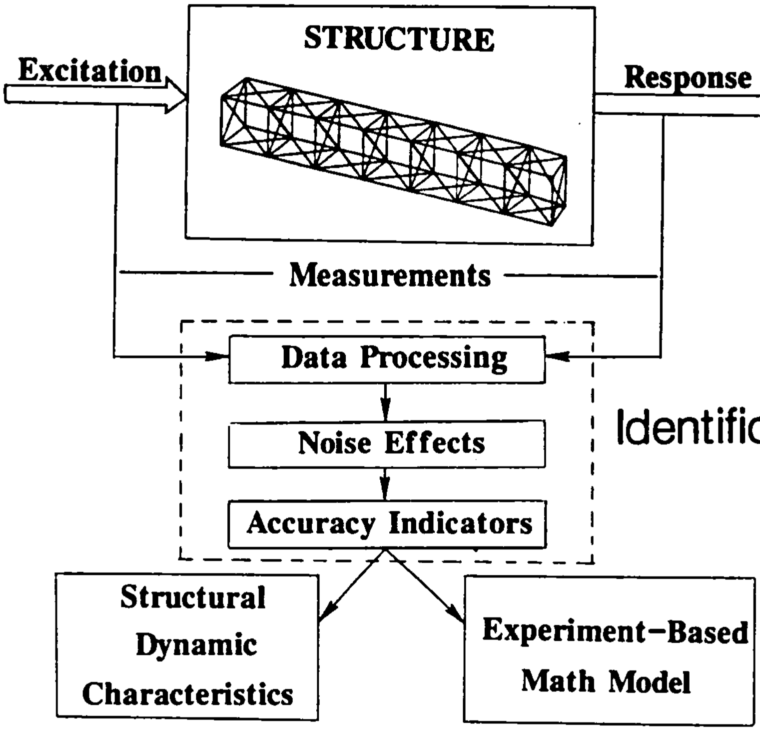




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TABLE OF CONTENTS

# I. INTRODUCTION

When formulating and solving an identification problem, it is important to have the purpose of the identification clearly in mind. In control problems such as control of large space structures, the final goal is to design control strategies for a particular structural system. On the other hand, there are situation where the primary interest is to analyze the properties of a dynamic system, such as stiffness, damping, frequencies, etc. The control problem might require a fairly accurate model of the system dynamics which will adequately describe the system's motion. Fundamentally, one seeks to find a set of parameters that builds a mathematical model to best reproduce, according to some criteria, the test data. The mathematical model for a linear finite-dimensional dynamic system typically includes a state matrix, an output influence matrix and an input influence matrix. For flexible structures, damping and frequencies constitute the state matrix, mode shapes produce the output influence matrix and modal participation factors yield the input influence matrix. In the controls field, the process of constructing a model (state space representation) from experimental data is called system realization (Refs. 1-6). The choice of model structure is one of the basic ingredients in the formulation of the identification problem. The choice will influence the character of the realization problem, the computational effort, the possibility for having a minimum order model, etc. The accuracy of identified modal parameters from the system realization are affected by the choice of the model.

The purpose of this report is to present methods using experimental data to estimate dynamic properties such as damping, frequencies, mode shapes and modal participation factors which are referred to as modal parameters. The identification of modal characteristics is perhaps the most rapidly developing technology in the area of structural system identification today. This results from advances in digital data acquisition and processing, and the availability of small portable minicomputers and microprocessors. Modal surveys which once took months now have the potential of being completed in days. The task of modal parameter identification is treated in several ways by different researchers (Refs. 7-8). There remains considerable controversy about the best technique to use. New methods are suggested en masse and, as a result, the field appears to look more like a bag of tricks than a unified subject. Therefore, a difficult assignment is to organize the bewilderingly large number of ideas. Many so-called different methods are in fact quite similar in the sense that they are mathematically equivalent but are presented in different ways. Thus a unified mathematical framework to treat modal parameter identification is needed to achieve some unification of the field. Although it is very difficult to get a complete overview of the field which is in rapid development, the reader is presented with a basic mathematical foundation which provides insight into the field. The objective of this report is to offer a unified approach using system realization theory to correlate several modal parameter identification methods, particulary for flexible structures.

The tuned sine dwell approach is the earliest method of modal parameter identification. The method requires the employment of sine-wave inputs having variable frequencies for frequency reponse identification (Refs. 9-11), which makes it possible to determine frequencies and, for lightly damped systems, damping coefficients. On the other hand, the recently developed methods for modal parameter identification are based on parametric models in terms of state equations. Fundamentally, two approaches can be classified with respect to the basic elements for construction of the data matrix such as the Hankel matrix (Ref. 1). The first approach, which uses the impulse response function to construct the data matrix to realize a model for modal parameter identification, is called the time-domain identification methods (Refs. 12-24). The second approach, which uses the transfer function matrix to realize a model and then identify the modal parameters, are referred to as the frequency-domain methods (Refs. 26-29). The first part of this report presents the time-domain identification methods using system realization theory. The frequency-domain identification methods will then be addressed. The relation among several existing methods is established and discussed.

## II. TIME-DOMAIN MODAL PARAMETER IDENTIFICATION (Refs. 12-24)

Many different time-domain methods and techniques in structures field were developed, analyzed and tested for modal parameter identification (Refs. 12-24). The question arises whether there exists relationship among these methods. The answer is positive. Indeed, a unified mathematical framework can be developed to present and discuss these methods. The time-domain methods for modal parameter identification in the structures field can start with the transfer function matrix, which yields Markov parameters (generalized impulse response samples). The knowledge of Markov parameters makes it possible to construct a Hankel matrix (Refs. 1-6) as the basis for the realization of a state space discrete-time model. This section will start with the derivation of discrete-time models from the continuous-time models which are usually used by structural engineers and follow by the basic concept of minimum realization which was developed by Ho and Kalman (see Ref. 1). Since the Eigensystem Realization Algorithm (ERA) (Refs. 12-15) for modal parameter identification was developed using minimum realization theory, it will be presented and discussed first. The Polyreference Technique (Refs. 16-18) and the Least Squares Regression method will then be derived using the mathematical framework developed in deriving the ERA.

### II-1. State Equations: Continuous-time and Discrete-time models (Refs. 2-6)

The equations of motion for a finite-dimensional linear dynamical system are a set of $n_2$ second-order differential equations, where $n_2$ is the number of independent coordinates. Let M, D, and K be the mass, damping and stiffness matrices, respectively. The state equations can be expressed in matrix notation as

$$M\ddot{w} + D\dot{w} + Kw = f(w,t) \tag{1}$$

where $\ddot{w}$, $\dot{w}$ and $w$ are vectors of generalized acceleration, velocity, and displacement respectively and f(w,t) is the forcing function over the period of interest at certain specific locations. Eq. (1) can be rewritten as a first order system of differential equations in a number of ways. Certain reformulations are more suitable for computation than others. One reformulation begins with the following definition

$$x = \begin{bmatrix} w \\ \dot{w} \end{bmatrix} \quad \bar{A} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad f(w,t) = B_f u(t) \tag{2}$$

$$\bar{B} = \begin{bmatrix} 0 \\ M^{-1}B_f \end{bmatrix}$$

where $B_f$ is an $n$ $(n = 2n_2)$ by $m$ input influence matrix. The integer $m$ is the number of inputs. Equation (1) can thus be written in a more compact form as

$$\dot{x} = \bar{A}x + \bar{B}u \tag{3}$$

If the dynamic system is excited and measured by the $\ell$ output quantities in the output vector y(t) using sensors such as strain gages, accelerometers, etc., a matrix output equation can be formulated as

$$y = Cx \tag{4}$$

where C is thus an $\ell \times n$ output influence matrix.

Eqs. (3) and (4) constitute a continuous-time model for finite-dimensional dynamic systems. The matrix $\bar{A}$ in Eq. (3) is a representation of mass, stiffness, and damping properties. The input matrix B characterizes the locations and type of input u, whereas the output matrix C describes the relationship between the state vector x and the output measurement vector y.

Using the initial conditions $x(t_o)$ at time $t = t_o$ gives the solution to the continuous-time equation with an input u(t)

$$x(t) = e^{\bar{A}(t-t_o)}x(t_o) + \int_{t_o}^{t} e^{\bar{A}(t-\tau)}\bar{B}u(\tau)\, d\tau \tag{5}$$

for $t \geq t_o$. This equation describes the change of state variable x with respect to the initial conditions $x(t_o)$ and the input u(t). It will be shown in the following that the evaluation of x(t) at equally spaced intervals of time t can be obtained by a discrete-time representation of Eq. (5).

Let the equally spaced times be given by $0, \Delta t, 2\Delta t, \ldots, k\Delta t, \ldots$ where $\Delta t$ is a constant interval. Substitution of $t = (k+1)\Delta t$ and $t_o = k\Delta t$ into Eq. (5) yields

$$x[(k+1)\Delta t] = e^{\bar{A}\Delta t}\, x\,(k\Delta t) + \int_{k\Delta t}^{(k+1)\Delta t} e^{\bar{A}[(k+1)\Delta t-\tau]}\bar{B}u(\tau)d\tau \tag{6}$$

If $u(\tau)$ is assumed to be constant over the interval $k\Delta t \leq \tau \leq (k+1)\Delta t$ and has the value $u(k\Delta t)$, Eq. (6) with a constant matrix $\bar{B}$ becomes

$$x[(k+1)\Delta t] = e^{\bar{A}\Delta t}\, x\,(k\Delta t) + [\int_{0}^{\Delta t} e^{\bar{A}\tau'}d\tau'\bar{B}]\, u(k\Delta t) \tag{7}$$

where the variable $\tau$ in Eq. (6) has been changed by letting $\tau'=(k+1)\Delta t-\tau$. Now, define

$$A = e^{\bar{A}\Delta t},\ B = \int_{0}^{\Delta t} e^{\bar{A}\tau'}d\tau'\bar{B},\ x(k+1) = x[(k+1)\Delta t],\ u(k) = u(k\Delta t), \tag{8}$$

Note that, for impulse responses, $B = \bar{B}$. Eq. (7) can be then expressed in a compact form

$$x(k+1) = A\,x(k) + Bu(k)\ ;\quad k = 0, 1, 2, \ldots \tag{9}$$

and Eq. (4) becomes

$$y(k) = C\,x(k) \tag{10}$$

Combination of Eqs. (9) and (10) is the discrete-time representation of a dynamical system. This set of equations are the basic formulations for system identification of linear, time-invariant dynamical systems, because experimental data are discrete in nature. If experimental data are recorded in a digital computer, continuous physical measurements will be converted into numbers in some format.

What are the response characteristics of the discrete model, i.e., Eqs. (9) and (10)? To observe the response to an impulse in one of the input variables, $u_i(0) = 1$ $(i = 1, \ldots, m)$ and $u_i(k)=0$ $(k=1, 2, \ldots)$ are substituted into Eq. (9). When the substitution is performed for each input element, the results can be assembled into an impulse response function matrix Y with dimensions $\ell$ by $m$ as follows:

$$Y(0)=CB,\ Y(1)=CAB,\ Y(2)=CA^2B,\ \ldots,\ Y(k)=CA^kB,\ Y(k+1)=CA^{k+1}B,\ \ldots \tag{11}$$

This sequence of constant matrices are known as Markov parameters which can be obtained from the experimental data through the transfer function or impulse responses. The Markov parameters can thus be used as the basis for building mathematical models for dynamical systems.

## II-2. Basic Concepts of Realization (Refs. 1-6)

The triple of constant matrices [A, B, C], which represents the system characteristics can be used to determine the system's response at any of the $\ell$ output points to any input at any of the input points. Such a representation is called a realization of the system. Any system has an infinite number of realizations which will predict the identical response for any particular input.

Let a new vector $z$ be defined such that

$$x = Tz \tag{12}$$

where $T$ is any nonsingular square matrix. Substitution of Eq. (12) into Eqs. (9) and (10) yields

$$z(k+1) = T^{-1}A\,T\,z(k) + T^{-1}Bu(k); \quad k = 0, 1, 2, \ldots \tag{13}$$

$$y(k) = CTz(k) \tag{14}$$

It is obvious that the effect of input u(k) on y(k) will be the same for this new system of equations (13) and (14). Thus, the triple $[T^{-1}AT,\ T^{-1}B,\ CT]$ will also be a realization for the same system. The predicted responses using the realization $[T^{-1}AT,\ T^{-1}B,\ CT]$ will be identical to those predicted using [A, B, C]. Because there exists an infinite number of nonsingular matrices $T$, there are an infinite number of such realizations.

By minimum realization is meant a model with the smallest state space dimension among systems realized that has the same input-output relations. All minimum realizations have the same set of eigenvalues which are parameters of the system itself.

Assume that the state matrix $A$ of order $n_o$ has a complete set of linearly independent eigenvectors $(\psi_1, \psi_2, \ldots, \psi_{n_o})$ with corresponding eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_{n_o})$ which are not necessarily distinct. Define $\Lambda$ as the diagonal matrix of eigenvalues and $\psi$ the matrix of eigenvectors, i.e.

$$\Lambda = \text{diag.}\ (\lambda_1, \lambda_2, \ldots, \lambda_{n_o}) \tag{15}$$

and

$$\psi = [\psi_1, \psi_2, \ldots, \psi_{n_o}] \tag{16}$$

The triple $[A, B, C]$ can then be transformed to the realization $[\Lambda, \psi^{-1}B, C\psi]$ by choosing $T = \psi$. The diagonal matrix $\Lambda$ contains the information of modal damping rates and damped natural frequencies. The matrix $\psi^{-1}B$ is called initial modal amplitudes and the matrix $C\psi$ mode shapes. All the modal parameters of a dynamic system can thus be identified by the triple $[\Lambda, \psi^{-1}B, C\psi]$. The desired modal damping rates and damped natural frequencies are simply the real and imaginary parts of the eigenvalues $\bar{\Lambda}$, after transformation from the discrete-time domain to the continuous-time domain using the relation $\bar{\Lambda} = \ln(\Lambda)/\Delta t$.

Time-domain analysis for identification of modal parameters begins by forming the generalized $(r+1)$ by $(s+1)$ Hankel matrix, composed of the Markov parameters (see Eq. 11).

$$H(k) = \begin{bmatrix} Y(k) & Y(k+1) & \ldots & Y(k+s) \\ Y(k+1) & Y(k+2) & \ldots & Y(k+s+1) \\ \vdots & \vdots & & \vdots \\ Y(k+r) & Y(k+r+1) & \ldots & Y(k+r+s) \end{bmatrix} \tag{17}$$

If $r+1 \geq n_o$ and $s+1 \geq n_o$ (the order of the system), the matrix $H(k)$ is of rank $n_o$. To confirm this point, observe from Eq. (11) that

$$H(k) = V_r A^k W_s \tag{18}$$

where

$$V_r = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^r \end{bmatrix} \quad \text{and} \quad W_s = [\, B \;\; AB \;\; A^2B \ldots A^sB \,] \tag{19}$$

The block matrix $V_r$ is called the observability matrix, whereas the block matrix $W_s$ is called the controllability matrix. If the order of the system is $n_o$, then the minimum dimension of the state matrix is $n_o \times n_o$.

If the system is controllable and observable, the block matrices $V_r$ and $W_s$ are of rank $n_0$. Therefore the Hankel matrix is of rank $n_0$ by Eq. (18). Based on the properties of the Hankel matrix composed of the Markov parameters (pulse response function), several methods for modal parameter identification are discussed in the following sections.

## II-3. The Eigensystem Realization Algorithm (ERA) (Refs. 12-15)

The basic development of the time-domain (state-space) concept is attributed to Ho and Kalman (Ref. 1) who introduced the important principles of minimum realization theory. The Ho-Kalman procedure uses the Hankel matrix (Eq. 17) to construct a state-space representation of a linear system from noise-free data. The methodology has been recently modified and extended to develop the Eigensystem Realization Algorithm (Refs. 12-15) to identify modal parameters from noisy measurement data.

In contrast to classical system realization methods which use the Hankel matrix (Eq. 17), the ERA algorithm begins by forming a block data matrix which is obtained by deleting some rows and columns of the Hankel matrix (Eq. 17), but maintaining the first block matrix intact. Furthermore, the standard ordering of entries in the Hankel matrix does not need to be maintained.

Let $B = [b_1, b_2, \ldots, b_m]$ and $C^T = [c_1^T, c_2^T, \ldots c_\ell^T]$ where the column vector $b_i$ is the control influence vector for the ith control input and the row vector $c_j$ is the measurement influence vector for the jth measurement sensor. Denote that columns of $B_{I_i}$ $(i = 0, 1, \ldots, \eta)$ and the rows of $C_{J_j}$ $(j = 0, 1, \ldots, \xi)$ as arbitrary subsets of $(b_1, b_2, \ldots, b_m)$ and $(c_1, c_2, \ldots, c_\ell^T)$, respectively. If the column index $I_0$ is defined as the set of integers $(1, 2, \ldots, m)$, the column index $I_i (i = 1, 2, \ldots, \eta)$ will be any arbitrary subset of $I_0$. On the other hand, if the row index $J_0$ is the set of integers $(1, 2, \ldots, \ell)$, the row index $J_j (j = 1, 2, \ldots, \xi)$ will be any arbitrary subset of $J_0$. The ERA block data matrix can then be expressed by

$$H(k) = [Y_{J_j I_i}(s_i+k+t_i)]; \quad Y_{J_j I_i}(s_i+k+t_j) = C_{J_j} A^{s_i+k+t_j} B_{I_i} \tag{20}$$

where $s_0 = t_0 = 0$, and $s_j$ and $t_i$ are arbitrary integers. When $i = j = 0$, $Y_{J_0 I_0}(k) = Y(k) = CA^k B$.

The ERA block data matrix (Eq. 20) allows one to include only good or strongly measured signals without losing any capability. This is useful since some measurement data may be noisier than others or sensors may malfunction during the test. The advantage of this capability is the potential to minimize the distortion of the identified parameters caused by noise. A judicious choice of data and its proper arrangement in the block matrix H(k) can also be used to minimize the computational requirements of the method. For example, the columns of H(k) may be made as independent as possible by properly selecting the data samples to use as entries of the matrix. This effort could substantially reduce the order of the matrix for large problems. For sufficiently low noise data, the order can be the same as that of the true system state matrix A. This fact results from examination of the controllability and observability matrices, to be discussed next.

From Eqs. (20), it can be shown that

$$H(k) = V_\xi A^k W_\eta \;;\; V_\xi = \begin{bmatrix} C \\ C_{J_1} A^{s_1} \\ \vdots \\ C_{J_\xi} A^{s_\xi} \end{bmatrix} \;;\; \text{and } W_\eta = [B \quad A^{t_1} B_{I_1} \quad \ldots \quad A^{t_\eta} B_{I_\eta}] \tag{21}$$

where $V_\xi$ and $W_\eta$ are generalized observability and controllability matrices.

Assume that there exists a matrix $H^\#$ satisfying the relation

$$W_\eta H^\# V_\xi = I_r \tag{22}$$

where $I_r$ is an identity matrix of r. It will be shown that the matrix $H^\#$ plays a major role in deriving the ERA. What is $H^\#$? Observe that

$$H(0) H^\# H(0) = V_\xi W_\eta H^\# V_\xi W_\eta = V_\xi W_\eta = H(0) \tag{23}$$

The matrix $H^\#$ is thus the pseudoinverse of the matrix H(0) in a general sense.

The ERA process starts with the factorization of the block data matrix (Eq. 20), for k=0, using singular value decomposition (Ref. 29):

$$H(0) = P\, D\, Q^T \tag{24}$$

where the columns of matrices P and Q are orthonormal and D is a rectangular matrix $D = \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix}$ with $D_r = \text{diag}\,[d_1, d_2, \ldots, d_n, d_{n+1}, \ldots, d_r]$ and monotonically non-increasing $d_i (i=1, 2, \ldots, r)$

$$d_1 \geq d_2 \geq \ldots \geq d_n \geq d_{n+1} \geq \ldots \geq d_r \geq 0.$$

Next, let $P_r$ and $Q_r$ be the matrices formed by the first r columns of P and Q, respectively. Hence the matrix H(0) and its pseudoinverse become

$$H(0) = P_r D_r Q_r^T \quad \text{where } P_r^T P_r = I_r = Q_r^T Q_r \tag{25}$$

and

$$H^{\#} = Q_r D_r^{-1} P_r^T \qquad (26)$$

Eq. (26) can be readily proved by observing Eq. (23).

Define $O_\ell$ as a null matrix of order $\ell$, $I_\ell$ an identity matrix of order $\ell$, and $E_\ell^T = [I_\ell \ O_\ell \ldots O_\ell]$. Using Eqs. (20), (21), (22), (25), and (26), a minimum order realization can be obtained as follows:

$$\begin{aligned}
Y(k) &= E_\ell^T H(k) E_m = E_\ell^T V_\xi A^k W_\eta E_m && \text{(use Eqs.(20) \& (21))} \\
&= E_\ell^T V_\xi [W_\eta H^{\#} V_\xi] A^k [W_\eta H^{\#} V_\xi] W_\eta E_m && \text{(use Eqs. (22)} \\
&= E_\ell^T H(0) [Q_r D_r^{-1} P_r^T] V_\xi A^k W_\eta [Q_r D_r^{-1} P_r^T] H(0) E_m && \text{(use Eqs. (21) \& (26))} \\
&= E_\ell^T H(0) Q_r D_r^{-1/2} [D_r^{-1/2} P_r^T H(1) Q_r D_r^{-1/2}]^k D_r^{-1/2} P_r^T H(0) E_m && \text{(use Eq. (25))} \\
&= E_\ell^T P_r D_r^{1/2} [D_r^{-1/2} P_r^T H(1) Q_r D_r^{-1/2}]^k D_r^{1/2} Q_r^T E_m && (27)
\end{aligned}$$

This is the basic formulation of realization for the ERA. The triple

$$\tilde{A} = D_r^{-1/2} P_r^T H(1) Q_r D_r^{-1/2} \ , \ \tilde{B} = D_r^{1/2} Q_r^T E_m \ , \ \tilde{C} = E_\ell^T P_r D_r^{1/2} \qquad (28)$$

is a minimum realization. The order of the matrix $A$ is $r$ which is equal to $n_o$ (the order of the system) for sufficiently low noise data. When the matrices $P_r$ and $Q_r$ are obtained through other factorization methods such that $P_r^T P_r \neq I_r$ and $Q_r^T Q_r \neq I_r$, Eq. (28) is still valid if the matrices $P_r^T$ and $Q_r^T$ are replaced by $P_r^{\#}$ and $Q_r^{\#}$ respectively.

Due to measurement noise, nonlinearity, and computer roundoff, the block matrix H(k) will usually be of full rank which does not, in general, equal the true order of the system under test. It should not be the aim to obtain a system realization which exactly reproduces the noisy sequence of data. A realization which produces a smoothed version of the sequence, and which closely represents the underlying linear dynamics of the system, is more desirable. Several accuracy indicators have been investigated for quantitatively partitioning the realized model into pure (principal) and noise (perturbational) portions so that the noise portion can be disregarded. Two principal indicators now available are the singular values of the block data matrix and a parameter referred to as Modal Amplitude Coherence. The number of retained singular values determines the order of the realization, and Modal Amplitude Coherence is used to assess the resulting degree of modal purity.

If Eqs. (21) and (22) are examined as a whole the equality

$$H(0) = V_\xi W_\eta = [P_r D_r^{1/2}] \ [D_r^{1/2} Q_r^T] \qquad (29)$$

defines the controllability and observability Grammians as

$$W_\eta W_\eta^T = D_r \quad \text{and} \quad V_\xi^T V_\xi = D_r . \tag{30}$$

The fact that the controllability and observability Grammians are equal and diagonal implies that the realized system [A,B,C] is as controllable as it is observable. This property is called an internally balanced realization. It means that the signal transfer from the input to the state and then from the state to the output are similar and balanced.

Some singular values, say $d_{n+1},...d_r$, may be relatively small and negligible in the sense that they contain more noise information than system information. In other words, the directions determined by the singular values $d_{n+1},...d_r$, have less significant degrees of controllability and observability relative to the noise. It would be unwise to require a realization including these directions. The reduced model of order n after deleting singular values $d_{n+1},...,d_r$ is then considered as the robustly controllable and observable part of the realized system. Reference 13 provide the mathematical framework for establishing the relationship between these accuracy indicators and the characteristics of the noise.

## An Alternate ERA Formulation for Implementation in a Small Computer.

For a large flexible dynamic system under test, the number of sensors may be on the order of one hundred which correspondingly can result in the number of rows for the block matrix H(k) (Eq. 20)) being significantly larger than the order of system. This then requires a large amount of computer storage and computational efforts. To circumvent this difficulty, an alternate formulation is derived in the following.

Let $T_r$ be a rectangular matrix of rank at least r which operates on the rows of the block matrix H(k) (Eq. 20)) such that

$$\tilde{H}(k) = T_r H(k) = [T_r V_\xi] A^k W_\eta = \tilde{V}_\xi A^k W_\eta \tag{31}$$

The operator of $T_r$ on H(0) may include: (1) multiplying a row by a nonzero number; (2) interchanging two rows and (3) adding the product of one row to another row. As shown in Eq. (22), the pseudoinverse $\tilde{H}^{\#}$ of the matrix $\tilde{H}(0)$ satisfies the relation

$$W_\eta \tilde{H}^{\#} \tilde{V}_\xi = I \tag{32}$$

Now, factorize the data matrix $\tilde{H}(0)$, using singular value decomposition,

$$\tilde{H}(0) = \tilde{P}_n \tilde{D}_n \tilde{Q}_n^T \quad \text{where } \tilde{P}_n^T \tilde{P}_n = I_n = \tilde{Q}_n^T \tilde{Q}_n \tag{33}$$

and thus

$$\tilde{H}^{\#} = \tilde{Q}_n \tilde{D}_n^{-1} \tilde{P}_n^T \tag{34}$$

The integer n is determined by retaining only the non-negligible singular values. Following the similar procedure shown in Eq. (27), a minimum order realization can be obtained as

$$
\begin{aligned}
\tilde{V}_\xi A^k W_\eta &= \tilde{V}_\xi [W_\eta \tilde{H}^\# \tilde{V}_\xi] A^k [W_\eta \tilde{H}^\# \tilde{V}_\xi] W_\eta && \text{(use Eq. (32))} \\
&= \tilde{H}(0)\tilde{H}^\# [\tilde{V}_\xi A^k W_\eta] \tilde{H}^\# \tilde{H}(0) && \text{(use Eq. (31))} \\
&= [\tilde{P}_n \tilde{D}_n^{1/2}][\tilde{D}_n^{-1/2} \tilde{P}_n^T \tilde{V}_\xi A^k W_\eta \tilde{Q}_n \tilde{D}_n^{-1/2}][\tilde{D}_n^{1/2} \tilde{Q}_n^T] && \text{(use Eqs. (33) \& (34))} \\
&= [\tilde{P}_n \tilde{D}_n^{1/2}][\tilde{D}_n^{-1/2} \tilde{P}_n^T \tilde{H}(1) \tilde{Q}_n \tilde{D}_n^{-1/2}] [\tilde{D}_n^{1/2} \tilde{Q}_n^T] && (35)
\end{aligned}
$$

The triple

$$\tilde{V}_\xi = \tilde{P}_n \tilde{D}_n^{1/2} \quad , \quad A = \tilde{D}_n^{-1/2} \tilde{P}_n^T \tilde{H}(1) \tilde{Q}_n \tilde{D}_n^{-1/2} \quad , \quad W_\eta = \tilde{D}_n^{1/2} \tilde{Q}_n^T \tag{36}$$

is a minimum realization of order n in which the state transition matrix A, the controllability matrix $W_\eta$ and the modified observability matrix $\tilde{V}_\xi$ after the row operation by $T_r$ are realized.

From the definition of the controllability matrix $W_\eta$ in Eq. (23) and the matrix $E_m^T = [I_m \; 0_m \; \dots \; 0_m]$, the input matrix B can be expressed as

$$B = \tilde{D}_n^{1/2} \tilde{Q}_n^T E_m \tag{37}$$

which takes the first m columns of the matrix $W_\eta$. To recover the output matrix C, a simple computation can be derived by observing from Eqs. (21) and (36) that

$$H(0) = V_\xi W_\eta = V_\xi \tilde{D}_n^{1/2} \tilde{Q}_n^T \tag{38}$$

which produces the original observability matrix as

$$V_\xi = H(0) \tilde{Q}_n \tilde{D}_n^{-1/2} \tag{39}$$

The output matrix C can thus be obtained from the observability matrix by

$$C = E_\ell^T V_\xi = E_\ell^T H(0) \tilde{Q}_n \tilde{D}_n^{-1/2} \tag{40}$$

As a result, the triple

$$A = \tilde{D}_n^{-1/2} \tilde{P}_n^T \tilde{H}(1) \tilde{Q}_n \tilde{D}_n^{-1/2} \quad , \quad B = \tilde{D}_n^{1/2} \tilde{Q}_n^T E_m \quad , \quad C = E_\ell^T H(0) \tilde{Q}_n \tilde{D}_n^{-1/2} \tag{41}$$

is a minimum realization of order n for the system.

The advantage of this formulation is that the modified matrix $\tilde{H}(0)$ can represent a reduced-in-size version of the block data matrix H(0). The number of rows for the matrix $\tilde{H}(0)$ can be set to a size only moderately larger than the order of the system without loosing the information of parameters in the data. For example, if the number of rows for the matrix $\tilde{H}(0)$ is set to r, all the measurements represented by a number larger than r can be added to a particular row or distributively to the first r rows. In doing this row operation, the matrix $\tilde{H}(0)$ will be reduced in size, and also the measurement data may be averaged to reduce the noise level of the data. Note that all the measurements may be added together and then shifted to form the matrix $\tilde{H}(0)$. The number r can not be chosen smaller than twice the anticipated number of system modes, or some modes would not be identified. If the controllability and observability Grammians are examined for this realization, it is seen that the realization is not internally balanced. Thus, if the system based on this realization is reduced for controller design, the reduced system may not be as robust as the one based on the realization of Eq. (28).

### II-4. The Polyreference Technique (Canonical-Form Realization) (Refs. 2, 6, 16-18)

During the past two decades, several algorithms for the construction of a canonical-form representation of linear systems have appeared in the controls literature (Refs. 2 & 6). Controls researchers are mainly concerned with, for an example, determining a passive or an active network that has a prescribed impedance or transfer function. Although techniques of canonical-form realization are available in the controls literature, direct application to modal parameter identification for flexible structures has not been addressed. Among several available methods for canonical-form realization, one based on Hankel matrices will be addressed in this section.

Recently in the structures field, a method, similar if not identical, to a canonical-form realization, was developed in Refs. 16-18 using frequency-response functions for identification of modal parameters from multi-input and multi-output measurement data. The method is referred to as the Polyreference technique. Mathematical background of the Polyreference technique will be presented in this section using a new approach which provides insights of correlation between the Polyreference technique and the Canonical-Form Realization.

Form the (r+1)x(s+1) block Hankel matrix

$$H(0) = \begin{bmatrix} Y(0) & Y(1) & \dots Y(s) \\ Y(1) & Y(2) & \dots Y(s+1) \\ \vdots & \vdots & \vdots \\ Y(r) & Y(r+1) & \dots Y(r+s) \end{bmatrix} \tag{42}$$

where r and s are integers which are chosen to be larger than the order of the system. Using the singular value decomposition, find the nonsingular matrices P and Q such that

$$H(0) = PDQ^T = [P_n \quad P_o] \begin{bmatrix} D_n & 0 \\ 0 & 0 \end{bmatrix} [Q_n \quad Q_o]^T \tag{43}$$

where $D_n$ is a diagonal matrix containing monotonically non-increasing non-negligible singular values. The integer n is determined by the characteristics of the system noise as discussed in Refs. 13 and 29. All singular values numbered after n are considered as zero values. The matrices $P_n$ and $P_o$ denote respectively the first n and the last remaining columns of the orthonormal matrix P. Similarly, $Q_n$ and $Q_o$ denote respectively the first n and the remaining columns of the orthonormal matrix Q.

Now observe, from the definition of Markov parameters, that

$$H(0) = VW; \; V = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^r \end{bmatrix} \text{ and } W = [B \; AB \; \dots \; A^sB] \tag{44}$$

where V and W are observability and controllability matrices respectively. Since matrices P and Q are orthonormal, i.e., $P^TP = Q^TQ = I$, Eqs. (43) and (44) lead to

$$P^TH(0)Q = P^TVWQ = \begin{bmatrix} P_n^TVWQ_n & P_n^TVWQ_o \\ P_o^TVWQ_n & P_o^TVWQ_o \end{bmatrix} = \begin{bmatrix} D_n & 0 \\ 0 & 0 \end{bmatrix} \tag{45}$$

which yields

$$P_n^TVWQ_n = D_n \, , \tag{46}$$

$$P_n^TVWQ_o = 0, \; P_o^TVWQ_n = 0 \, , \tag{47}$$

and

$$P_o \; VWQ_o = 0. \tag{48}$$

Note that the ERA algorithm is developed using the matrices $P_n$, $Q_n$ and $D_n$ as shown in Eq. (46). If the system is assumed controllable and observable, each of the five matrices $P_n$, V, W, $Q_n$ and $D_n$ is of rank n. This means that the ranks of matrices $P_n^TV$ and $WQ_n$ are n. Thus Eqs. (47) and (48) imply

$$P_o^TV = 0 \text{ and } WQ_o = 0 \tag{49}$$

The matrix $P_o$ provides the left orthonormal basis for the null subspace which is orthogonal to the observability matrix, whereas the matrix $Q_o$ gives the right orthonormal basis for the null subspace which is orthogonal to the controllability matrix. Now partition the matrices $P_o$ and $Q_o$ as

$$P_o^T = [P_{o0}^T \quad P_{o1}^T \ldots P_{or}^T] \text{ and } Q_o^T = [Q_{o0}^T \quad Q_{o1}^T \ldots Q_{os}^T] \tag{50}$$

Substitution of Eqs. (44) and (50) into Eqs. (49) yields

$$\sum_{i=0}^{r} P_{oi}^T C A^i = P_{o0}^T C + P_{o1}^T CA + \ldots + P_{or}^T CA^r = 0 \tag{51a}$$

$$\sum_{i=0}^{s} A^i B Q_{oi} = BQ_{o0} + ABQ_{o1} + \ldots + A^s BQ_{os} = 0 \tag{51b}$$

Eqs. (51) are the basic formulation for the Polyreference technique and the Canonical-Form Realization. In fact, Eq. (51a) is the basis for an observable canonical-form realization whereas Eq. (51b) is the basis for a controllable canonical-form realization (see Ref.4, pp. 321, Problems 6-19 and 6-21). Both equations (51) should produce the same results. The question arises as to whether Eq. (51a) is more favorable than Eq. (51b) or vice versa. The answer is given in the following.

Observe that each submatrix $P_{oi}(i=0,\ldots,r)$ must have more columns than rows (the number of outputs $\ell$). Similarly, each submatrix $Q_{oi}$ ($i = 0, \ldots, s$) has more columns than rows (the number of inputs $m$). Form square matrices $\bar{P}_{oi}$ of order $\ell$ and $\bar{Q}_{oi}$ of order $m$ respectively from the last $\ell$ columns of the matrix $P_{oi}$ and the last m columns of the matrix $Q_{oi}$, and rewrite Eqs. (51) such that

$$-\sum_{i=0}^{r-1} [\bar{P}_{or}^T]^{-1} \bar{P}_{oi}^T C A^i = C A^r \triangleq \sum_{i=0}^{r-1} \tilde{P}_{oi}^T C A^i \tag{52a}$$

and

$$-\sum_{i=0}^{s-1} A^i B \bar{Q}_{oi}[\bar{Q}_{os}]^{-1} = A^s B \triangleq \sum_{i=0}^{s-1} A^i B \tilde{Q}_{oi} \tag{52b}$$

with $\tilde{P}_{oi}^T = [\bar{P}_{or}^T]^{-1}\bar{P}_{oi}^T$ and $\tilde{Q}_{oi} = \bar{Q}_{oi}[\bar{Q}_{os}]^{-1}$

Eqs. (52) can be rearranged into companion matrix form as

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-2} \\ CA^{r-1} \end{bmatrix} A = \begin{bmatrix} 0 & I_\ell & 0 & \ldots & 0 \\ 0 & 0 & I_\ell & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & I_\ell \\ -\tilde{P}_{o0}^T & -\tilde{P}_{o1}^T & -\tilde{P}_{o2}^T & \ldots & -\tilde{P}_{o(r-1)}^T \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-2} \\ CA^{r-1} \end{bmatrix} \tag{53a}$$

and

$$A\,[B\ AB\ \dots\ A^{s-1}B\,] = [\,B\ AB\ \dots\ A^{s-1}B\,]\begin{bmatrix} 0 & 0 & \dots & 0 & \tilde{Q}_{o0} \\ I_m & 0 & \dots & 0 & \tilde{Q}_{o1} \\ 0 & I_m & \dots & 0 & \tilde{Q}_{o2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & I_m & \tilde{Q}_{o(s-1)} \end{bmatrix} \tag{53b}$$

The matrix $I_\ell$ is an identity matrix of order $\ell$ and the matrix $I_m$ is an identity matrix of order $m$. Now it is claimed from Eq. (53a) that the triple

$$\tilde{A} = \begin{bmatrix} 0 & I_\ell & 0 & \dots & 0 \\ 0 & 0 & I_\ell & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & I_\ell \\ -\tilde{P}^T_{o0} & -\tilde{P}^T_{o1} & -\tilde{P}^T_{o2} & \dots & -\tilde{P}^T_{o(r-1)} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} Y(0) \\ Y(1) \\ \vdots \\ Y(r-2) \\ Y(r-1) \end{bmatrix}, \quad \tilde{C} = [I_\ell\ 0\ \dots\ 0\ 0] \tag{54a}$$

is an $\ell r$-dimensional realization of the system. Indeed, it can be readily verified that

$$Y(0) = \tilde{C}\tilde{B},\ Y(1) = \tilde{C}\tilde{A}\tilde{B},\ \dots,\ Y(r) = \tilde{C}\tilde{A}^r\tilde{B}.$$

Because of the structure of $\tilde{A}$ and $\tilde{C}$, it is easy to show that the realization is observable. However, it is not in general controllable. This realization is called an observable canonical-form realization. It is not a minimum realization, because, in general, it is not both controllable and observable.

Similarly, from Eq. (53b), it can be verified that the triple

$$\tilde{A} = \begin{bmatrix} 0 & 0 & \dots & 0 & \tilde{Q}_{o0} \\ I_m & 0 & \dots & 0 & \tilde{Q}_{o1} \\ 0 & I_m & \dots & 0 & \tilde{Q}_{o2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & I_m & \tilde{Q}_{o(s-1)} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} I_m \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \tilde{C} = [Y(0)\ Y(1)\ \dots\ Y(s-2)\ Y(s-1)] \tag{54b}$$

is an ms-dimensional realization of the system. This is a controllable canonical-form realization which is not in general observable. Again this realization is not of minimum order.

Eqs. (54) can be reduced to a minimum order realization by applying the reduction procedure shown in Ref. 6, Chapter 5. However, the canonical form will be destroyed after the application of the reduction procedure.

The order of either observable or controllable canonical-form realization, i.e., $\ell r$ or $ms$, is required to be equal to or larger than the order of the system. From a computational point of view, one should choose the one with smaller dimension to work for modal parameter identification. The numerical problem for the eigensolution of the canonical-form realization can be solved in various ways. A technique suitable and efficient for mini-computer systems has been implemented and shown in Ref. 31. It should be remarked that only a subset of eigenvalues in the realized state matrix $\tilde{A}$ (see Eq. (54)) belongs to the actual state matrix $A$, since the matrix $\tilde{A}$ is generally oversized for multi-input and multi-output measurements.

The method used here to obtain the canonical-form realization is different from that shown in Ref. 16. Orthonormal matrices $P_o$ and $Q_o$ are computed through the application of the singular value decomposition to realize a companion-form state matrix. Since orthonormal matrices are very close to identity matrices, this thus generates a computationally well-behaved canonical-form realization.

## II-5. An Alternate Method for the ERA and the Polyreference Techniques (Ref. 19)

A minimum order of canonical-form realization is generally impossible for multi-input, multi-output systems due to the constraint that the realized state matrix is a companion form. If the constraint is released, a minimum order realization can be obtained from Eq. (53).

In view of Eqs. (43) and (44), the controllability and observability matrices can be expressed by the following equations

$$V = P_n D_n^{1/2} \quad \text{and} \quad W = D_n^{1/2} Q_n^T \tag{55}$$

Define $0$ as a null matrix, $I_{\ell r}$ an identity matrix of order $\ell r$ and $E_{\ell r} = [I_{\ell r}\ 0]$ of dimension $\ell r \times \ell(r+1)$.

Hence, Eq. (53a) can be written with the aid of Eq.(54a) as

$$[E_{\ell r} P_n] D_n^{1/2} A = \tilde{A} [E_{\ell r} P_n] D_n^{1/2} \quad \Rightarrow \quad A = D_n^{-1/2} [E_{\ell r} P_n]^{\#} \tilde{A} [E_{\ell r} P_n] D_n^{1/2} \tag{56}$$

where # means pseudoinverse. This is a minimum realization of order $n$. To compute Eq. (56), a simple procedure can be developed as follows. Define $\sigma_\ell$ as a shift operator which shifts $\ell$ columns of a matrix, for an example, $\sigma_\ell E_{\ell r} = [0 \quad I_{\ell r}]$. In view of the definition of the shift operator $\sigma_\ell$, the companion matrix $\tilde{A}$ in Eq. (54a) and Eq. (55), Eq. (56) becomes

$$A = D_n^{-1/2} [E_{\ell r} P_n]^{\#} [\sigma_\ell E_{\ell r} P_n] D_n^{1/2} \tag{57}$$

Here, $[\sigma_\ell E_{\ell r} P_n]$ simply means the matrix obtained by deleting the last $\ell$ rows of the matrix $P_n$ and $[E_{\ell r}\ P_n]$ represents the matrix obtained by deleting the first $\ell$ rows of the matrix $P_n$. This equation was first presented in Ref. 19. Since $P_n$ is an orthonormal matrix, a special and efficient procedure can be developed to compute the pseudoinverse of the matrix $E_{\ell r}P$ using the matrix inversion lemma (Ref. 19).

Similarly, an equation for the state transition matrix A can be derived from the controllable form (Eqs. (53b)) as

$$A = D_n^{1/2}[\sigma_m E_{ms} Q_n]^T [E_{ms} Q_n]^{T\#} D_n^{-1/2} \tag{58}$$

where $[E_{ms}Q_n]$ means the matrix obtained by deleting the first m rows of the matrix $Q_n$ and $[\sigma_m E_{ms} Q_n]$ represents the matrix by deleting the last rows of the matrix $Q_n$.

Eqs. (57) and (58) can also be derived by the ERA procedure. Let an oversized Hankel matrix $\hat{H}$ be formed such that

$$H(0) = E_{\ell r}\hat{H} \quad \text{and} \quad H(1) = \sigma_\ell E_{\ell r}\hat{H} \tag{59}$$

The Hankel matrix H(0) is formed by deleting the last $\ell$ rows of the Hankel matrix $\hat{H}$ whereas the Hankel matrix H(1) is obtained by deleting the first $\ell$ rows of the matrix $\hat{H}$.

Find the orthonormal matrices $P_n$ and $Q_n$, and a diagonal matrix $D_n$ such that

$$\hat{H} = P_n D_n Q_n^T \tag{60}$$

Eq. (59) thus becomes

$$H(0) = [E_{\ell r} P_n] D_n Q_n^T \quad \text{and} \quad H(1) = [\sigma_\ell E_{\ell r} P_n] D_n Q_n^T \tag{61}$$

Substituting this equation into the ERA basic formulation (41) and noting that $Q_n^T Q_n = I_n$ yields the triple

$$A = D_n^{-1/2}[E_{\ell r}P_n]^{\#}[\sigma_\ell E_{\ell r} P_n] D_n^{1/2} \quad , \; B = D_n^{1/2} Q_n^T E_m \quad , \; C = E_\ell^T P_n D_n^{1/2} \tag{62}$$

The state transition matrix A is indeed identical to that in Eq.(57).

Similarly, if

$$H(0) = \hat{H}E_{ms}^T = P_n D_n [E_{ms} Q_n]^T \quad \text{and} \quad H(1) = \hat{H}[\sigma_m E_{ms}]^T = P_n D_n [\sigma_m E_{ms} Q_n]^T \tag{63}$$

where the Hankel matrix H(0) is obtained by deleting the last m columns of the oversized Hankel matrix $\hat{H}$ and the Hankel matrix H(1) is obtained by deleting the first m columns of the same matrix $\hat{H}$. Substitution of Eq. (63) into the ERA basic formulation, with the aid of $P^T_n P_n = I_n$, produces the triple

$$A = D_n^{1/2}[\sigma_m E_{ms} Q_n]^T [E_{ms} Q_n]^{T\#} D_n^{-1/2} \quad , \quad B = D_n^{1/2} Q_n^T E_m \quad , \quad C = E_\ell^T P_n D_n^{1/2} \tag{64}$$

The state transition matrix A is again identical to that in Eq. (58).

Realizations (62) and (64) preserve the same features as for the ERA, including a good numerical performance, internal balancedness, and flexibility in determining order-error tradeoff. Based on formulations (62) and (64), a close link between the ERA and the Polyreference techniques is established through the singular value decomposition and the generalized Hankel matrix.

## II-6. Least Squares Regression Techniques (Refs. 20-24)

The least square regression technique for a discrete-time dynamic model has been derived and used for system identification for more than two decades (see Ref. 20, Chapter 5, pp. 97-99). The same technique was rederived and further developed for modal parameter identification in the structures field (see Ref. 23-24). Here, the least squares regression technique will be formulated using system realization theory which provides a good basis for the comparison with other methods.

In view of Eqs. (21) and (22), the measurement function Y(k) can be obtained through either of two other algorithms as follows:

$$\begin{aligned} Y(k) &= E_\ell^T H(k) E_m = E_\ell^T V_\xi A^k [W_\eta H^\# V_\xi] W_\eta E_m \\ &= E_\ell^T [V_\xi A W_\eta H^\#]^k V_\xi W_\eta E_m \\ &= E_\ell^T [H(1) H^\#]^k H(0) E_m \end{aligned} \tag{65}$$

or

$$\begin{aligned} Y(k) &= E_\ell^T H(k) E_m = E_\ell^T V_\xi [W_\eta H^\# V_\xi] A^k W_\eta E_m \\ &= E_\ell^T V_\xi W_\eta [H^\# V_\xi A W_\eta]^k E_m \\ &= E_\ell^T H(0) [H^\# H(1)]^k E_m \end{aligned} \tag{66}$$

Hence, the triple $[H(1)H^\#, H(0)E_m, E_\ell^T]$ or the triple $[H^\# H(1), E_m, E_\ell^T H(0)]$ is a realization. The matrix $H(1)H^\#$ or $H^\# H(1)$ constitutes the basis for the least square regression technique (see Ref. 20, pp. 97-99).

The matrix $H^\#$ is the pseudoinverse of the matrix H(0). For a single input, there exists a case where the rank of H(0) equals the column number of H(0), then

$$H^\# = [H(0)^T H(0)]^{-1} H(0)^T$$

On the other hand, there exist a case for a single output where the rank equals the row number, then

$$H^{\#} = H(0)^T[H(0)H(0)^T]^{-1}$$

The matrix $H(1)H^{\#}$ has been used in the structural dynamics field to identify system modes and frequencies (see Refs. 23-24). This is a special case representing a single input which cannot realize a system that has repeated eigenvalues, or using sufficiently low noise data unless the system order is known a priori.

These realizations (Eqs.(65) and (66)) are not of minimum order, since the dimension of the state vector, x, is the number of either columns or rows of the matrix H(0) which is larger than the order of the state matrix A for multi-input and multi-output cases. Examination of Eqs. (65) and (66) reveals that these two equations are special cases of ERA. Eq.(65) is formulated by inserting the identity matrix (Eq. (22)) on the right-hand side of the state transition matrix A. On the other hand, Eq. (66) is obtained by inserting the identity matrix (Eq. (22)) on the left-hand side of the state transition matrix A. However, the ERA is formed by inserting the identity matrix (Eq. (22)) on both sides of the state transition matrix A as shown in Eq. (27). Because of these differences, the least squares regression method does not minimize the order of the system. Mathematically, if the singular value decomposition technique or other rank detection technique is not included in the computational procedures, the realized triple obtained from Eq. (65) or (66) cannot be numerically implemented, unless a certain degree of artificial noise and/or system noise is present. Noise tends to make up the rank deficiency of the data matrix H(0). Since noise contamination is not quaranteed, the least squares regression technique (Eq. (65) or (66)) is subject to uncertain results.

# III. FREQUENCY-DOMAIN MODAL PARAMETER IDENTIFICATION

Time and frequency are two fundamental bases of description for linear dynamic systems (Ref. 25). For example, given a single input and single output linear dynamic system, a three-dimensional space can be constructed for the output response with amplitude as one axis, and time and frequency as the other axes. A sinusoidal time history for each individual frequency (mode) can be treated as a projection on the time plane, existing at some distance from the origin. This distance is measured along the frequency axis. Similarly, the dynamic output response has a projection onto the frequency plane. This projection takes the form of an impulse with an amplitude. The position of the impulse coincides with the corresponding frequency. Summing multiple time plane projections produces the time history of the dynamic response. Similarly, connecting all the frequency components in the frequency plane yields the spectrum. The duality of the time and frequency description of the dynamic response for a linear system becomes evident.

This section exposes the close conceptual connection between time-domain and frequency-domain approaches to identification of modal parameters for linear dynamic systems. To identify modal parameters including damping ratios, frequencies, mode shapes, and modal participation factors, many methods have been developed using classical frequency spectra (transfer function) analysis or sophisticated time-domain methods. For many years, the multi-shaker sine-dwell approach was the primary experimental method employed in aerospace structural mode surveys. The major advantage is that each mode is experimentally isolated by careful tuning of a series of shakers and verified as part of the experimental process. The disadvantage lies in the long elapsed testing time and the requirement for considerable test engineering expertise.

The current section presents some recently developed frequency-domain techniques. All of the methods will be correlated using system realization theory. There are a number of important features in the frequency-domain analysis, including, overlap averaging and zooming (Ref. 25). The overlap averaging is used to smooth the transfer function, while the zooming is used to concentrate all the spectral lines into a narrow band in the frequency range of interest.

The transfer functions are basic elements for frequency-domain techniques. This section starts with the discussion of the transfer function characteristics. It then follows by the presentation of Eigensystem Realization Algorithm in frequency domain and the Polyreference technique in frequency domain, for the purpose of comparison with time-domain identification techniques. The conceptual connections between time-domain and frequency-domain approaches are exposed and discussed.

## III-1. Characteristics of the Transfer Functions.

As pointed out in the last section, the linear, constant, finite-dimensional dynamical systems can be represented by the continuous-time model

$$\dot{x} = \bar{A}x + \bar{B}u \tag{67}$$

$$y = Cx \tag{68}$$

Taking the Laplace transform of Eqs. (67) and (68) yields

$$y(s) = C[sI - \bar{A}]^{-1}(x(t_o) + \bar{B}u(s)) \tag{69}$$

where I is an identity matrix and s is the parameter of the Laplace transform. For $x(t_o) = 0$,

$$y(s) = C[sI - \bar{A}]^{-1}\,\bar{B}u(s) \tag{70}$$

This equation shows the direct relation between the input spectrum u(s) and the output spectrum y(s). The function defined by

$$Y(s) = C[sI - \bar{A}]^{-1}\,\bar{B} \tag{71}$$

is called the transfer function matrix of dimension $\ell$ (the number of outputs) by m (the number of inputs).

Let Y(t) be the impulse response function matrix

$$Y(t) = C\,e^{\bar{A}t}\,\bar{B} \tag{72}$$

which is then related to the transfer function matrix Y(s) by

$$Y(s) = L[Y(t)] = \int_0^\infty Y(t)e^{-st}dt = \int_0^\infty Ce^{\bar{A}t}\bar{B}\,e^{-st}dt = C[sI - \bar{A}]^{-1}\,\bar{B} \tag{73}$$

or

$$Y(t) = L^{-1}[Y(s)] = \int_{-\infty}^{0} Y(s)e^{st}ds = \int_{-\infty}^{0} C[sI - \bar{A}]^{-1}\,\bar{B}\,e^{st}ds = C\,e^{\bar{A}t}\,\bar{B} \tag{74}$$

where the eigenvalues of $\bar{A}$ are assumed to have negative real parts and $L^{-1}[\ ]$ is the inverse of the Laplace transform operator $L[\ ]$. The Laplace transform of the ith derivative of the impulse response function matrix becomes

$$L[Y^{(i)}(t)] = \int_0^\infty C\bar{A}^{i}e^{\bar{A}t}\bar{B}\,e^{-st}dt = C\bar{A}^{i}\int_0^\infty e^{\bar{A}t}\bar{B}\,e^{-st}dt = C\bar{A}^{i}[sI-\bar{A}]^{-1}\bar{B} = C\bar{A}^{i}G(s)\bar{B}$$

$$= s^{i}Y(s) - s^{i-1}Y(t{=}0) - s^{i-2}Y^{(1)}(t{=}0) - \ldots - Y^{(i-1)}(t{=}0)$$

$$= s^{i}Y(s) - \sum_{k=0}^{i-1} Y^{(k)}(t{=}0)\,s^{i-1-k} \quad ; \quad i = 1, 2, \ldots \tag{75}$$

where $Y^{(i)} = d^{i}Y(t)/dt^{i}$ and $G(s) = [sI-\bar{A}]^{-1}$

Let the equally spaced time be given by 0, $\Delta t$, $2\Delta t$, ..., $k\Delta t$, ... where $\Delta t$ is a constant interval. The numerical Fourier transform of the impulse response function $Y(j\omega)$ has the approximation as follows:

$$Y(j\omega) = \int_0^{\infty} Y(t)e^{-j\omega t}dt \approx \sum_{k=0}^{\infty} Y(k\Delta t)e^{-j\omega k\Delta t}\Delta t = \sum_{k=0}^{\infty} Ce^{\bar{A}k\Delta t}\bar{B}e^{-j\omega k\Delta t}\Delta t$$

$$= \sum_{k=0}^{\infty} CA^kBe^{-j\omega k\Delta t}\Delta t = \sum_{k=0}^{\infty} Y(k)e^{-j\omega k\Delta t}\Delta t = CG(j\omega)B \qquad (76)$$

where $A=e^{\bar{A}\Delta t}$(Eq. 8), $G(j\omega) = \sum_{k=0}^{\infty} A^k e^{-j\omega k\Delta t}\Delta t$ and $Y(k) = CA^kB$ (Eq. 11). The Fourier transform of the ith derivative of the impulse response matrix can correspondingly be approximated by

$$(j\omega)^i Y(j\omega) - \sum_{k=0}^{i-1} Y^{(k)}(t=0)\,(j\omega)^{i-1-k} \approx \sum_{k=0}^{\infty} C\bar{A}^i e^{\bar{A}k\Delta t}\bar{B}e^{-j\omega k\Delta t}\Delta t = C\bar{A}^i G(j\omega)\bar{B} \qquad (77)$$

Eqs. (76) and (77) constitute the bases for the frequency-domain parameter identification methods which will be presented in the following.

## III-2. The Eigensystem Realization Algorithm in Frequency Domain (ERA-FD) (Ref. 26)

This subsection presents a brief review of a recently developed technique for modal parameter identification which is referred to as the Eigensystem Realization Algorithm in Frequency-Domain (ERA-FD). Similar to the Eigensystem Realization Algorithm in time-domain, the ERA-FD starts with a complex data matrix formed by transfer functions and corresponding shifted transfer functions. Using singular value decomposition on the data matrix, a state space triple [A, B, C] is realized to match the transfer functions of the system. Both discrete-time and continuous-time models are considered for comparisons with other methods.

Consider a linear, time invariant system initially at rest with an m-dimensional input signal u(t) and an $\ell$-dimensional output y(t), subject to an additive disturbance n(t),

$$y(t) = \sum_{k=0}^{\infty} Y(k)u(t-k) + n(t) \;;\; t=0,1,2,\ldots \qquad (78)$$

where Y(k) is the impulse response matrix. This system has the transfer function as shown in Eq. (76)

$$Y(j\omega_i) = \sum_{k=0}^{\infty} Y(k)e^{-jk\Delta t\omega_i}\Delta t \;;\; i=0,1,2,\ldots \qquad (79)$$

The identification problem now is the following. Generate and/or measure an input signal u(k) (k=0,...,N) and measure the corresponding output signal y(k) (k=0,...,N). Based on these measurements, form an estimate of the transfer function, subject to the additive noise n(t),

$$Y_0(j\omega_i) = \sum_{k=0}^{N} Y(k)e^{-jk\Delta t\omega_i}\Delta t = \sum_{k=0}^{N} CA^kBe^{-jk\Delta t\omega_i}\Delta t = CG(j\omega_i)B;\; i=0,\ldots,N \qquad (80)$$

where $G(j\omega_i) = \sum_{k=0}^{N} A^k e^{-jk\Delta t\omega_i}\Delta t$. Our aim is that, given an estimated transfer function $Y_0(j\omega_i)$, find a triple [A, B, C] of minimum order such that identities of Eq.(80) hold. The triple with minimum order will minimize the effect of noise on the identified modal parameters.

Define a shifted transfer function by

$$Y_\tau(j\omega_i) \triangleq \sum_{k=0}^{N} Y(k+\tau)e^{-jk\Delta t\omega_i}\Delta t = \sum_{k=0}^{N} CA^{k+\tau}Be^{-jk\Delta t\omega_i}\Delta t = CA^\tau G(j\omega_i)B;\ \tau=0,1,2,\ldots \quad (81)$$

A special recursive formula to compute the shifted transfer function from $Y_0(j\omega_i)$ has been developed in Ref. 26. The idea of the shifted transfer function is derived from the basic concept of system realization, i.e. Hankel matrix.

The Eigensystem Realization Algorithm in frequency-domain begins by forming the $r\times(N+1)$ complex block matrix

$$H_g(k) = \begin{bmatrix} Y_k(j\omega_o) & Y_k(j\omega_1) & \cdots & Y_k(j\omega_N) \\ Y_{k+t_1}(j\omega_o) & Y_{k+t_1}(j\omega_1) & \cdots & Y_{k+t_1}(j\omega_N) \\ \vdots & \vdots & & \vdots \\ Y_{k+t_{r-1}}(j\omega_o) & Y_{k+t_{r-1}}(j\omega_1) & \cdots & Y_{k+t_{r-1}}(j\omega_N) \end{bmatrix} = V A^k W \quad (82)$$

where

$$V = \begin{bmatrix} C \\ CA^{t_1} \\ \vdots \\ CA^{t_{r-1}} \end{bmatrix}, \quad W = [G(j\omega_o)B \quad G(j\omega_1)B \ \ldots \ G(j\omega_N)B]$$

and $t_i (i=1,\ldots,r-1)$ are arbitrary integers. The matrix V is the observability matrix and the matrix W is the controllability matrix in frequency domain. Now find the singular value decomposition for the matrix $H_g(0)$

$$H_g(0) = P_n D_n Q_n^* \ ; \ * = \text{complex conjugate transpose} \quad (83)$$

where $P_n$ and $Q_n$ are orthonormal matrices in complex domain, and $D_n$ is a diagonal matrix with positive elements $[d_1, d_2, \ldots, d_n]$ referred to as singular values of $H_g(0)$. The rank of $H_g(0)$ is determined by testing the singular values for zero. The pseudo-inverse of the matrix $H_g(0)$ can then be given by

$$H_g^{\#} = Q_n D_n^{-1} P_n^* \quad (84)$$

Now observe that, from Eq.(82),

$$H_g(0) = H_g(0)H_g^{\#}H_g(0) = VWH_g^{\#}VW \tag{85}$$

which implies

$$WH_g^{\#}V = I_n \tag{86}$$

where $I_n$ is an identity matrix of order n

Define $0_\ell$ as a null matrix of order $\ell$, $I_\ell$ as identity matrix of order p and $E_{\ell i}^T = [0_\ell, \ldots, I_\ell, \ldots, 0_\ell]$ where $I_\ell$ is located at the ith position. With the aid of Eqs. (83) - (86), a minimum order-realization can be obtained from

$$\begin{aligned} Y_k(j\omega_i) &= E_{\ell 1}^T H_g(k)E_{mi} = E_{\ell 1}^T VA^k WE_{mi} \\ &= E_{\ell 1}^T V[WH_g^{\#}V]A^k[WH_g^{\#}V]WE_{mi} \\ &= E_{\ell 1}^T H_g(0)H_g^{\#}VA^k WH_g^{\#}H_g(0)E_{mi} \\ &= E_{\ell 1}^T P_n D_n^{1/2}[D_n^{-1/2}P_n^* VA^k WQ_n D_n^{-1/2}]D_n^{1/2}Q_n^* E_{mi} \\ &= E_{\ell 1}^T P_n D_n^{1/2}[D_n^{-1/2}P_n^* H_g(1)Q_n D_n^{-1/2}]^k D_n^{1/2}Q_n^* E_{mi} \end{aligned} \tag{87}$$

Examination of Eqs.(81) and (87) shows that the triple

$$A = D_n^{-1/2}P_n^* H_g(1)Q_n D_n^{-1/2}, \quad B = G^{-1}(j\omega_i)D_n^{1/2}Q_n^* E_{mi}, \quad C = E_{\ell 1}^T P_n D_n^{1/2} \tag{88}$$

is a minimum realization derived from frequency domain analysis. To compute the matrix B, the integer i can be any value from 0 to N and $G^{-1}(j\omega_i)$ can be approximated by $[j\omega_i I - (\ln A)/\Delta t]$ (see Eqs. (8) & (75)). For simplicity, it is set to i = 0. Based on the special characteristics of the forms B and C, accuracy indicators were developed in Ref. 39 to quantify the system and noise modes. Although the transfer function matrices $Y_k(j\omega_i)$ (i = 0, ..., N) are in the complex domain, the block matrix H(k) can be implemented in the real domain by putting the real part of each individual matrix $Y_k(j\omega_i)$ in one block and its imaginary part in a consecutive block. In doing this way, all the computations required for the system realization become real arithmetic.

Eq. (88) is developed using the discrete-time dynamic model as the basis so that the realized matrix A represents the state transition matrix (see Eq.(9)). The question arises whether a realization can be derived using the continuous-time model (Eq.(3)) to identify a state matrix $\bar{A}$ directly. Recall Eq. (76)

$$Y_0(j\omega_i) = \sum_{k=0}^{N} Ce^{\bar{A}k\Delta t}\bar{B}e^{-j\omega_i k\Delta t}\Delta t = CG(j\omega_i)\bar{B} \; ; \; i=0,\ldots,N \tag{89}$$

Redefine the shifted transfer functions $\bar{Y}_\tau(j\omega_i)$ with the aid of Eq. (77) as

$$\bar{Y}_\tau(j\omega_i)=(j\omega_i)^\tau Y_0(j\omega_i) - \sum_{k=0}^{\tau-1} Y^{(k)}(t=0)\ (j\omega)^{i-1-k} = C\bar{A}^\tau G(j\omega_i)\bar{B};\ i=0,\ldots,N \quad (90)$$

The shifted transfer functions in this case are obtained by differentiating the impulse response function matrix $\tau$ times. For example, if deflection sensors are used for dynamic measurements, $Y_0(j\omega_i)$ represent the transfer function matrix for the deflection responses, $\bar{Y}_1(j\omega_i)$ describe the velocity responses and $\bar{Y}_2(j\omega_i)$ describe the acceleration responses. It is generally inadvisable to differentiate a measurement signal because noise effects are magnified. On the other hand, if accelerometers are used for the dynamic measurements, $\bar{Y}_2(j\omega_i)$ represents the corresponding transfer function matrices. Then matrices $\bar{Y}_1(j\omega_i)$ and $Y_0(j\omega_i)$ can be obtained by integrating the acceleration signals once and twice to respectively describe the velocity and deflection signals. The information of high frequency modes may be lost due to the integration process, however.

Now substituting these shifted transfer functions (Eq. (90)) into the block matrix (Eq. (82) and performing the same procedures (Eqs. (83) - (88)) such as singular value decomposition, etc., a minimum realization identical to Eq. (88) will be obtained except that the state transition matrix A is replaced by the state matrix $\bar{A}$. The detailed description is omitted. Instead, a simple example is discussed.

Let the block matrix $\bar{H}_g(0)$ be formed by

$$\bar{H}_g(0) = \begin{bmatrix} Y_0(j\omega_0) & Y_0(j\omega_1) & \ldots & Y_0(j\omega_N) \\ \bar{Y}_1(j\omega_0) & \bar{Y}_1(j\omega_1) & \ldots & \bar{Y}_1(j\omega_N) \end{bmatrix} = V\ W \quad (91)$$

where $V = \begin{bmatrix} C \\ C\bar{A} \end{bmatrix}$ and $W = [G(j\omega_0)\bar{B}\ \ G(j\omega_1)\bar{B}\ \ldots\ G(j\omega_N)\bar{B}]$.

Note that the controllability matrix W is identical to the one shown in Eq. (82) except that B is replaced by $\bar{B}$. Observe that

$$\bar{H}_g(1) = \begin{bmatrix} \bar{Y}_1(j\omega_0) & \bar{Y}_1(j\omega_1) & \ldots & \bar{Y}_1(j\omega_N) \\ \bar{Y}_2(j\omega_0) & \bar{Y}_2(j\omega_1) & \ldots & \bar{Y}_2(j\omega_N) \end{bmatrix} = V\ \bar{A}\ W \quad (92)$$

and assume that the number of rows for the matrix $H_g(0)$ is greater than the order of the system. Find the singular value decomposition for the block matrix $\bar{H}_g(0) = P_n D_n Q_n^*$ (93)

which is similar to Eq. (83). The triple

$$\bar{A} = [D_n^{-1/2} P_n^{*} \bar{H}_g(1) Q_n D_n^{-1/2}]^{-1}, \quad \bar{B} = G^{-1}(j\omega_i) D_n^{1/2} Q_n^{*} E_{pi}, \quad C = E_{m1}^T P_n D_n^{1/2} \tag{94}$$

is a direct minimum realization for the continuous-time model (Eqs.(3) & (4)). This simple example will be used to derive and discuss with other methods.

## III-3. The Polyreference Technique in Frequency Domain (Refs. 27-28)

This subsection presents the Polyreference technique in frequency-domain for modal parameter identification using system realization theory. The methods make use of a set of transfer function matrices and shifted transfer function matrices to form a complex Hankel-like data matrix. By employing singular value decomposition of the data matrix, an orthonormal matrix is computed to derive an observable canonical-form realization which is in parallel to the canonical-form realization obtained for the Polyreference technique in time-domain.

The first part of this subsection will show the direct canonical-form realization of the discrete-time model. The close relationship of the time-domain and frequency-domain Polyreference techniques is established. The second part of this subsection will show the direct observable-form realization of the continuous-time model. The close relationship between this approach and other existing approaches (Refs. 27 & 28) is discussed.

Form the (r+1)x(s+1) block Hankel-like matrix

$$H_g(0) = \begin{bmatrix} Y_0(j\omega_0) & Y_0(j\omega_1) & \dots & Y_0(j\omega_N) \\ Y_1(j\omega_0) & Y_1(j\omega_1) & \dots & Y_1(j\omega_N) \\ \vdots & \vdots & & \vdots \\ Y_k(j\omega_0) & Y_k(j\omega_1) & \dots & Y_k(j\omega_N) \end{bmatrix} = V\ W \tag{95}$$

where

$$C = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^r \end{bmatrix}, \quad W = [G(j\omega_0)B \quad G(j\omega_1)B \ \dots \ G(j\omega_N)B] \tag{96}$$

and r is an integer chosen such that the number $\ell r$ ($\ell$ = the number of outputs) is greater than the order of the system. Using the same procedures shown in Eqs.(44) - (48), first find the singular value decomposition of $H_g(0)$ such that

$$\begin{bmatrix} P_n^T \\ P_o^T \end{bmatrix} H(0)[Q_n \quad Q_o] = \begin{bmatrix} D_n & 0 \\ 0 & 0 \end{bmatrix} \tag{97}$$

which yields

$$P_o^T VWQ_n = 0 \;\; \Rightarrow \;\; P_o^T V = 0 \tag{98}$$

where $P_n$, $Q_n^T$, $P_o$, and $Q_o^T$ are orthonormal matrices and $D_n$ is a diagonal matrix. The integer n is determined by the characteristics of the system noise as discussed in Ref. 25. All singular values after n are considered as zero values.

Now partition the matrices $P_o$ such that $P_o^T = [P_{o0}^T \;\; P_{o1}^T \ldots \; P_{or}^T]$ and choose square matrices $\bar{P}_{oi}$ of order $\ell$ from matrices $P_{oi}$ (i = 0, 1, ..., r). Substitution of the observability matrix defined in Eq. (96) into Eq. (98) with the partitioned matrices $P_{oi}$ produces

$$-\sum_{i=0}^{r-1} [\bar{P}_{or}^T]^{-1} \bar{P}_{oi}^T C A^i = C A^r \triangleq \sum_{i=0}^{r-1} \tilde{P}_{oi}^T C A^i \tag{99}$$

where $\tilde{P}_{oi}^T = [\bar{P}_{or}^T]^{-1} \bar{P}_{oi}^T$

Eqs. (99) can be rearranged into companion matrix form as

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-2} \\ CA^{r-1} \end{bmatrix} A = \begin{bmatrix} 0 & I_\ell & 0 & \ldots & 0 \\ 0 & 0 & I_\ell & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & I_\ell \\ -\tilde{P}_{o0}^T & -\tilde{P}_{o1}^T & -\tilde{P}_{o2}^T & \ldots & -\tilde{P}_{o(r-1)}^T \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-2} \\ CA^{r-1} \end{bmatrix} \tag{100}$$

The matrix $I_\ell$ is an identity matrix of order $\ell$. Now it is claimed from Eq. (100) that the triple

$$\tilde{A} = \begin{bmatrix} 0 & I_\ell & 0 & \ldots & 0 \\ 0 & 0 & I_\ell & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & I_\ell \\ -\tilde{P}_{o0}^T & -\tilde{P}_{o1}^T & -\tilde{P}_{o2}^T & \ldots & -\tilde{P}_{o(r-1)}^T \end{bmatrix}, \; G(j\omega_i)\tilde{B} = \begin{bmatrix} Y_0(j\omega_i) \\ Y_1(j\omega_i) \\ \vdots \\ Y_{r-2}(j\omega_i) \\ Y_{r-1}(j\omega_i) \end{bmatrix}, \; \tilde{C} = [I_\ell \; 0 \ldots 0 \; 0] \tag{101}$$

is an $\ell r$-dimensional realization of the system. Indeed, it can be readily verified that

$$Y_0(j\omega_i) = \tilde{C}G(j\omega_i)\tilde{B}, \; Y_1(j\omega_i) = \tilde{C}\tilde{A}G(j\omega_i)\tilde{B}, \; \ldots, \; Y_r(j\omega_i) = \tilde{C}\tilde{A}^r G(j\omega_i)\tilde{B}.$$

Because of the structure of $\tilde{A}$ and $\tilde{C}$, it is easy to show that the realization is observable. However, it is not in general controllable. This realization is called an observable canonical-form realization. Therefore, it is not a minimum realization. Since the controllability matrix

W defined in Eq. (96) does not have the form which explicitly involves the state transition matrix as the observability matrix does, there does not exist an explicit controllable-form realization as that for the time-domain Polyreference technique.

Eq. (101) is developed using the discrete-time dynamic model as the basis such that the realized matrix $\bar{A}$ represents the state transition matrix. It is natural to question whether a realization similar to Eq. (101) can be derived directly using the continuous-time model. The answer is affirmative. Replacing the shifted transfer function matrices in Eq. (95) by the ones shown in Eq. (90) and performing the same procedures (Eqs.(96) - (101)), an observable canonical-form realization identical to Eq. (101) will be obtained except that the triple $[\tilde{A}, \tilde{B}, \tilde{C}]$ is replaced by $[\bar{A}, \bar{B}, \bar{C}]$ where $\bar{A}$ represents the state matrix for the continous-time model. The detailed description is omitted. Instead, a simple example is discussed for comparison with other existing methods (Refs. 40 - 41).

Let the Hankel-like matrix $\bar{H}_g(0)$ be formed as

$$\bar{H}_g(0) = \begin{bmatrix} Y_0(j\omega_0) & Y_0(j\omega_1) & \dots & Y_0(j\omega_N) \\ \bar{Y}_1(j\omega_0) & \bar{Y}_1(j\omega_1) & \dots & \bar{Y}_1(j\omega_N) \\ \bar{Y}_2(j\omega_0) & \bar{Y}_2(j\omega_1) & \dots & \bar{Y}_2(j\omega_N) \end{bmatrix} = V\,W \qquad (102)$$

where

$$C = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}, \quad W = [G(j\omega_0)B \;\; G(j\omega_1)B \;\dots\; G(j\omega_N)B] \qquad (103)$$

Assume that the number of rows of the matrix $H_g(0)$ is greater than the order of the system. Following the same procedures shown from Eqs. (97) to Eq. (100) and using the same notations produces a polynomial equation

$$C\,\bar{A}^2 + \tilde{P}_{o1}^T\, C\,\bar{A} + \tilde{P}_{o0}^T\, C = 0 \qquad (104)$$

Eqs. (99) can be rearranged into companion matrix form as

$$\begin{bmatrix} C \\ C\bar{A} \end{bmatrix} \bar{A} = \begin{bmatrix} 0 & I_\ell \\ -\tilde{P}_{o0}^T & -\tilde{P}_{o1}^T \end{bmatrix} \begin{bmatrix} C \\ C\bar{A} \end{bmatrix} \qquad (105)$$

The matrix $I_\ell$ is an identity matrix of order $\ell$. The triple

$$\tilde{A} = \begin{bmatrix} 0 & I_\ell \\ -\tilde{P}_{o0}^T & -\tilde{P}_{o1}^T \end{bmatrix}, \quad G(j\omega_i)\bar{B} = \begin{bmatrix} Y_o(j\omega_i) \\ Y_1(j\omega_i) \end{bmatrix}, \quad \tilde{C} = [I_\ell \;\; 0] \qquad (106)$$

is a $2\ell$-dimensional realization of the system. Examination of Eqs. (2) and (106) reveals that the mass matrix M, the damping matrix D and the stiffness matrix k for a finite-dimensional system can be related by

$$M^{-1}K = \tilde{P}_{o0}^T \quad \text{and} \quad M^{-1}D = \tilde{P}_{o1}^T \tag{107}$$

Modal parameters of the estimated system can be obtained by solving the eigenvalue problem of Eq. (106). If real and undamped modes are sought, just solve the eigenvalue problem with the absence of the matrix $\tilde{P}_{o1}^T$ in the state matrix $\tilde{A}$. The approach used here to obtain the observable canonical-form realization is different from that shown in Ref. 40 which first introduced the companion-form state matrix $\tilde{A}$ shown in Eq. (106) for modal parameter identification. However, both methods which use the same transfer function matrices to build the state space model are conceptually similar.

The orthonormal matrix $P_o$ is computed through the application of the singular value decomposition to realize a companion-form state matrix. Since the orthonormal matrix is very close to the unity matrix, the method presented here thus generates a computationally well-behaved realization.

Now consider a simplest case where the Hankel-like matrix $\bar{H}_g(0)$ is formed by only two series of transfer function matrices $Y_0(j\omega_i)$ and $\bar{Y}_1(j\omega_i)$ ($i = 0, 1, \ldots, N$) as shown in Eq.(92). Assume that the number $2\ell$ of rows of the Hankel-like matrix is at least twice the order of the system. Following the same procedures as shown from Eqs. (97) to (99) yields a first degree polynomial equation as

$$C\bar{A} + \tilde{P}_{o0}^T C = 0 \tag{108}$$

The triple

$$\tilde{A} = -\tilde{P}_{o0}^T , \quad G(j\omega_i)\tilde{B} = Y_0(j\omega_i) , \quad \tilde{C} = I_\ell \tag{109}$$

is a $\ell$-dimensional realization of the system. Eq. (108) was introduced in Ref. 41 for modal parameter identification for flexible structures. In contrast to the minimum realization shown in Eq. (94), this realization (Eq. (109)) is not of minimun order in the sense that the identified state matrix is usually oversized if the order of the system is not known a priori. Furthermore, the number of sensors $\ell$ must be greater than the order of the system.

## III-4. An Alternate method for the ERA-FD and the Polyreference Technique in Frequency Domain

A minimum order of canonical-form realization in frequency domain is generally impossible for multi-input and multi-output systems due to the constraint that the realized state matrix is a companion form. If the constraint is released, a minimum order realization in frequency-domain can be obtained from Eq. (101). The procedures to derive the minimum order realization are identical to that shown in Section III-5 for the time domain case. Using the block Hankel-like matrix (Eq.(95)) and notations defined in Section II-5, the triple

$$A = D_n^{-1/2}[E_{\ell r}P_n]^{\#}[\sigma_\ell E_{\ell r}P_n]D_n^{1/2}, \quad B = G^{-1}(j\omega_i)D_n^{1/2}Q_n^{*}E_m, \quad C = E_\ell^T P_n D_n^{1/2} \tag{110}$$

is a minimum realization of order n. Here, $[\sigma_\ell E_{\ell r}P_n]$ simply means the matrix obtained by deleting the last $\ell$ rows of the matrix $P_n$ and $[E_{\ell r}\ P_n]$ represents the matrix obtained by deleting the first $\ell$ rows of the matrix $P_n$. Since $P_n$ is an orthonormal matrix, a special and efficient procedure can be developed to compute the pseudoinverse of the matrix $E_{\ell r}P$ using the matrix inversion lemma (Ref. 19).

Based on Eq. (108), a close link between between the ERA-FD and the Polyrefernce technique in Frequency domain is established. A minimum realization similar to Eq. (108) can also be derived for the direct realization of the continuous-time model.

After several methods in frequency-domain for modal parameter identification are derived using system realization theroy, the derivation of the least squares regression technique in frequency domain becomes trivial and thus is omitted. Following the same procedure as shown in the section II-6, the reader is encouraged to derive it on his own.

## IV. CONCLUDING REMARKS

In this report, several methods for modal parameter identification have been presented and derived using system realization theory. The relations between different techniques are reasonably well understood and the choice of methods can be done largely on the basis of the final purpose of the identification, for example control of flexible structures. Most methods are claimed to work well on simulated and test data. In spite of a large body of literature on identification, there are few papers which compare different techniques using experimental data. Unfortunately, conclusive results have not been obtained. It is more fun to dream up new methods than to work with somebody else's scheme. However, for a person engaged in application, it would be highly desirable to have comparisons available. This report illustrates the mathematical relations among several recently developed methods via system realization theory, which provides a basis and insight for comparison and evaluation. The practicing engineer would still not be satisfied, however. It would be necessary to have a selection of test data from real systems to which several techniques are tried and compared for evaluation.

It is hoped and expected, through combined efforts in the control and structural dynamics disciplines, that the field of modal parameter identification is moving towards more unification and that there will be more comparisions of different methods. Two purposes of this report is to contribute to the goal of unification and to serve as a starting point to stimulate more research toward this goal. It is believed that interaction with other disciplines such as controls, artificial intelligence, etc. is essential for accomplishment.

For a report like this size, it is difficult to be complete. There are many other techniques available in the fields of modal parameter identification. The reader is directed to the References, Bibliography on System Realization Theory, and other literature for further information.

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16. Abstract

A unified approach is introduced using system realization theory to derive and correlate modal parameter identification methods for flexible structures. Several different time-domain and frequency-domain methods are analyzed and treated. A basic mathematical foundation is presented which provides insight into the field of modal parameter identification for comparison and evaluation. The relation among various existing methods is established and discussed. This report serves as a starting point to stimulate additional research towards the unification of the many possible approaches for modal parameter identification.

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