NASA CR 178,082

NASA Contractor Report 178082

ICASE REPORT NO. 86-16

NASA-CR-178082
19860015696

ICASE

SHIFTING THE CLOSED-LOOP SPECTRUM IN THE OPTIMAL LINEAR QUADRATIC REGULATOR PROBLEM FOR HEREDITARY SYSTEMS

J. S. Gibson

and

I. G. Rosen

Contract Nos. NAS1-17070 and NAS1-18107

March 1986

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

NASA

National Aeronautics and Space Administration

Langley Research Center
Hampton, Virginia 23665

SHIFTING THE CLOSED-LOOP SPECTRUM IN THE OPTIMAL LINEAR QUADRATIC REGULATOR PROBELM FOR HEREDITARY SYSTEMS[(†)]

by

J.S. Gibson*
Department of Mechanical, Aerospace
and Nuclear Engineering
University of California, Los Angeles
Los Angeles, CA 90024

and

I.G. Rosen**
Department of Mathematics
University of Southern California
Los Angeles, CA 90089

ABSTRACT

In the optimal linear quadratic regulator problem for finite dimensional systems, the method known as an α-shift can be used to produce a closed-loop system whose spectrum lies to the left of some specified vertical line; that is, a closed-loop system with a prescribed degree of stability. This paper treats the extension of the α-shift to hereditary systems. As in finite dimensions, the shift can be accomplished by adding α times the identity to the open-loop semigroup generator and then solving an optimal regulator problem. However, this approach does not work with a new approximation scheme for hereditary control problems recently developed by Kappel and Salamon. Since this scheme is among the best to date for the numerical solution of the linear regulator problem for hereditary systems, an alternative method for shifting the closed-loop spectrum is needed. An α-shift technique that can be used with the Kappel-Salamon approximation scheme is developed. Both the continuous-time and discrete-time problems are considered. A numerical example which demonstrates the feasibility of the method is included.

(*)This research was supported in part by The Air Force Office of Scientific Research under contract No. AFOSR-84-0309.

(**)This research was supported in part by the The Air Force Office of Scientific Research under contract No. AFOSR-84-0393.

(†)Part of this research was carried out while the authors were visiting scientists at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA, which is operated under NASA contracts No. NAS1-17070 and No. NAS1-18107.

1. INTRODUCTION

In this paper we consider the problem of computing optimal linear state feedback control laws for linear hereditary systems which yield a resulting optimal state trajectory that exhibits a prescribed degree of stability. This problem is sometimes referred to as the linear quadratic regulator problem with α-shift, α being the desired degree of stability, since it involves a linear state constraint, the minimization of a quadratic payoff functional and the shifting of the closed-loop spectrum to the left of the line Re $z = -\alpha$ in the complex z-plane. A solution is a control law of the form

$$(1.1) \qquad u^*(t) = -K(x^*(t),\ x_t^*), \qquad t > 0$$

where K is a linear function of the optimal trajectory $x^*(t)$ and its past history x_t^* at time t, $x^* = x^*(u^*)$ is the solution to the underlying hereditary system, u^* minimizes a performance index which is quadratic in the state and the control and x^* satisfies a uniform exponential bound of the form

$$(1.2) \qquad |x^*(t)| < Me^{-\alpha t}, \qquad t > 0 \ .$$

These ideas will be made precise in the subsequent Section 2.

In finite dimensions, i.e. when the state is given by a linear ordinary differential equation of the form

$$(1.3) \qquad x(t) = Ax(t) + Bu(t), \qquad t > 0,$$

the linear quadratic regulator problem with α-shift and its solution are well known (see [1], [2]). The matrix A is simply replaced by the matrix

$A + \alpha I$ and the resulting standard linear quadratic regulator problem (the shifted problem) is solved in the usual fashion.

A hereditary system, on the other hand, is infinite dimensional. Instead of the matrix A being replaced by $A + \alpha I$, it is the infinitesimal generator $\mathcal{A}$ of the solution semigroup which is replaced by $\mathcal{A} + \alpha I$. The solution of the resulting shifted regulator problem requires the use of some form of finite dimensional approximation. One is tempted to take the general approach which by now has become standard in the control of infinite dimensional or distributed systems. That is, approximate the unshifted system using one of the currently available schemes for the regulator problem for hereditary systems and then apply the standard finite dimensional theory and techniques to the finite dimensional approximating systems to obtain approximations to the shifted system. However, as we discovered, this approach may not work.

The linear spline-based approximation scheme for the linear quadratic regulator problem for hereditary systems recently developed by Kappel and Salamon in [14] has been shown to be, in many respects, one of the most attractive approximation methods currently available for this class of problems. However, the finite dimensional approximating systems possess eigenvalues which do not converge to eigenvalues of the original underlying hereditary system. Those eigenvalues are stable and hence do not cause difficulties when the unshifted problems are solved. However, they are extremely difficult, if not impossible to shift. Consequently, even if the poles of the hereditary system which are to the right of the line $\mathrm{Re}\, z = -\alpha$ in the complex z-plane can be shifted (i.e. the finite dimensional subspace spanned by the eigenvectors corresponding to the eigenvalues with real part greater than or equal to $-\alpha$ is controllable), when the α-shift is applied to the approximating systems the solution of the resulting finite dimensional regulator problems fail.

We have found a relatively simple and straight forward way to overcome this difficulty. It involves a modification of the infinite dimensional shifted system which permits the extraneous eigenvalues introduced by the approximation to remain stable while the true eigenvalues of the hereditary system are forced to the left of the line Re $z = -\alpha$.

In the paper we have treated both the continuous-time and the discrete-time or sampled problems. The Kappel-Salamon approximation and the α-shift are discussed in Section 3. In Section 4 we provide an example with numerical results.

2. THE OPTIMAL LINEAR QUADRATIC REGULATOR PROBLEM FOR HEREDITARY SYSTEMS WITH A PRESCRIBED DEGREE OF STABILITY

We consider linear hereditary control systems of the form

$$(2.1) \qquad x(t) = Lx_t + B_0 u(t), \quad t \geq 0,$$

$$(2.2) \qquad x(0) = \eta, \quad x_0 = \phi ,$$

where $x(t) \varepsilon R^n$, $u \varepsilon L_2(0,t_f;R^m)$ for each t_f with $0 < t_f < \infty$, $B_0 \varepsilon \mathcal{L}(R^m,R^n)$, $\eta \varepsilon R^n$, $\phi \varepsilon L_2(-r,0;R^n)$ and for each $t \geq 0$, $x_t \varepsilon L_2(-r,0;R^n)$ denotes the past history of the state x on the interval $[t-r, t]$. That is, $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. The linear transformation L is assumed to be of the form

$$(2.3) \qquad L\psi = \sum_{i=0}^{\nu} A_i\psi(-r_i) + \int_{-r}^{0} A(\theta)\psi(\theta)d\theta$$

with $A_i \varepsilon \mathcal{L}(R^n,R^n)$, $i = 0,1,2,\ldots,\nu$, $(\nu < \infty)$, $A \varepsilon L_2(-r,0; \mathcal{L}(R^n,R^n))$ and $0 = r_0 < r_1 < r_2 \quad < r_\nu = r$.

Standard arguments yield the existence of a unique solution $x(\ ; \eta,\phi,u)$ to (2.1), (2.2) which is absolutely continuous with $x(\ ; \eta,\phi,u) \varepsilon L_2(0,t_f;R^n)$ for any t_f, $0 < t_f < \infty$, which satisfies (2.1) for almost every $t \geq 0$ and which depends continuously on η,ϕ, and u.

A one parameter family of solution operators for the homogeneous system corresponding to (2.1), (2.2) can be defined by

$$(2.4) \qquad \mathcal{T}(t)(\eta,\phi) = (x(t;\eta,\phi,0),\ x_t(\eta,\phi,0)).$$

If we let $Z = R^n \times L_2(-r,0;R^n)$ together with the usual inner product

$$(2.5) \qquad \langle(\xi,\psi),(\zeta,\chi)\rangle_Z = \xi^T\zeta + \int_{-r}^{0}\psi(\theta)^T\chi(\theta)d\theta,$$

then the family of operators, $\{\ \mathcal{T}(t) : t \geq 0\}$ forms a C_0 semigroup of bounded linear operators on the Hilbert space Z. The infinitesimal generator is given by

$$\mathrm{Dom}(\mathcal{A}) = \{(\xi,\psi) \in Z : \psi \in H^1(-r,0;R^n),\ \xi = \psi(0)\}$$

$$(2.6) \qquad \mathcal{A}(\psi(0),\psi) = (L\psi, D\psi)\ .$$

If we define the operator $\mathcal{B}$: $R^m \to Z$ by

$$(2.7) \qquad \mathcal{B}u = (B_0u,0)$$

then an equivalence exists between solutions to (2.1), (2.2) and mild or generalized solutions to the abstract evolution equation

$$(2.8) \qquad z(t) = \mathcal{A}z(t) + \mathcal{B}u(t), \quad t \geq 0$$

with initial condition

$$(2.9) \qquad z(0) = (\eta,\phi)\ .$$

That is, $z(t) = (x(t;\eta,\phi u),\ x_t(\eta,\phi,u))$, where

$$(2.10) \qquad z(t) = \mathcal{T}(t)(\eta,\phi) + \int_0^t \mathcal{T}(t-s)\,\mathcal{B}u(s)ds, \qquad t > 0 .$$

In the subsequent discussion, when the solution to the system (2.8), (2.9) is referred to, it should be understood to imply the mild solution given by (2.10).

2.1 THE CONTINUOUS-TIME PROBLEM

The control problem which is of interest to us here is the infinite time horizon linear quadratic regulator (LQR) problem given by

Find $u^* \varepsilon L_2(0,\infty;R^m)$ which minimizes the performance index

$$(2.11) \qquad J(u) = \int_0^\infty x(t)^T Q_0 x(t) + u(t)^T Ru(t)dt$$

where x is the solution to (2.1), (2.2) corresponding to u.

The matrix $Q_0 \; \varepsilon \; \mathcal{L}(R^n,R^n)$ is assumed to be nonnegative, symmetric and the matrix $R \; \varepsilon \; \mathcal{L}(R^m,R^m)$ is assumed to be positive definite, symmetric.

Defining the nonnegative, symmetric operator $Q : Z \rightarrow Z$ by

$$(2.12) \qquad Q(\xi,\psi) = (Q_0\xi,0),$$

we treat the equivalent LQR problem given by

(P1) Find $u^* \varepsilon L_2(0,\infty;R^m)$ which minimizes

$$(2.13) \qquad J(u) = \int_0^\infty \langle Qz(t),z(t)\rangle_Z + u(t)^T Ru(t)dt$$

where z is the solution to (2.8), (2.9) corresponding to u.

We summarize the results from [7] concerning the solution of problem (P1). An admissible control for the initial state $z(0) = (\eta,\phi) \in Z$ is a function $u \in L_2(0,\infty;R^m)$ for which $J(u) < \infty$. Under the assumptions

(A1) for each initial state, $z(0) = (\eta,\phi) \in Z$ there exists an admissible control,

and

(B1) the operators L, B_0 and Q_0 are such that any admissible control u drives the state z(t), $t > 0$ to zero, asymptotically as $t \to \infty$,

there exists a unique nonnegative, self-adjoint solution $P \in L(Z,Z)$ to the Riccati algebraic equation

$$A^* P + PA - PBR^{-1} B^* P + Q = 0. \tag{2.14}$$

The unique solution $u^* \in L_2(0,\infty;R^m)$ to problem (P1) is given in feedback form by

$$u^*(t) = -R^{-1} B^* Pz^*(t), \quad t > 0 \tag{2.15}$$

and

$$J(u^*) = \langle P(\eta,\phi),(\eta,\phi)\rangle_Z . \tag{2.16}$$

The optimal trajectory, z^*, is given by

$$(2.17) \qquad z^*(t) = \mathcal{S}(t)(\eta,\phi)$$

where $\{\mathcal{S}(t) : t \geq 0\}$ is the C_0 semigroup generated by $\mathcal{A} - \mathcal{B}R^{-1}\mathcal{B}^*\mathcal{P}$. The semigroup $\{\mathcal{S}(t) : t \geq 0\}$ is uniformly exponentially stable, i.e. there exist positive constants M and ω for which

$$(2.18) \qquad |\mathcal{S}(t)| \leq Me^{-\omega t}, \quad t \geq 0 .$$

The operator $\mathcal{A}^*$ is given by

$$\begin{aligned} \mathrm{Dom}(\mathcal{A}^*) = \{(\xi,\psi) \in Z : D\psi \in L_2(-r,0;R^n), \psi \text{ absolutely} \\ \text{continuous on } [-r,0] \text{ except at the points } -r_1,\ldots,-r_{\nu-1} \\ \text{where } \psi((-r_i)^+)-\psi((-r_i)^-) = A_i^T\xi,\ 1 \leq i \leq \nu - 1 \\ \text{and } \psi(-r) = A_\nu^T\xi\} \ , \end{aligned}$$

$$(2.19) \qquad \mathcal{A}^*(\xi,\psi) = (A_0^T\xi + \psi(0),\ A^T\xi - D\psi)$$

and is the infinitesimal generator of the C_0 semigroup $\{\mathcal{T}^*(t) : t \geq 0\}$. We have $\mathcal{P}\,Z \subset \mathrm{Dom}\,(\mathcal{A}^*)$.

The operator $\mathcal{P}$ can be represented by a matrix of operators

$$(2.20) \qquad \mathcal{P} = \begin{pmatrix} \mathcal{P}^{00} & \mathcal{P}^{01} \\ \mathcal{P}^{10} & \mathcal{P}^{11} \end{pmatrix}$$

where $\mathcal{P}^{00} \in \mathcal{L}(R^n,R^n)$ is a nonnegative symmetric matrix,

$P^{10} \varepsilon \; L_2 \; (-r,0; \; L(R^n,R^n)), \quad P^{01} = P^{10^*}$ with

$$(2.21) \qquad P^{01}\psi = \int_{-r}^{0} P^{10}(\theta)^T\psi(\theta)d\theta, \quad \psi \; \varepsilon \; L_2(-r,0;R^n),$$

and $P^{11} \; \varepsilon \;\; L(L_2(-r,0;R^n),L_2(-r,0;R^n))$ is nonnegative and self-adjoint. We have

$$(2.22) \qquad u^*(t) = - p^0 x^*(t) - \int_{-r}^{0} p^1(\theta)x_t^*(\theta)d\theta, \qquad t \geq 0$$

where $p^0 = R^{-1}B_0^T \, P^{00}$ and $p^1(\theta) = R^{-1}B_0^T \, P^{10}(\theta)^T, \quad -r \leq \theta \leq 0.$

We note that Assumption (A1) is satisfied if the unstable subspace of L (which is finite dimensional, see [9], [23]) is controllable. Assumption (B1) is certainly satisfied if Q_0 is positive definite.

The Continuous-Time Problem with α-Shift

An α-shift, we recall, is a technique which is used in conjuction with the standard LQR theory to obtain an optimal feedback control which yields not just an asymptotically stable closed-loop system, but rather, one which exhibits a prescribed degree of stability. that is, one for which the state z(t) decays at least as fast as $e^{-\alpha t}$, i.e.

$$(2.23) \qquad |z(t)|_Z \leq \hat{M} \, e^{-\alpha t}|z(0)|_Z$$

where $\hat{M}$ is a positive constant and $\alpha > 0$ is the desired degree of stability. For the hereditary systems of interest to us here, this is completely equivalent to requiring that the eigenvalues of the closed-loop system have real part less than $-\alpha$. A discussion of this problm and its solution for finite dimensional systems first appeared in [1] and can also be found in [2].

One approach to solving this problem involves the inclusion of the multiplicative factor $e^{2\alpha t}$ under the integral sign in the performance index given in (2.13). However, making the change of variables

$$\hat{z}(t) = e^{\alpha t} z(t) \tag{2.24}$$

$$\hat{u}(t) = e^{\alpha t} u(t) \tag{2.25}$$

it is easily seen that if one solves the modified LQR problem

($\hat{P}1$) Find $\hat{u}^* \in L_2(0,\infty;R^m)$ which minimizes

$$\hat{J}(\hat{u}) = \int_{-\infty}^{\infty} \langle Q\hat{z}(t),\hat{z}(t)\rangle_Z + \hat{u}(t)^T R\hat{u}(t)dt \tag{2.26}$$

where $\hat{z}$ is the (mild) solution to the abstract evolution system

$$\frac{d}{dt}\hat{z}(t) = (\mathcal{A}+\alpha I)\hat{z}(t) + \mathcal{B}\hat{u}(t), \quad t > 0 \tag{2.27}$$

$$\hat{z}(0) = (\eta,\phi) \tag{2.28}$$

and applies

$$u_\alpha^*(t) = e^{-\alpha t}\hat{u}^*(t), \quad t > 0 \tag{2.29}$$

to the original control system (2.8), (2.9), the resulting optimal trajectory,

z_α^*, will satisfy (2.23).

Strictly speaking (2.27), (2.28) is not a hereditary system. However, the results outlined above concerning the solution of problem (P1) in closed-loop form are in fact derived from a more general abstract theory (see [6]). This more general theory can be applied directly to problem $(\hat{P1})$. Under assumptions (A1) and (B1) (with z, u, and J replaced by $\hat{z}$, $\hat{u}$ and $\hat{J}$ respectively) the unique solution to problem $(\hat{P1})$ is given in state feedback form by

$$\hat{u}^* = -R^{-1} B^* \hat{P} \hat{z}^*(t), \quad t > 0 \tag{2.30}$$

where $\hat{P} \in L(Z,Z)$ is the unique nonnegative self-adjoint solution to the Riccati algebraic equation (2.14) with A and A^* replaced by $A + \alpha I$ and $A^* + \alpha I$ respectively. The operator $\hat{P}$ can be represented by a matrix of operators analogous to the one given in (2.20). From (2.24) and (2.29) we obtain

$$u_\alpha^*(t) = -R^{-1} B^* \hat{P} z_\alpha^*(t), \quad t > 0 . \tag{2.31}$$

It then follows that

$$z_\alpha^*(t) = S_\alpha(t)(\eta,\phi), \quad t > 0 \tag{2.32}$$

with

$$|S_\alpha(t)| < \hat{M} e^{-\alpha t}, \quad t > 0 \tag{2.33}$$

where $\{S_\alpha(t) : t > 0\}$ is the C_0 semigroup generated by $\mathcal{A} - \mathcal{B}R^{-1}\mathcal{B}^*\hat{P}$.

Since the shifted system (2.27) is not a hereditary system, it would seem that to obtain the estimate (2.33) from the general theory presented in [6], the coercivity of Q would be required. (In the case of a hereditary system, assumption (B1) is sufficient.) However, as will become clear in the next section, (2.27) is in fact related to a hereditary system through a bounded similarity transformation. Consequenly, assumption (B1) is sufficient to obtain the uniform exponential bound (2.33) for the shifted system as well.

The controllability of the finite dimensional generalized eigenspaces corresponding to the eigenvalues of $\mathcal{A}$ with real part greater than or equal to $-\alpha$ is sufficent to conclude that assumption (A1) holds for the shifted system.

2.2 THE DISCRETE-TIME PROBLEM

The discrete-time or sampled analog of problem (P1) is given by

(P2) Find $u^* = \{u_k^*\}_{k=0}^{\infty} \varepsilon\ \ell_2(0,\infty;R^m)$ which minimizes

(2.34) $$J(u) = \sum_{j=0}^{\infty} \langle Qz_j, z_j\rangle_Z + u_j^T Ru_j$$ where $z = \{z_k\}_{k=0}^{\infty}$ satisfies the recurrence

(2.35) $$z_{k+1} = Tz_k + Bu_k, \quad k = 0,1,2,\ldots,$$

with

(2.36) $$z_0 = (\eta,\phi).$$

The operators $T \varepsilon\ L(Z,Z)$ and $B \varepsilon\ L(R^m;Z)$ are defined by

$$(2.37) \qquad \mathrm{T} = \mathcal{T}(\tau) \qquad \text{and} \qquad \mathrm{B} = \int_0^\tau \mathcal{T}(s)\,\mathcal{B}ds,$$

respectively, where τ denotes the length of the sampling interval.

The characterization of the solution to the discrete-time LQR problem in state feedback form for infinite dimensional systems is treated in [18] and [24]. The application of the general theory to problems involving hereditary systems is discussed in [8]. The results are completely analogous to those given above for the continuous-time problem. We briefly summarize them here.

An admissible control sequence $u \in \ell_2(0,\infty;R^m)$ for the initial condition $z_0 = (\eta,\phi) \in Z$ is one for which $J(u) < \infty$. If the assumptions

(A2) for each initial condition $z_0 = (\eta,\phi) \in Z$ there exists an admissible control

and

(B2) the operators L, B_0 and Q_0 are such that if u is an admissible control for the initial condition $z_0 = (\eta,\phi)$ then the state $z = \{z_k\}_{k=0}^{\infty}$ given by (2.35) satisfies $\lim_{k\to\infty} |z_k|_Z = 0$

hold, then there exists a unique solution to problem (P2) which is given in linear state feedback form by

$$(2.38) \qquad u_k^* = -Fz_k^*, \quad k = 0,1,2,\ldots$$

where

(2.39) $$F = \tilde{R}^{-1}B^{*}PT,$$

(2.40) $$\tilde{R} = R + B^{*}PB$$

and P is the unique nonnegative self-adjoint solution to the Riccati algebraic equation

(2.41) $$P = T^{*}(P - PB(R + B^{*}PB)^{-1}B^{*}P)T + Q \ .$$

The minimum value of the performance index can be computed from

(2.42) $$J(u^{*}) = \langle P(\eta,\phi),(\eta,\phi)\rangle_{Z}$$

and the optimal trajectory z^{*} satisfies

(2.43) $$z_{k}^{*} = S^{k}(\eta,\phi), \quad k = 0,1,2,\ldots$$

where $S \in L(Z,Z)$ is given by

(2.44) $$S = T-BF \ .$$

We also have the following result.

Theorem 2.1 If assumptions (A2) and (B2) hold then the operator S has spectral radius less than 1 and there exist positive constants M and ρ with $\rho < 1$ for which

$$(2.45) \qquad |S^k| \leq M\rho^k, \quad k = 0,1,2,\ldots$$

Proof

Since $\{T(t) : t \geq 0\}$ is the solution semigroup for the hereditary system (2.8) and the operator BF is of finite rank, the operators $S^k = (T - BF)^k$ are compact for all k sufficiently large. It follows therefore (see [5], Chapter VII, Section 4, Theorem 6) that the spectrumn of S, $\sigma(S)$, contains at most a countable number of points with no accumulation points in the complex plane except possibly $\lambda = 0$. The non-zero elements in $\sigma(S)$ are in the point spectrum of S; that is, they are eigenvalues of S.

Now suppose $\lambda \in \sigma(S)$, $\lambda \neq 0$ and $S(\xi,\psi) = \lambda(\xi,\psi)$ with $(\xi,\psi) \neq 0$. If $\xi \neq 0$, then for $z_0 = (\xi,\psi)$ the optimal trajectory is

$$(2.46) \qquad z_k^* = S^k(\xi,\psi) = \lambda^k(\xi,\psi), \quad k = 0,1,2,\ldots .$$

Consequently assumption (B2) implies $\lambda < 1$. If $\xi = 0$ and $(0,\psi) \notin N(F)$, the null space of F, then $z_0 = (0,\psi)$ implies

$$(2.47) \quad u_k^* = -Fz_k^* = -FS^k(0,\psi) = -\lambda^k F(0,\psi), \quad k = 0,1,2,\ldots$$

Since $u^* \in \ell_2(0,\infty;R^m)$, (2.47) implies $\lambda < 1$. Finally, if $(0,\psi) \in N(F)$, then

$$(2.48) \qquad \lambda(0,\psi) = S(0,\psi) = T(0,\psi) = (x(\tau;0,\psi,0),\ x_\tau(0,\psi,0))$$

which implies $\psi = 0$ and consequently that $\lambda \notin \sigma(S)$.

Therefore, we conclude that the spectral radius of S is less than 1 and that $|S^k| < \rho^k$ for all k sufficiently large for some positive $\rho < 1$. The uniform exponential bound (2.45) immediately follows.

The operator $F \in L(Z,R^m)$ can be represented by a matrix of operators, (F^0,F^1) where $F^0 \in L(R^n,R^m)$ can be represented by an m×n matrix f^0 and $F^1 \in L(L_2(-r,0;R^n),R^m)$ can be represented by a square integrable m×n matrix valued function f^1 defined on the interval $[-r,0]$. We have

$$(2.49) \qquad u_k^* = -f^0 x^*(k\tau) - \int_{-r}^{0} f^1(\theta) x_{k\tau}^*(\theta)d\theta, \quad k = 0,1,2,\ldots$$

with $(x^*(k\tau),\ x_{k\tau}^*) = z_k^*, \quad k = 0,1,2,\ldots$.

The Discrete-Time Problem with α-Shift

The shifted problem in discrete-time involves the finding of an optimal control u_α^* for which the resulting optimal trajectory z_α^* satisfies

$$(2.50) \qquad |z_{\alpha,k}^*| < \hat{M}\alpha^k, \quad k = 0,1,2,\ldots$$

where α, the prescribed degree of stability is a positive number less than 1. The modified discrete-time problem (analogous to problem $(\hat{P}1)$) takes the form

$(\hat{P}2)$ Find $\hat{u}^* \in \ell_2(0,\infty;R^m)$ which minimizes

$$(2.51) \qquad \hat{J}(\hat{u}) = \sum_{j=0}^{\infty} \langle Q\hat{z}_j,\hat{z}_j\rangle_Z + \hat{u}_j^T R\hat{u}_j \quad \text{where } \hat{z} = \{\hat{z}_k\}_{k=0}^{\infty} \text{ satisfies the recurrence}$$

$$(2.52) \qquad \hat{z}_{k+1} = \frac{1}{\alpha} T\hat{z}_k + \frac{1}{\alpha} B\hat{u}_k, \quad k = 0,1,2,\ldots$$

$$(2.53) \qquad \hat{z}_0 = (\eta,\phi) .$$

Assumptions analogous to (A2) and (B2) yield

$$(2.54) \qquad \hat{u}_k^* = -\hat{F}\hat{z}_k^*, \quad k = 0,1,2,...$$

where $\hat{F}$ is given by (2.39) - (2.41) with T and B replaced by $\frac{1}{\alpha}T$ and $\frac{1}{\alpha}$ B respectively. It follows that

$$(2.55) \qquad u_{\alpha,k}^* = -\hat{F}z_{\alpha,k}^*, \quad k = 0,1,2,...$$

with the optimal trajectory given by

$$(2.56) \qquad z_{\alpha,k}^* = S_\alpha^k(\eta,\phi), \quad k = 0,1,2,...$$

where $S_\alpha \varepsilon L(Z,Z)$ is defined by

$$(2.57) \qquad S_\alpha = T - B\hat{F} .$$

The operator S_α has spectral radius less than 1 and is uniformly exponentially bounded;

$$(2.58) \qquad |S_\alpha^k| < \hat{M}\alpha^k, \quad k = 0,1,2,..$$

where $\hat{M}$ is a positive constant which does not depend on k.

3. APPROXIMATION

The infinite dimensionality of problems (P1) (or $(\hat{P1})$) and (P2) (or $(\hat{P2})$) necessitates the use of some form of finite dimensional approximation to solve them. The standard approach involves the use of finite element (Rayleigh-Ritz, Galerkin, etc.) techniques to discretize the state equations (2.8) (or (2.27)) and (2.35) (or (2.52)). A sequence of finite dimensional LQR problems result, each of which can be solved in linear state feedback form using standard techniques and readily available software. The averaging , or AVE scheme, which uses piecewise constant elements with finite differencing, and its application to the continuous time problem is carefully studied in [7]. A linear spline based Galerkin method is treated in [4]. More recently, methods using piecewise linear elements [21] and Legendre polynomials [13] and a method based upon Lanczos' τ-method for partial differential equations which also uses Legendre polynomials [12] have yielded promising results.

While AVE yields strong L_2 convergence of the approximating feedback kernals, the observed rate of convergence is relatively slow. The spline based scheme, by some measures, offers superior performance. However, it appears that only weak L_2 convergence of the approximating feedback kernals can be obtained. Kappel and Salamon [14] have developed a new linear spline based method which performs at the level of the original spline scheme and which seems to yield strong L_2 convergence of the approximating functional feedback gains. It is this approximation scheme which is the focus of our discussions below.

3.1 AN APPROXIMATION SCHEME FOR LINEAR HEREDITARY SYSTEMS

We briefly outline the details of the formulation of the Kappel-Salamon scheme. Fundamental to their approach (and unlike the standard Galerkin approach) is the choosing of the approximating spaces so that they are not subspaces of either Dom(A) or Dom(A^*). Herein lies the key to obtaining strong L_2 convergence of the approximating feedback kernals.

For each N = 1,2,... let

$$\theta_N^{k,j} = -r_{k-1} - j\frac{\Delta r_k}{N}, \quad k = 1,2,\ldots,\nu, \quad j = 0,1,2,\ldots,N \tag{3.1}$$

where

$$\Delta r_k = r_k - r_{k-1}, \quad k = 1,2,\ldots,\nu \,. \tag{3.2}$$

Let $\{\phi_N^{k,j}\}_{j=0}^{N}$ denote the usual linear B-spline elements with respect to the mesh $\{\theta_N^{k,N},\ldots,\theta_N^{k,0}\}$ on the interval $[-r_k, -r_{k-1})$, k = 1,2,...N and extended to be zero elsewhere on the interval $[-r,0]$. That is for each k = 1,2,...,N

$$\phi_N^{k,0}(\theta) = \begin{cases} \frac{N}{\Delta r_k}(\theta - \theta_N^{k,1}), & \theta \in [\theta_N^{k,1}, \theta_N^{k,0}) \\ 0 & \text{elsewhere} \end{cases}$$

$$\phi_N^{k,j}(\theta) = \begin{cases} \frac{-N}{\Delta r_k}(\theta - \theta_N^{k,j-1}), & \theta \in [\theta_N^{k,j}, \theta_N^{k,j-1}] \\ \frac{N}{\Delta r_k}(\theta - \theta_N^{k,j+1}), & \theta \in [\theta_N^{k,j+1}, \theta_N^{k,j}] \\ 0 & \text{elsewhere} \end{cases} \tag{3.3}$$

$j = 1,2,\ldots,N-1$ and

$$\phi_N^{k,N}(\theta) = \begin{cases} \frac{-N}{\Delta r_k}(\theta - \theta_N^{k,N-1}), & \theta \in [\theta_N^{k,N}, \theta_N^{k,N-1}] \\ 0 & \text{elsewhere} . \end{cases}$$

Defining

$$(3.4) \qquad e_N^0 = (I_n, 0) \text{ and } e_N^{k,j} = (0, \phi_N^{k,j} I_n)$$

in $R^{n\times n} \times L_2(-r,0;R^{n\times n})$ where I_n denotes the $n\times n$ identity matrix, we let

$$(3.5) \qquad Z_N = \{(\xi,\psi_N) \in Z: (\xi,\psi_N) = e_N^0 a_0 + \sum_{k=1}^{\nu} \sum_{j=0}^{N} e_N^{k,j} a_{k,j},\ a_0,\ a_{k,j} \in R^n\}.$$

The collection $\{e_N^0, e_N^{k,j}\}$ is a basis for the $K_N = n((N+1)\nu + 1)$ dimensional subspace of Z, Z_N and $a = (a_0, a_{1,0},\ldots,a_{\nu,N})^T$ is referred to as the coordinate vector with respect to the basis $\{e_N^0, e_N^{k,j}\}$ for the element $(\xi,\psi_N) \in Z_N$. Defining

$$(3.6) \qquad E_N = (e_N^0, e_N^{1,0},\ldots,e_N^{\nu,N}),$$

we have $(\xi,\psi_N) = E_N a$.

Let $p_N: Z \to Z_N$ denote the orthogonal projection of Z onto Z_N. It is immediately clear that $p_N(\xi,\psi) = (\xi,\pi_N\psi)$ where π_N is the orthogonal projection of $L_2(-r,0;R^n)$ onto $\text{span}\{\phi_N^{k,j} I_n\}$.

Noting that $Z_N \not\subset \mathrm{Dom}(\mathcal{A})$, approximations $\mathcal{A}_N: Z_N \to Z_N$ to the operator $\mathcal{A}$ are defined by first extending $\mathcal{A}$ to all of Z_N. for $(\xi,\psi_N) \in Z_N$, define

$$(3.7) \qquad \mathring{\mathcal{A}}(\xi,\psi_N) = (\mathring{L}(\xi,\psi_N),\mathring{D}(\xi,\psi_N))$$

where

$$(3.8) \qquad \mathring{L}(\xi,\psi_N) = A_0\xi + \sum_{k=1}^{\nu} A_k\psi_N(-r_k) + \int_{-r}^{0} A(\theta)\psi_N(\theta)d\theta$$

and

$$(3.9) \qquad \mathring{D}(\xi,\psi_N) = D^+\psi_N + \delta_0(\xi - \lim_{\theta\to 0^-}\psi_N(\theta)) + \sum_{k=1}^{\nu-1} \delta_k(\psi_N(-r_k) - \lim_{\theta\to -r_k^-}\psi_N(\theta))$$

with δ_i the Dirac delta impulse centered at $-r_i$, $i = 0,1,2,\ldots\nu-1$ and $D^+\psi_N$ the derivative from the right of ψ_N.

Let $M_N \in \mathcal{L}(R^{K_N}, R^{K_N})$ be given by

$$(3.10) \qquad M_N = \langle E_N^T,E_N\rangle_Z$$

and define $\delta_N^{k,+}$, $\delta_N^{k,-} \in \mathcal{L}(R^n,Z_N)$ by

$$(3.11) \qquad \delta_N^{k,+}\xi = E_N\gamma_N^{k,+}\xi, \qquad k = 1,2,\ldots,\nu,$$

$$(3.12) \qquad \delta_N^{k,-}\xi = E_N\gamma_N^{k,-}\xi, \qquad k = 0,1,2,\ldots,\nu-1 ,$$

where

$$(3.13) \qquad \gamma_N^{k,+} = M_N^{-1}(0,\phi_N^{1,0}(-r_k),\ldots,\phi_N^{\nu,N}(-r_k))^T \otimes I_n,$$

$$(3.14) \qquad \gamma_N^{k,-} = M_N^{-1}(0, \lim_{\theta\to -r_k^-} \phi_N^{1,0}(\theta),\ldots, \lim_{\theta\to -r_k^-} \phi_N^{\nu,N}(\theta))^T \otimes I_n$$

and $\otimes$ denotes the Kronecker product. The approximating operators $A_N : Z_N \to Z_N$ and their adjoints are given by

$$(3.15) \qquad A_N(\xi,\psi_N) = (\mathring{L}(\xi,\psi_N),\pi_N D^+\psi_N) + \delta_N^{0,-} (\xi - \lim_{\theta\to 0^-} \psi_N(\theta))$$
$$+ \sum_{k=1}^{\nu-1} \delta_N^{k,-} (\psi_N(-r_k) - \lim_{\theta\to -r_k^-} \psi_N(\theta))$$

and

$$(3.16) \qquad A_N^*(\xi,\psi_N) = (\lim_{\theta\to 0^-} \psi_N(\theta) + A_0^T\xi, \ \pi_N(A^T\xi - D^+\psi_N))$$
$$+ \sum_{k=1}^{\nu-1} \delta_N^{k,+} (A_k^T\xi + \lim_{\theta\to -r_k^-} \psi_N(\theta) - \psi_N(-r_k))$$
$$+ \delta_N^{\nu,+} (A_\nu^T\xi - \psi_N(-r))$$

respectively.

3.2 THE APPROXIMATE SOLUTION OF THE REQULATOR PROBLEMS

The Continuous-Time Problem

$$(3.17) \qquad \mathcal{B}_N = p_N \mathcal{B} |_{Z_N} = \mathcal{B}|_{Z_N}$$

and

(3.18) $$Q_N = p_N Q|_{Z_N} = Q|_{Z_N}$$

and assume that problem (P1) with A, B, Q and (η,ϕ) replaced by A_N, B_N, Q_N and $p_N(\eta,\phi)$ satisfies assumptions analogous to (A1) and (B1). (Under certain conditions, if the original system satisfies (A1) and (B1) so too will the approximating systems if N is sufficiently large, see [14].) The approximating solutions to problem (P1) are then given in feedback form by

(3.19) $$u_N^*(t) = -R^{-1} B_N^* P_N p_N z_N^*(t), \quad t \geq 0$$

where P_N is the unique nonnegative self-adjoint solution to the Riccati algebraic equation

(3.20) $$A_N^* P_N + P_N A_N + P_N B_N R^{-1} B_N^* P_N + Q_N = 0$$

and z_N^* is given by

(3.21) $$z_N^*(t) = S^N(t)(\eta,\phi), \quad t \geq 0$$

where $\{S^N(t) : t \geq 0\}$ is the C_0 semigroup with infinitesimal generator $A - BR^{-1} B_N^* P_N p_N$.

In practice, the approximating feedback gains are computed by solving the K_N dimensional matrix Riccati algebraic equation

(3.22) $$[A_N]^T \Pi_N + \Pi_N [A_N] + \Pi_N [B_N] R^{-1} [B_N]^T \Pi_N + Q_N = 0$$

where brackets denote the operator's matrix representation with respect to the basis $\{e_N^0, e_N^{k,j}\}$ and the matrix Π_N is given by

$$(3.23) \qquad \Pi_N = M_N[P_N].$$

Then, if we write

$$(3.24) \qquad [P_N] = \left(\begin{array}{c|c} P_N^0 & \\ P_N^{1,0} & \\ \vdots & \tilde{P}_N \\ P_N^{\nu,N-1} & \\ P_N^{\nu,N} & \end{array}\right)$$

where P_N^0 and $P_N^{k,j}$, $k = 1,2,\ldots,\nu$, $j = 0,1,2,\ldots,N$ are $n\times n$ matrices, we have

$$(3.25) \qquad u_N^*(t) = p_N^0\, x_N^*(t) - \int_{-r}^{0} p_N^1(\theta)(x_N^*)_t(\theta)d\theta, \quad t > 0$$

with

$$(3.26) \qquad p_N^0 = R^{-1}B_0^T P_N^0 \;,\; p_N^1(\theta) = \sum_{k=1}^{\nu} \sum_{j=0}^{N} R^{-1}B_0^T P_N^{k,j} \phi_N^{k,j}(\theta), \quad -r < \theta < 0$$

and $(x_N^*(t),(x_N^*)_t) = z_N^*(t)$, $\quad t > 0$.

The Discrete-Time Problem

For the discrete-time problem, we let

(3.27) $\quad T_N = \mathcal{T}_N(\tau) \quad \text{and} \quad B_N = \int_0^\tau \mathcal{T}_N(t)\, \mathcal{B}_N dt$

where $\mathcal{B}_N$ is given by (3.17) and $\{\mathcal{T}_N(t) : t \geq 0\}$ is the C_0 semigroup with infinitesimal generator $\mathcal{A}_N$. The approximating solutions to problem (P2) are then given in feedback form by

(3.28) $\quad u^*_{N,k} = - F_N P_N z^*_{N,k}, \quad k = 0,1,2,\ldots$

where

(3.29) $$F_N = \tilde{R}_N^{-1} B_N^* P_N T_N ,$$

(3.30) $$\tilde{R}_N = R + B_N^* P_N B_N,$$

P_N is the unique, nonnegative, self-adjoint solution to the Riccati algebraic equation

(3.31) $\quad P_N = T_N^* \, (P_N - P_N B_N (R + B_N^* P_N B_N)^{-1} B_N^* \, P_N) T_N + Q_N$

and z^*_N is given by

(3.32) $$z^*_{N,k} = (S^N)^k(\eta,\phi), \quad k = 0,1,2,\ldots$$

with

$$S^N = T - BF_N P_N . \tag{3.33}$$

The approximating feedback kernals can be computed from

$$[F_N] = [\tilde{R}_N]^{-1}[B_N]^T \Gamma_N [T_N] , \tag{3.34}$$

and

$$[R_N] = R + [B_N]^T \Gamma_N [B_N] \tag{3.35}$$

where

$$[T_N] = \exp\,([A_N]\tau), \quad [B_N] = \int_0^\tau \exp([A_N]t)[B_N]dt \tag{3.36}$$

and Γ_N is the unique nonnetative symmetric solution to the K_N dimensional matrix Riccati algebraic equation

$$\Gamma_N = [T_N]^T(\Gamma_N - \Gamma_N[B_N](R + [B_N]^T\Gamma_N[B_N])^{-1}[B_N]^T\Gamma_N[T_N] + [Q_N] . \tag{3.37}$$

If we set

$$[F_N]M_N^{-1} = (F_N^0 \, , \; F_N^{1,0}, \ldots, F_N^{\nu,N}) \tag{3.38}$$

where F_N^0 and $F_N^{k,j}$, $k = 1,2,\ldots,\nu$, $j = 0,1,2,\ldots,N$ are m×n matrices, we have

$$u^*_{N,k} = -f^0_N\, x^*_N(k\tau) - \int_{-r}^{0} f^1_N(\theta)(x^*_N)_{k\tau}(\theta)d\theta, \quad k = 0,1,2,\ldots \tag{3.39}$$

with

$$f^0_N = F^0_N\ ,\ f^1_N(\theta) = \sum_{k=1}^{\nu} \sum_{j=0}^{N} F^{k,j}_N\ \phi^{k,j}_N(\theta), \quad -r \leq \theta \leq 0 \tag{3.40}$$

and $(x^*_N(k\tau),\ (x^*_N)_{k\tau}) = z^*_{N,k}$, $\quad k = 0,1,2,\ldots$.

3.3 CONVERGENCE

The Continuous-Time Problem

Elementary approximation properties of spline functions and the Trotter-Kato Theorem on the approximation of semigroups (stability together with consistency imply convergence, see [15], [20]) can be used to argue that

$$\mathcal{T}_N(t) \to \mathcal{T}(t) \text{ and } \mathcal{T}^*_N(t) \to \mathcal{T}^*(t), \quad t \geq 0 \tag{3.41}$$

strongly on Z as $N \to \infty$, uniformly in t for t in bounded sub-intervals, where $\{\mathcal{T}^*_N(t) : t \geq 0\}$ and $\{\mathcal{T}^*(t) : t \geq 0\}$ are the C_0 semigroups with infinitesimal generators $\mathcal{A}^*_N$ and $\mathcal{A}^*$ respectively. Observing (numerically) that $|\mathcal{P}_N|$ is bounded in N, it follows (see [7], Theorem 6.7) that $\mathcal{P}_N$ converges weakly to $\mathcal{P}$ as $N \to \infty$ and consequently that $p^0_N \to p^0$ in $R^{m\times n}$ and $p^1_N \to p^1$ weakly in $L_2(-r_0;\ R^{m\times n})$ as $N \to \infty$. To obtain strong convergence of $\mathcal{P}_N$ to $\mathcal{P}$ and strong L_2 convergence of p^1_N to p^1, the only known result (see [7], Theorem 6.9) requires the existence of positive constants M and ω, independent of N, for which

$$(3.42) \qquad |S_N(t)| \leq Me^{-\omega t}, \quad t \geq 0,\ N \geq 1$$

where $\{ S_N(t) : t \geq 0\}$ is the C_0 semigroup with infinitesimal generator $A_N - B_N R^{-1} B_N^* P_N p_N$. While all numerical results indicate that strong convergence of the approximating feedback kernals holds, analysis in [14] and numerical studies point to the fact that a uniform exponential bound of the form (3.42) can not be obtained for the Kappel-Salamon scheme. Indeed, both the open and closed-loop approximating systems yield a sequence of extraneous eigenvalues (i.e. ones which do not appear to be converging to an element of the spectrum of the original open or closed-loop herediatary system) $\{\lambda_N\}_{N=1}^{\infty}$ for which Re $\lambda_N \to 0^-$ and Im $\lambda_N \to +\infty$ (or $-\infty$) as $N \to \infty$.

We note that the N-independent uniform exponential bound (3.42) which is sufficient for strong (in fact trace norm, see [7]) convergence of P_N to P has been shown to hold for the AVE scheme in [22] and for the Legendre-tau method in [11].

The Discrete-Time Problem

For the discrete-time problem (see [8], Theorem 3.12) $|P_N|$ bounded in N implies $P_N \to P$ weakly, $F_N \to F$ strongly, $f_N^0 \to f^0$ in $R^{m\times n}$ and $f_N^1 \to f^1$ weakly in $L_2(-r,0;R^{m\times n})$ as $N \to \infty$. The existence of positive constants M and ρ which do not depend on N, with $\rho < 1$ and for which

$$(3.43) \qquad |S_N^k| \leq M\rho^k, \ k = 0,1,2,\ldots, \qquad N \geq 1,$$

where

$$(3.44) \qquad S_N = T_N - B_N F_N p_N$$

is sufficient (see [8], Theorem 3.10) to conclude strong convergence of P_N to P, uniform norm convergence of F_N to F and strong L_2 convergence of f_N^1 to f^1 as $N \to \infty$. Although numerical studies indicate that the stronger modes of convergence hold, the approximating open and closed-loop discrete-time systems constructed using the Kappel-Salamon scheme yield a sequence of extraneous eigenvalues $\{\mu_N\}_{N=1}^{\infty}$ with $|\mu_N| \to 1^-$ as $N \to \infty$.

3.4 THE APPROXIMATE SOLUTION OF THE REGULATOR PROBLEMS WITH α-SHIFT

One obvious approach for approximating the solutions to the LQR problems with α-shift is to replace the operators $\mathcal{A}$,$\mathcal{B}$ and Q in problem $(\hat{P}1)$ with the operators $\mathcal{A}_N$, $\mathcal{B}_N$ and Q_N or the operators T, B and Q in problem $(\hat{P}2)$ with T_N, B_N and Q_N and then solve the finite dimensional shifted problems with states given either by

$$(3.45) \qquad \frac{d}{dt}\hat{z}_N(t) = (\mathcal{A}_N + \alpha I)\hat{z}_N(t) + \mathcal{B}_N\hat{u}(t), \quad t > 0$$

or

$$(3.46) \qquad \hat{z}_{N,k+1} = \frac{1}{\alpha} T_N \hat{z}_{N,k} + \frac{1}{\alpha} B_N\hat{u}_k, \quad k = 0,1,2,\ldots .$$

However, if the Kappel-Salamon scheme outlined above is used, this approach will not work. Indeed, our numerical studies indicate that as a result of the extraneous eigenvalues introduced by the approximation scheme, for N and α sufficiently large the resulting finite dimensional systems are, at best, marginally stabilizable. The solutions to the matrix Riccati algebraic equations begin to deteriorate. Eventually the eigenvalue-eigenvector or Schur vector methods used to solve them fail completely. We observe this independently of the stabilizability of the original underlying hereditary system. Since only the poles of the original hereditary system are to be shifted, this situation can be remedied by observing that the operator $\mathcal{A} + \alpha I$ is related to the intinitesimal generator for a hereditary system (semigroup) through a bounded similarity transformation.

For γ a real number, define $U_\gamma \in \mathcal{L}(Z,Z)$ by

$$(3.47) \qquad U_\gamma(\xi,\psi) = (\xi, e^{\gamma\cdot}\psi) ,$$

where the function $e^{\gamma\cdot}\psi$ evaluated at θ is $e^{\gamma\theta}\psi(\theta)$, $r \leq \theta \leq 0$. Then

$$(3.48) \qquad U_\gamma^{-1}(\xi,\psi) = (\xi, e^{-\gamma\cdot}\psi).$$

The Continuous-Time Problem

For the continuous-time problem, set $\hat{w}(t) = U_\alpha \hat{z}(t)$. Then $\hat{w}$ satisfies

$$(3.49) \qquad \frac{d}{dt}\hat{w}(t) = \mathcal{A}_\alpha \hat{w}(t) + \mathcal{B}\hat{u}(t), \qquad t > 0$$

$$(3.50) \qquad \hat{w}(0) = (\eta, e^{\alpha\cdot}\phi)$$

where for $\gamma \in R$, $\mathcal{A}_\gamma$: $\mathrm{Dom}(\mathcal{A}) \subset Z \to Z$ is given by

$$(3.51) \qquad \mathcal{A}_\gamma(\psi(0), \psi) = U_\gamma(\mathcal{A} + \gamma I)\, U_\gamma^{-1}(\psi(0), \psi) = (L_\gamma\psi, D\psi)$$

with

$$(3.52) \qquad L_\gamma \psi = (A_0 + \gamma I)\psi(0) + \sum_{i=1}^{\nu} e^{\gamma r_i} A_i \psi(-r_i) + \int_{-r}^{0} e^{-\gamma\theta} A(\theta)\psi(\theta)d\theta \ .$$

Since U_γ^{-1} is self-adjoint and $U_\gamma^{-1} Q\, U_\gamma^{-1} = Q$ for all $\gamma \in R$, the Kappel-Salamon scheme can be applied to problem $(\hat{P}1)$ with $\mathcal{A} + \alpha I$ replaced with $\hat{\mathcal{A}}_\alpha$ and $\hat{z}$ replaced with $\hat{w}$. The extraneous eigenvalues will now remain stable and therefore cause no problems when the approximating feedback laws are computed. The approximating solutions to the continuous-time problem with α-shift are given by

$$(3.53) \qquad u_{\alpha N}^*(t) = -R^{-1}\, \mathcal{B}_N^*\, \hat{\mathcal{P}}_{\alpha N} p_N z_{\alpha N}^*(t) = -R^{-1}\, \mathcal{B}_N^*\, \mathcal{P}_{\alpha N} p_N U_\alpha z_{\alpha N}^*(t), \qquad t > 0$$

where $\mathcal{P}_{\alpha N}$ is the solution to the Riccati algebraic equation (3.20) with $\mathcal{A}_N$ and $\mathcal{A}_N^*$ replaced by $\mathcal{A}_{\alpha N}$ and $\mathcal{A}_{\alpha N}^*$ respectively and $z_{\alpha N}^*$ is given by

$$(3.54) \qquad z_{\alpha N}^*(t) = \mathcal{S}_\alpha^N(t)(\eta,\phi), \qquad t > 0$$

where $\{\, \mathcal{S}_\alpha^N(t) : t > 0\}$ is the C_0 semigroup with infinitesimal generator $\mathcal{A} - \mathcal{B}R^{-1}\, \mathcal{B}_N^*\, \hat{\mathcal{P}}_{\alpha N} p_N$. We have

$$(3.55) \qquad u_{\alpha N}^*(t) = -\hat{p}_{\alpha N}^0 x_{\alpha N}^*(t) - \int_{-r}^{0} \hat{p}_{\alpha N}^1(\theta)(x_{\alpha N}^*)_t(\theta)d\theta$$

$$= -p_{\alpha N}^0\, x_{\alpha N}^*(t) - \int_{-r}^{0} p_{\alpha N}^1(\theta)e^{\alpha\theta}(x_{\alpha N}^*)_t(\theta)d\theta, \qquad t > 0$$

where $p_{\alpha N}^0$ and $p_{\alpha N}^1$ are obtained from $\mathcal{P}_{\alpha N}$ in the same manner that p_N^0 and p_N^1 are obtained from $\mathcal{P}_N$ (as in (3.2.6)) and $(x_{\alpha N}^*(t), (x_{\alpha N}^*)_t) = z_{\alpha N}^*(t), \quad t > 0$.

The Discrete-Time Problem

For the discrete-time problem, we set $\hat{w}_k = U_\beta \hat{z}_k$. Then

$$(3.56) \qquad \hat{w}_{k+1} = T_\alpha \hat{w}_k + B_\alpha \hat{u}_k, \quad k = 0,1,2,\ldots$$

$$(3.57) \qquad \hat{w}_0 = (\eta, e^{\beta \cdot} \phi)$$

where

$$(3.58) \qquad T_\alpha = \frac{1}{\alpha} U_\beta T U_\beta^{-1} = \mathcal{T}_\beta(\tau),$$

$\{\mathcal{T}_\beta(t) : t \geq 0\}$ the C_0 semigroup whose infinitesimal generator is $\mathcal{A}_\beta$,

$$(3.59) \qquad B_\alpha = \frac{1}{\alpha} \int_0^\tau e^{-\beta t}\, \mathcal{T}_\beta(t)\, \mathcal{B}\, dt$$

and $\beta = -(\ln\alpha)/\tau$. The approximating solutions to the discrete-time problem with α-shift are given by

$$(3.60) \quad u^*_{\alpha N,k} = -\hat{F}_{\alpha N} p_N z^*_{\alpha N,k} = -F_{\alpha N} p_N U_\beta z^*_{\alpha N,k}, \quad k = 0,1,2,\ldots$$

where $F_{\alpha N}$ is computed according to (3.29)-(3.31) with F_N, $\tilde{R}_N$, B_N, T_N and P_N replaced by $F_{\alpha N}$, $\tilde{R}_{\alpha N}$, $B_{\alpha N}$, $T_{\alpha N}$ and $P_{\alpha N}$ respectively and $z^*_{\alpha N}$ is given by

$$(3.61) \qquad z^*_{\alpha N,k} = (S^N_\alpha)^k(\eta,\phi), \quad k = 0,1,2,\ldots$$

with

$$(3.62) \qquad S_\alpha^N = T - B\hat{F}_{\alpha N} p_N.$$

Finally, we have

$$(3.63) \quad \overset{*}{u}_{\alpha N,k} = -\hat{f}^0_{\alpha N} \overset{*}{x}_{\alpha N}(k\tau) - \int_{-r}^0 \hat{f}^1_{\alpha N}(\theta)(\overset{*}{x}_{\alpha N})_{k\tau}(\theta)d\theta$$

$$= -f^0_{\alpha N}\ \overset{*}{x}_{\alpha N}(k\tau) - \int_{-r}^0 f^1_{\alpha N}(\theta)e^{\beta\theta}(\overset{*}{x}_{\alpha N})_{k\tau}(\theta)d\theta, \quad k = 0,1,2,\ldots$$

where $f^0_{\alpha N}$ and $f^1_{\alpha N}$ are obtained from $F_{\alpha N}$ as were f^0_N and f^1_N from F_N in (3.40) and $(\overset{*}{x}_{\alpha N}(k\tau),\ (\overset{*}{x}_{\alpha N})_{k\tau}) = \overset{*}{z}_{\alpha N,k}$, $k = 0,1,2,\ldots$.

4. AN EXAMPLE AND NUMERICAL RESULTS

We consider the second order linear harmonic oscillator with delayed damping given by

$$(4.1)\qquad \ddot{y}(t) + \dot{y}(t-1) + y(t) = u(t), \quad t > 0 .$$

We take the continuous-time performance index to be

$$(4.2)\qquad J(u) = \int_0^\infty y(t)^2 + \dot{y}(t)^2 + u(t)^2 dt .$$

Setting $x(t) = (y(t), \dot{y}(t))^T$, we rewrite (4.1) as a first order system;

$$(4.3)\qquad \dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x(t-1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \quad t > 0 .$$

For this example we have $n = 2$, $m = 1$, $r = 1$, $\nu = 1$, $A \equiv 0$,

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $R = 1$.

We computed the optimal feedback gains for the shifted and unshifted, continuous and discrete-time control problems on an IBM PC personal computer using the Kappel-Salamon approximation scheme and the α-shift technique outlined in the previous section. The matrix Riccati algebraic equations (3.22) and (3.37) were solved using either a standard eigenvalue/eigenvector (see [16]) or Schur vector (see [17], [19]) decomposition of the Hamiltonian matrix. For the discrete-time problem, matrix exponentials were computed using an eigenvalue/eigenvector decomposition.

A Scheme to Compute Eigenvalues of Linear Hereditary Systems

In order to evaluate the performance of the method, we had to be able to compute approximations to the closed-loop eigenvalues of continuous-time and sampled hereditary systems. To do this we used a spline-based scheme developed in [10]. In the case of the continuous-time problem, the closed-loop system is a homogeneous hereditary system. Let $\mathcal{A}^N = \mathcal{A} - \mathcal{B}R^{-1}\mathcal{B}_N^* \hat{P}_{\alpha N}p_N$ denote the infinitesimal generator of the closed-loop semigroup $\{S^N(t) : t \geq 0\}$ and let $\{Z^M\}$ denote a sequence of finite dimensional spline-based subspaces of Z which are contained in Dom($\mathcal{A}$). Let $q^M : Z \to Z^M$ denote the orthogonal projection of Z onto Z^M with respect to the inner product

$$(4.4) \qquad \langle\langle(\xi,\psi), (\zeta,\chi)\rangle\rangle_Z = \langle\mathcal{A}^N(\xi,\psi), \mathcal{A}^N(\zeta,\chi)\rangle_Z \ .$$

An approximation to the spectrum of $\mathcal{A}^N$ is obtained by computing the eigenvalues of the matrix representation of the inverse of the operators

$$(4.5) \qquad q^M(\mathcal{A}^N)^{-1}\big|_{Z^M} \ .$$

Spectral convergence is argued in [10] using the theory of collectively compact families of operators (see [3]).

The approach outlined above is used to obtain approximations T^M and B^M to the discrete-time open-loop state transition operator T and input operator B. The feedback gains $\hat{F}_{\alpha N}p_N$ are projected (with respect to the standard Z inner product) onto Z^M to obtain the operators $\hat{F}^M_{\alpha N}$. The eigenvalues of the operator $T^M - B^M\hat{F}^M_{\alpha N}$ are taken to be an approximation to the closed-loop spectrum of the discrete-time system.

Numerical Findings

The eigenvalues in the examples which follow were computed using the method we have described above with quintic splines and M taken large enough to declare a sufficient number of the eigenvalues converged. Typically, taking M = 30, which results in a 70 dimensional eigenvalue problem, sufficed to yield 21 converged eigenvalues. The resulting matrix eigenvalue problems were solved using IMSL routines EIGRF (the QR method for the standard eigenvalue problem) or EIGZF (the QZ method for the generalized problem). These computations were performed in double precision on the IBM 3081 at the University of Southern California.

The eigenvalues of $\mathcal{A}_N$, the Nth Kappel-Salamon approximation to $\mathcal{A}$, with real part greater than -3.5 (ordered by decreasing real part) are given in Tables 4.2 and 4.3 for various values of N. The first nine "true" continuous-time open-loop eigenvalues (eigenvalues of the operator $\mathcal{A}$) can be found in Table 4.1. Upon careful inspection of Tables 4.1, 4.2 and 4.3, one can easily discern the true eigenvalues of the hereditary system emerging and observe the behavior of the extraneous, artifactual eigenvalues which was described in the previous section as N increases.

For the present example, we used the schemes described in Section 3 to compute the continuous-time feedback gains, $\hat{\mathcal{P}}^0_{\alpha N}$ and $\hat{\mathcal{P}}^1_{\alpha N}$ for α = 0, 2.0 and 3.0, and the discrete-time gains $\hat{f}^0_{\alpha N}$ and $\hat{f}^1_{\alpha N}$ for α = 1.0, .98 and .975. As α is increased, larger values of N are necessary to ensure that the approximating optimal feedback laws have essentially converged. The results presented below were computed with N = 10. The scalar gains $\hat{\mathcal{P}}^0_{\alpha N}$ and $\hat{f}^0_{\alpha N}$ are given in Table 4.4 and 4.6 respectively. The kernals or functional gains $(\hat{\mathcal{P}}^1_{\alpha N})_2$ and $(\hat{f}^1_{\alpha N})_2$ (where $(\ )_j$, j = 1,2 denotes the j^{th} component) are plotted in Figures 4.5 and 4.7. Note that the initial conditions

(4.6) $\quad \eta = (0,0)^T,\ (\phi)_1$ arbitrary, $(\phi)_2 = 0$

for the system (4.3) yield $x(t) = 0,\ t > 0$ and consequently that the optimal control is $u(t) = 0,\ t > 0$. This will also be true for the corresponding "α-shifted" systems. It therefore immediately follows from this observation and the basic structure of the finite dimensional approximating systems that the true and approximating, continuous and discrete-time optimal control laws do not feedback displacement history; that is

(4.7) $$(\hat{\mathcal{P}}^1_{\alpha N})_1 = (\hat{\mathcal{P}}^1_{\alpha})_1 = (\hat{f}^1_{\alpha N})_1 = (\hat{f}^1_{\alpha})_1 = 0$$

for all N.

The resulting closed-loop eigenvalues for the continuous-time systems are plotted in Figure 4.8 and are tabulated for the discrete-time systems in Table 4.9. In the discrete-time example, the length of the sampling interval τ was taken to be .01.

i	λ_i
1,2	.0219 ± 1.6020i
3	-.7384
4.5	-2.0469 ± 7.5820i
6,7	-2.6484 ± 13.9477i
8,9	-3.0179 ± 20.2719i

TABLE 4.1: EIGENVALUES OF A

	N = 4	N = 6	N = 8
1,2	.0220 ± 1.6017i	.0219 ± 1.6019i	.0219 ± 1.6019i
			-.5833 ± 12.7756i
			-.6503 ± 12.7819i
3	-.7384	-.7384	-.7384
	-1.2679 ± 6.1063i	-.9824 ± 9.2136i	
	-1.5000 ± 5.8095i	-1.1564 ± 9.1174i	
	-3.4286 ± 2.9692i		
4,5		-2.0876 ± 7.0492i	-2.1251 ± 7.4637i
		-2.8600 ± 6.3173i	-2.1392 ± 9.9917i
			-2.8084 ± 10.0431i

TABLE 4.2: EIGENVALUES OF A_N

	N = 10	N = 20	N = 30
1,2	.0219 ± 1.6019i	.0219 ± 1.6020i	.0219 ± 1.6020i
	-.4513 ± 16.3731i	- .1227 ± 34.0989i	- .0575 ± 51.5897i
	-.4681 ± 16.4023i	- .1265 ± 34.0977i	- .0576 ± 51.5885i
		- .4868 ± 32.5168i	- .2258 ± 50.4897i
		- .5013 ± 32.5050i	- .2264 ± 50.4946i
			- .4869 ± 48.6959i
			- .4938 ± 48.7064i
3	- .7384	- .7384	- .7384
	-1.3706 ± 13.7131i	-1.0176 ± 30.0843i	- .8459 ± 46.3080i
	-1.5800 ± 13.8084i	-1.0330 ± 30.0323i	- .8707 ± 46.3184i
		-1.6310 ± 26.7096i	-1.2170 ± 43.3964i
		-1.7094 ± 26.8311i	-1.2651 ± 43.3778i
			-1.7362 ± 40.0062i
			-1.7657 ± 39.9081i
4,5	-2.0765 ± 7.5469i	-2.0483 ± 7.5804i	-2.0471 ± 7.5817i
	-2.9685 ± 10.3102i	-2.4723 ± 23.1169i	-2.2485 ± 36.2831i
			-2.3628 ± 36.4404i
			-2.4687 ± 32.3647i
6,7		-2.5840 ± 13.9956i	-2.6381 ± 13.9567i
		-2.7313 ± 19.6455i	-2.7968 ± 32.2934i
		-2.8700 ± 23.2028i	
		-3.3004 ± 19.0864i	
8,9			-2.9769 ± 20.1502i
			-3.3039 ± 27.2171i
			-3.3823 ± 28.1149i

TABLE 4.3: EIGENVALUES OF A_N

	$\alpha = 0.0$	$\alpha = 2.0$	$\alpha = 3.0$
$(\hat{p}^{0}_{\alpha 10})_1$	.4142	31.8304	129.1725
$(\hat{p}^{0}_{\alpha 10})_2$	1.4291	10.8678	21.6132

TABLE 4.4: SCALAR GAINS - CONTINUOUS-TIME

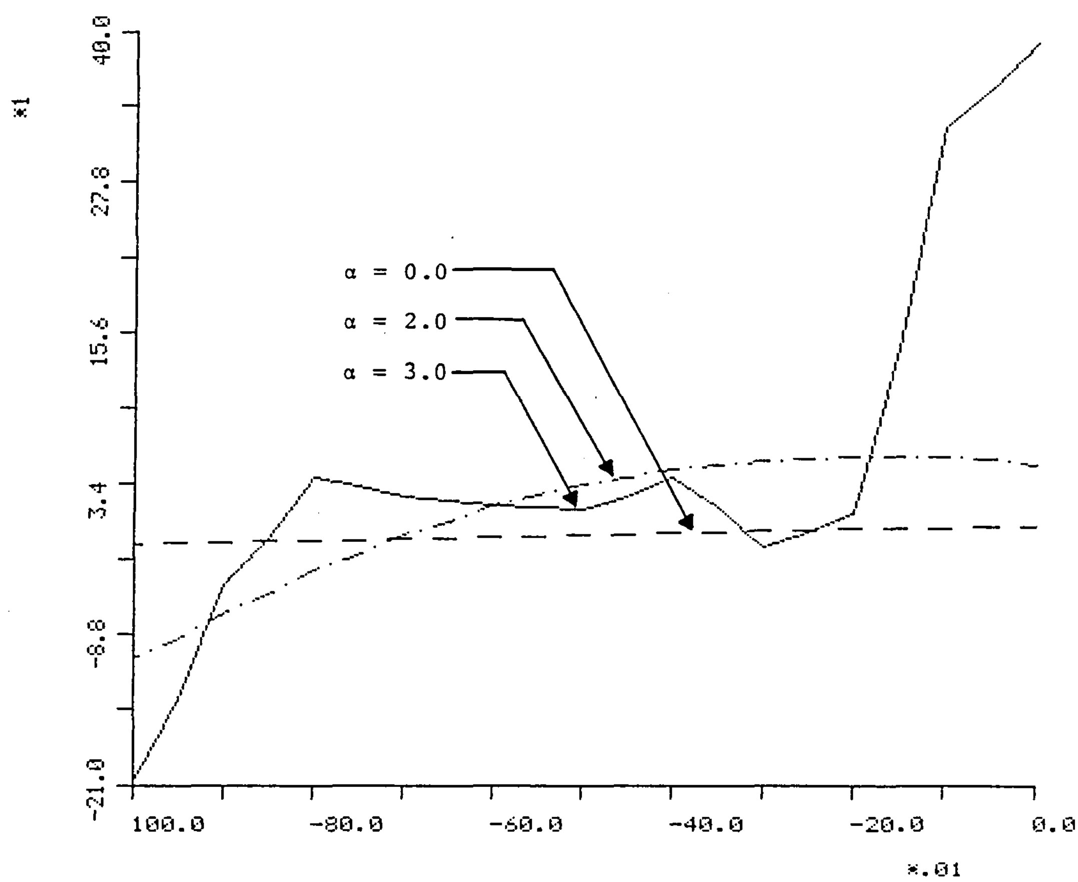

FIGURE 4.5: FUNCTIONAL GAINS - CONTINUOUS-TIME

	α = 1.0	α = .98	α = .975
$(\hat{f}^0_{\alpha 10})_1$	.4041	30.9821	63.8527
$(\hat{f}^0_{\alpha 10})_2$	1.4215	10.5734	14.9272

TABLE 4.6: SCALAR GAINS - DISCRETE-TIME

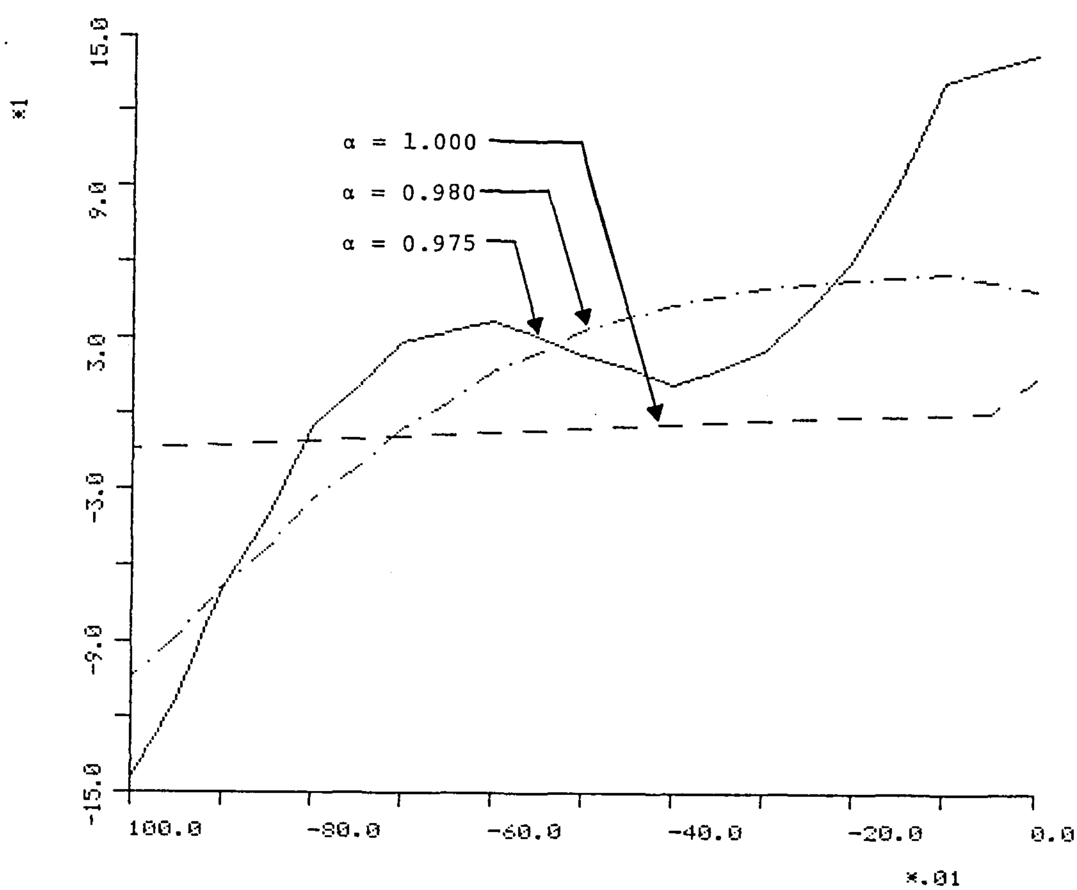

FIGURE 4.7: FUNCTIONAL GAINS - DISCRETE-TIME

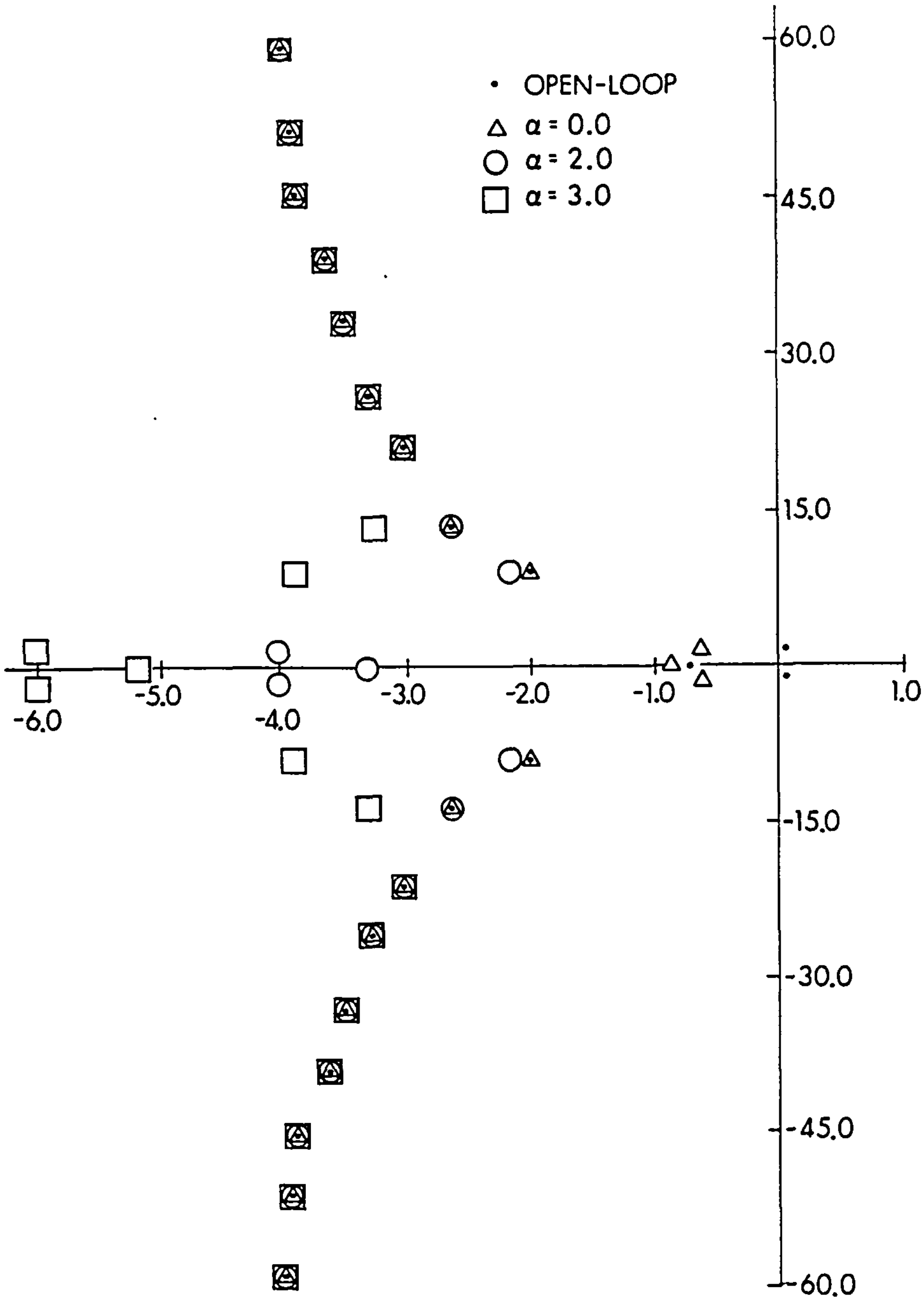

FIGURE 4.8: OPEN AND CLOSED-LOOP SPECTRUM - CONTINUOUS TIME; EIGENVALUES OF A AND $A - BR^{-1}B_{10}^{*}\hat{P}_{\alpha 10}$

OPEN-LOOP		$\alpha = 1.0$		$\alpha = .98$		$\alpha = .975$	
MAG	ARG	MAG	ARG	MAG	ARG	MAG	ARG
.9926	0	.9919	0	.9693	0	.9630	0
1.0002	± .0160	.9938	± .0170	.9594	± .0177	.9484	± .0211
.9797	± .0758	.9796	± .0759	.9787	± .0758	.9702	± .0757
.9739	± .1395	.9738	± .1395	.9738	± .1395	.9737	± .1395
.9703	± .2027	.9702	± .2027	.9703	± .2027	.9703	± .2027
.9677	± .2658	.9677	± .2658	.9677	± .2658	.9678	± .2657
.9656	± .3288	.9656	± .3288	.9656	± .3288	.9657	± .3288
.9639	± .3918	.9639	± .3918	.9639	± .3918	.9639	± .3918
.9625	± .4547	.9625	± .4547	.9625	± .4538	.9625	± .4546
.9612	± .5176	.9612	± .5176	.9613	± .5176	.9613	± .5175
.9599	± .5806	.9599	± .5806	.9600	± .5806	.9602	± .5805

TABLE 4.9: OPEN AND CLOSED-LOOP SPECTRUM - DISCRETE-TIME;
EIGENVALUES OF T AND $T - B\hat{F}_{\alpha 10}$

REFERENCES

[1] B.D.O. Anderson and J.B. Moore, Linear system optimization with prescribed degree of stability, Proc. IEEE, 116 (1969), 2083-2087.

[2] B.D.O. Anderson and J.B. Moore, Linear Optimal Control, Prentice Hall, Englewood Cliffs, N.J., 1971.

[3] P.M. Anselone, Collecltively Compact Operator Approximation Theory and Applications to Integral Equations, Prentice Hall, Englewood Cliffs, N.J., 1971.

[4] H.T. Banks, I.G. Rosen and K. Ito, A spline based technique for computing Riccati operators and feedback controls in regulator problems for delay equations, SIAM J. Sci. Stat. Comput., 5 (1984), 830-855.

[5] N. Dunford and J.T. Schwartz, Linear operators, Part I: General Theory, Wiley-Interscience, New York, 1957.

[6] J.S. Gibson, The Riccati integral equations for optimal control problems in Hilbert spaces, SIAM J. Control and Opt. 17 (1979), 537-565.

[7] J.S. Gibson, Linear quadratic optimal control of hereditary differential systems: Infinite dimensional Riccati equations and numerical approximations, SIAM J. Control and Opt. 21(1983), 95-139.

[8] J.S. Gibson and I.G. Rosen, Numerical approximation for the infinite-dimensional discrete-time optimal linear-quadratic regulator problem, ICASE Report No. 86-15, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA, 1986. (Available as NASA CR-178081.)

[9] J. Hale, Theory of Functional Differential Equations, Springer Verlag, New York, 1977.

[10] K. Ito, On the approximation of eigenvalues associated with functional differential equations, J. Diff. Equ., 60 (1985), 285-300.

[11] K. Ito, Legendre-tau approximation for functional differential equations, Part III: Eigenvalue approximations and uniform stability, ICASE Report No. 84-42, Institute for Computer Applications in Science and Engineering , NASA Langley Research Center, Hampton, VA, 1984. (Available as NASA CR-172447.)

[12] K. Ito and R. Teglas, Legendre-tau approximation for functional differential equations, Part II: The linear quadratic optimal control problem, ICASE Report No. 84-31, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA. 1984. (Available as NASA CR-172397.)

[13] F. Kappel and G. Probst, Approximation of Feedback controls for delay systems using Legendre polynomials, Proceedings, Conferenze del Seminario Di Matematica Dell'Universita Di Bari, Bari, Italy 1984.

[14] F. Kappel and D. Salamon, Spline Approximation for retarded systems and the Riccati Equation, SIAM J. Control and Opt., (1986), to appear.

[15] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1966.

[16] H. Kwakernaak and R. Sivan, Linear Optimal Control Systems, Wiley-Interscience, New York, 1972.

[17] A.J. Laub, A Schur method for solving algebraic Riccati equations, IEEE Trans. Automat. Contr., AC-24 (1979), 914-921.

[18] K.Y. Lee, S. Chow and R.O. Barr, On the control of discrete-time distributed parameter systems, SIAM J. Control and Opt., 10(1972), 361-376.

[19] T. Pappas, A.J. Laub and N.R. Sandell, Jr., On the numerical solution of the discrete-time algebraic Riccati equation, IEEE Trans. Automat. Contr., AC-25, (1980), 631-641.

[20] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differntial Equations, Springer-Verlag, New York 1983.

[21] G. Probst, Piecewise linear approximation for hereditary control problems, Technical Report No. 60-1985, Institute fur Mathematik, Technische Universitat Graz, Universitat Graz, Graz, Austria, 1985.

[22] D. Salamon, Structure and stability of finite dimensional approximation for functional differential equations, TSR #2586, Mathematics Research Center, University of Wisconsin, Madison, Wisconsin, 1983.

[23] R.B. Vinter, On the evolution of state of linear differential delay equations in M^2: Properties of generators, J. Inst. Math. and Appl, 21 (1978), 13-23.

[24] J. Zabczyk, Remarks on the control of discrete-time distributed parameter systems, SIAM J. Control and Opt. 12(1974), 721-735.

1. Report No. NASA CR-178082
ICASE Report No. 86-16

2. Government Accession No.

3. Recipient's Catalog No.

4. Title and Subtitle
SHIFTING THE CLOSED-LOOP SPECTRUM IN THE OPTIMAL LINEAR QUADRATIC REGULATOR PROBLEM FOR HEREDITARY SYSTEMS

5. Report Date
March 1986

6. Performing Organization Code

7. Author(s)
J. S. Gibson and I. G. Rosen

8. Performing Organization Report No.
86-16

9. Performing Organization Name and Address
Institute for Computer Applications in Science and Engineering
Mail Stop 132C, NASA Langley Research Center
Hampton, VA 23665-5225

10. Work Unit No.

11. Contract or Grant No.
NAS1-17070, NAS1-18107

12. Sponsoring Agency Name and Address
National Aeronautics and Space Administration
Washington, D.C. 20546

13. Type of Report and Period Covered
Contractor Report

14. Sponsoring Agency Code
505-31-83-01

15. Supplementary Notes
Langley Technical Monitor:
C. South

Submitted to IEEE Trans. Auto. J. Control

Final Report

16. Abstract

In the optimal linear quadratic regulator problem for finite dimensional systems, the method known as an α-shift can be used to produce a closed-loop system whose spectrum lies to the left of some specified vertical line; that is, a closed-loop system with a prescribed degree of stability. This paper treats the extension of the α-shift to hereditary systems. As in finite dimensions, the shift can be accomplished by adding α times the identity to the open-loop semigroup generator and then solving an optimal regulator problem. However, this approach does not work with a new approximation scheme for hereditary control problems recently developed by Kappel and Salamon. Since this scheme is among the best to date for the numerical solution of the linear regulator problem for hereditary systems, an alternative method for shifting the closed-loop spectrum is needed. An α-shift technique that can be used with the Kappel-Salamon approximation scheme is developed. Both the continuous-time and discrete-time problems are considered. A numerical example which demonstrates the feasibility of the method is included.

17. Key Words (Suggested by Authors(s))
linear quadratic regulator, α-shift, hereditary system, approximation

18. Distribution Statement
64 - Numerical Analysis
67 - Theoretical Mathematics

Unclassified - unlimited

19. Security Classif.(of this report)	20. Security Classif.(of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	47	A03

For sale by the National Technical Information Service, Springfield, Virginia 22161