NASA Contractor Report 178081

ICASE REPORT NO. 86-15

ICASE

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Contract Nos. NAS1-17070 and NAS1-18107

March 1986

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
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NUMERICAL APPROXIMATION FOR THE INFINITE-DIMENSIONAL DISCRETE-TIME OPTIMAL LINEAR-QUADRATIC REGULATOR PROBLEM[†]

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ABSTRACT

An abstract approximation framework is developed for the finite and infinite time horizon discrete-time linear-quadratic regulator problem for systems whose state dynamics are described by a linear semigroup of operators on an infinite dimensional Hilbert space. The schemes included in the framework yield finite dimensional approximations to the linear state feedback gains which determine the optimal control law. Convergence arguments are given. Examples involving hereditary and parabolic systems and the vibration of a flexible beam are considered. Spline-based finite element schemes for these classes of problems, together with numerical results, are presented and discussed.

(*)This research was supported in part by the Air Force Office of Scientific Research under contract No. AFOSR-84-0309.

(**)This research was supported in part by the Air Force Office of Scientific Research under contract No. AFOSR-84-0393.

(†)Part of this research was carried out while the authors were visiting scientists at the Institute for Computer Applications in Science and Engineering (ICASE), NASA, Langley Research Center, Hampton, VA, 23665 which is operated under NASA contracts No. NAS1-17070 and No. NAS1-18107.

1. Introduction

Recent advances in micro-processor technology have led to increased interest in digital or discrete-time control systems. In addition, because many current application areas involve complex systems which are most appropriately modelled using functional and/or partial differential equations, it has become important to study digital control techniques in the context of infinite dimensional or distributed systems.

A great deal of attention has been given to the continuous-time infinite-dimensional linear-quadratic regulator problem. The general theory and characterization of the linear state feedback form of the optimal control are discussed in [5], [6], [8], [9], [21] and [22], while its application to hereditary, parabolic and hyperbolic systems with emphasis on approximation is treated in [2], [3], [7], [10], [11], [14] and [17] to mention just some of the work that has been done.

On the other hand, relatively little can be found in the literature concerning the corresponding discrete-time problem. The major contributions in this area can be found in the papers by Lee, Chow and Barr [20] and Zabczyk [28]. In these studies the Riccati difference equations that characterize the linear feedback form of the optimal control for the finite time problem are given and limiting properties as the length of the time horizon tends to infinity are discussed. However, the issue of approximation is not considered.

In the present paper, we develop numerical approximation schemes that yield finite dimensional approximations to the feedback gain operators which determine the discrete-time optimal control law. We consider control systems whose dynamics can be described in terms of a linear semigroup of operators on an infinite dimensional Hilbert space. The basis of our approach is the construction of a sequence of finite dimensional (presumably finite element based) state approximations which in turn leads to a sequence of finite dimensional discrete-time linear-quadratic regulator problems each of which can be solved using standard techniques.

Under appropriate assumptions on the nature of the original problem and the convergence of the state approximation, we are able to prove that the approximating optimal controls and feedback gains converge to the true optimal control sequences and feedback laws for the original infinite dimensional system. Depending upon the convergence properties of the state approximation, we are able to establish strong or uniform norm convergence of the approximating gain operators and the corresponding weak or strong convergence of the approximating feedback kernals which are used in the implementation of the optimal control. We treat both the finite and infinite-time horizon problems.

We have tested our schemes on a wide variety of examples. This paper includes numerical results for problems with state dynamics given by hereditary and parabolic (heat/diffusion) differential equations and a hybrid system of partial and ordinary differential equations for the vibration of an Euler-Bernoulli beam connected to a rigid body and a lumped mass. We implemented and tested the methods on an IBM Personal Computer.

We give a brief outline of the remainder of the paper. In section 2 we breifly outline previous results concerning the characterization of the optimal control and feedback gains for both the finite and infinite time horizon discrete-time regulator problem for distributed systems. The Riccati difference and algebraic equations whose solutions determine the optimal feedback control law are discussed. In section 3 we develop the abstract approximation framework and convergence arguments. Section 4 contains a discussion of particular schemes for the classes of problems mentioned above together with the results of our numerical studies. Some concluding remarks are given in Section 5.

We employ standard notation throughout. For an interval (a,b), we denote by $H^k(a,b)$ the usual Sobolev spaces of real-valued functions defined on (a,b) whose (k-1)st derivatives are absolutely continuous and whose k^{th} derivatives are L_2. The standard Sobolev inner product on $H^k(a,b)$ is denoted by $\langle\cdot,\cdot\rangle_k$. For X and Y normed linear spaces we denote by $L(X,Y)$ the space of bounded linear operators from X into Y. When Y = X, we use the shorthand notation $L(X)$.

2. The Optimal Control Problem

2.1 Optimal Control on a Finite Interval

Let Z and U be Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_Z$ and $\langle\cdot,\cdot\rangle_U$, respectively, with U finite dimensional. For $\{H, \langle\cdot,\cdot\rangle_H\}$ a Hilbert space, let $\ell^2(t_0,t_f;H)$ denote the usual Hilbert space of sequences $x = \{x(t)\}_{t=t_0}^{t_f}$ with $x(t) \in H$ together with the inner product

$$(2.1)\qquad \langle x,y\rangle_{\ell^2} = \sum_{t=t_0}^{t_f} \langle x(t), y(t)\rangle_H.$$

The discrete-time linear quadratic regulator problem on the finite time interval $[t_0,t_f]$ is

(P1) Choose $u \in \ell^2(t_0,t_f;U)$ to minimize the quadratic performance index

$$J(G;t_0,t_f,z(t_0),u) =$$

(2.2) $$\sum_{t=t_0}^{t_f-1} [\langle Qz(t), z(t)\rangle_Z + \langle Ru(t),u(t)\rangle_U] + \langle Gz(t_f),z(t_f)\rangle_Z$$

subject to the discrete-time control system

(2.3) $$z(t+1) = Tz(t) + Bu(t), \quad t \geq t_0$$
$$z(t_0) \in Z,$$

where T and B are bounded linear operators from Z into Z and U into Z, respectively, Q and G are bounded, nonnegative self-adjoint operators on Z, and R is a positive definite self-adjoint operator on U.

Of primary concern to us will be applications where (2.3) is the sampled form of the continuous-time control system

(2.4) $$\dot{z}(s) = \mathcal{A} z(s) + \mathcal{B}u(s)$$

where $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of bounded linear operators $\mathcal{T}(s)$, $s \geq 0$, on Z, and $\mathcal{B}$ is a possibly unbounded linear operator from U into Z. In this case we have

(2.5) $$T = \mathcal{T}(\tau) \quad \text{and} \quad B = \int_0^\tau \mathcal{T}(s)\mathcal{B}\, ds,$$

where τ is the sampling interval. If, as in our subsequent example discussed in Section 4.1 where u is a boundary control in a heat equation, $\mathcal{B}$ is unbounded (more precisely, $\mathcal{B}$ maps U not into Z but into some larger space), then the integral in (2.5) is not interpreted literally.

The solution to Problem ($\mathcal{P}1$) has been given for infinite dimensional control systems in [20],[28], and the equations representing the solution have the same form as in the finite dimensional case. We will give now the version of the solution that is most useful for our purposes.

For given $z(t_0)$, $J(G;t_0,t_f,z(t_0),u)$ is a bounded linear-quadratic functional on $\ell^2(t_0,t_f;U)$ with coercive quadratic part. Therefore, for each $z(t_0)$, there exists a unique optimal control sequence in $\ell^2(t_0,t_f;U)$. Also, the minimum value of the performance index is a quadratic functional of $z(t_0)$, so that there exists a unique nonnegative, self-adjoint $\Pi(t_0) \in L(Z)$ such that

$$(2.6) \qquad J_* = \min J(G;t_0,t_f,z(t_0),u) = \langle \Pi(t_0)z(t_0),z(t_0)\rangle_Z.$$

Application of the principle of dynamic optimality establishes that the optimal control has the feedback form

$$(2.7) \qquad u_*(t) = -F(t)z_*(t), \qquad t_0 \le t \le t_f-1$$

where

$$(2.8) \qquad F(t) = \hat{R}(t)^{-1}B^*\Pi(t+1)T,$$

$$(2.9) \qquad \hat{R}(t) = R + B^*\Pi(t+1)B$$

and $\Pi(t)$ satisfies the Riccati difference equation

$$(2.10) \quad \Pi(t) = T^*[\Pi(t+1) - \Pi(t+1)B\hat{R}(t)^{-1}B^*\Pi(t+1)]T + Q, \qquad t \le t_f-1,$$

with the final condition

(2.11) $\Pi(t_f) = G.$

The optimal trajectory z_* is given by

(2.12) $z_*(t+1) = S(t)z_*(t), \quad t \geq t_0$

where

(2.13) $S(t) = T-BF(t).$

2.2 Control on the Infinite Interval

Here, $t_f = \infty$ and $G = 0$. To simplify notation, we will write $J(t_0,\infty,z(t_0),u)$ instead of $J(0;t_0,\infty,z(t_0),u)$.

Definition 2.1. A control sequence $u \in \ell^2(0,\infty;U)$ is an admissible control for the initial condition z if $J(0,\infty,z,u) < \infty$.

The discrete-time linear-quadratic regulator problem on the infinite interval is

($\mathcal{P}2$) Choose an admissible control u_* to minimize $J(0,\infty,z,u)$, if an admissible control exists for the initial condition z.

That a unique optimal control u_* exists whenever at least one admissible control exists follows from the fact that the quadratic part of $J(0,\infty,z,u)$ is coercive on a subspace of $\ell^2(0,\infty;U)$. See the discussion following Definition 4.1 of [9].

Definition 2.2. A bounded linear operator Π on Z is a solution to the Riccati algebraic equation if

$$(2.14)\qquad \Pi = T^{*}[\Pi - \Pi B(R+B^{*}\Pi B)^{-1}B^{*}\Pi]T + Q.$$

The following theorem summarizes results from Zabczyk [28].

Theorem 2.3. The following are equivalent:

(i) There exists an admissible control for each $z \in Z$;

(ii) for each $z \in Z$, $\sup_{t<t_f} \langle \Pi(t)z,z\rangle_Z < \infty$, where $\Pi(t)$ is the Riccati operator in (2.10) and $\Pi(t_f) = 0$ for fixed t_f;

(iii) as $t \to -\infty$, $\Pi(t)$ converges strongly to a nonnegative self-adjoint solution to the Riccati algebraic equation;

(iv) there exists a nonnegative self-adjoint solution to the Riccati algebraic equation.

For uniqueness of the solution to the Riccati algebraic equation and characterization of the optimal control, Zabczyk treated two cases: when Q is coercive, and when the spectral radius of T is less than 1 (i.e., the open-loop system is uniformly exponentially stable). Since neither is the case in the example we discuss in Section 4.2 and other applications in which we are interested, we will need the following hypothesis and theorem.

Hypothesis 2.4. The operators T, B and Q are such that, if $z(0) \in Z$ and u is an admissible control for $z(0)$, then

$$(2.15)\qquad \lim_{t\to\infty} |z(t)|_Z = 0.$$

Theorem 2.5. When Hypothesis 2.4 holds, there exists at most one nonnegative self-adjoint solution to the Riccati algebraic equation. If such a solution Π exists, then there exists a unique solution to problem ($\mathcal{P}$2) for each initial condition $z(0) \in Z$, the minimum value of the performance index is

$$(2.16)\qquad J_* = \min_{u \text{ admissible}} J(0,\infty,z(0),u) = \langle \Pi z(0), z(0)\rangle_Z,$$

the optimal control has the feedback form

$$(2.17)\qquad u_*(t) = -Fz_*(t), \quad t \geq 0,$$

where

$$(2.18)\qquad F = \tilde{R}^{-1}B^*\Pi\ T,$$

$$(2.19)\qquad \tilde{R} = R+B^*\Pi\ B$$

and the optimal trajectory $z_*(t)$ satisfies

$$(2.20)\qquad z_*(t+1) = Sz_*(t), \quad t \geq 0,$$

with

$$(2.21)\qquad S = T-BF.$$

Proof. Let Π be such a solution and note that, for any finite t_f, Π is a constant solution to (2.10) and (2.11) with $G = \Pi$. Then the corresponding

$F(t)$ and $\tilde{R}(t)$ defined by (2.8) and (2.9) are the constant operators in (2.18) and (2.19). For $z(0) \in Z$, define $\bar{z}(0) = z(0)$,

$$\bar{z}(t+1) = (T-BF)\bar{z}(t), \quad t \geq 0, \tag{2.22}$$

and

$$\bar{u}(t) = -F\bar{z}(t), \quad t \geq 0. \tag{2.23}$$

Now suppose that u is an admissible control for z(0) and that z(t) is the corresponding solution to (2.3). For $t_f > 0$, the preceding results about the solution to Problem ($\mathcal{P}$1) with $G = \Pi$ imply

$$\begin{aligned} J(\Pi;0,t_f,z(0),\bar{u}) &\leq J(0;0,t_f,z(0),u) + \langle \Pi z(t_f),z(t_f)\rangle_Z \\ &\leq J(0;0,\infty,z(0),u) + \langle \Pi z(t_f),z(t_f)\rangle_Z. \end{aligned} \tag{2.24}$$

Also,

$$\begin{aligned} J(\Pi;0,t_f,z(0),\bar{u}) &= \langle \Pi z(0),z(0)\rangle_Z \\ &= J(0;0,t_f,z(0),\bar{u}) + \langle \Pi \bar{z}(t_f),\bar{z}(t_f)\rangle_Z. \end{aligned} \tag{2.25}$$

Since $z(t_f) \to 0$ as $t_f \to \infty$, (2.24) shows that $\bar{u}$ is both admissible and optimal for Problem ($\mathcal{P}$2). Since $\bar{z}(t_f) \to 0$ as $t_f \to \infty$, (2.25) shows (2.16). As we see now, (2.16) must hold for any nonnegative self-adjoint solution of the Riccati algebraic equation; therefore, such a solution is unique.

Remark 2.6. When Hypothesis 2.4 does not hold, the Riccati algebraic equation may have more than one nonnegative self-adjoint solution. In this case, the minimal such solution -- there will be one -- gives the solution to Problem

(P2) as in Theorem 2.5. Throughout this paper, we assume that Hypothesis 2.4 holds.

Lemma 2.7. Suppose that $Q \geq m$ for some positive constant m, and set $C_n = \sum_{t=0}^{n} (T^*)^t Q T^t$, for $n = 1,2, \ldots$. Then $|C_n z|_Z$ is bounded in n for each $z \in Z$ if and only if C_n converges in norm to the operator

$$(2.26) \qquad C = \sum_{t=0}^{\infty} (T^*)^t Q T^t$$

and

$$(2.27) \qquad |T^t| \leq (|C|/m)(1 - m/|C|)^t, \quad t = 1,2, \ldots .$$

Proof. Since C_n is an increasing sequence of bounded self-adjoint linear operators, C_n converges strongly to some bounded self-adjoint C if and only if $\langle C_n z, z\rangle_Z$ is bounded in n for each z, if and only if $|C_n z|_Z$ is bounded in n for each z. This is a standard result. The proof of the Lemma is then a standard exercise using the Lyapunov functional $\langle Cz(t), z(t)\rangle_Z$ for the homogeneous part of (2.3).

Corollary 2.8. If $Q \geq m > 0$ and the Riccati algebraic equation has a nonnegative self-adjoint solution Π, then the spectral radius of the operator S in (2.21) is less than 1, and

$$(2.28) \qquad |S^t| \leq (|\Pi|/m)(1 - m|\Pi|)^t, \quad t = 1,2, \ldots .$$

Proof. This follows from Lemma 2.7 and

$$(2.29) \qquad \langle \Pi z,z \rangle_Z = \sum_{t=0}^{\infty} (S^*)^t[Q + F^*RF]S^t.$$

For Q coercive, Zabczyk proved a stronger result than part (iii) of Theorem 2.3: if a nonnegative self-adjoint solution to the Riccati algebraic equation exists, then $|\Pi(t) - \Pi| \to 0$ geometrically fast as $t \to -\infty$ (Also, see [13]). We will need such a result, along with an explicit convergence rate, for the approximation theory in Section 3.2. Since Zabczyk's proof does not yield an explicit convergence rate, we give the following.

Theorem 2.9. Suppose that there exists a nonnegative self-adjoint solution Π to (2.14) and that

$$(2.30) \qquad |S^t| \leq Mr^t, \; t = 1,2, \ldots ,$$

where M and r are positive constants with $r < 1$ and S is the optimal closed-loop operator in Theorem 2.5. If $\Pi(\cdot)$ is the operator in (2.10) with $t_f = 0$ and

$$(2.31) \qquad \Pi(0) \geq \Pi,$$

then

$$(2.32) \quad \langle \Pi z,z \rangle_Z \leq \langle \Pi(-t)z,z \rangle_Z \leq \langle \Pi z,z \rangle_Z + (Mr^t)^2|\Pi(0)|, \qquad t = 1,2, \ldots .$$

Proof. For t_0 a negative integer, let u_0 be the optimal control sequence for the finite-time Problem ($\mathcal{P}$1) on the interval $[t_0,0]$ with initial condition

$z(t_0) \in Z$, with z_0 the corresponding optimal trajectory. Also, let u_* be the optimal control sequence on the infinite interval for Problem ($P2$) with initial condition $z(t_0)$, with z_* the corresponding optimal trajectory. Since Π is a constant solution to (2.10) for the final condition $G = \Pi$, we have

$$\begin{aligned}
\langle \Pi z(t_0), z(t_0)\rangle_Z &= J(\Pi; t_0, 0, z(t_0), u_*) \\
&\le J(0; t_0, 0, z(t_0), u_0) + \langle \Pi z_0(0), z_0(0)\rangle_Z \\
&\le J(0; t_0, 0, z(t_0), u_0) + \langle \Pi(0) z_0(0), z_0(0)\rangle_Z \\
&= \langle \Pi(t_0) z(t_0), z(t_0)\rangle_Z .
\end{aligned} \tag{2.33}$$

On the other hand (note that $z_*(t_0) = S^{-t} z(t_0)$),

$$\begin{aligned}
&\langle \Pi(t_0) z(t_0), z(t_0)\rangle_Z \\
&\le J(0; 0, -t_0, z(t_0), u_*) + \langle \Pi(0) z_*(-t_0), z_*(-t_0)\rangle_Z \\
&\le J(0; 0, \infty, z(t_0), u_*) + \langle \Pi(0) z_*(t_0), z_*(t_0)\rangle_Z \\
&\le \langle \Pi z(t_0), z(t_0)\rangle_Z + |\Pi(0)|(|S^{-t_0}||z(t_0)|_Z)^2 .
\end{aligned} \tag{2.34}$$

3. Approximation Theory

3.1 The finite time interval problem

In this section we develop a general approximation framework for the finite time interval problem (P1) and describe associated convergence results.

For each $N = 1,2, \ldots$, let $Z_N \subset Z$ be a finite dimensional subspace of Z and let $P_N: Z \to Z_N$ denote the orthogonal projection of Z onto Z_N with respect to the $\langle\cdot,\cdot\rangle_Z$ inner product. We require the following hypotheses.

Hypothesis 3.1 There exist operators $T_N: Z_N \to Z_N$, $B_N: U \to Z_N$ $Q_N: Z_N \to Z_N$ and $G_N: Z_N \to Z_N$ which satisfy

$$
\begin{aligned}
T_N P_N &\to T && \text{strongly,}\\
T_N^* P_N &\to T^* && \text{strongly,}\\
B_N &\to B && \text{strongly,}\\
Q_N P_N &\to Q && \text{strongly,}\\
G_N P_N &\to G && \text{strongly,}
\end{aligned}
$$

as $N \to \infty$ with T_N and B_N bounded and Q_N and G_N bounded, self-adjoint and nonnegative.

Hypothesis 3.2 The spaces Z_N are approximating subspaces in the sense that the projections P_N satisfy $P_N \to I$ strongly on Z as $N \to \infty$.

We note that since U has been assumed to be finite dimensional, Hypothesis 3.1 above necessarily implies that $B_N \to B$ and $B_N^* P_N \to B^*$ in the uniform norm topology on $\mathcal{L}(U,Z)$ and $\mathcal{L}(Z,U)$ respectively.

We define a sequence of approximating discrete-time linear quadratic regulator problems on the finite time interval $[t_0,t_f]$ as follows:

$(\mathcal{P}1_N)$

Find $u_*^N \in \ell^2(t_0,t_f-1;U)$ which minimizes

(3.1) $$J_N(G_N;t_0,t_f,z(t_0),u) = \sum_{t=t_0}^{t_f-1} [\langle Q_N z_N(t), z_N(t)\rangle_Z + \langle Ru(t),u(t)\rangle_U] + \langle G_N z_N(t_f), z_N(t_f)\rangle_Z$$

subject to

(3.2) $$z_N(t+1) = T_N z_N(t) + B_N u(t), \quad t \geq t_0$$

$$z_N(t_0) = P_N z(t_0).$$

The results stated in Section 2.1 concerning the existence and uniqueness of solutions to Problem $(\mathcal{P}1)$ apply to the Problems $(\mathcal{P}1_N)$ as well. Indeed, there exists a unique solution $u_*^N \in \ell^2(t_0,t_f-1;U)$ to Problem $(\mathcal{P}1_N)$ which is given in feedback form by

(3.3) $$u_*^N(t) = -F_N(t)\, z_*^N(t), \quad t_0 \leq t \leq t_f-1$$

where

(3.4) $$F_N(t) = \hat{R}_N(t)^{-1} B_N^* \Pi_N(t+1) T_N$$

with

$$(3.5) \qquad \hat{R}_N(t) = R + B_N^* \Pi_N(t+1)B_N$$

and the operators $\{\Pi_N(t)\}_{t=t_0}^{t_f}$ on Z_N satisfying the Riccati difference equation

$$(3.6) \qquad \Pi_N(t) = T_N^*[\Pi_N(t+1) - \Pi_N(t+1)B_N\hat{R}_N(t)^{-1}B_N^*\Pi_N(t+1)]T_N + Q_N$$

with terminal condition

$$(3.7) \qquad \Pi_N(t_f) = G_N.$$

The optimal trajectory z_*^N is given by

$$(3.8) \qquad \begin{aligned} z_*^N(t+1) &= S_N(t)z_*^N(t), \quad t > t_0, \\ z_*^N(t_0) &= P_N z(t_0) \end{aligned}$$

where

$$(3.9) \qquad S_N(t) = T_N - B_N F_N(t), \quad t > t_0.$$

The operators $\{\Pi_N(t)\}_{t=t_0}^{t_f}$ are bounded, self-adjoint and nonnegative. The minimum value of the performance index (3.1) is given by

$$(3.10) \qquad J_*^N = J_N(G_N;t_0,t_f,z(t_0),u_*^N) = \langle\Pi_N(t_0)z_*^N(t_0),\ z_*^N(t_0)\rangle_Z \ .$$

The fundamental convergence result is given in the following theorem.

Theorem 3.3 Let u_*^N and u_* be the unique solutions to problems $(\mathcal{P}1_N)$ and $(\mathcal{P}1)$, respectively, with z_*^N and z_* the corresponding optimal trajectories generated by (3.8) and (2.12). Let J_N, Π_N and F_N and J, Π and F be given by (3.1), (3.6) and (3.4) and (2.2), (2.10) and (2.8). Then, if Hypotheses 3.1 and 3.2 hold, we have

(i) $\lim_{N\to\infty} |u_*^N - u_*|_{\ell^2} = 0,$

(ii) $\lim_{N\to\infty} |z_*^N - z_*|_{\ell^2} = 0,$

(iii) $\lim_{N\to\infty} |J_*^N - J_*| = 0,$

(iv) $\lim_{N\to\infty} |\Pi_N(t)P_N z - \Pi(t)z|_Z = 0, \; z \in Z, \; t_0 \leq t \leq t_f$

and

(v) $\lim_{N\to\infty} |F_N(t)P_N - F(t)| = 0, \quad t_0 \leq t \leq t_f - 1.$

Proof

We first note that $\Pi_N(t)$ being nonnegative implies that $|\hat{R}_N(t)| \geq |R|$ and consequently that $|\hat{R}_N(t)^{-1}| \leq |R|^{-1}$. It follows therefore that for $u \in U$.

(3.11) $|(\hat{R}_N(t)^{-1} - \hat{R}(t)^{-1})u|_U$

$$= |\hat{R}_N(t)^{-1}(\hat{R}(t) - \hat{R}_N(t))\hat{R}(t)^{-1}u|_U$$

$$\leq |R|^{-1}|(\hat{R}(t) - \hat{R}_N(t))\hat{R}(t)^{-1}u|_U.$$

The above estimate together with (2.9), (2.11), (3.5), (3.7) and Hypothesis 3.1 imply that

(3.12) $$\hat{R}_N(t_f-1)^{-1} \to \hat{R}(t_f-1)^{-1}$$

as $N \to \infty$ strongly on U. Since U is finite dimensional the convergence in (3.13) is in fact uniform. It then follows immediately from (2.8), (3.4) and Hypothesis 3.1 that

(3.13) $$F_N(t_f - 1)\ P_N \to F(t_f - 1),$$

uniformly as $N \to \infty$, and from (2.10) and (3.6) that

(3.14) $$\Pi_N(t_f - 1)P_N \to \Pi(t_f-1)$$

strongly on Z as $N \to \infty$. A simple induction yields (iv) and (v) from which (i), (ii) and (iii) then follow trivially.

Remark It will, on occassion, be the case that in constructing a particular approximation scheme $T_N P_N \to T$ strongly but $T_N^* P_N \to T^*$ only weakly (see, for example, [3]). However, by using the fact that

(3.15) $$(T_N^* \Pi_N(t+1))^* = \Pi_N(t+1)T_N$$

implies that $T_N^* \Pi_N(t+1) \to T^* \Pi\ (t+1)$ weakly if $\Pi_N(t+1) \to \Pi(t+1)$ weakly, we conclude that Theorem 3.3 continues to hold under these somewhat weaker hypotheses with the strong convergence in (iv) replaced by weak and the uniform convergence in (v) replaced by strong.

Under certain additional hypotheses it can be shown that the operators $\Pi(t)$, $t_0 < t < t_f$ given by (2.10), (2.11) are trace class (see [15]) and that

$$\lim_{N\to\infty} \|\Pi_N(t)P_N - \Pi(t)\|_1 = 0, \quad t_0 < t < t_f, \tag{3.16}$$

where $\|\cdot\|_1$ denotes the trace norm, the strongest of all common operator norms. We require the following lemmas.

<u>**Lemma 3.4**</u> If $\{a_i\}_{i=1}^{\infty}$ is an absolutely summable sequence of real numbers then there exist sequences $\{b_i\}_{i=1}^{\infty}$ and $\{c_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} b_i = 0$, $\{c_i\}_{i=1}^{\infty}$ is absolutely summable and $a_i = b_i c_i$.

<u>**Proof**</u>

Let

$$\alpha = \sum_{i=1}^{\infty} |a_i| \tag{3.17}$$

and for $j = 0,1,2,\ldots$ define nonnegative integers n_j as follows. Let $n_0 = 0$ and let n_j denote the first index for which

$$\sum_{i=1}^{n_j} |a_i| > \alpha - \frac{1}{j^3}, \tag{3.18}$$

$j = 1,2,\ldots$. Set

$$(3.19) \qquad b_i = \frac{1}{j}, \; c_i = ja_i, \quad i = n_{j-1}+1,\ldots,n_j, \; j = 1,2,\ldots$$

Then $b_i c_i = a_i$, $i = 1,2,\ldots$, $\lim_{i\to\infty} b_i = 0$ and

$$(3.20) \qquad \sum_{i=1}^{\infty} |c_i| = \sum_{j=1}^{\infty} j \sum_{k=n_{j-1}+1}^{n_j} |a_k| < \alpha + \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty .$$

Lemma 3.5 If L is a self-adjoint trace class operator on a separable Hilbert space H, then L can be written as L^1L^2 where L^1 is compact and L^2 is trace class.

Proof

Let $\{\lambda_i\}_{i=1}^{\infty}$ denote the eigenvalues of L repeated according to multiplicity and let $\{\phi_i\}_{i=1}^{\infty}$ denote the corresponding eigenvectors. Then $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence of real numbers, each of finite multiplicity, and

$$(3.21) \qquad \sum_{i=1}^{\infty} |\lambda_i| = |L|_1 < \infty .$$

Applying the previous lemma there exist sequences $\{\mu_i\}_{i=1}^{\infty}$ and $\{\nu_i\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} \mu_i = 0$, $\sum_{i=1}^{\infty} |\nu_i| < \infty$ and $\lambda_i = \mu_i \nu_i$. Defining L^1 and L^2 by

$$(3.22) \qquad L^1\phi = \sum_{i=1}^{\infty} \mu_i \langle \phi, \phi_i \rangle_H \phi_i, \quad \phi \in H$$

and

$$(3.23) \qquad L^2\phi = \sum_{i=1}^{\infty} \nu_i \langle \phi, \phi_i \rangle \, \phi_i, \quad \phi \in H$$

respectively, the lemma immediately follows.

__Lemma 3.6__ Let $\{S_N\}_{N=1}^{\infty}$ be a sequence of bounded linear operators on a seperable Hilbert space H which converges strongly to a bounded linear operator S. Let $\{L_N\}_{N=1}^{\infty}$ be a sequence of trace class operators on H which converges in trace norm to an operator L. If L can be written as $L = L^1L^2$ with L^1 compact and L^2 trace class then the sequence $\{S_NL_N\}_{N=1}^{\infty}$ converges in trace norm to SL.

__Proof__

The result follows immediately from

$$(3.24) \qquad \|S_NL_N - SL\|_1 \leq \|S_N(L_N - L)\|_1 + \|(S_N - S)L^1L^2\|_1$$

$$\leq |S_N|\|L_N - L\|_1 + |(S_N - S)L^1|\|L^2\|_1.$$

__Theorem 3.7__ If Q and G are trace class operators then the operators $\{\Pi(t)\}_{t=t_0}^{t_f}$ given by (2.10) and (2.11) are trace class. Moreover, if Hypotheses 3.1 and 3.2 hold and $Q_NP_N \to Q$ and $G_NP_N \to G$ in trace norm as $N \to \infty$ then we have

$$(3.25) \qquad \lim_{N\to\infty} \|\Pi_N(t)P_N - \Pi(t)\|_1 = 0, \quad t_0 \leq t \leq t_f.$$

__Proof__

That the operators $\Pi(t)$, $t_0 \leq t \leq t_f$ are trace class is an immediate consequence of the hypotheses of the theorem, (2.10), (2.11) and the fact that the trace class operators form a two sided ideal of $L(Z)$, the space of bounded linear operators on Z (see [15]).

The trace norm convergence stated in (3.25) will follow once we have shown that

$$(3.26) \qquad \lim_{N\to\infty} \|\Pi_N(t+1)P_N - \Pi(t+1)\|_1 = 0$$

implies

$$\text{(i)} \quad \lim_{N\to\infty} \|T_N^*\Pi_N(t+1)T_N P_N - T^*\Pi(t+1)T\|_1 = 0$$

and

$$\text{(ii)} \quad \lim_{N\to\infty} \|T_N^*\,\Pi_N(t+1)\,B_N\hat{R}_N(t)^{-1}B_N^*\Pi_N(t+1)T_N P_N - T^*\Pi(t+1)B\hat{R}(t)^{-1}B^*\Pi(t+1)T\|_1 = 0.$$

To argue (i) we first note that Hypothesis 3.1 and Lemmas 3.5 and 3.6 imply

$$(3.27) \qquad \lim_{N\to\infty} \|T_N^*\,\Pi_N(t+1)P_N - T^*\Pi(t+1)\|_1 = 0.$$

Taking adjoints we obtain

$$(3.28) \qquad \lim_{N\to\infty} \|\Pi_N(t+1)T_N P_N - \Pi(t+1)T\|_1 = 0.$$

Another application of the previous two lemmas yields

$$(3.29) \qquad \lim_{N\to\infty} \|T_N^*\,\Pi_N(t+1)T_N P_N - T^*\Pi(t+1)T\|_1 \le \lim_{N\to\infty} |T_N^*|\,\|\Pi_N(t+1)T_N P_N - \Pi(t+1)T\|_1$$

$$+ \lim_{N\to\infty} |(T_N^* P_N - T^*)\, \Pi^1(t+1)|\,\|\Pi^2(t+1)T\|_1$$

where $\Pi(t+1) = \Pi^1(t+1)\Pi^2(t+1)$ is the factorization of $\Pi(t+1)$ described in Lemma 3.6.

The verification of (ii) is analogous and the theorem is proven.

We note that if Hypotheses 3.1 and 3.2 hold and if the operators Q and G are trace class with Q_N and G_N defined by

(3.30) $$Q_N = P_N Q$$

and

(3.31) $$G_N = P_N G,$$

then Lemmas 3.5 and 3.6 imply that the trace norm convergence hypotheses in Theorem 3.7 hold. The significance of this observation will become apparent when examples are discussed in Section 4. Indeed, it is frequently the case in practice that Q and G have finite rank (and consequently are trace class) and the operators Q_N and G_N are defined as in (3.30), and (3.31).

3.2 Approximation on the Infinite Interval

Problem $(\mathcal{P}2_N)$ is Problem $(\mathcal{P}2)$ for the control system in (3.2) and the performance index

(3.32) $$J_N(0,\infty,z_N(0),u) = \sum_{t=0}^{\infty} [\langle Q_N z_N(t), z_N(t)\rangle_Z + \langle Ru(t),u(t)\rangle_U].$$

Hypothesis 3.8. For each N, there exists exactly one nonnegative self-adjoint solution to the Riccati algebraic equation

$$(3.33)\qquad \Pi_N = T_N^*[\Pi_N - \Pi_N B_N(R + B_N^*\Pi_N B_N)^{-1} B_N^*\Pi_N]T_N + Q_N.$$

By Theorem 2.3, this implies that

$$(3.34)\qquad \lim_{t\to -\infty} |\Pi_N - \Pi_N(t)| = 0$$

for each N, since $\dim(Z_N) < \infty$.

As in Theorem 2.5, we write

$$(3.35)\qquad F_N = \tilde{R}_N^{-1} B_N^* \Pi_N T_N,$$

$$(3.36)\qquad \tilde{R}_N = R + B_N^*\Pi_N B_N,$$

and

$$(3.37)\qquad S_N = T_N - B_N F_N.$$

From here on, Π will be the nonnegative self-adjoint solution to the infinite dimensional Riccati algebraic equation (2.14) -- when it exists -- F will be the corresponding feedback operator in (2.18) and S will be the corresponding closed-loop operator in (2.21).

Theorem 3.9. If $\Pi_N P_N$ converges strongly to some bounded linear operator Π, then Π is a nonnegative self-adjoint solution to (2.14), $F_N P_N$ converges in norm to F and $S_N P_N$ converges strongly to S.

Proof. This follows from Hypotheses 3.1 and 3.2, (3.33) and (3.35) - (3.37), and the fact that the control space U has fixed finite dimension.

Theorem 3.10. Suppose that there exist positive constants M and r, independent of N, with $r < 1$, such that

(3.38) $\quad \Pi_N \leq M, \qquad N = 1,2, \ldots ,$

and

(3.39) $\quad |S_N^t| \leq Mr^t, \qquad t = 1,2, \ldots , \qquad N = 1,2, \ldots .$

Then a nonnegative self-adjoint solution Π to (2.14) exists, and as $N \to \infty$,

(3.40) $\quad \Pi_N P_N \to \Pi$ strongly.

If there exists a positive m, independent of N, such that

(3.41) $\quad Q_N \geq m, \ N = 1,2, \ldots,$

then (3.38) implies the existence of an r less that one and independent of N for which (3.39) holds.

Proof. For each N, let $\Pi_N(\cdot)$ satisfy (3.6) with $t_f = 0$ and $\Pi_N(0) = MI$, where I denotes the identity operator on Z_N. From (2.32),

$$(3.42) \qquad |\Pi_N - \Pi_N(-t)| \to 0 \text{ as } t \to \infty,$$

uniformly in N. Now, for $z \in Z$, write

$$(3.43) \qquad \langle(\Pi_N - \Pi_{N'})z,z\rangle_Z = \langle(\Pi_N - \Pi_N(-t))z,z\rangle_Z + \langle(\Pi_N(-t) - \Pi_{N'}(-t))z,z\rangle_Z$$

$$+ \langle(\Pi_{N'}(-t) - \Pi_{N'})z,z\rangle_Z.$$

For $\varepsilon > 0$ choose $t > 0$ such that $|(\Pi_N - \Pi_N(-t))z|_Z < \varepsilon$ and $|(\Pi_{N'} - \Pi_{N'}(-t))z| < \varepsilon$. Then, for N and N' large enough, $|(\Pi_N(-t) - \Pi_{N'}(-t))z|_Z < \varepsilon$. This shows that $\Pi_N z$ is a Cauchy sequence in Z for each z. Therefore, Π_N converges strongly to a nonnegative self-adjoint solution to (2.14).

An important application of this theorem is when the approximating open-loop operators T_N have an exponential decay rate independent of N, Q is coercive and $Q_N = P_N Q|_{Z_N}$. In this case, the zero control gives an upper bound, independent of N, on Π_N. Such is the case in the example discussed in Section 4.1 and in applications to flexible structures with no rigid-body modes and coercive structural damping.

Theorem 3.11. Suppose that $\Pi_N P_N$ converges strongly to Π, $Q_N P_N$ converges in trace norm to Q (hence Q is trace class), and (3.39) holds for positive M and r independent of N with r less than one. Then $\Pi_N P_N$ converges in trace norm to Π.

Proof. Since $\|S_N Q_N S_N^*\|_1 \leq |S_N|^2 \|Q_N\|_1$, the series in (2.26) converges in trace norm, uniformly in N. The current result follows then from Lemmas 3.5 and 3.6.

Note that $Q_N P_N$ converges in trace norm to Q if Q is trace class and $Q = P_N Q P_N |_{Z_N}$.

Theorem 3.12 if $|\Pi_N|$ is bounded in N, then a nonnegative self-adjoint solution Π to (2.14) exists, $\Pi_N P_N$ converges weakly to Π, and $F_N P_N$ and $S_N P_N$ converge strongly to F and S, respectively.

Proof. According to [11 Theorem 6], $\Pi_N P_N$ converges weakly to some nonnegative self-adjoint bounded Π. It follows from (3.33) and Hypotheses 3.1 and 3.2 that Π satisfies (2.14) and that F_N and S_N converge as indicated.

Note that Theorem 3.12 holds if $S_N P_N$ converges strongly but $S_N^* P_N$ converges only weakly.

3.3 Implementation of the Approximation Schemes

In constructing the approximating operators T_N, B_N, Q_N and G_N a standard Galerkin approach is often taken; that is, $T_N = P_N T$, $B_N = P_N B$, $Q_N = P_N Q$ and $G_N = P_N G$. We note however that explicit representations for the operators T and B are frequently not available. In particular, this can occur when the discrete-time system (2.3) arises from the sampling of an infinite dimensional continuous-time system of the form (2.4). In this case it is the operators $\mathcal{A}$ and $\mathcal{B}$ which are approximated by a sequence of finite dimensional operators A_N and B_N on Z_N, from which an approximation to the semigroup $\{T(s) : s \geq 0\}$ is obtained as $T_N(s) = \exp(A_N s)$, $s \geq 0$. The operators T_N and B_N are then $T_N = T_N(\tau)$ and $B_N = \int_0^\tau T_N(s) B_N ds$, respectively. The strong convergence $T_N P_N \to T$ and $B_N \to B$ is then usually argued using an appropriate formulation of the Trotter-Kato theorem, a well known semigroup approximation result (see [15] [23]).

The expressions given by (3.3) - (3.7) are operator equations and although they are finite dimensional, they are not appropriate for computations. To make use of our approximation framework, we must first determine equivalent matrix formulations. Toward this end we assume, without loss of generality, that $U = R^m$ with the standard basis and inner product and let $\{\phi_N^i\}_{i=1}^{K_N}$ be a basis for Z_N. Define the $K_N \times K_N$ Gram matrix M_N by

$$(3.44) \qquad [M_N]_{ij} = \langle \phi_N^i, \phi_N^j \rangle_Z.$$

For an operator A we denote its matrix representation with respect to the bases defined above by [A]. Similarly, for an element $z \in Z$ or $u \in U$, we let its vector representation be given by [z] or [u] respectively. Standard calculations yield

$$(3.45) \qquad [T_N^*] = M_N^{-1} [T_N]^T M_N$$

and

$$(3.46) \qquad [B_N^*] = [B_N]^T M_N.$$

Defining

(3.47) $$\hat{\Pi}_N(t) = M_N[\Pi_N(t)],$$

(3.48) $$\hat{Q}_N = M_N[Q_N],$$

and

(3.49) $$\hat{G}_N = M_N[G_N]$$

we obtain

(3.50) $$[u_*^N(t)] = -[F_N(t)][z_*^N(t)], \qquad t_0 \le t \le t_f - 1,$$

(3.51) $$[F_N(t)] = [\hat{R}_N(t)]^{-1}[B_N]^T\hat{\Pi}_N(t+1)[T_N],$$

(3.52) $$[\hat{R}_N(t)] = [R] + [B_N]^T\hat{\Pi}_N(t+1)[B_N],$$

(3.53) $$\hat{\Pi}_N(t) - [T_N]^T(\hat{\Pi}_N(t+1) -$$

$$\hat{\Pi}_N(t+1)[B_N\ ][\hat{R}_N(t)]^{-1}[B_N]^T\hat{\Pi}_N(t+1))[T_N] + \hat{Q}_N, \qquad t_0 \le t \le t_f-1,$$

(3.54) $$\hat{\Pi}_N(t_f) = \hat{G}_N.$$

Note that since Q_N and G_N are self-adjoint and nonnegative so too are $\hat{Q}_N$ and $\hat{G}_N$. Equations (3.50) - (3.54) are therefore in the form of the standard ones obtained for the feedback law for a discrete-time linear-quadratic regulator problem in R^{K_N}. Consequently they can be solved using conventional techniques. The minimum value of the performance index is given by

$$(3.55)\qquad J_*^N = [z_*^N(t_0)]^T\hat{\Pi}_N(t_0)[z_*^N(t_0)].$$

Analogously, for the infinite time horizon problem, (3.33), (3.35) and (3.36) yield

$$(3.56)\qquad [u_*^N(t)] = -[F_N][z_*^N(t)], \quad t > t_0,$$

$$(3.57)\qquad [F_N] = [\tilde{R}_N]^{-1}[B_N]^T\hat{\Pi}_N[T_N]$$

$$(3.58)\qquad [\tilde{R}_N] = [R] + [B_N]^T\hat{\Pi}_N[B_N]$$

where $\hat{\Pi}_N$ is the solution to the matrix algebraic Riccati equation

$$(3.59)\qquad \hat{\Pi}_N = [T_N]^T(\hat{\Pi}_N - \hat{\Pi}_N[\tilde{R}_N]^{-1}[B_N]^T\hat{\Pi}_N)[T_N] + \hat{Q}_N,$$

with $\hat{Q}_N$ given by (3.48). The minimum value of the performance index is given by

$$(3.60)\qquad J_N^* = [z_*^N(t_0)]^T\hat{\Pi}_N[z_*^N(t_0)].$$

4. Examples and Numerical Results

In this section we describe the application of the general approximation framework developed above to a variety of examples. In addition to theoretical considerations, in each of the examples below, we discuss some numerical results for an infinite-time horizon problem of the form given in Problem (P2). All numerical studies were performed on an IBM Personal Computer. The machine we used was equipped with an Intel 8086 math co-processor chip and 640K bytes of random access memory (of which less than 384K was required).

Matrix exponentials were computed from eigenvalue-eigenvector decompositions obtained using the QR algorithm. The matrix Riccati equations (3.59) were solved using a Schur-vector decomposition of the Hamiltonian matrix (see [18][24]). It should be noted that if the eigenvalue pairs of the Hamiltonian matrix for a continuous-time linear-quadratic regulator problem are asymptotic to $\pm\gamma(n)$ as $n \to \infty$, then the eigenvalue pairs of the Hamiltonian matrix for the corresponding discrete-time problem will be asymptotic to $e^{\pm\gamma(n)\tau}$ as $n \to \infty$. Consequently, for all but very small τ, conditioning problems arise more quickly than in the continuous-time case when the approximating matrix algebraic Riccati equations are solved.

4.1. The Heat Equation with Boundary Input

In this example we consider the scalar parabolic system with boundary control given by

$$(4.1) \qquad \frac{\partial}{\partial s} w(s,x) = \frac{\partial}{\partial x} a(x) \frac{\partial}{\partial x} w(s,x), \quad s > 0,\ x \in (0,1)$$

(4.2) $w(s,0) = 0, \quad w(s,1) = v(s)$

(4.3) $w(0,x) = \phi(x)$

with $a \in H^1(0,1)$, $a(x) > 0$, $x \in [0,1]$, $\phi \in H^0(0,1) = L_2(0,1)$ and $v \in L_2(0,\infty)$.

To formulate the discrete-time state equation for this system we let τ denote the sampling interval and consider only piecewise constant controls v given by

(4.4) $v(s) = u(t) \qquad s \in [t\tau, (t+1)\tau),$

$t = 0, 1,2,\ldots$. We choose as our state space Z the Sobolev space $H^0(0,1)$ with the usual inner product

(4.5) $\langle\phi,\psi\rangle_Z = \langle\phi,\psi\rangle_0 = \int_0^1 \phi(\theta)\psi(\theta)d\theta.$

The state $z(t) \in Z$ is

(4.6) $z(t) = \lim_{s \to t\tau^-} w(s,\cdot), \quad t = 1,2, \ldots$

(4.7) $z(0) = \phi.$

For $t \in \{0,1,2\ldots\}$, we define $y(s) \in Z$ by

(4.8) $y(s) = w(s,\cdot) - \psi_0 u(t), \quad s \in (t\tau, (t+1)\tau)$

(4.9) $y(t\tau) = z(t) - \psi_0 u(t),$

where $\psi_0 \varepsilon Z$ is given by $\psi_0(x) = x$, $x \varepsilon [0,1]$. A straight forward calculation reveals that $y(s) = y(s,\cdot)$ satisfies

(4.10) $\dot{y}(s) = DaDy(s) + a'u(t), \quad s \varepsilon (t\tau, (t+1)\tau)$

(4.11) $y(s)|_0 = 0, \; y(s)|_1 = 0, \quad s \varepsilon (t\tau,(t+1)\tau)$

(4.12) $y(t\tau) = z(t) - \psi_0 u(t),$

where D denotes the differentiation operator on $H^1(0,1)$.

Let A: $\mathrm{dom}(A) \subset Z \to Z$ be given by

$$\mathrm{Dom}(A) = H^2(0,1) \cap H_0^1(0,1) = \{\phi \varepsilon H^0(0,1): \phi \varepsilon H^2(0,1), \\ \phi(0) = \phi(1) = 0\}$$

(4.13) $A\psi = DaD\psi$.

The operator A is densely defined and self-adjoint. It satisfies

(4.14) $\langle Az,z\rangle_Z \leq -\omega|z|_Z^2$, $z \varepsilon \mathrm{Dom}(A)$

for some $\omega > 0$ and has compact resolvent. Also, A is the infinitesimal generator of an analytic semigroup of contractions $\{T(s) : s \geq 0\}$ on Z which, in light of (4.14), satisfies $|T(s)| \leq e^{-\omega s}$, $s \geq 0$. It follows therefore, that

(4.15) $$y(s) = T(s-t\tau)y(t\tau) + \int_{t\tau}^{s} T(s-\sigma)a'd\sigma u(t), \quad s \in [t\tau, (t+1)\tau).$$

The continuity of y, (4.6), (4.8) and (4.9) imply

(4.16) $$z(t) = y(t\tau) + \psi_0 u(t)$$

and

(4.17) $$z(t+1) = y((t+1)\tau) + \psi_0 u(t),$$

and hence that

(4.18) $$z(t+1) = y((t+1)\tau) + \psi_0 u(t)$$

$$= T(\tau)(z(t) - \psi_0 u(t)) + \int_{t\tau}^{(t+1)\tau} T((t+1)\tau-\sigma)a'd\sigma u(t) + \psi_0 u(t).$$

Defining the operators $T \in L(Z)$ and $B \in L(R^1,Z)$ by

(4.19) $$Tz = T(\tau)z, \quad z \in Z$$

and

(4.20) $$Bu = [(I - T(\tau))\psi_0 + \int_0^{\tau} T(\sigma)a'd\sigma]u, \quad u \in R^1,$$

we obtain

(4.21) $$z(t+1) = Tz(t) + Bu(t), \; t = 0,1,2,..$$

(4.22) $z(0) = \phi$.

We take the performance index to be

(4.23) $$J(g_0;0,t_f,\phi,u) = \sum_{t=0}^{t_f-1} \{q_0|z(t)|_0^2 + ru(t)^2\} + g_0|z(t_f)|_0^2$$

with $q_0, g_0 \geq 0$ and $r > 0$.

Applying the theory developed in Section 2.1, we have, for the finite time interval problem, that the optimal control is given by

(4.24) $$u_*(t) = -F(t)z_*(t), \qquad t = 0, 1,2,\ldots,t_f-1,$$

where for each t, F(t) is the continuous linear functional on Z given by (2.8) - (2.11). It follows that F(t) has a representation $f(t,\cdot) \in H^0(0,1)$ and that

(4.25) $$F(t)\psi = \int_0^1 f(t,\cdot)\psi(\theta)d\theta \ ,$$

$\psi \in H^0(0,1)$, $t = 0,1,2,\ldots,t_f-1$.

For the infinite time interval problem ($t_f = \infty$, $g_0 = 0$), it is immediately clear that Hypothesis 2.4 is satisfied. It is also clear that (4.14) imples that for each $z(0) \in H^0(0,1)$, $u(t) = 0$, $t = 0,1,2,\ldots$ is an admissible control, and hence that there exists a unique nonnegative self-adjoint solution of the Riccati algebraic equation (2.14). From (2.17) - (2.19) we obtain

(4.26) $u_*(t) = -Fz_*(t), \quad t = 0,1,2,\ldots$

where F is a continuous linear functional on Z and

(4.27) $F\psi = \int_0^1 f(\theta)\psi(\theta)d\theta, \quad \psi \in H^0(0,1),$

with $f \in H^0(0,1)$.

We define an approximation scheme using a standard Ritz-Galerkin approach. We note that the operator $-A$ is coercive and that it can be written as $-A = L^*L$ where the operator L and its adjoint L^* are given by

(4.28) $L\psi = a^{1/2}D\psi, \quad \psi \in H_0^1(0,1),$

and

(4.29) $L^*\psi = -Da^{1/2}\psi, \quad \psi \in H^1(0,1),$

respectively. We define the space $V = H_0^1(0,1)$ together with the inner product

(4.30) $\langle\phi,\psi\rangle_V = \langle L\phi, L\psi\rangle_Z, \quad \phi,\psi \in V.$

We note that V is the energy space associated with the operator $-A$, $V = \mathrm{Dom}((-A)^{1/2})$, and it is the closure of Dom(A) with respect to the energy norm $|\cdot|_V$, which satisfies

$$(4.31)\qquad |\phi|_V^2 = \langle L\phi, L\phi\rangle_Z = \langle -A\phi,\phi\rangle_Z, \qquad \phi \in \mathrm{Dom}(A).$$

For each N = 2,3,... let Δ_N denote the uniform partition of the interval [0,1] given by $\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1\}$. Let $\{e_N^j\}_{j=1}^{N-1}$ denote the usual linear B-splines on [0,1] corresponding to the partition Δ_N and which satisfy $e_N^j(0) = e_N^j(1) = 0$, j = 1,2,... N-1. The e_N^j are given by

$$(4.32)\quad e_N^j(\theta) = \begin{cases} N(\theta - \frac{(j-1)}{N}) & \theta \in [\frac{j-1}{N}, \frac{j}{N}] \\ N(\frac{(j+1)}{N} - \theta) & \theta \in [\frac{j}{N}, \frac{j+1}{N}] \\ 0 & \text{elsewhere} \end{cases}$$

j = 1,2,...,N-1 and are depicted in the figure below.

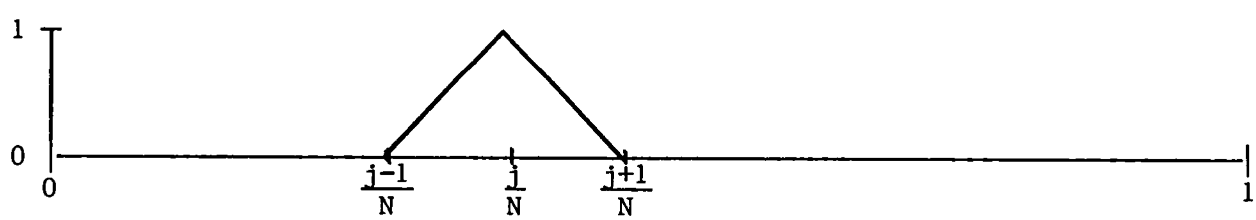

Figure 4.1

Let $Z_N = \mathrm{span}\{e_N^j\}_{j=1}^{N-1}$. We note that $Z_N \subset V$ and dim Z_N = N-1, N = 2,3,... . Define $P_N : Z \to Z_N$ to be the orthogonal projection of Z onto Z_N with respect to the $\langle\ ,\ \rangle_Z$ inner product and $\mathcal{P}_N : V \to Z_N$ to be the orthogonal projection of V onto Z_N with respect to the $\langle\ ,\ \rangle_V$ inner product.

We define the operator $A_N : Z_N \to Z_N$ as the inverse of the operator given by

$$(4.33) \qquad A_N^{-1} = P_N \, A^{-1}|_{Z_N}.$$

The invertibility of A of course follows from the coercivity of $-A$ (see (4.14)). On the other hand, a straight forward calculation yields

$$(4.34) \qquad \langle A_N^{-1} z_N, z_N \rangle_V = |z_N|_Z^2 , \qquad z_N \in Z_N.$$

Consequently the operator A_N^{-1} is invertible and the operator A_N is well defined. It is also self-adjoint. Indeed

$$(4.35) \qquad \langle A_N z_N, y_N \rangle_Z = - \langle z_N, y_N \rangle_V , \qquad z_N, y_N \in Z_N.$$

From (4.35) it also follows that

$$(4.36) \qquad \langle A_N z_N, z_N \rangle_Z \leq -\omega |z_N|_Z^2 , \qquad z_N \in Z_N.$$

It can be concluded therefore, that A_N is the infinitesimal generator of a semigroup of bounded linear operators on Z_N, $\{ T_N(s) : s \geq 0\}$ with $T_N(s) = \exp(A_N s)$, $s \geq 0$ and satisfying

$$(4.37) \qquad |T_N(s)|_Z \leq e^{-\omega s}, \; s \geq 0.$$

Elementary properties of spline functions (see [27]) imply $P_N \to I$ as $N \to \infty$, strongly on V. Furthermore, A^{-1} compact and

$$(4.38) \qquad |P_N A^{-1}z - A^{-1}z|_Z < |P_N A^{-1}z - A^{-1}z|_V = |(P_N - I) A^{-1}z|_V$$

imply that $P_N A^{-1} \to A^{-1}$ as $N \to \infty$ in the uniform operator topology on $L(Z)$. We have therefore, that

$$(4.39) \qquad |A_N^{-1} P_N - A^{-1}|_Z \to 0$$

as $N \to \infty$.

From (4.37) and (4.39) we conclude (see [4], [15])

$$(4.40) \qquad T_N(s)P_N z \to T(s)z$$

and

$$(4.41) \qquad T_N^*(s)P_N z \to T^*(s)z$$

as $N \to \infty$ for each $z \in Z$, uniformly in s for s in compact subintervals.

With $T_N = T_N(\tau)$, $Q_N = q_0 P_N$, $G_N = g_0 P_N$ and

$$(4.42) \qquad B_N = (I - T_N)P_N\psi + \int_0^\tau T_N(\sigma)P_N a' d\sigma,$$

(4.40), (4.41) and elementary approximation properties of spline functions guarantee that Hypotheses 3.1 and 3.2 hold and hence that the convergence results for the finite time interval problem given in Theorem 3.3 apply.

For the infinite time interval Problem, (4.37) implies that Hypothesis 3.8 is satisfied. Moreover, if $q_0 > 0$ the conditions given in the statement of Theorem 3.10 are satisfied (see the Remark following the proof of Theorem 3.10) and consequently the convergence results for the infinite time horizon problem given in Theorem 3.9 hold.

Define the R^{N-1} vector valued function E_N on $[0,1]$ by

$$E_N(\theta)^T = (e_N^1(\theta),\ e_N^2(\theta),\dots,\ e_N^{N-1}(\theta)), \tag{4.43}$$

$\theta \in [0,1]$ and the $(N-1) \times (N-1)$ Gram matrix associated with the basis elements $\{e_N^j\}_{j=1}^{N-1}$

$$M_N = \langle E_N,\ E_N^T\rangle_Z \ . \tag{4.44}$$

A straight forward calulation yields

$$M_N = \frac{1}{N}\begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 & \cdots & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \cdots & 0 & \frac{1}{6} & \frac{2}{3} \end{bmatrix} \tag{4.45}$$

Let the $(N-1) \times (N-1)$ matrix H_N be given by

$$H_N = \langle E_N,\ A_N E_N^T\rangle_Z \ . \tag{4.46}$$

Then

$$\begin{aligned} [H_N]_{ij} &= \langle e_N^i,\ A_N\, e_N^j\rangle_Z \\ &= -\langle e_N^i\ ,\ e_N^j\rangle_V \end{aligned} \tag{4.47}$$

$$= - \langle Le_N^i, Le_N^j \rangle_Z$$

$$= - \langle aDe_N^i, De_N^j \rangle_Z .$$

In the case $a(x) = a$, $x \in [0,1]$, a constant, we have

$$(4.48) \qquad H_N = aN \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}$$

The matrix representation $[A_N]$ for the operator A_N is given by

$$(4.49) \qquad [A_N] = M_N^{-1} H_N,$$

while for the operators T_N, Q_N and G_N we have

$$(4.50) \qquad [T_N] = \exp([A_N]\tau),$$

$$(4.51) \qquad [Q_N] = q_0 I_N$$

and

$$(4.52) \qquad [G_N] = g_0 I_N$$

where I_N denotes the $(N - 1) \times (N - 1)$ identity matrix. If we define ψ_{0N}, $a'_N \in R^{N-1}$ by

(4.53) $$\psi_{0N} = \langle E_N, \psi_0 \rangle_Z$$

and

(4.54) $$a'_N = \langle E_N, a' \rangle_Z,$$

j = 1,2,...,N-1 we obtain

(4.55) $$[B_N] = (I_N - [T_N])M_N^{-1}\psi_{0N} + \int_0^\tau \exp([A_N]\sigma)M_N^{-1}a'_N d\sigma$$

$$= (I_N - [T_N])M_N^{-1}\psi_{0N} + [A_N]^{-1}([T_N] - I_N)M_N^{-1}a'_N .$$

When the finite dimensional approximating gain matrices $[F_N(t)]$, t = 0,1,2,...t_f-1 for the finite time interval problem have been computed using (3.51)-(3.54), an approximation $f_N(t,\cdot)$ to the feedback kernal $f(t,\cdot)$ can be obtained from

(4.56) $$f_N(t,\theta) = E_N(\theta)^T M_N^{-1}[F_N(t)]^T,$$

$t = 0,1,2,\ldots t_f-1$, $\theta \in [0,1]$. We have

(4.57) $$f_N(t,\cdot) \to f(t,\cdot)$$

in $H^0(0,1)$ as $N \to \infty$ for each $t = 0,1,2,...t_f-1$.

Similarly, for the infinite time interval problem, an approximation f_N to f is given by

$$(4.58) \qquad f_N(\theta) = E_N(\theta)^T M_N^{-1} [F_N]^T,$$

$\theta \in [0,1]$ where the matrix $[F_N]$ is computed from (3.57) - (3.59). We have

$$(4.59) \qquad f_N \to f$$

in $H^0(0,1)$ as $N \to \infty$.

We demonstrate the feasibility of our schemes on an infinite time interval problem of the form discussed above. Taking $q_0 = 1.0$, $r = 1.0$, $a(x) = a = 1.0$, $x \in [0,1]$ and $\tau = .01$ we obtained the approximating feedback kernals given in Table 4.2 and shown in Figure 4.3 below.

θ	$f_3(\theta)$	$f_5(\theta)$	$f_9(\theta)$	$f_{11}(\theta)$	$f_{13}(\theta)$
0.00	.0000	.0000	.0000	.0000	.0000
0.05	.0250	.0228	.0224	.0222	.0221
0.10	.0500	.0456	.0449	.0445	.0443
0.15	.0750	.0684	.0673	.0668	.0664
0.20	.1000	.0911	.0898	.0891	.0887
0.25	.1250	.1147	.1122	.1115	.1109
0.30	.1500	.1382	.1346	.1338	.1331
0.35	.1727	.1617	.1571	.1561	.1554
0.40	.1908	.1853	.1798	.1787	.1778
0.45	.2088	.2129	.2028	.2013	.2005
0.50	.2269	.2405	.2286	.2261	.2245
0.55	.2450	.2681	.2543	.2513	.2496
0.60	.2630	.2958	.2819	.2793	.2769
0.65	.2811	.2956	.3097	.3056	.3035
0.70	.2584	.2954	.3246	.3275	.3275
0.75	.2153	.2953	.3331	.3352	.3385
0.80	.1723	.2951	.3169	.3260	.3263
0.85	.1292	.2213	.2697	.2163	.2170
0.90	.0861	.1475	.2097	.2163	.2170
0.95	.0431	.0738	.1049	.1125	.1175
1.00	.0000	.0000	.0000	.0000	.0000

Table 4.2

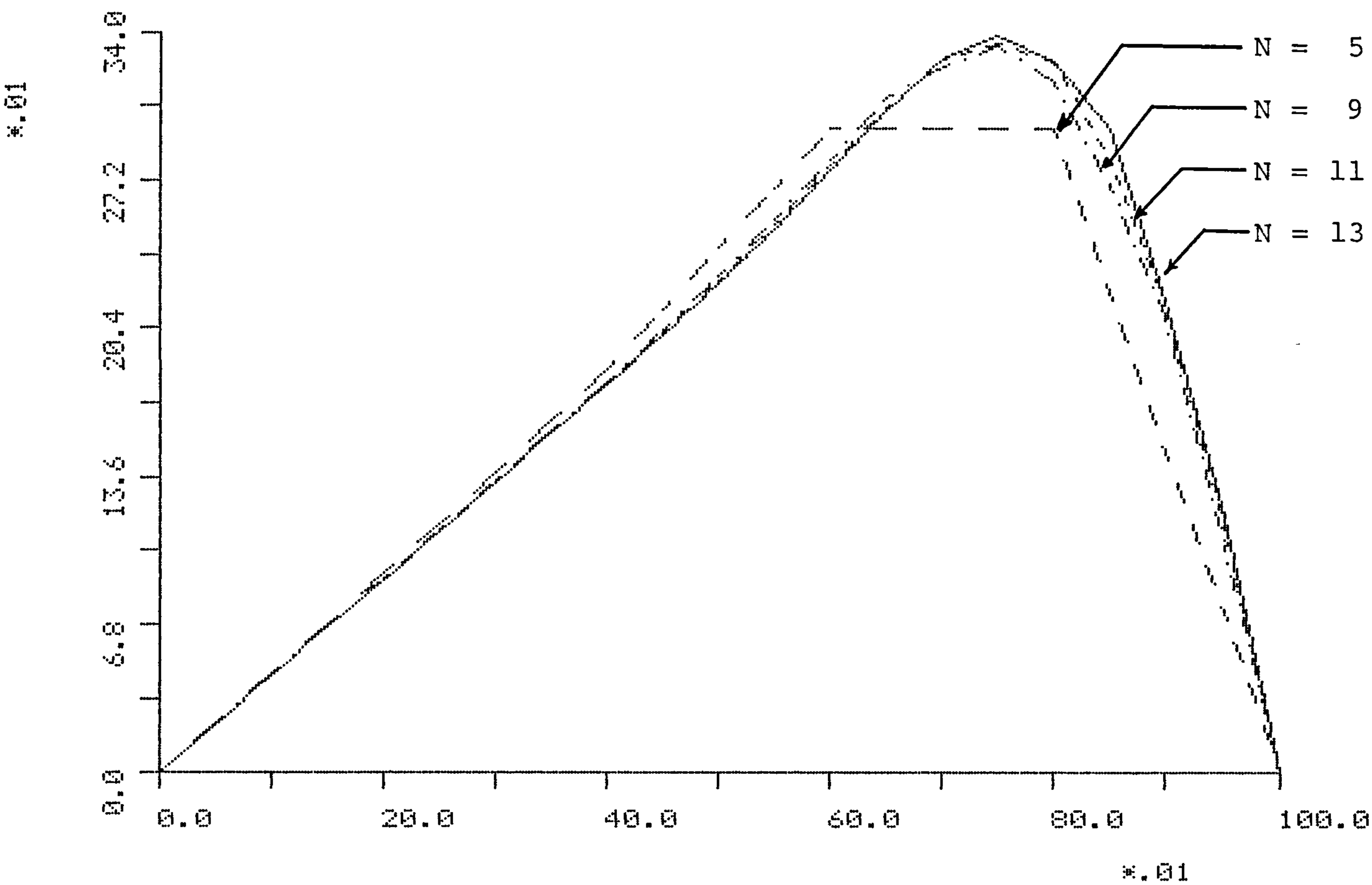

Figure 4.3

4.2. Hereditary or Time Delay Systems

In this example we consider linear hereditary control systems of the form

$$(4.60) \qquad \dot{x}(s) = Lx_s + B_0 v(s) \qquad s > 0$$

where $x(s) \in R^n$, $x_s \in L_2((-r,0);R^n)$ for some $r > 0$, $v \in L_2((0,s_1);R^m)$ for all $s_1 < \infty$ and $B_0 \in L(R^m,R^n)$. The function x_s represents the history on the interval $[s-r,s]$; that is $x_s(\theta) = x(s+\theta)$, $\theta \in [-r,0]$. The operator L is assumed to be of the form

$$(4.61) \qquad L\phi = \sum_{i=0}^{\nu} A_i\phi(-r_i) + \int_{-r}^{0} A(\theta)\phi(\theta)d\theta$$

where $A_i \in L(R^n)$, $i = 0,1,2,\ldots,\nu$, $A(\cdot) \in L_2((-r,0);\ L(R^N))$ and $0 = r_0 < r_1 < \cdots < r_\nu = r$. The initial data is given by

$$(4.62) \qquad x(0) = \eta, \quad x_0 = \phi,$$

where $\eta \in R^n$ and $\phi \in L_2((-r,0);\ R^n)$.

Once again, we formulate the discrete-time control problem by letting τ denote the sampling interval and considering piecewise constant control inputs of the form

$$(4.63) \qquad v(s) = u(t) \qquad s \in [t\tau,\ (t+1)\tau),$$

$t = 0,1,2,\ldots$, where $u(t) \in R^m$ for each t. The state space is

$$(4.64) \qquad Z = R^n \times L_2((-r,0);R^n)$$

with the inner product

$$(4.65) \quad \langle(\eta,\phi),\ (\xi,\psi)\rangle_Z = \langle\eta,\xi\rangle_{R^n} + \langle\phi,\psi\rangle_{L_2}.$$

The state of the discrete-time control system is

$$(4.66) \quad z(t) = (x(t\tau;\eta,\phi,v),\ x_{t\tau}(\eta,\phi,v)), \qquad t = 0,1,2,\ldots,$$

where $x(s;\eta,\phi,v)$ denotes the solution at time s to the system (4.60) - (4.62) and $x_s(\eta,\phi,v)$ its history on $[s-r,s]$. Then

$$(4.67) \quad z(t+1) = \mathcal{T}(\tau)z(t) + \int_0^\tau \mathcal{T}(\sigma)\,\mathcal{B}\,d\sigma u(t), \qquad t = 0,1,2,\ldots,$$

$$(4.68) \qquad z(0) = (\eta,\phi)$$

where $\{\mathcal{T}(s) : s \geq 0\}$ is the C_0 semigroup of bounded linear operators on Z with infinitesimal generator $\mathcal{A}$: $\text{Dom}(\mathcal{A}) \subset Z \to Z$ given by

$$\text{Dom}\,(\mathcal{A}) = \{(\xi,\psi) \in Z : \psi \in H^1((-r,0);\ R^n),\ \xi = \psi(0)\}$$

$$(4.69) \qquad \mathcal{A}(\psi(0),\ \psi) = (L\psi,\ D\psi),$$

and the operator $\mathcal{B}$: $R^m \to Z$ is defined by

$$(4.70) \qquad \mathcal{B}u = (B_0u,\ 0),\ u \in R^m.$$

Letting $T = \mathcal{T}(\tau)$ and $B = \int_0^\tau \mathcal{T}(\sigma)\,\mathcal{B}\,d\sigma$, we obtain

$$z(t+1) = Tz(t) + Bu(t), \quad t = 0,1,2,\ldots$$
$$z(0) = (\eta,\phi). \tag{4.71}$$

Let Q_0, $G_0 \in L(R^n)$ be symmetric and nonnegative and let $R \in L(R^m)$ be symmetric and positive definite. We consider the discrete-time linear-quadratic regulator problem with state given by (4.71) and performance index

$$J(G;0,t_f,\eta,\phi,u) = \sum_{t=0}^{t_f-1} \{\langle Qz(t),z(t)\rangle_Z + \langle Ru(t),u(t)\rangle_{R^m}\} + \langle Gz(t_f),z(t_f)\rangle_Z \tag{4.72}$$

where the operators $Q : Z \to Z$ and $G : Z \to Z$ are given by

$$Q(\xi,\psi) = (Q_0\xi,0) \tag{4.73}$$

and

$$G(\xi,\psi) = (G_0\xi,0) \tag{4.74}$$

respectively. For the infinite time problem we of course have $t_f = \infty$ and $G_0 = 0$.

For the finite time problem with the operators, T,B,Q and G as defined above, we apply the theory developed in Section 2.1 and conclude that the optimal control is given by

$$u_*(t) = -F(t)z_*(t), \qquad t = 0,1,2,\ldots,t_f-1, \tag{4.75}$$

where the operators $F(t) \varepsilon L(Z,R^n)$ are given by (2.8) - (2.11). The operator $F(t)$ can be represented by a matrix of operators, $[F^0(t), F^1(t)]$ with $F^0(t) \varepsilon L(R^n,R^m)$ and $F^1(t) \varepsilon L(L_2((-r,0); R^n); R^m)$. It follows from (4.70) therefore that

$$(4.76) \quad u_*(t) = -f^0(t)x_*(t\tau) - \int_{-r}^{0} f^1(t,\theta)(x_*)_{t\tau}(\theta)d\theta, \quad t = 0,1,2,\ldots,t_f-1,$$

where $f^0(t)$ is an $m \times n$ matrix and $f^1(t,\cdot)$ is a square integrable $m \times n$ matrix valued function on $(-r,0)$.

For the infinite time problem we assume that our original hereditary system and Q_0 are such that there exists an admissible control for each $z(0) = (\eta,\phi) \varepsilon Z$ and that Hypothesis 2.4 is satisfied. Then Theorems 2.3 and 2.5 imply that there exists a unique nonnegative self-adjoint solution Π to the Riccati algebraic equation (2.14) with the optimal control u_* given by

$$(4.77) \qquad u_*(t) = -Fz_*(t), \qquad t = 0,1,2,\ldots,$$

where $F \varepsilon L(Z,R^m)$ is given by (2.18) - (2.19). The feedback gain F can be represented by a matrix of operators $[F^0,F^1]$ with $F^0 \varepsilon L(R^n,R^m)$ and $F^1 \varepsilon L(L_2((-r,0);R^n),R^m)$. We have

$$(4.78) \quad u_*(t) = -f^0x_*(t\tau) - \int_{-r}^{0} f^1(\theta)(x_*)_{t\tau}(\theta)d\theta, \qquad t = 0,1,2,\ldots,$$

where f^0 is an $m \times n$ matrix and f^1 is a square integrable $m \times n$ matrix valued function on $(-r,0)$.

Numerical methods for the approximate solution of the continuous time linear quadratic regulator problem for hereditary systems in closed-loop form have been studied extensively (see [7],[16],[19],[25],[26]). Most closely related to the approximation framework which we have developed here for the discrete-time problem are the treatments for the continuous-time problem in [2],[11],[14] and [17]. The first approximation scheme applied to the continuous-time linear-quadratic control problem for hereditary systems was the AVE scheme used in [1],[11],[25] and [26] which approximates the history by piecewise constant functions and derivatives with finite differences. In [1], Banks and Burns cast AVE in its modern semigroup approximation form, proved strong convergence of the approximating open-loop semigroups and used the scheme to compute the open-loop optimal control. Gibson [11] used the convergence results in [1] to prove strong convergence of the solutions to the approximating continuous-time Riccati equations and uniform norm convergence of the approximating feedback control laws. However, the rate of convergence as it was observed in numerical studies was relatively slow. Spline based schemes for continuous- time systems were developed for the finite time interval problem in [17] and for the infinite time interval problem in [2]. With regard to the minimization of the performance index these schemes represented an improvement over the AVE scheme. However, they appeared to yield only weak convergence of the solutions to the approximating Riccati equations and strong operator convergence of the approximating feedback gains.

Recently, a new spline-based state approximation for hereditary systems has been proposed in [14]. When applied to the continuous time control problem, this new method performs at the level of the earlier spline-based schemes and yields strong convergence of the approximating feedback gains. We have chosen this scheme to describe and implement here for the discrete-time problem.

To simplify the presentation, we consider systems of the form (4.60) having only a single discrete delay ($\nu = 1$) and no distributed delay term ($A(\theta) \equiv 0$). A detailed description of the scheme in complete generality can be found in [14].

For each $N = 1,2,\ldots$ define

$$\hat{e}_N^0 = (I_n, 0) \quad \text{and} \quad \hat{e}_N^j = (0, e_N^j), \quad j = 1,2,\ldots N+1, \tag{4.79}$$

where I_n denotes the $n \times n$ identity matrix and the e_N^j are the $n \times n$ matrix valued piecewise linear elements on $[-r,0)$ given by

$$e_N^1(\theta) = \begin{cases} \frac{N}{r}(\theta + \frac{r}{N})I_n & \theta \in [\frac{-r}{N}, 0) \\ 0 & \text{elsewhere} \end{cases}$$

$$e_N^j(\theta) = \begin{cases} \frac{N}{r}(\theta + j\frac{r}{N})I_n & \theta \in [-j\frac{r}{N}, -(j-1)\frac{r}{N}) \\ \frac{-N}{r}(\theta + (j-2)\frac{r}{N})I_n & \theta \in [-(j-1)\frac{r}{N}, -(j-2)\frac{r}{N}) \\ 0 & \text{elsewhere} \end{cases} \tag{4.80}$$

$j = 2,3,4,\ldots N$

$$e_N^{N+1}(\theta) = \begin{cases} \frac{-N}{r}\,(\theta + (N-1)\,\frac{r}{N})I_n & \theta \,\varepsilon\, [-r,-(n-1)\,\frac{r}{N}) \\ 0 & \text{elsewhere}\,. \end{cases}$$

Let

$$(4.81) \qquad Z_N = \{z \,\varepsilon\, Z : z = \sum_{j=0}^{N+1} \alpha_j\, \hat{e}_N^j \;,\; \alpha_j \,\varepsilon\, R^n\}.$$

Note that dim $Z_N = n(N+2)$. We shall refer to the collection $\{\hat{e}_N^j\}_{j=0}^{N+1}$ as a "basis" for Z_N and a vector $\alpha \,\varepsilon \times_{j=0}^{N+1} R^n$ as being a "coordinate vector" for an element in Z_N. Defining

$$(4.82) \qquad \hat{E}_N^T = (\hat{e}_N^0,\, \hat{e}_N^1, \cdots\, \hat{e}_N^{N+1})$$

we have

$$(4.83) \qquad Z_N = \{z \,\varepsilon\, Z : z = \hat{E}_N^T \alpha,\, \alpha \,\varepsilon \times_{j=0}^{N+1} R^n\}.$$

Let M_N denote the Gram matrix corresponding to the basis $\{\hat{e}_N^j\}_{j=0}^{N+1}$. A straight forward computation yields

$$(4.84) \qquad M_N = \text{diag}\,(I_n,\, \frac{r}{N}\, m_N \otimes I_n)$$

where the $(N+1) \times (N+1)$ matrix m_N is given by

(4.85) $$m_N = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & 0 & \cdots & & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \cdots & & 0 & \frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

and $\otimes$ denotes the Kronecker product.

Let P_N denote the orthogonal projection of Z on to Z_N with respect to the inner product (4.65). It follows that

(4.86) $$P_N(\xi,\psi) = (\xi, p_N\psi)$$

where p_N is the orthogonal projection of $L_2((-r,0);R^n)$ on to $\text{span}\{e_N^j\}_{j=1}^{N+1}$ with respect to the usual L_2 inner product. We have

(4.87) $$P_N(\xi,\psi) = \hat{E}_N^T \alpha_N(\xi,\psi)$$

where $\alpha_N(\xi,\psi) \in \overset{N+1}{\underset{j=0}{\times}} R^n$ is given by

(4.88) $$\alpha_N(\xi,\psi) = M_N^{-1} \text{ col } (\xi, \psi_N^1, \psi_N^2, \ldots \psi_N^{N+1})$$

with

(4.89) $$\psi_N^j = \int_{-r}^0 e_N^j(\theta)\psi(\theta)d\theta.$$

We shall set $T_N = \exp(A_N\tau)$ where $A_N : Z_N \to Z_N$ is an appropriately defined finite dimensional approximation to the operator A given by (4.69). Noting that $Z_N \not\subset \mathrm{Dom}(A)$, we motivate the definition of A_N by first formally extending the operator A to an operator defined on Z_N.

For $z_N = (\xi_N, \psi_N) \in Z_N$ define

$$(4.90) \quad A^N z_N = (A_0\xi_N + A_1\psi_N(-r),\ D^+\psi_N + \delta(\xi_N - \lim_{\theta\to 0^-}\psi_N(\theta)))$$

where δ is the Dirac delta impulse concentrated at zero and $D^+\psi$ denotes the right hand derivative of ψ. For each N = 1,2... let $A_N : Z_N \to Z_N$ be given by

$$(4.91) \quad A_N z_N = (A_0\xi_N + A_1\psi_N(-r),\ p_N D^+\psi_N) + \delta_N(\xi_N - \lim_{\theta\to 0^-}\psi_N(\theta))$$

where

$$(4.92) \quad \delta_N = \hat{E}_N^T \gamma_N$$

with

$$(4.93) \quad \gamma_N = M_N^{-1} \operatorname{col}\,(0, \lim_{\theta\to 0^-} e_N^1(\theta), \cdots, \lim_{\theta\to 0^-} e_N^{N+1}(\theta)).$$

To compute $[A_N]$, the matrix representation for the operator A_N, we let $[z_N]$ denote the coordinate vector representation for an element $z_N \in Z_N$. Then from

$$(4.94) \quad [A_N z_N] = [A_N][z_N]$$

and

(4.95) $$[A_N z_N] = M_N^{-1}\ \alpha_N(A_N z_N) = M_N^{-1}\ \langle \hat{E}_N, A_N z_N \rangle_Z = M_N^{-1}\ \langle \hat{E}_N, A_N \hat{E}_N^T \rangle_Z [z_N]$$

we obtain

(4.96) $$[A_N] = M_N^{-1}\ H_N$$

where

(4.97) $$H_N = \langle \hat{E}_N, A_N \hat{E}_N^T \rangle_Z \ .$$

Using the definitions of A_N and $\hat{E}_N$ a straight forward calculation (see [14]) yields

(4.98) $$H_N = \left[\begin{array}{c|ccc} A_0 & 0 & 0 & A_1 \\ \hline I_n & & & \\ 0 & & h_N \otimes I_n & \\ & & & \end{array}\right]$$

where the $(N+1) \times (N+1)$ matrix h_N is given by

$$(4.99) \qquad h_N = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & \cdots & \cdots & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \cdots & \cdots & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} .$$

The matrix representation for the operator $T_N = \exp(\mathcal{A}_N \tau)$ can then be computed from

$$(4.100) \qquad [T_N] = \exp([\mathcal{A}_N]\tau).$$

We define the operators $\mathcal{B}_N : R^m \to Z_N$, $Q_N : Z_N \to Z_N$ and $G_N : Z_N \to Z_N$ by

$$(4.101) \qquad \mathcal{B}_N = P_N \mathcal{B}$$

$$(4.102) \qquad Q_N = P_N Q$$

and

$$(4.103) \qquad G_N = P_N G$$

from which we obtain

(4.104) $$[\mathcal{B}_N] = \text{col}\ (B_0,\ 0,\ldots,0)\ \varepsilon\ R^{n(N+2)\times m}$$

(4.105) $$[Q_N] = \begin{bmatrix} Q_0 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{bmatrix} \varepsilon\ R^{n(N+2)\times n(N+2)}$$

and

(4.106) $$[G_N] = \begin{bmatrix} G_0 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{bmatrix} \varepsilon\ R^{n(N+2)\times n(N+2)}$$

Finally, defining

(4.107) $$B_N = \int_0^\tau \exp\ (\mathcal{A}_N s)\ \mathcal{B}_N ds$$

we have

(4.108) $$[B_N] = \int_0^\tau \exp\ ([\mathcal{A}_N]s)[\mathcal{B}_N]ds.$$

Once the matrix representations for the approximating feedback gains have been computed, $[F_N(t)]$, $t = 0,1,2,\ldots\ t_f-1$ from (3.51) - (3.54) for the finite time interval problem and $[F_N]$ from (3.57) - (3.59) (assuming, for the moment, that solutions to (3.34) exist) for the infinite time interval problem, approximations for f^0, $f^1(t,\cdot)$, $t = 01,2,\ldots t_f-1$ and $f^0, f^1(\cdot)$ can be computed from

$$(4.109)\quad ((f_N^0(t))^T,\ (f_N^1(t,\cdot))^T) = \hat{E}_N^T M_N^{-1}[F_N(t)]^T, \qquad t = 0,1,2,\ldots,t_f-1,$$

and

$$(4.110)\quad ((f_N^0)^T,\ (f_N^0(\cdot))^T) = \hat{E}_N^T M_N^{-1}[F_N]^T$$

respectively.

For the approximation scheme defined above, it is shown in [14] that $P_N \to I$ strongly on Z. Using a Trotter-Kato like result it is also shown that

$$(4.111)\qquad \exp\ (\mathcal{A}_N s)P_N \to \mathcal{T}(s)$$

and

$$(4.112)\qquad \exp(\mathcal{A}_N^* s)P_N \to \mathcal{T}^*(s)$$

strongly on Z and uniformly in s for s in compact intervals. Hypothesis 3.1 is a simple consequence of these results. The present scheme, therefore, satisfies all of the hypotheses of Theorem 3.3 and we may conclude that the convergence results for the finite time interval problem given in the statement of the theorem hold. In particular, we have

$$(4.113)\quad f_N^0(t) \to f^0(t)$$

in $R^{m \times n}$ and

$$(4.114) \quad f_N^1(t,\cdot) \to f^1(t,\cdot)$$

in $L_2((-r,0);R^{m \times n})$ for each $t = 0,1,2,\ldots t_f - 1$.

With the operators Q and G given by (4.73) and (4.74) and the operators Q_N and G_N defined as in (4.102) and (4.103) it is clear that the hypotheses given in the statement of Theorem 3.7 are satisfied. We have therefore that for the present example the operators $\{\Pi(t)\}_{t=0}^{t_f}$ are trace class and

$$(4.115) \qquad \lim_{N\to\infty} \|\Pi_N(t)P_N - \Pi(t)\|_1 = 0, \qquad t = 0,1,2,\ldots,t_f.$$

For the infinite time problem and the approximation scheme discussed here, the situation with regard to convergence is much the same as it is for the continuous time problem (see [14]). We are unable to demonstrate the existence of an M and an $r < 1$ for which (3.38) and (3.39) hold. In fact, our numerical studies point to the conclusion that condition (3.39) is violated by the present scheme. We observe the existence of a sequence of closed-loop eigenvalues of the approximating discrete-time control problems $(\mathcal{P}2_N)$ which tend toward the unit circle as $N \to \infty$.

On the other hand, upon solving the approximating problems it is also apparent that $|\Pi_N|$ remains bounded in N. Consequently we may apply Theorem 3.12 to conclude that a solution Π to (2.14) does in fact exist, $\Pi_N P_N \to \Pi$ weakly and $F_N P_N \to F$ strongly as $N \to \infty$. We have therefore that

$$(4.116) \qquad f_N^0 \to f^0$$

in $R^{m\times n}$ and

$$(4.117) \qquad f_N^1 \to f^1$$

weakly in $L_2((-r,0);R^{m\times n})$ as $N \to \infty$.

We applied the scheme to the infinite-time problem with state

$$(4.118) \qquad \ddot{y}(s) + y(s-1) = v(s).$$

Transforming (4.118) to an equivalent first order system we obtain a system of the form given in (4.60), (4.61) with $n = 2$, $r = 1$, $m = 1$,

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A = 0 \text{ and } x(s) = \begin{bmatrix} y(s) \\ \dot{y}(s) \end{bmatrix}.$$

Taking $Q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ the performance index takes the form

$$(4.119) \quad J(0;0,\infty,y(0),\dot{y}(0),y_0,\dot{y}_0,u) = \sum_{t=0}^{\infty} y(t\tau)^2 + \dot{y}\,(t\tau)^2 + Ru(t)^2$$

where τ is the length of the sampling interval. The optimal feedback control is given by

$$(4.120) \quad \begin{aligned} u_*(t) = \; & -[f^0]_1 y(t\tau) - [f^0]_2 \dot{y}(t\tau) \\ & - \int_{-1}^{0} \{[f^1(\theta)]_1 y(t\tau+\theta) + [f^1(\theta)]_2 \dot{y}(t\tau+\theta)\} d\theta \end{aligned}$$

where $[f^0]_i$ and $[f^1(\theta)]_i$, $i = 1,2$ are the i^{th} components of the 1×2 matrices f^0 and $f^1(\theta)$.

Since by taking the initial conditions

$$(4.121) \quad y(0) = 0, \quad \dot{y}(0) = 0, \quad y_0(\theta) = 0, \ -1 \leq \theta \leq 0$$

we have $y(s) = 0$, $s \geq 0$ regardless of how $\dot{y}_0(\theta)$, $-1 \leq \theta \leq 0$ is chosen, it follows that $[f^1(\theta)]_2 = 0$, $-1 \leq \theta \leq 0$. Indeed, the optimal control corresponding to the initial conditions (4.121) with $\dot{y}_0$ arbitrary is $u(t) = 0$, $t = 0,1,2,\ldots$. Furthermore, the nature of the approximation scheme is such that we must have $[f_N^1(\theta)]_2 = 0$, $-1 \leq \theta \leq 0$, $N \geq 1$.

Setting $\tau = .01$ we obtained the results given in Tables 4.4 and 4.5 and Figure 4.8 below when R = .05. With R = 1.0, the results given in Tables 4.6 and 4.7 and Figure 4.9 were obtained. As the cost of control increases the effect that the optimal control for the infinite dimensional problem has on higher modes decreases. Consequently, the finite dimensional approximations are more effective and convergence is more rapid.

N	2	4	8	10
$[f_N^0]_1$	4.5483	4.5452	4.5451	4.5451
$[f_N^0]_2$	5.2954	5.2948	5.2948	5.2948

Table 4.4

θ	$[f_2^1(\theta)]_1$	$[f_4^1(\theta)]_1$	$[f_8^1(\theta)]_1$	$[f_{10}^1(\theta)]_1$
.00	.1277	.0942	.0878	.0872
-.05	.0948	.0690	.0659	.0658
-.10	.0619	.0437	.0439	.0445
-.15	.0289	.0185	.0163	.0154
-.20	-.0040	-.0068	-.0171	-.0137
-.25	-.0369	-.0321	-.0506	-.0556
-.30	-.0698	-.1046	-.1054	-.0976
-.35	-.1027	-.1772	-.1603	-.1650
-.40	-.1357	-.2497	-.2375	-.2324
-.45	-.1686	-.3223	-.3371	-.3361
-.50	-.2015	-.3949	-.4367	-.4399
-.55	-.5891	-.6156	-.6090	-.6054
-.60	-.9767	-.8363	-.7813	-.7709
-.65	-1.3643	-1.0570	-1.0200	-1.0283
-.70	-1.7519	-1.2777	-1.3251	-1.2858
-.75	-2.1395	-1.4984	-1.6301	-1.6899
-.80	-2.5271	-2.1836	-2.1603	-2.0941
-.85	-2.9147	-2.8688	-2.6905	-2.7232
-.90	-3.3023	-3.5540	-3.4158	-3.3523
-.95	-3.6899	-4.2391	-4.3361	-4.3269
-1.00	-4.0775	-4.9243	-5.2565	-5.3017

Table 4.5

N	2	4	8	10
$[f_N^0]_1$	1.4050	1.4054	1.4054	1.4054
$[f_N^0]_2$	1.9477	1.9479	1.9479	1.9479

Table 4.6

θ	$[f_2^1(\theta)]_1$	$[f_4^1(\theta)]_1$	$[f_8^1(\theta)]_1$	$[f_{10}^1(\theta)]_1$
.00	-.2813	-.2831	-.2835	-.2835
-.05	-.3433	-.3402	-.3383	-.3379
-.10	-.4052	-.3972	-.3931	-.3923
-.15	-.4052	-.4543	-.4522	-.4534
-.20	-.5292	-.5114	-.5156	-.5146
-.25	-.5911	-.5684	-.5790	-.5798
-.30	-.6531	-.6439	-.6478	-.6450
-.35	-.7151	-.7195	-.7165	-.7179
-.40	-.7770	-.7950	-.7902	-.7907
-.45	-.8390	-.8705	-.8689	-.8685
-.50	-.9010	-.9460	-.9476	-.9462
-.55	-1.0044	-1.0347	-1.0328	-1.0324
-.60	-1.1078	-1.1233	-1.1181	-1.1185
-.65	-1.2112	-1.2120	-1.2089	-1.2103
-.70	-1.3146	-1.3007	-1.3053	-1.3021
-.75	-1.4180	-1.3894	-1.4016	-1.4030
-.80	-1.5214	-1.5013	-1.5057	-1.5040
-.85	-1.6248	-1.6132	-1.6098	-1.6112
-.90	-1.7282	-1.7251	-1.7198	-1.7185
-.95	-1.8316	-1.8370	-1.8357	-1.8352
-1.00	-1.9350	-1.9489	-1.9516	-1.9519

Table 4.7

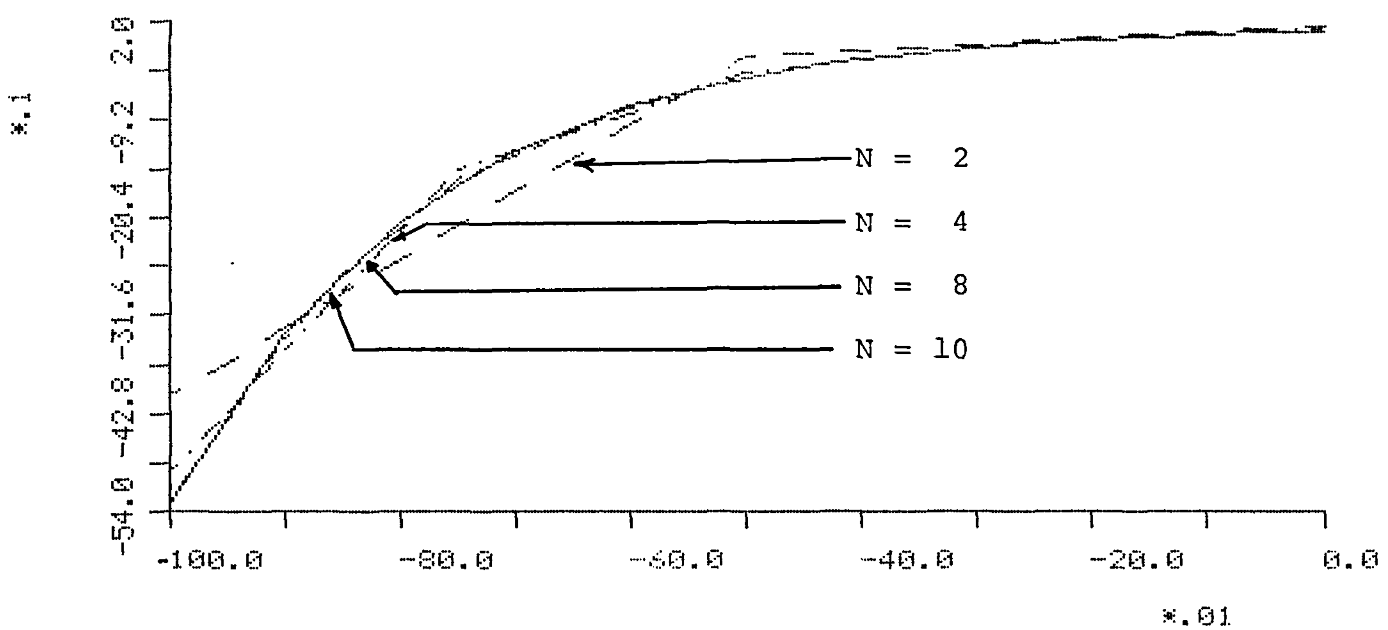

Figure 4.8

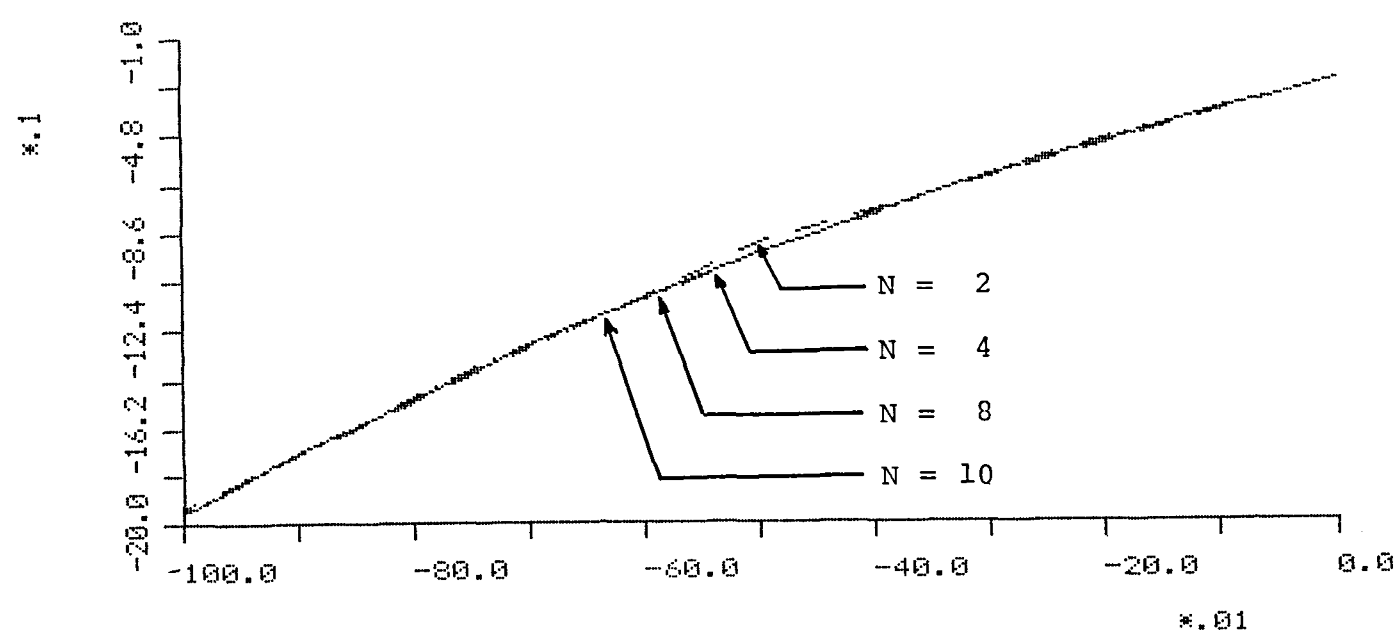

Figure 4.9

4.3 Control of a Flexible Structure

We consider an Euler-Bernoulli beam cantilevered to a rigid hub which is free to rotate about its fixed center, point 0. Also, a point mass m_1 is attached to the other end of the beam. The control is a torque u applied to the hub, and all motion is in the plane. See Figure 4.10 and Table 4.11.

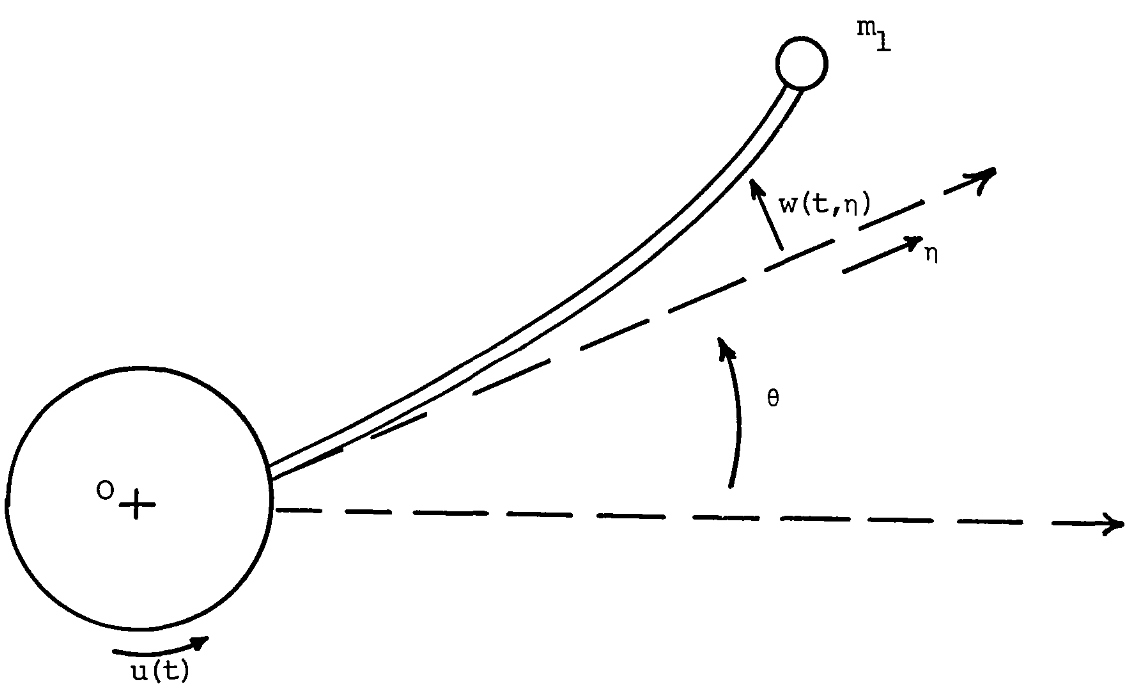

Figure 4.10

r = hub radius	10 in
ℓ = beam length	100 in
I_0 = hub moment of inertia about axis perpendicular to page through 0	100 slug in^2
m_b = beam mass per unit length	.01 slug/in
m_1 = tip mass	1 slug
EI = product of elastic modulus and second moment of cross section for beam	13,333 slg in^3/sec^2
fundamental frequency of undamped structure	.9672 rad/sec

Table 4.11

The angle θ represents the rotation of the hub (the rigid-body mode), $w(t,\eta)$ is the elastic deflection of the beam from the rigid-body position, and $w_1(t)$ is the displacement of m_1 from the rigid-body position. For technical reasons, we do not yet impose the condition $w_1(t) = w(t,\ell)$.

The control problem is to stabilize rigid-body motions and linear (small) transverse elastic vibrations about the state $\theta = 0$ and $w = 0$. Our linear model assumes not only that the elastic deflection of the beam is linear but also that the axial inertial force produced by the rigid-body angular velocity has negligible effect on the bending stiffness of the beam. The rigid-body angle need not be small.

For this example, it is a straight forward exercise to derive the coupled ordinary and partial differential equtions of motion in θ, w and w_1. However, rather than writing these equations explicitly, it is easier and more useful for our purposes to derive an abstract second order evolution equation for the structure. To do this, we define the generalized displacement vector

(4.122) $$x = (\theta, w, w_1) \in H = R^1 \times L_2(0,\ell) \times R^1.$$

The kinetic energy in the system is then

(4.123) $$\text{Kinetic Energy} = 1/2 \langle M_0\dot{x},\dot{x}\rangle_H$$

where M_0 is the unique bounded self-adjoint linear mass operator M_0 on H such that

(4.124) $$\langle M_0 x,\hat{x}\rangle = I_0\theta\hat{\theta} + m_b\langle w + \psi_0\theta, \hat{w} + \psi_0\hat{\theta}\rangle_{L_2} + m_1(w_1 + \psi_0(\ell)\theta)(\hat{w}_1 + \psi_0(\ell)\hat{\theta}),$$

where $\psi_0 \in L_2(0,\ell)$ is given by $\psi_0(\eta) = r + \eta$. It is easy to show that M_0 is also coercive. The elastic strain energy is

(4.125) $$\text{Strain Energy} = 1/2\ a(x,x)$$

with

(4.126) $$a(x,\hat{x}) = EI\ \langle D^2 w, D^2\hat{w}\rangle_{L_2}.$$

We make $a(\cdot,\cdot)$ into an inner product by setting

(4.127) $$\langle x,\hat{x}\rangle_V = a(x,\hat{x}) + \theta\hat{\theta}$$

and define the strain-energy space

$$(4.128)\qquad V = \{x = (\theta,\phi,\phi(\ell)) : \phi \in H^2(0,\ell),\ \phi(0) = D\phi(0) = 0\}.$$

The last term in (4.127) is necessary for the V-inner product because there is no strain energy associated with the rotation of the hub.

We define the stiffness operator A_0 by

$$(4.129)\quad \mathrm{Dom}(A_0) = \{x = (\theta,\phi,\phi(\ell)) \in V : \phi \in H^4(0,\ell),\ D^2\phi(\ell) = 0\}$$

and

$$(4.130)\qquad A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & EI\,D^4 & 0 \\ 0 & -EI\,D^3 & 0 \end{bmatrix}.$$

This operator is self-adjoint with compact resolvent and all positive eigenvalues except the one zero eigenvalue corresponding to the rigid-body mode. Note that V is the domain of the square root of A_0.

With these mass and stiffness operators, we can write the equations of motion as

$$(4.131)\qquad M_0\ddot{x}(s) + c_0A_0\dot{x}(s) + A_0x(s) = B_0u(s),\ s > 0,$$

where c_0 is a positive constant and the term $c_0A_0\dot{x}$ represents viscoelastic damping in the beam. The input operator is

(4.132) $$B_0 = (1,0,0).$$

Letting $Z = V \times H$ with inner product $\langle(v,h),(\hat{v},\hat{h})\rangle_Z = \langle v,\hat{v}\rangle_V + \langle M_0 h,\hat{h}\rangle_H$, the first order form of this system is given by

(4.133) $$\dot{z}(s) = \mathcal{A}z(s) + \mathcal{B}u(s), \quad s > 0,$$

where $z = (x,\dot{x}) \in Z$ and $\mathcal{A}$ is the unique extension of the operator

(4.134) $$\mathring{\mathcal{A}} = \begin{bmatrix} 0 & I \\ -M_0^{-1}A_0 & -M_0^{-1}A_0 \end{bmatrix}, \qquad \mathrm{Dom}(\mathring{\mathcal{A}}) = \mathrm{Dom}(A_0) \times \mathrm{Dom}(A_0),$$

that generates a C_0-semigroup on the space Z. Of course, $\mathcal{B}$ is

(4.135) $$\mathcal{B} = \begin{bmatrix} 0 \\ M_0^{-1}B_0 \end{bmatrix} .$$

See [10] and [12]. The hub-beam-tip mass structure here is discussed in more detail in [12], along with the continuous-time problem.

The discrete-time control system for sampling interval τ is

(4.136) $$z(t+1) = Tz(t) + Bu(t), \quad t = 0,1,2,\ldots,$$

where

(4.137) $$T = \mathcal{T}(\tau), \qquad B = \int_0^\tau \mathcal{T}(s)\mathcal{B}\, ds$$

and $\{\mathcal{T}(s): s > 0\}$ is the semigroup generated by the $\mathcal{A}$ in (4.133).

As in the previous examples, we will solve a discrete-time optimal control problem on the infinite interval. In the performance index, we take the state weighting operator Q to be the identity on Z. This means that $\langle Qz,z\rangle_Z$ is twice the total energy in the structure plus the square of the rigid-body rotation. Since there is one input, the control weighting R is a scalar. The optimal control has the feedback form

(4.138) $$u_*(t) = -\langle f,x(t)\rangle_V - \langle M_0 g,\dot{x}(t)\rangle_H$$

where x(t) has the form (4.122) and

(4.139) $$f = (f^1,f^2,f^3) \varepsilon V$$

(4.140) $$g = (g^1,g^2,g^3) \varepsilon H .$$

Our approximation of the structure is based on a finite element approximation of the beam which uses Hermite cubic splines as basis functions ([27]). We define the sequence of spaces $V_N = \text{span}\, \{e_N^j\}_{j=1}^{J_N}$ with

(4.141) $$e_N^1 = (1,0,0),$$

(4.142) $$e_N^j = (0,\phi_N^j,\phi_N^j(\ell)),\ j = 2,3,\ldots J_N,$$

where the ϕ_N^j's are the cubic splines. Each V_N is a subspace of V, and our approximation scheme is a Ritz-Galerkin approximation obtained by projecting (4.131) onto V_N. See [12] for details. Writing

$$x_N(s) = \sum_{j=1}^{J_N} [x_N(s)]_j e_N^j, \tag{4.143}$$

we have

$$M_N\,[\ddot{x}_N(s)] + c_0 K_N\,[\dot{x}_N(s)] + K_N[x_N(s)] = B_{ON}u(s) \tag{4.144}$$

to solve for the vector $[x_N(s)]$ of time-dependent coefficients $[x_N(s)]_j$. The mass matrix M_N and the stiffness matrix K_N are given by

$$[M_N]_{ij} = \langle M_0 e_N^i,\ e_N^j\rangle_H, \qquad [K_N]_{ij} = \langle e_N^i,\ e_N^j\rangle_V \tag{4.145}$$

and the input matrix is

$$B_{ON} = [1\ 0\ 0\ \ldots\ 0]^T. \tag{4.146}$$

With $z_N = (x_N,\dot{x}_N) \in V_N \times V_N$, (4.144) is the matrix representation of an evolution equation

$$\dot{z}_N(s) = A_N z_N(s) + B_N u(s) \tag{4.147}$$

where A_N and B_N approximate A and B. It is shown in [12] that, as N increases, the semigroup $\{T_N(s):\ s > 0\}$ generated by A_N converges strongly to the semigroup $\{T(s) : s > 0\}$ and that the adjoint semigroup $\{T_N^*(s):\ s > 0\}$ converges strongly as well. Since B_N is the Z-projection of B onto $V_N \times V_N$, it converges strongly to B.

For the approximating discrete-time control systems, we replace z(t), T, B, $T(\cdot)$ and $\mathcal{B}$ in (4.136) and (4.137) with z_N, T_N, B_N, $T_N(\cdot)$ and $\mathcal{B}_N$, respectively. For each N, the solution to the infinite-time optimal control problem is based on the N^{th} Riccati operator equation (3.33). As in the previous examples, we solve the Riccati matrix equation (3.59) for $\hat{\Pi}_N$, which is related to $[\Pi_N]$ (the matrix representation of the operator Π_N) as in Section 3.3, except here we have

$$(4.148) \qquad \hat{\Pi}_N = W_N[\Pi_N],$$

where

$$(4.149) \qquad W_N = \begin{bmatrix} \tilde{K}_N & 0 \\ 0 & M_N \end{bmatrix}$$

and $\tilde{K}_N$ is the stiffness matrix with 1 added to the first element. Since Q = I in the infinite dimensional problem, Q_N is the identity on $V_N \times V_N$ and it follows from (3.48) that the matrix $\hat{Q}_N$ for (3.59) is W_N.

The optimal feedback control for the N^{th} problem is given by (3.56) with the matrices in (3.57) and (3.58), and it has the equivalent representation

$$(4.150) \qquad u_*^N(t) = - \langle f_N, x_N(t)\rangle_V - \langle M_0 g_N, \dot{x}_N(t)\rangle_H$$

where

$$(4.151) \qquad f_N = (f_N^1, f_N^2, f_N^3) \in V,$$

$$(4.152) \qquad g_N = (g_N^1, g_N^2, g_N^3) \in H$$

as in (4.139) and (4.140). From (3.56), (4.143) and (4.150), it follows that

$$(4.153)\qquad \begin{bmatrix} f_N \\ g_N \end{bmatrix} = \begin{bmatrix} E_N^T & 0 \\ 0 & E_N^T \end{bmatrix} W_N^{-1}[F_N]^T ,$$

where $E_N^T = (e_N^1, e_N^2, \ldots, e_N^{J_N})$.

For the sampling interval $\tau = .01$, the damping coefficient $c_0 = .001$ and the control weighting $R = 1$, Tables 4.12-4.15 give the values of the corresponding scalar and functional gains, f_N^i, g_N^i, $i = 1,2,3$ for various values of N. The values of the functional gains $D^2 f_N^2$ and g_N^2 along the length of the beam also are plotted in Figures 4.16 and 4.17. We tabulated and plotted $D^2 f_N^2$ because this is what appears in the V inner product in (4.150) and also to show the H^2 convergence. We note that the form of the V inner product given by (4.127) is such that f_N^3 does not appear in the feedback law.

N	3	4	5	7
f_N^1	.9991	.9992	.9990	.9992
g_N^1	.1030	.1040	.1043	.1044

Table 4.12

N	3	4	5	7
f_N^3	.1750	.1769	.1774	.1777
g_N^3	-18.1231	-18.3385	-18.3902	-18.4158

Table 4.13

η	$D^2f_3^2(\eta)\times10^5$	$D^2f_4^2(\eta)\times10^5$	$D^2f_5^2(\eta)\times10^5$	$D^2f_7^2(\eta)\times10^5$
0.0	6.0266	7.0718	7.6281	8.1441
5.0	4.8991	5.4251	5.6142	5.6817
10.0	3.7717	3.7784	3.6003	3.2193
15.0	2.6442	2.1317	1.5865	1.1727
20.0	1.5167	.4850	.2048	.7757
25.0	.3893	.1353	.4057	.3786
30.0	-.7382	.4896	.6066	.4481
35.0	1.0908	.8438	.8075	.8303
40.0	1.2767	1.1980	1.3790	1.2126
45.0	1.4626	1.5523	1.5713	1.6121
50.0	1.6484	2.0984	1.7636	1.8049
55.0	1.8343	1.9858	1.9559	1.9976
60.0	2.0202	1.8731	2.0543	1.9613
65.0	2.2061	1.7605	1.8179	1.8003
70.0	1.6333	1.6478	1.5815	1.6392
75.0	1.3545	1.3288	1.3450	1.3391
80.0	1.0757	1.0562	1.0303	1.0505
85.0	.7970	.7837	.7667	.7620
90.0	.5182	.5112	.5031	.4921
95.0	.2394	.2387	.2395	.2400
100.0	-.0394	-.0339	-.0242	-.0122

Table 4.14

η	$g_3^2(\eta)$	$g_4^2(\eta)$	$g_5^2(\eta)$	$g_7^2(\eta)$
0.0	.0000	.0000	.0000	.0000
5.0	-.2348	-.2576	-.2673	-.2745
10.0	-.8710	-.9445	-.9727	-.9903
15.0	-1.8063	-1.9320	-1.9715	-1.9854
20.0	-2.9383	-3.0915	-3.1190	-3.1171
25.0	-4.1649	-4.2941	-4.2940	-4.3031
30.0	-5.3837	-5.4478	-5.4540	-5.4733
35.0	-6.4982	-6.5450	-6.5720	-6.5842
40.0	-7.5155	-7.5840	-7.6210	-7.6289
45.0	-8.4647	-8.5629	-8.5918	-8.6032
50.0	-9.3609	-9.4797	-9.5044	-9.5179
55.0	-10.2188	-10.3493	-10.3761	-10.3910
60.0	-11.0535	-11.1963	-11.2240	-11.2407
65.0	-11.8797	-12.0341	-12.0666	-12.0838
70.0	-12.7146	-12.8759	-12.9158	-12.9327
75.0	-13.5719	-13.7350	-13.7790	-13.7988
80.0	-14.4502	-14.6201	-14.6632	-14.6861
85.0	-15.3469	-15.5278	-15.5720	-15.5946
90.0	-16.2595	-16.4533	-16.5000	-16.5232
95.0	-17.1858	-17.3917	-17.4414	-17.4659
100.0	-18.1231	-18.3385	-18.3902	-18.4158

Table 4.15

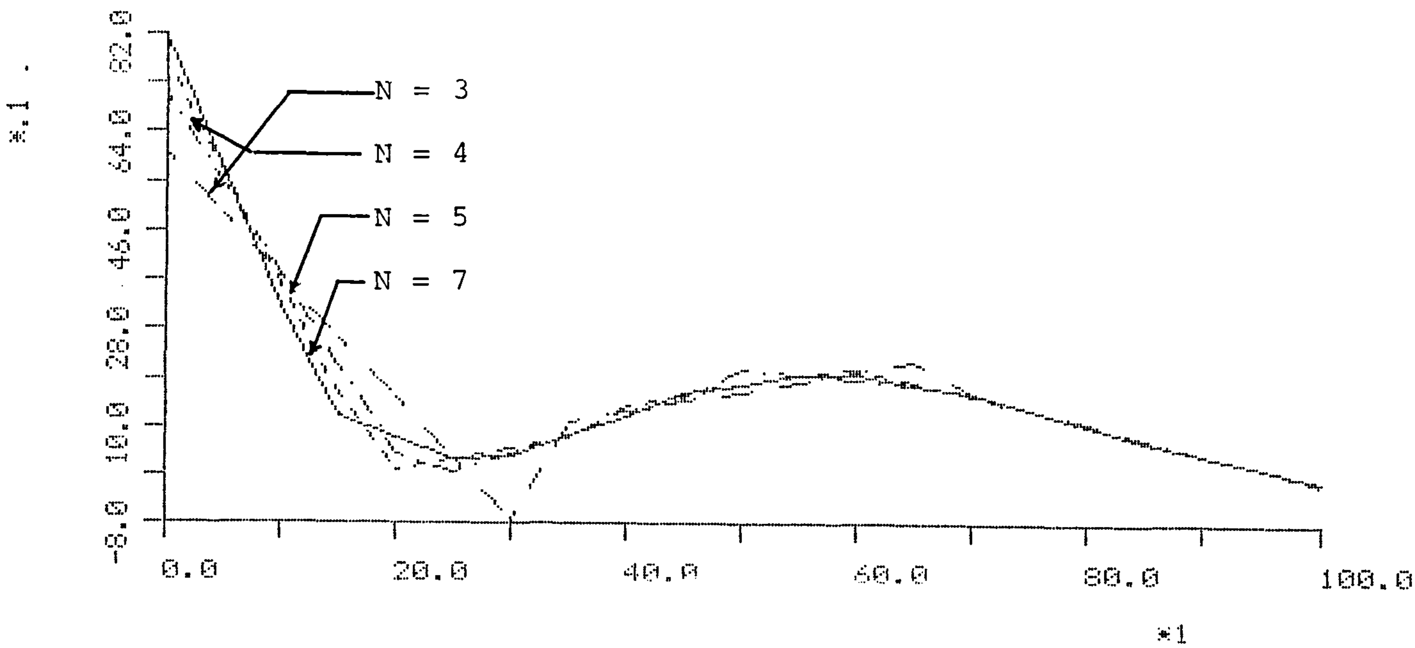

Figure 4.16

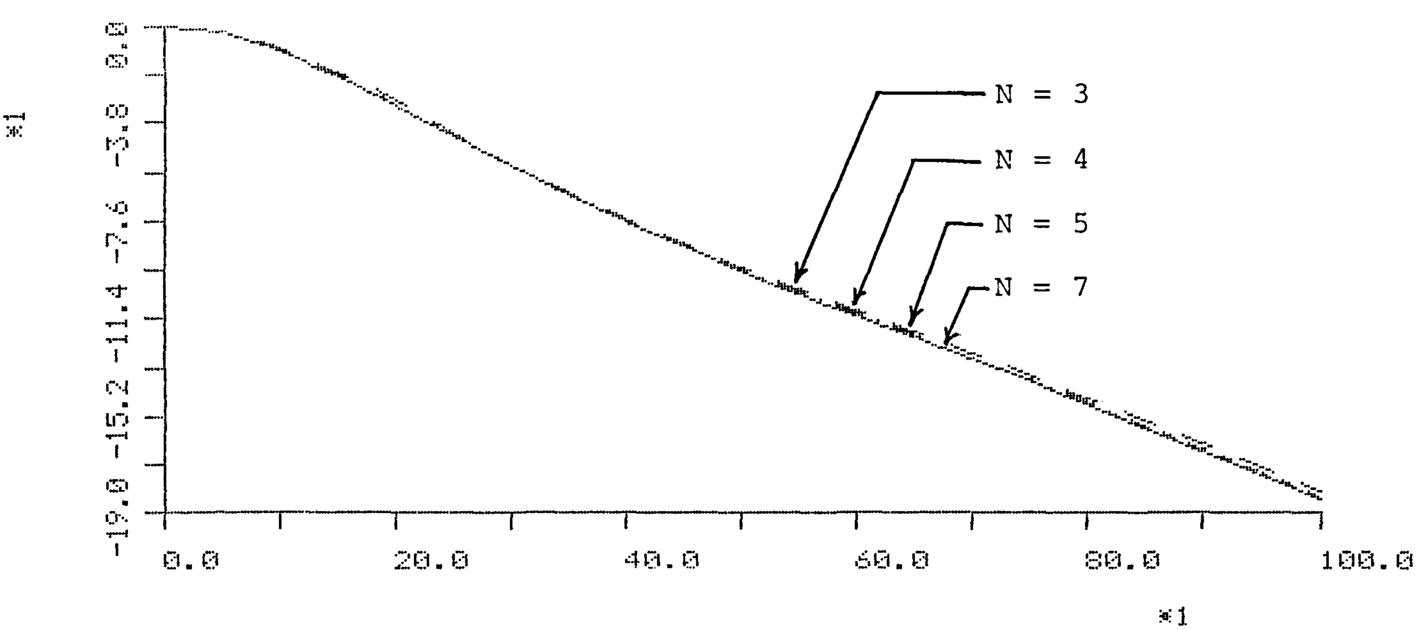

Figure 4.17

5. Concluding Remarks

We have presented an approximation theory for numerical solution of the discrete-time optimal linear regulator problem in Hilbert space, on both finite and infinite time intervals. The motivation for this theory comes from optimal control problems for systems involving diffusion equations, hereditary differential equations and distributed models of flexible structures. We have demonstrated the application of the theory to examples from all three areas

The solution to the infinite dimensional optimal control problem is based on an infinite dimensional Riccati operator equation -- a difference equation in the finite-time problem and an algebraic equation in the infinite-time problem. We have shown that the solution to the infinite dimensional problem can be approximated by the solutions to a sequence of finite dimensional problems each of which involves a finite dimensional Riccati matrix equation to be solved numerically. The finite dimensional problems are just the corresponding optimal control problems for finite element approximations to the infinite dimensional control system. For the infinite-time problem, the finite dimensional Riccati equations usually are solved via eigenspace decomposition of the Hamiltonian matrix.

In both continuous and discrete-time optimal regulator problems for distributed systems, the numerical solution often involves solution of large Riccati matrix equations. As we observed at the beginning of Section 4, the asymptotic relationship between the eigenvalues of a continuous-time Hamiltonian system and the eigenvalues of the corresponding discrete-time Hamiltonian system is exponential. This means that the approximating finite dimensional discrete-time Riccati equations for a given distributed system invariably are not as well conditioned as the corresponding continuous-time Riccati equations. Nonetheless, as our examples should illustrate, the numerical solution of such problems is well within the reach of current computing. To emphasize this, we obtained all of the numerical results in this paper on an IBM Personal Computer (not an XT or AT) with 640K of random access memory and an Intel 8087 math coprocessor chip. The largest Riccati matrix equation that we solved here was a 30 × 30 steady state equation for the hub-beam-tip mass example. This solution takes 15 to 20 minutes on the PC. We have solved much larger Riccati equations easily on larger mainframe computers.

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1. Report No. NASA CR-178081 ICASE Report No. 86-15	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle NUMERICAL APPROXIMATION FOR THE INFINITE-DIMENSIONAL DISCRETE-TIME OPTIMAL LINEAR-QUADRATIC REGULATOR PROBLEM		5. Report Date March 1986 6. Performing Organization Code
7. Author(s) J. S. Gibson and I. G. Rosen		8. Performing Organization Report No. 86-15
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225		10. Work Unit No. 11. Contract or Grant No. NAS1-17070, NAS1-18107
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Contractor Report 14. Sponsoring Agency Code 505-31-83-01

15. Supplementary Notes
Langley Technical Monitor: J. C. South
Submitted to SIAM J. Cont. Opt.

Final Report

16. Abstract

An abstract approximation framework is developed for the finite and infinite time horizon discrete-time linear-quadratic regulator problem for systems whose state dynamics are described by a linear semigroup of operators on an infinite dimensional Hilbert space. The schemes included in the framework yield finite dimensional approximations to the linear state feedback gains which determine the optimal control law. Convergence arguments are given. Examples involving hereditary and parabolic systems and the vibration of a flexible beam are considered. Spline-based finite element schemes for these classes of problems, together with numerical results, are presented and discussed.

17. Key Words (Suggested by Authors(s))	18. Distribution Statement
linear quadratic regulator, discrete time, distributed system, approximation	64 - Numerical Analysis 67 - Theoretical Mathematics Unclassified - unlimited

19. Security Classif.(of this report)	20. Security Classif.(of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	82	A05

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