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Two Time Scale Output Feedback Regulation for Ill-Conditioned Systems

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REGULATION FOR ILL-CONDITIONED SYSTEMS

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SUMMARY

Issues pertaining to the well-posedness of a two time scale approach to the output feedback regulator design problem are examined. An approximate quadratic performance index which reflects a two time scale decomposition of the system dynamics is developed. It is shown that, under mild assumptions, minimization of this cost leads to feedback gains providing a second-order approximation of optimal full system performance. A simplified approach to two time scale feedback design is also developed, in which gains are separately calculated to stabilize the slow and fast sybsystem models. By exploiting the notion of combined control and observation spillover suppression, conditions are derived assuring that these gains will stabilize the full-order system.

A sequential numerical algorithm is described which obtains output feedback gains minimizing a broad class of performance indices, including the standard LQ case. It is shown that the algorithm converges to a local minimum under nonrestrictive assumptions. This procedure is adopted to and demonstrated for the two time scale design formulations.

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SECTION 1

INTRODUCTION

This report examines the continuous time optimal output feedback
regulator problem for linear, time-invariant, deterministic systems with
ill-conditioned dynamics. In the sequel, an output feedback controller
will be characterized as one in which the feedback law is based on a set
of system outputs, rather than the full internal state. In the
time-invariant regulator case, the simplest form of the controller is
that of a matrix of constant gains. This is referred to as static gain
output feedback. The extension of output feedback to the case of
fixed-order dynamic compensation is also considered, where an arbitrary
number of dynamic elements are included in the feedback structure.

Output feedback control laws offer the important advantage of
simplicity in implementation over controllers which are based on
full-state feedback. Since the control designer only rarely has access
to all of the system states, implementing a full-state feedback
controller requires an observer or Kalman filter in order to reconstruct
the states unavailable directly from the output. In contrast, the
structure of an output feedback controller can be kept as simple as is
consistent with the constraint of output feedback stabilizability, or
that of meeting closed-loop design criteria. This has motivated the
study of LQ optimal output feedback regulation problems [1-7]. In these
problems, given the prespecified feedback structure, the controller gains
are calculated to minimize an infinite-time integral quadratic perfor-
mance index on the state and control. This formulation is particularly

advantageous in that it directly addresses the issue of RMS control activity, allowing the designer to make a well-defined compromise between a measure of the system performance and one of the control expenditure required to attain it.

The necessary conditions for optimality [1] for the optimal output feedback problem take the form of coupled nonlinear matrix equations. For realistic problems, solutions are obtained numerically through the use of iterative procedures. This fact lies at the root of the two major difficulties which have impeded the application of optimal output feedback design techniques to practical design problems:

    i.)   the lack of simple, computationally inexpensive, convergent numerical algorithms for solution of the necessary conditions,

    ii.)   the fact that realistically detailed system models tend to be of large order, and often contain slow and fast modes. This leads to ill-conditioning in computations related to controller design.

1.1 Numerical Procedures for Optimal Gain Calculation

Many algorithms have been suggested for numerically solving the necessary conditions for optimality, falling into one of two broad categories. The first category comprises gradient [8-10] and nongradient based search procedures [11,12]. These algorithms will converge to a stationary point [8], but are computationally expensive. The second category of algorithms consist of those in which either the nonlinear necessary conditions [1] or a related system of linear equations [3,4,13] are solved sequentially. When they do converge to a solution, these

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methods are recognized as being considerably faster than the search procedures [12]. Of this latter class of algorithms, only that of [13] for the discrete LQ stochastic output feedback regulator has been shown to converge.

Section 2 of this report formulates the continuous time LQ output feedback regulator problem. A simple sequential algorithm is described which calculates optimal output feedback gains for a broad class of problems which includes the standard LQ case. Unrestrictive conditions are stated under which this algorithm provides a monotonically improving sequence of gains converging to a stationary point.

1.2 Output Feedback Design for Systems with Ill-Conditioned Dynamics

Even given the practical and reliable algorithm in Section 2, numerical calculation of optimal gains for system design models which include slow and fast modes can be difficult or impossible due to the numerical ill-conditioning of such models. In addition, the sensitivity of numerical procedures to ill-conditioning increases with the dimensionality of the system model.

These considerations have motivated the use of singular perturbation theory (SPT) [14] for decomposing ill-conditioned linear systems into well-conditioned slow and fast subsystems. Loosely speaking, the singularly perturbed approximation of a system with asymptotically fast dynamics consists of approximating the fast modes as infinitely fast. Under this assumption, fast transients decay instantaneously, so that the fast states are replaced by an algebraic function of the slow states. Similarly, if one wishes a well-conditioned approximate model for the

fast dynamics, one approximates the slow states as being infinitely slow, compared with the fast. The slow states are then replaced by constant values at some boundary condition near which the fast behavior is of interest. We thus obtain two subsystems, each approximating a portion of the original model. This is also referred to as a "two time scale" approximation. It should be noted that this theory has been extended to systems where the fast dynamics are marginally stable [15]. For the problem considered here, however, interest centers on the case of closed-loop asymptotic stability.

The singularly perturbed LQ full-state feedback regulator problem has attracted considerable attention [16-21]. Here, an SPT decomposition of the system dynamics leads to a complete separation of the regulator design into slow and fast subproblems. This very convenient feature does not exist in the case of singularly perturbed output feedback systems, occuring naturally only for a highly restrictive class of output structures [22,23]. In full-state feedback, each subsystem is stabilized through a dedicated gain matrix feeding back only the subsystem states. In output feedback, the slow and fast subsystems must both share a single gain matrix based on the system output. This requires that, in general, the dynamics of both subsystems must be accomodated simultaneously in the design. In fact, designing an output feedback controller based only on a low frequency "design model" may destabilize neglected fast states [24].

Section 3 provides a detailed development of the SPT decomposition of a closed-loop system with output feedback, and addresses various issues relating to the well-posedness of the design problem and of the

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approximation. Section 4 describes the SPT approximation of the optimal output feedback problem and states a design procedure. Gains designed by this method provide a second order approximation to closed-loop integral quadratic performance. Unfortunately, the form of the necessary conditions for optimality dictate that systems of equations be solved for the dynamics of both subsystems simultaneously.

The complication of simultaneously designing for the slow and fast subsystems can be circumvented when the input/output structures of the slow and fast subsystems exhibit rank deficiency. This situation is not as restrictive as it may sound. Subsystem I/O rank deficiency can occur even when none exists in the full-order system, since both lower order subsystems have the same number of inputs and outputs as the full system. In fact, the phenomenon is commonplace in models of systems which have many sensors and actuators. When this is the case, the use of combined control and observation spillover suppression can be employed in separating the subsystem control designs into separate tasks. This is examined in Section 5, and a two-step LQ design procedure is developed. The spillover suppression constraints are enforced through the use of penalty functions, so the theory can also be applied in situations where subsystem I/O matrices are only "nearly rank deficient."

Section 6 briefly examines treatment of two time scale design of fixed-order dynamic compensators as an extension of the static gain case. A number of questions are raised which, hopefully, will help to motivate further work in this area.

The work is summarized in Section 7.

SECTION 2

OPTIMAL OUTPUT FEEDBACK DESIGN

In this section, the optimal output feedback problem is formulated for a class of problems which includes the standard LQ case. A convergent sequential numerical algorithm for solving the necessary conditions for optimality is described. Because the algorithm provides a sequence of monotonically improving gains, the solution obtained at convergence is locally optimal.

2.1 Problem Formulation and Necessary Conditions for Optimality

We consider systems of the form

$$\dot{x} = Ax + Bu \qquad x(0) = x_0 \qquad (2.1)$$

where $x \in {}^n$ and $u \in {}^m$, with output

$$y = Cx \qquad (2.2)$$

where $y \in {}^p$. The control has the form

$$u = -Gy \qquad (2.3)$$

The gain G is to be chosen to minimize

$$J = \int_0^\infty x^T Q x + u^T R u \, dt + \gamma(G) \qquad (2.4)$$

where $Q = \Gamma^T \Gamma$ such that the pair (Γ, A) is detectable, and $R > 0$. In addition, it will be seen that, in order to avoid singularity in the necessary conditions for optimization problem, we must have

$$\rho(C) = p \qquad (2.5)$$

In (2.4), $\gamma(G)$ is any scalar function having a continuous gradient in G, and for which J is bounded below, for all G which render the closed loop dynamics (2.1-2.3) asymptotically stable. This class of performance index will find use in Section 5, when it is used to enforce conditions

leading to a two-stage two time scale output feedback design procedure.

Also, in section 2.3 we illustrate how a performance index of the form

(2.4) allows individual gain elements in an output feedback gain matrix

to be zeroed. Many other applications doubtless exist.

It is well known that the integral portion of J satisfies the

relation

$$\int_0^\infty x^T Q x + u^T R u \ dt = tr\{K x_0 x_0^T\} \tag{2.6}$$

where $K > 0$ is the unique solution of

$$S(G,K) = \hat{A}^T K + K\hat{A} + Q + C^T G^T R G C = 0 \tag{2.7}$$

$$\hat{A} = A - BGC \tag{2.8}$$

and A is asymptotically stable. As suggested in [1], it is customary to

relieve (2.6) of its dependence on x_0 by assuming that it is uniformly

distributed on the unit sphere; then the problem statement is modified

slightly to that of minimizing $E\{J\}$. This amounts to replacing $x_0 x_0^T$ in

(2.6) by I.

The minimization of (2.4) is now cast, as in [5], as a static

optimization problem, in which the Lagrangian

$$(G,K,L) = tr\{K\} + \gamma(G) + tr\{S(G,K)L^T\} \tag{2.9}$$

is minimized with respect to G, K and L, where L is a matrix of Lagrange

multipliers. If the system (2.1-2.3) can be stabilized by output

feedback, the first order necessary conditions for optimality are

$$\frac{\partial \mathscr{L}}{\partial G}\bigg|_* = 0 \qquad \frac{\partial \mathscr{L}}{\partial K}\bigg|_* = 0 \qquad \frac{\partial \mathscr{L}}{\partial L}\bigg|_* = 0 \tag{2.10}$$

where the *'s mean that the gradients are evaluated at the optimal values

of G, K and L. In the sequel, the * notation is suppressed since the

gradients are assumed evaluated at their optimal values unless specified

otherwise. Defining the gradient of $\gamma(G)$

$$\frac{\partial \gamma(G)}{\partial G} = \gamma_G(G) \qquad (2.11)$$

the expansion of (2.10) is

$$RGCLC^T - B^TKLC^T + \frac{1}{2}\gamma_G(G) = 0 \qquad (2.12)$$

$$\hat{A}L + L\hat{A}^T + I = 0 \qquad (2.13)$$

$$S(G,K) = 0 \qquad (2.14)$$

From (2.12), the optimal value of G will satisfy

$$G^* = R^{-1}[B^TKLC^T - \gamma_G(G)](CLC^T)^{-1} \qquad (2.15)$$

where $(CLC^T)^{-1}$ exists because of (2.5) and the fact that $L > 0$ in (2.13).

2.2 A Convergent Numerical Algorithm

The following algorithm suggests itself for solving (2.12-2.14):

0. Choose any G such that $\hat{A}$ is Hurwitz.   Set $i = 0$.

1. Solve (2.13,2.14) for K_i and L_i.

2. On the basis of (2.15), evaluate

$$\Delta G_i = R^{-1}[B^TK_iL_iC^T - \frac{1}{2}\gamma_G(G_i)](CL_iC^T)^{-1} - G_i \qquad (2.16)$$

3. Set

$$G_{i+1} = G_i + \alpha\Delta G_i \qquad (2.17)$$

where $\alpha \in (0,1]$ is chosen to ensure that

$$J_{i+1} < J_i = tr\{K_i\} + \gamma(G_i) \qquad (2.18)$$

4. Set $i = i + 1$ and go to 1.

This is a very simple procedure to implement, since it only involves the solution of two Lyapunov equations. The unfortunate necessity of supplying an initial stabilizing gain for step 0 is shared by other sequential algorithms currently available. In [25], a simple procedure for obtaining an initial stabilizing gain is given.

In Appendix B, the following theorem is proven:

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Theorem 2.1: For the optimal output feedback problem defined in (2.1-2.4), let the following conditions be satisfied:

i) $\mathscr{G} = \Big[G : A \text{ is Hurwitz} \Big] \neq \emptyset$

ii) $\rho\{C\} = p$

iii) $Q = \Gamma^T\Gamma$ such that (Γ,A) is detectable; $R > 0$

iv) $\gamma(G)$ is $\mathscr{C}^1$ for all $G \in \mathscr{G}$

v) If $\gamma(G) \to -\infty$ for all $\|G \in \mathscr{G}\| \to \infty$, then it does so in such a way that $|\gamma(G)|/\text{tr}\{K\} < 1$

If (i-v) are true, then the sequence $\Big[G_i : \mathscr{G}, \ i = 0,1,\ldots \Big]$ of stabilizing gains defined by (17) exists for any $G_o \in \mathscr{G}$, such that (2.18) is satisfied at each iteration. Moreover, the sequence converges to a stationary point in J.

Note that (i-iii) are the standard conditions required for solving the LQ optimal output feedback problem. Loosely speaking, (v) means that, in choosing $\gamma(G)$, one must be certain that it does not become negatively unbounded at a faster rate than $\text{tr}\{K\}$ becomes positively unbounded for $\|G\| \to \infty$. Recall that, because of (iv), $\gamma(G)$ cannot assume unbounded values for finite G. It should also be noted that, while the theorem does not rule out the theoretical possibility of convergence to a saddle point in J, encountering a saddle in practice would only slow the convergence to a local minimum of J, since the saddle point would be unstable in G.

2.3 Numerical Example

This example illustrates the breadth of the class of problems which can be solved using this algorithm. A legitimate criticism of modern control theory is that multivariable techniques stress feeding back all of the outputs to all of the inputs. Often, this is needless and costly

from an engineering standpoint. Recently, in the context of the eigen-structure assignment problems [26-29], and in the context of constrained optimization theory [30], attention has been paid to zeroing selected elements of the multivariable gain matrix. This feature provides a considerable measure of real-world practicality, insofar as it permits the designer to balance the dynamic performance of the system against the structural complexity of the controller.

In the context of our theory, the ij^{th} element of of G is zeroed by defining

$$\gamma(G) = \frac{\nu}{2} g_{ij}^2 \tag{2.19}$$

where $\nu > 0$ is sufficiently large to result in suppression of the gain element. The gradient of (2.19) is

$$\gamma_G(G) = \{a_{qr}g_{qr}\} \tag{2.20}$$

$$a_{qr} = \delta_{qi}\delta_{rj} \tag{2.21}$$

This penalty function was applied to the problem of designing a constrained output feedback regulator for the lateral dynamics of an L-1011 aircraft at cruise flight condition, taken from [29]. The state vector is

$$x = \begin{bmatrix} \delta_r \\ \delta_a \\ \phi \\ r \\ p \\ \beta \\ x_7 \end{bmatrix} \quad \begin{matrix} \text{rudder deflection (rad)} \\ \text{aileron deflection (rad)} \\ \text{bank angle (rad)} \\ \text{yaw rate (rad/sec)} \\ \text{roll rate (rad/sec)} \\ \text{sideslip angle (rad)} \\ \text{washout filter state} \end{matrix}$$

The system matrices are:

$$
A = \begin{bmatrix}
-20 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -25 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-0.744 & -0.032 & 0 & -0.154 & -0.0042 & 1.54 & 0 \\
0.337 & -1.12 & 0 & 0.249 & -1.0 & -5.2 & 0 \\
0.02 & 0 & 0.0386 & -0.996 & -0.000295 & -0.117 & 0 \\
0 & 0 & 0 & 0.5 & 0 & 0 & -0.5
\end{bmatrix}
$$

$$
B^T = \begin{bmatrix}
20 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 25 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
C = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

The system inputs and outputs are:

$$
u = \begin{bmatrix} \delta r_c \\ \delta a_c \end{bmatrix}
\begin{array}{l} \text{rudder command (rad)} \\ \text{aileron command (rad)} \end{array}
$$

$$
y = \begin{bmatrix} r_{wo} \\ \psi \\ \theta \\ \phi \end{bmatrix}
\begin{array}{l} \text{washed out yaw rate} \\ \text{roll rate} \\ \text{sideslip angle} \\ \text{bank angle} \end{array}
$$

The eigenvalues of the A matrix are:

$\lambda_1 \quad = -20.0$ rudder mode

$\lambda_2 \quad = -25.0$ aileron mode

$\lambda_{3,4} \quad = -0.0884 \pm j1.272$ dutch roll mode

$\lambda_5 \quad = -1.085$ roll subsidence mode

λ_6 = -0.00911   spiral mode

λ_7 = -0.5   washout filter mode

For the penalty matrices,

$$Q = \text{diag}[1 \ 1 \ 30 \ 30 \ 5 \ 5 \ 1] \qquad R = \text{diag} \ [1 \ 1]$$

the optimal output feedback gain is

$$G^* = \begin{bmatrix} -2.60 & -.396 & 2.72 & -.053 \\ -.998 & -2.41 & 4.36 & -3.74 \end{bmatrix}$$

resulting in the closed-loop eigenvalues

λ_1 = -18.0   rudder mode

λ_2 = -22.0   aileron mode

$\lambda_{3,4}$ = -1.20 ± j1.42   dutch roll mode

$\lambda_{5,6}$ = -1.81 ± j.734   roll mode

λ_7 = -.746   washout filter mode

which closely approximate the values in [29]. Now, optimality aside, due to the near-decoupling of yaw-related ($\delta r, r, \theta$) and roll-related ($\delta a, \psi, \phi$) states, there is not much to be gained in performance by feeding r and β to δ_{ac}, or p and ϕ to δ_{rc}. The gain elements corresponding to these feedback loops - the (1,2), (1,4), (2,1) and (2,3) positions of G - were suppressed by employing (2.19). This structure corresponds to F(4) in [29]. In this case

$$\gamma(G) = \frac{\nu}{2}(g_{12}^2 + g_{14}^2 + g_{21}^2 + g_{23}^2)$$

$$\gamma_G(G) = \nu \begin{bmatrix} 0 & g_{12} & 0 & g_{14} \\ g_{21} & 0 & g_{23} & 0 \end{bmatrix}$$

The variation of integral quadratic performance with ν is shown in Figure 1. For ν = 1000, the optimal suppressed gain matrix is:

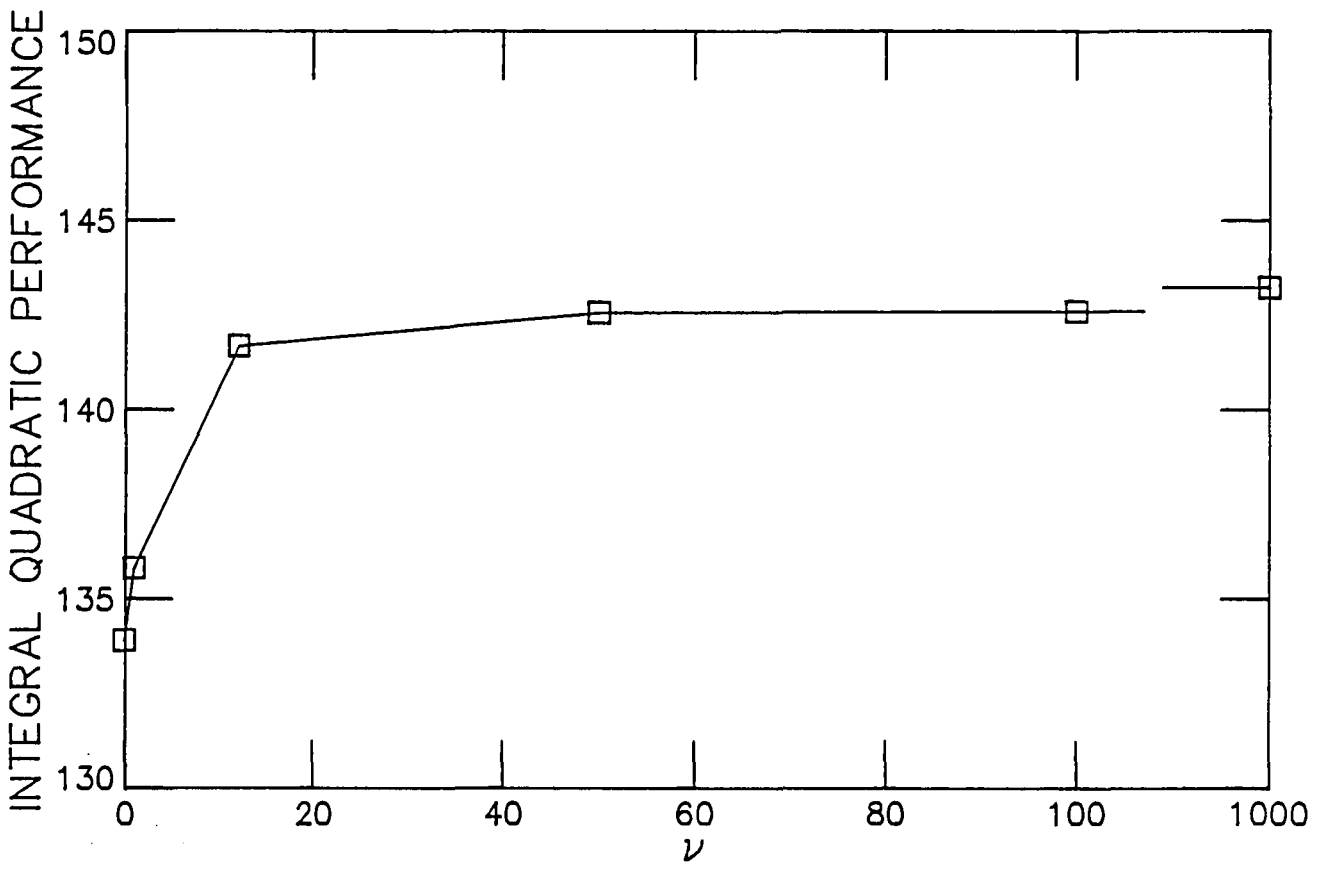$$G_s^* = \begin{bmatrix} -2.78 & -.001 & 3.26 & -.004 \\ -.0009 & -4.70 & .003 & -5.87 \end{bmatrix}$$

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FIGURE 1. INCREASE IN INTEGRAL QUADRATIC PERFORMANCE FOR

INCREASING VALUES OF ν.

resulting in the closed-loop eigenvalues:

$$\lambda_1 \quad = -17.7 \qquad\qquad \text{rudder mode}$$

$$\lambda_2 \quad = -17.7 \qquad\qquad \text{aileron mode}$$

$$\lambda_{3,4} = -1.19 \pm j1.38 \qquad \text{dutch roll mode}$$

$$\lambda_5 \quad = -1.37 \qquad\qquad \text{roll mode}$$

$$\lambda_6 \quad = -6.95 \qquad\qquad \text{roll mode}$$

$$\lambda_7 \quad = -.687 \qquad\qquad \text{washout filter mode}$$

Note the actuator and the washout filter modes are close to their
open-loop values. This illustrates one major advantage in output feed-
back, in that it does not speed up actuator modes, which is a problem
commonly encountered in full state feedback. The dutch roll mode is
relatively unaffected by gain suppression. The roll mode is overdamped
by gain suppression; however, the roll response is dominated by λ_5,
which results in approximately the same settling time as the complex
modes $\lambda_{5,6}$ without gain suppression. Thus, the impulse responses of both
closed-loop systems are essentially the same. The minor degradation in
the integral quadratic cost (7%) indicates that this is accomplished with
little increase in control effort. Simply zeroing (1,2), (1,4), (2,1)
and (2,3) elements in G^* has little effect on the closed-loop
eigenvalues; however, the integral quadratic performance is 157, which
corresponds to a 17.4% increase.

From this example it can be seen that this approach to design
permits total control over the feedback structure while optimizing the
individual gains for an integral quadratic cost. Insofar as this proce-
dure is simple to implement, it represents a significant step forward in
the flexibility and applicability of optimal output feedback design.

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SECTION 3

SPT IN OUTPUT FEEDBACK

In this section SPT is employed to decompose an ill-conditioned closed-loop output feedback system into its slow and fast subsystems. In the process of doing so, we gain some insight into the well-posedness of the SPT-approximate design problem.

3.1 Problem Formulation

Consider the system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u \quad x_1(0) = x_{10} \quad x_1 \in R^{n_1} \quad (3.1)$$

$$\epsilon\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u \quad x_2(0) = x_{20} \quad x_2 \in R^{n_2} \quad (3.2)$$

where $0 < \epsilon \ll 1$, with output

$$y = C_1x_1 + C_2x_2 \qquad\qquad y \in R^p \qquad (3.3)$$

The feedback law is

$$u = -Gy \qquad\qquad u \in R^m \qquad (3.4)$$

If A_{22} is invertible, a reduced order approximation of (3.1-3.3) can be obtained by setting $\epsilon = 0$ in (3.2):

$$\dot{\xi} = A_0 + B_0u \qquad\qquad \xi \in R^{n_1} \qquad (3.5)$$

$$\overline{y} = C_0\xi + D_0u \qquad\qquad (3.6)$$

where

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21} \qquad B_0 = B_1 - A_{12}A_{22}^{-1}B_2$$

$$C_0 = C_1 - C_2A_{22}^{-1}A_{21} \qquad D_0 = -C_2A_{22}^{-1}B_2 \qquad (3.7)$$

Substituting (3.4) in (3.1,3.2) and setting $\epsilon = 0$, the reduced feedback control is expressed as

$$\overline{u} = -G^0C_0\xi \qquad\qquad (3.8)$$

$$G^0 = (I + GD_0)^{-1}G \qquad\qquad (3.9)$$

which necessitates the assumption

$$\rho(I + GD_o) = m \qquad (3.10)$$

The inverse of (9) is

$$G = G^o(I - D_oG^o)^{-1} \qquad (3.11)$$

The following lemma states that satisfaction of the invertibility

conditions for (3.9) and (3.11) is simultaneous, and that this guarantees

local one-to-one correspondence between G^o and G. The proof is given in

Appendix C.

Lemma 3.1:

$$\rho(I - D_oG^o) = p \quad \text{iff} \quad \rho(I + GD_o) = m;$$

furthermore, these conditions are necessary and sufficient for G^o

and G to be locally one-to-one.

The next lemma, also proven in Appendix C, assures that (3.10) will hold

for any G not rendering the fast closed-loop subsystem singular.

Lemma 3.2: Given that A_{22} is nonsingular,

$$\rho(I + GD_o) = m \quad \text{iff} \quad \rho(A_{22} - B_2GC_2) = n_2$$

In summary, Lemmas 3.1 and 3.2 assure that the inverses in

(3.9,3.11) exist for any realistic design problem. Indeed, if

$A_{22} - B_2GC_2$ were singular, the fast subsystem dynamics would not be

"fast". It should be noted that if (3.9) and (3.11) did not define a

unique correspondence between G^o and G, reduced order approximations

would have very little utility in output feedback design.

3.2 Asymptotic Properties

The closed-loop system matrix for (3.1-3.4) takes the form

$$\hat{A} = \begin{bmatrix} A_{11} - B_1GC_1 & A_{12} - B_1GC_2 \\ (A_{21} - B_2GC_1)/\varepsilon & (A_{22} - B_2GC_2)/\varepsilon \end{bmatrix} \qquad (3.12)$$

Following [31], construct an invertible transformation which block diagonalizes A:

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = T(\varepsilon) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (3.13.a)$$

$$T(\varepsilon) = \begin{bmatrix} I-\varepsilon HN & -\varepsilon H \\ N & I \end{bmatrix} \qquad T^{-1}(\varepsilon) = \begin{bmatrix} I & \varepsilon H \\ -N & I-\varepsilon NH \end{bmatrix} \qquad (3.13.b)$$

In (3.13), ξ is exclusively the slowly varying portion of the closed loop state and η is the fast transient. After some algebra, it can be shown that

$$N(\varepsilon) = A_{22}^{-1}(A_{21} - B_2G^oC_o)$$
$$+ \varepsilon(I+A_{22}^{-1}B_2G^oC_2)A_{22}^{-2}(A_{21}-B_2G^oC_o)(A_o-B_oG^oC_o) + O(\varepsilon^2) \qquad (3.14)$$
$$H(\varepsilon) = (A_{12} - B_oG^oC_2)A_{22}^{-1} + O(\varepsilon) \qquad (3.15)$$

These expressions can easily be verified from [31], if one recalls the definitions in (3.7) and uses the fact that, if A_{22}^{-1} exists,

$$(A_{22} - B_2GC_2)^{-1} = (I + A_{22}^{-1}B_2G^oC_2)A_{22}^{-1} \qquad (3.16)$$

Expression (3.16) is obtained by a straightforward application of (A.5). Using (3.13) in (3.12), the dynamics are decoupled:

$$\dot{\xi} = [(A_o-B_oG^oC_o)+O(\varepsilon)]\xi \qquad \xi(0) = x_{10} \qquad (3.17)$$
$$\varepsilon\dot{\eta} = [(A_{22}-B_2GC_2)+O(\varepsilon)]\eta \qquad \eta(0) = x_{20}-A_{22}^{-1}(A_{21}-B_2G^oC_o)x_{10}+O(\varepsilon) \qquad (3.18)$$

so that, for ε sufficiently small,

$$\xi(t) = \exp[(A_o - B_oG^oC_o)t]\xi(0) + O(\varepsilon) \qquad (3.19)$$

$$\eta(t) = \exp[(A_{22} - B_2GC_2)t/\varepsilon]\eta(0) + O(\varepsilon) \qquad (3.20)$$

Employing $T^{-1}(\varepsilon)$ from (3.13) to transform back to x_1, x_2, we obtain

$$x_1(t) = \xi(t) + O(\varepsilon) \qquad (3.21)$$

$$x_2(t) = -A_{22}^{-1}(A_{21} - B_2G^oC_o)\xi(t) + \eta(t) + O(\varepsilon) \qquad (3.22)$$

Similarly, $T^{-1}(\varepsilon)$ transforms u as defined by (3.3,3.4):

$$u(t) = -G^oC_o\xi(t) - GC_2\eta(t) + O(\varepsilon) \qquad (3.23)$$

This development is summarized in the following theorem:

Theorem 3.1: If $A_{22} - B_2GC_2$ is Hurwitzian, then (3.21-3.23)

describe the full order system and control trajectories for all

finite $t > 0$. Additionally, if $A_o - B_oG^oC_o$ is Hurwitzian, then

(3.21-3.23) are true for all $t > 0$.

An immediate (and crucial) consequence of this theorem is that, for

sufficiently small ε, output feedback stabilizability of the full system

(3.1-3.4) is equivalent to joint output feedback stabilizability of both

subsystems. Note that the output feedback problem does not naturally

decompose into separate slow and fast designs, as in [18]; instead, G^o

and G must stabilize the separate systems (3.17,3.18) while satisfying

the hard constraint (3.9).

SECTION 4

NEAR-OPTIMAL OUTPUT FEEDBACK REGULATION

In this section, for the ill-conditioned system dynamics of Section 3, the block diagonalizing transformation $T(\varepsilon)$ from (3.13) is applied to the quadratic performance criterion of Section 2. If the slow subsystem measurements are nonredundant, then minimizing the transformed criterion at ε = 0 results in a gain solution which yields a second order approximation to optimal full system performance, while eliminating the dimensionality and ill-conditioning difficulties of minimizing directly for the full system dynamics.

4.1 Definition of the Approximate Problem

The performance index for the full order system (3.1-3.4) is

$$J = \int_0^\infty [x_1^T, x_2^T] \; Q \; \begin{array}{c} x_1 \\ x_2 \end{array} + u^T Ru \; dt \tag{4.1}$$

where $R > 0$ and $Q = \Gamma^T\Gamma$ such that (Γ, A) is detectable. Q is compatibly partitioned as

$$Q = \begin{bmatrix} Q_2 & Q_2^T \\ Q_2 & Q_3 \end{bmatrix} \tag{4.2}$$

Assuming that the closed-loop system matrix $\hat{A}$ in (3.12) is asymptotically stable, than (4.2) is equivalent to

$$J = tr\{Kx_o x_o^T\} \tag{4.3}$$

where x_o^T is $[x_{10}^T, x_{20}^T]$, and $K \geq 0$ is the unique solution of

$$\hat{A}^T K + K\hat{A} + \hat{Q} = 0 \tag{4.4}$$

$$K = \begin{bmatrix} K_1 & \varepsilon K_2^T \\ \varepsilon K_2 & \varepsilon K_3 \end{bmatrix} \tag{4.5}$$

$$\hat{Q} = \begin{bmatrix} Q_1 + C_1^T G^T RGC_1 & Q_2^T + C_1^T G^T RGC_2 \\ Q_2 + C_2^T G^T RGC_1 & Q_3^T + C_2^T G^T RGC_2 \end{bmatrix} \qquad (4.6)$$

The problem of minimizing (4.3) with respect to G can be decomposed by using $T^{-1}(\epsilon)$ from (3.13) to transform the coordinates from x_1, x_2 to ξ and η. After transformation, (4.4) decouples into:

$$S_1(G_o, K_1, \epsilon) = \bar{A}_o^T \bar{K}_1 + \bar{K}_1 \bar{A}_o + \bar{Q}_1 = 0 \qquad (4.7)$$

$$\bar{A}_{22}^T \bar{K}_2 + \bar{K}_2 \bar{A}_o + \bar{Q}_2 = 0 \qquad (4.8)$$

$$S_3(G, K_3, \epsilon) = \bar{A}_{22}^T \bar{K}_3 + \bar{K}_3 \bar{A}_{22} + \bar{Q}_3 = 0 \qquad (4.9)$$

$$\bar{A}_o = A_o - B_o G^o C_o + 0(\epsilon) \qquad (4.10)$$

$$\bar{A}_{22} = A_{22} - B_2 GC_2 + 0(\epsilon) \qquad (4.11)$$

$$\bar{Q}_1 = Q_1 - N^T Q_2 - Q_2^T N + N^T Q_3 N + C_o^T G^{o^T} RG^o C_o + 0(\epsilon) \qquad (4.12)$$

$$\bar{Q}_2 = Q_2 - Q_3 N + C_2^T G^T RG^o C_o + 0(\epsilon) \qquad (4.13)$$

$$\bar{Q}_3 = Q_3 + C_2^T G^T RGC_2 + 0(\epsilon) \qquad (4.14)$$

As suggested in [1], it is customary to remove the dependence of (4.3) on initial conditions by assuming that they are uniformly distributed on the unit sphere. The problem statement is then modified slightly to that of minimizing $E\{J\}$, which amounts to replacing $x_o x_o^T$ in (4.3) by the identity matrix. For the two time scale problem, we instead assume that $[\xi^T(0), \eta^T(0)]$ is uniformly distributed on the unit sphere. This is because, under transformation by $T(\epsilon)$ at $\epsilon = 0$, the former assumption leads to

$$E\left\{ \begin{bmatrix} \xi(0) \\ \eta(0) \end{bmatrix} [\xi^T(0)\eta^T(0)] \right\} = T(0)E\{x_o x_o^T\}T^T(0)$$

$$= \begin{bmatrix} I & N^T \\ N & I + NN^T \end{bmatrix} \qquad (4.15)$$

which is inconveniently complicated. It should be noted from (4.3,4.5) and (3.13) that the difference between the costs resulting from either assumption is only $O(\varepsilon)$; further, the results from this section can be extended to any assumption on the initial condition.

The transformed cost for this problem is

$$J = tr\{\overline{K}_1\} + \varepsilon tr\{\overline{K}_3\} \qquad (4.16)$$

Now, note that the fast subsystem performance measure is, not unexpectedly, $O(\varepsilon)$. At $\varepsilon = 0$, where we would like to approximate the system dynamics, there is no cost associated with fast dynamics. On the other hand, minimization of $tr\{\overline{K}_1(\varepsilon = 0)\}$ with respect to G^0 must be done over the set of gains which would also stabilize $\overline{A}_{22}$, subject to (3.11). In order to do this in an orderly way, we instead minimize

$$J^0 = tr\{\overline{K}_1(\varepsilon = 0)\} + \varepsilon^0 tr\{\overline{K}_3(\varepsilon = 0)\} \qquad (4.17)$$

where ε^0 is fixed as the value of ε in (3.2). In fact, minimizing (4.17) allows simultaneous near-optimization of the slow and fast dynamics for essentially the same level of computational effort that would have been required to minimize $tr\{\overline{K}_1(\varepsilon = 0)\}$ alone, subject to the asymptotic stability of the fast subsystem. This situation differs with that seen in the singularly perturbed state feedback optimization problem [18]. There, because of the complete decoupling of the slow and fast subsystems, the control designer has the option of only calculating gains for the slow dynamics, if the fast dynamics are open-loop stable and if an $O(\varepsilon)$ approximation to optimal system performance is satisfactory. Even if the fast dynamics require stabilization, this is done as a task totally divorced from the slow subsystem design, and without using

information about ε. Here, in the output feedback problem, the constraint (3.9) inseparably links the slow and fast subproblems.

It is fairly obvious that a gain $\tilde{G}$ minimizing J^O, when applied to the full-order dynamics (3.1-3.4), will provide an $O(\varepsilon)$ approximation to actual optimal performance. In cases where $\rho\{C_O\} = p$, however, it is possible to make a stronger statement about the near-optimality of the approximate gain:

> **Theorem 4.1:** Given that $\rho\{C\} = p$, assume that $\rho\{C_O\} = p$. Let G^* be such that $J(G^*) < J(G)$ for J given by (4.16) and the dynamics (3.1-3.4). Let $\tilde{G}$ be such that $J^O(\tilde{G}) < J^O(G)$ for J^O given by (4.17) and the dynamics (3.17,3.18) at $\varepsilon = 0$. Then,
>
> $$J(\tilde{G}) = J(G^*) + O(\varepsilon^2) \qquad (4.18)$$

Theorem 4.1 is proven in Appendix D.

4.2 Two Time Scale Necessary Conditions

Following [5], minimization of J^O is recast as minimization of the Lagrangian

$$\mathscr{L} = J_O + \text{tr}\{S_1(G^O,\bar{K}_1,0)L_1^T\} + \text{tr}\{S_3(G,K_3,0)L_3^T\} + \text{tr}\{S_G(G,G^O)L_G^T\} \qquad (4.19)$$

with respect to G, G^O, $\bar{K}_1$, $\bar{K}_3$, L_1, L_3 and L_G at $\varepsilon = 0$. In (4.19), L_1, L_3 and L_G are matrices of Lagrange multipliers, and

$$S_G(G,G^O) = G^O - G + GD_OG^O = 0 \qquad (4.20)$$

from (3.9). Since the rest of the development takes place at $\varepsilon = 0$, all notation relating to ε will be suppressed for simplicity. The necessary conditions for optimality are determined by employing trace gradient identities found in [5]:

$$\frac{\partial\mathscr{L}}{\partial G} = RGC_2L_3C_2^T - B_2\bar{K}_3L_3C_2^T - \frac{1}{2}L_G(I - D_OG^O)^T = 0 \qquad (4.21)$$

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$$\frac{\partial \mathscr{L}}{\partial G^o} = R_o G^o C_o L_1 C_o^T - [B_o^T \bar{K}_1 - \zeta] L_1 C_o^T + \frac{1}{2}(I + GD_o)^T L_G = 0 \qquad (4.22.a)$$

$$R_o = R + B_2^T A_{22}^{-1}{}^T Q_3 A_{22}^{-1} B_2 \qquad (4.22.b)$$

$$\zeta = B_2^T A_{22}^{-1}{}^T (Q_2 - Q_3 A_{22}^{-1} A_{21}) \qquad (4.22.c)$$

$$\frac{\partial \mathscr{L}}{\partial \bar{K}_1} = \hat{A}_o L_1 + L_1 \hat{A}_o^T + I = 0 \qquad (4.23)$$

$$\frac{\partial \mathscr{L}}{\partial \bar{K}_3} = \hat{A}_{22} L_3 + L_3 \hat{A}_{22} + \varepsilon^o I = 0 \qquad (4.24)$$

$$\frac{\partial \mathscr{L}}{\partial L_1} = S_1(G^o, \bar{K}_1, 0) = 0 \qquad (4.25)$$

$$\frac{\partial \mathscr{L}}{\partial L_3} = S_3(G, \bar{K}_3, 0) = 0 \qquad (4.26)$$

$$\frac{\partial \mathscr{L}}{\partial L_G} = S_G(G, G^o) = 0 \qquad (4.27)$$

As was mentioned in Section 2, singularity in the necessary conditions is avoided if one makes the assumption

$$\rho\{C\} = p \qquad (4.28)$$

Although this by no means assures that C_o and C_2 individually have full rank, note that

$$\rho\{[C_o|C_2]\} = \rho\{CT(0)\}\Big|_{G^o = 0} = p \qquad (4.29)$$

so that the full rank of the total system output is preserved under the block diagonalizing transformation of Section 3. This fact is exploited in the following subsection, which presents an adaptation of the numerical algorithm of Section 2.2 to the computation of near-optimal output feedback gains.

4.3 Computational Algorithm

The following algorithm can be used to calculate gains satisfying

the necessary conditions (4.21-4.27). This is a direct adaptation of the
algorithm of Section 2.2 and shares its convergence properties.

0. Choose any G such that $A_{22} - B_2GC_2$ is Hurwitzian and $A_o - B_oG^oC_o$

   is Hurwitz subject to (3.9). Set $i = 0$.

1. Solve (4.23-4.26) for L_1^i, L_3^i, $\bar{K}_1^i$, $\bar{K}_3^i$

2. If i is even, solve (4.22) and (4.21) for

$$L_G^i = 2(I + G^iD_o)^{-1T}[B_o^T\bar{K}_1^i + \zeta - R_o(G^o)^iC_o]L_1^iC_o^T \tag{4.30}$$

$$\Delta G^i = R^{-1}[B_2^T\bar{K}_3^iL_3^iC_2^T + \tfrac{1}{2}L_G^i(I - D_o(G^o)^i)^T](C_2L_3^iC_2^T)^+ - G^i \tag{4.31}$$

   If i is odd, solve (4.21) and (4.22) for

$$L_G^i = 2[B_2^T\bar{K}_3^i - RG^iC_2]L_3^iC_2^T(I - D_o(G^o)^i)^{-1T} \tag{4.32}$$

$$(\Delta G^o)^i = R_o^{-1}[(B_o^T\bar{K}_1^i+\zeta)L_1^iC_o^T - \tfrac{1}{2}(I+G^iD_o)^TL_G^i](C_oL_1^iC_o^T)^+ - (G^o)^i \tag{4.33}$$

3. If i is even, set

$$G^{i+1} = G^i + \alpha\Delta G^i \ , \ (G^o)^{i+1} = [I + G^{i+1}D_o]^{-1}G^{i+1} \tag{4.34}$$

   or, if i is odd, set

$$(G^o)^{i+1} = (G^o)^i + \alpha(\Delta G^o)^i \ , \ G^{i+1} = (Go)^{i+1}[I-D_o(G^o)^{i+1}]^{-1} \tag{4.35}$$

   where $\alpha \ \epsilon \ (0,1]$ is chosen to ensure that

$$J_{i+1}^o < J_i^o = tr\{\bar{K}_1^i\} + \epsilon^o tr\{\bar{K}_3^i\} \tag{4.36}$$

4. Set $i = i + 1$ and go to 1.

The only functional difference between this algorithm and that in Section
2.2 is that, because of the potential for subsystem rank deficiency,
columns of the transposed gain matrix which fall into $im\{C_2\}$ are

incrementally optimized on even iterations, and those which fall into $im\{C_0\}$ are incrementally optimized on odd iterations. Assuming that C has full rank, the optimization at convergence will extend over the entire p-dimensional range of C. If C_0 has full rank, then the algorithm simplifies to using (4.23,4.33) and (4.35) at each iteration with $(C_0 L_1^i C_0^T)^+$ replaced by $(C_0 L_1^i C_0)^{-1}$.

4.4 Numerical Example

In [18], a system of the form (3.1-3.3) was examined, where

$$A_{11} = \begin{bmatrix} 0 & .400 \\ 0 & 0 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 0 & 0 \\ .345 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & -.524 \\ 0 & 0 \end{bmatrix} \quad A_{22} = \begin{bmatrix} -.465 & .262 \\ 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad C_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

For the output feedback control structure (3.4), optimal and near-optimal gains (minimizing (4.16) and (4.17), respectively) were calculated for

$$Q = \text{diag}[0.5, 0, 0.5, 0] \qquad R = 0.5$$

at several values of ε. For J given by (4.16), the difference $J(G^*) - J(\tilde{G})$ is displayed as a function of ε in Figure 2. It can be seen that the error in performance optimality due to the approximate gain is, indeed, $O(\varepsilon^2)$, as stated in Theorem 4.1.
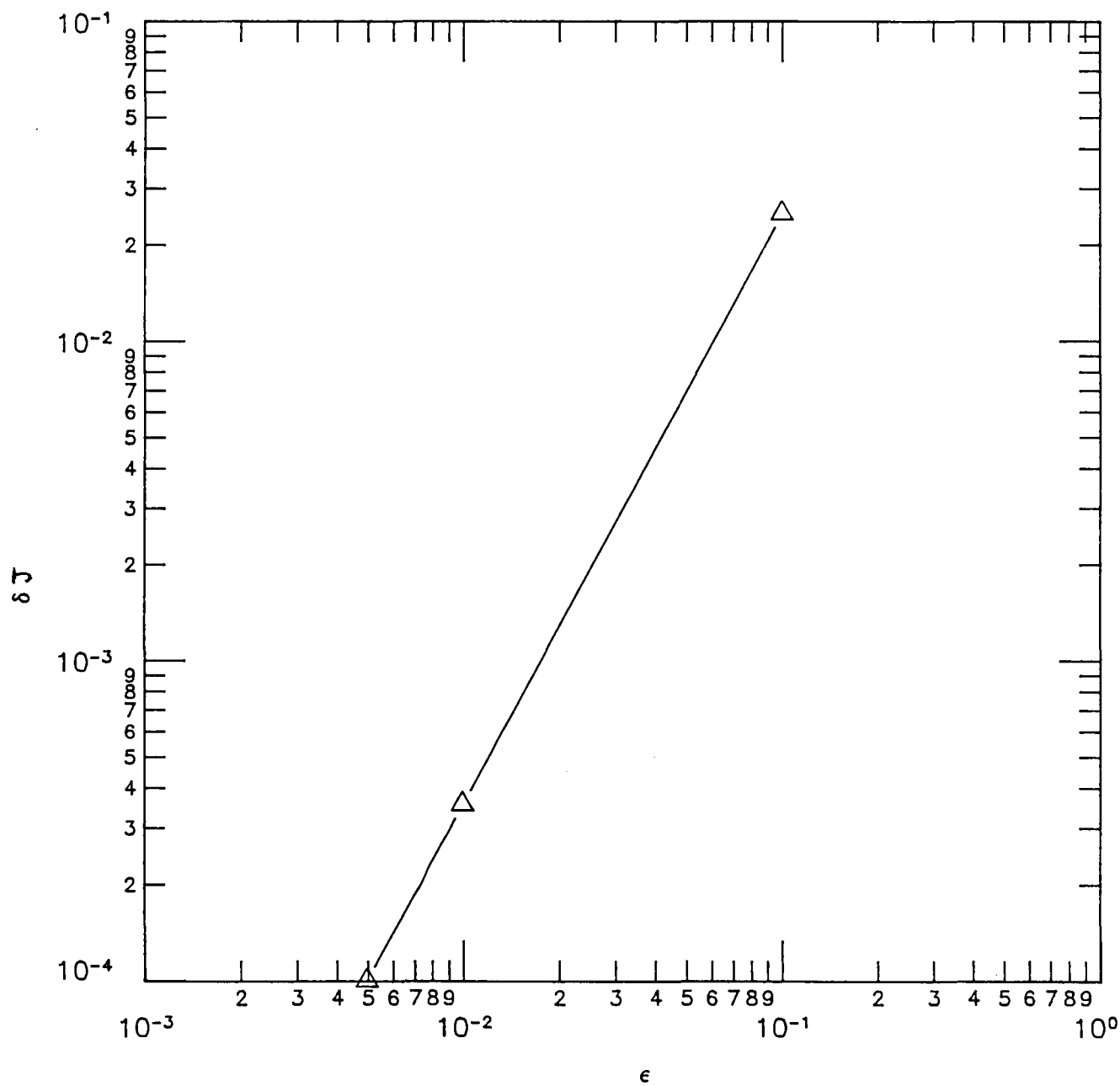
FIGURE 2. LOSS OF OPTIMALITY FOR NEAR-OPTIMAL GAIN AS A FUNCTION OF ϵ.

SECTION 5

GAIN SPILLOVER SUPPRESSION IN TWO TIME SCALE DESIGN

Though the LQ design procedure described in the last Section provides a second-order approximation to optimal closed-loop performance, it does so at some expense in complexity. This is primarily a result of the necessity of solving systems of equations for both subsystems at once. In this Section the design problem is decomposed into two separate subproblems through exploiting rank deficiency in the slow and fast subsystem input and output matrices. This is done by enforcing the two-way control and observation spillover suppression constraints. For brevity, this will be referred to as gain spillover suppression (GSS). The GSS constraints are enforced in an LQ-based design procedure by means of penalty functions adjoined to standard integral quadratic cost functions based on ξ and η.

The primary emphasis in this Section is on procedural simplicity; therefore our attention is focussed on stabilization of the slow and fast subsystems rather than designing toward a performance measure based on the full-order system. In other words, we design to satisfy the following goals:

$$(A_0 - B_0 G^0 C_0) \text{ is Hurwitz} \tag{5.1}$$

$$(A_{22} - B_2 G C_2) \text{ is Hurwitz} \tag{5.2}$$

subject to (3.9). From Theorem 3.1, we are assured that a gain design satisfying (5.1,5.2) will stabilize the full-order system (3.1-3.4) for sufficiently small ε and that, in fact, the closed-loop spectrum of the full-order system will be

$$\sigma = \sigma[(A_0 - B_0 G^0 C_0) + O(\varepsilon)] \bigcup \sigma[(A_{22} - B_2 G C_2) + O(\varepsilon)]/\varepsilon \tag{5.3}$$

5.1 Spillover Suppression Conditions

Here, conditions are given for using GSS to separate the design of G into a two step process: One gain matrix, G_1, is designed so as to satisfy (5.1) without disturbing the eigenstructure in (5.2). The other, G_2, is designed to satisfy (5.2) without affecting (5.1). The implemented gain takes the form

$$G = G_1 + G_2 \qquad (5.4)$$

It will be shown that this particular ordering of the design steps is necessary, and that it does not impose any additional restriction on the implemented gain.

Suppose that A_{22} is "sufficiently" stable. In this case, let $G_2 = 0$, so that $G = G_1$ in (5.4). In order to avoid gain spillover into the fast dynamics, we require

$$B_2 G_1 C_2 = 0 \qquad (5.5)$$

The following lemma provides an easily enforced constraint for satisfying (5.5). This lemma and Lemmas 5.2 and 5.3 are proven in the Appendix E.

Lemma 5.1: Condition (5.5) holds iff

$$B_2 G_1^o C_2 = 0 \qquad (5.6)$$

where

$$G_1^o = (I + G_1 D_o)^{-1} G_1 \qquad (5.7)$$

Moreover, if (5.6) holds, then the inverse in (5.7) exists and is given by

$$(I + G_1 D_o)^{-1} = (I - G_1 D_o) \qquad (5.8)$$

The slow susbsystem design thus consists of satisfying (5.1) and (5.6), where $G^o = G_1^o$ in (5.1). Once a satisfactory G_1^o is obtained, G_1 is calculated using

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$$G_1 = G_1^O(I - D_oG_1^O)^{-1} = G_1^O(I + D_oG_1^O) \qquad (5.9)$$

The second equality of (5.9) can easily be verified from (5.6) and the form of D_o in (3.7) Also, note from (5.8) and the form of D_o that, if one only exploits rank deficiency in the fast subsystem input matrix; that is, if one insists that

$$B_2G_1^O = 0 \qquad (5.10)$$

then $G_1^O = G_1$, so that G_1 may be directly designed to stabilize $A_o-B_oG_1C_o$ subject to the control spillover constraint (5.10).

Now, suppose that the fast dynamics require improvement. In this case, G_2 is designed to stabilize the fast dynamics without spilling over into the slow dynamics. The design criteria are (5.2) and

$$B_o[I+(G_1+G_2)D_o]^{-1}(G_1+G_2)C_o = B_oG_1^OC_o \qquad (5.11)$$

where the spillover condition (5.11) is obtained from (3.9), (5.1) and (5.4). One immediately notes that, if $D_o = 0$, (5.11) collapses to a form which mirrors the slow subsystem GSS condition:

$$B_oG_2C_o = 0 \qquad (5.12)$$

The following Lemma provides a necessary and sufficient condition on G_2 for satisfaction of (5.11)

Lemma 5.2: Condition (5.11) holds iff

$$B_o(I-G_1D_o)G_2(I-D_oG_1)C_o = 0 \qquad (5.13)$$

Despite the fact that (5.13) is dependent on G_1, the slow subsystem gain has no effect on the fast subsystem dynamics:

Lemma 5.3: Given that G_1 satisfies (5.5) and G_2 satisfies (5.13), $B_2G_2C_2$ is not a function of G_1.

Lemma 5.3 implies that no flexibility is lost by adopting a two-step

design procedure in which G_1 is treated as a constant during the design of G_2 satisfying (5.13). The set of admissible stabilizing G_2 is limited only by $\ker\{B_0\}$ and $\ker\{C_0{}^T\}$. On the basis of the preceding development, the following theorem is stated:

Theorem 5.1: Let G_1^0 be an asymptotically stabilizing feedback gain for the reduced system $\{A_0, B_0, C_0\}$ which satisfies the GSS constraint (5.2). Let G_2 be an asymptotically stabilizing feedback gain for the fast subsystem $\{A_{22}, B_2, C_2\}$ satisfying the GSS constraint (5.13). Then,

$$G = G_1^0(I + D_0 G_1^0) + G_2 \qquad (5.14)$$

stabilizes the full-order system (3.1-3.4) for sufficiently small ε. Moreover, the closed-loop spectrum will be given by (5.3).

An easily implemented approach to enforcing the GSS conditions (5.5) and (5.13) is developed in the next section.

5.2 LQ Design Procedure

In this section, LQ optimal control theory is applied to the problem of determining G that stabilizes the matrix

$$\hat{A} = A - BGC \qquad (5.15)$$

subject to the constraint

$$MGP = 0 \qquad (5.16)$$

which corresponds to the general form of the conditions stated in the theorem. Under the assumption that a solution exists, this is done by defining the performance index

$$J_0 = E_{x_0}\{\int_0^\infty x^T Q x + u^T R u \, dt\} + \nu \|MGP\|^2 \qquad (5.17)$$

for the dynamics

$$\dot{x} = Ax + Bu \quad x(0) = x_0 \qquad (5.18)$$

$$u = -GCx \qquad (5.19)$$

In (5.17), $Q = \Gamma^T \Gamma$ such that $\{\Gamma, \hat{A}\}$ is detectable and $R > 0$. The notation $E_{x_0}\{.\}$ denotes expectation with respect to the random initial state x_0 where, for simplicity, it is assumed that x_0 is uniformly distributed on the unit sphere, so that $E\{x_0 x_0^T\} = I$. The notation $\|.\|$ denotes the inner product matrix norm

$$\|W\|^2 = tr\{W^T W\} \qquad (5.20)$$

and $\nu > 0$ is chosen sufficiently large that $\|MGP\| \to 0$. Following [5], the Lagrangian is written:

$$\mathscr{L}(G,K,L) = tr\{K\} + tr\{S(G,K)L^T\} + \nu\|MGP\|^2 \qquad (5.21)$$

$$S(G,K) = \hat{A}^T K + K\hat{A} + Q + C^T G^T RGC = 0 \qquad (5.22)$$

The first-order necessary conditions for optimality are:

$$\frac{\partial \mathscr{L}}{\partial G}\bigg|_* = 0 \quad \frac{\partial \mathscr{L}}{\partial K}\bigg|_* = 0 \quad \frac{\partial \mathscr{L}}{\partial L}\bigg|_* = 0 \qquad (5.23)$$

where the $*$'s indicate that the gradients are evaluated at the optimal values of G, K and L. The $*$ notation is henceforth suppressed, since the gradients are assumed evaluated at their optimal values unless specified otherwise. Expanding the gradients in (5.23), we have:

$$-B^T KLC^T + RGCLC^T + \nu M^T MGPP^T = 0 \qquad (5.24)$$

$$\hat{A}L + L\hat{A}^T + I = 0 \qquad (5.25)$$

$$S(G,K) = 0 \qquad (5.26)$$

These necessary conditions can be solved by using the algorithm described below. This algorithm satisfies the sufficient conditions for numerical convergence given in Section 2, is simple to implement, and has demonstrated a "fast" rate of convergence in practice.

0. Choose any G rendering $\hat{A}$ Hurwitzian. Set $i = 0$. Set $G = GCC^+$

1. Solve (5.25,5.26) for L^i, K^i

2. On the basis of (5.24), evaluate

$$\Delta G^i = R^{-1}[B^T K^i L^i C^T - \nu M^T M G^i P P^T](CL^i C^T)^+ - G^i \tag{5.27}$$

3. Set

$$G_1^{i+1} = G^i + \alpha \Delta G^i \tag{5.28}$$

where $\alpha \in (0,1]$ is chosen to ensure that

$$J_{i+1} < J_i = tr\{K^i\} + \nu \| M G^i P \| \tag{5.29}$$

Note: In (5.27) and step 0, $(.)^+$ denotes the pseudoinverse. In the case where $\rho(C) = p$, $CC^+ = I$ and $(CLC^T)^+ = (CLC^T)^{-1}$; however, this is not generally the case. Using the characteristics of the pseudo-inverse in Appendix A, it can be shown that the columns of the transposed incremental gain $(\Delta G^i)^T$ will always lie wholly in $im\{C^T\}$. Because of this fact, the algorithm satisfies the requirements in Theorem 2.1 for convergence.

5.3 Numerical Example

The design procedure of Section 5.2 is demonstrated on a model of a large flexible space structure. Data for this system came from [32].

$$A_{11} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(.42)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -(.43)^2 & 0 \end{bmatrix} \qquad A_{12} = 0$$

$$
A_{22}/\epsilon = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(2.1)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -(2.2)^2 & 0 \end{bmatrix}
\qquad A_{21} = 0
$$

$$
\begin{bmatrix} B_1 \\ \hline B_2/\epsilon \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -.92 & -1.4 & .92 & -1.4 \\ 0 & 0 & 0 & 0 \\ .65 & 1.6 & .65 & -1.6 \\ \hline 0 & 0 & 0 & 0 \\ 1.4 & -1.0 & 1.4 & 1.0 \\ 0 & 0 & 0 & 0 \\ 2.05 & -.80 & -2.0 & -.80 \end{bmatrix}
$$

$$
C_1 = \begin{bmatrix} 0 & -1.8 & 0 & 1.3 \\ 0 & -2.7 & 0 & 3.2 \\ 0 & 1.8 & 0 & 1.3 \\ 0 & -2.7 & 0 & -3.2 \end{bmatrix}
$$

$$
C_2 = \begin{bmatrix} 0 & 2.9 & 0 & 4.1 \\ 0 & -2.1 & 0 & -1.6 \\ 0 & 2.9 & 0 & -4.1 \\ 0 & 2.1 & 0 & -1.6 \end{bmatrix}
$$

In this system, the third and fourth modes are approximately five times faster than the first and second modes. When describing the system in the form (1,2), one would naturally choose $\epsilon = 1/5$. In the reduced order

model $D_0 = 0$. In the absence of control feedthrough the GSS constraint

for the fast subsystem collapse to the form (5.12).

The penalty weights employed in the slow and fast subsystem designs

were:

$$Q_0 = \text{diag}[0,\ 0.065,\ 0,\ 0.065] \qquad R_0 = I$$

$$Q_2 = \text{diag}[0,\ 1.3,\ 0.,\ 1.0] \qquad R_2 = I$$

Figure 3 shows the upper half plane closed-loop eigenvalues due to G

formed from optimal subsystem designs without GSS ($v_0 = v_2 = 0$). The

Figure also displays the intended closed-loop eigenvalue locations for

the slow and fast subsystems. Note that the gain spillover distorts the

response of the full-order system, tending to destabilize two of the

modes. Figures 4 and 5 show the variation of integral quadratic

performance and spillover penalty for the slow and fast subsystems,

respectively. Figure 6 shows the upper half-plane closed-loop

eigenvalues for the gain resulting from choosing $v_0 = 10$ and $v_2 = .0001$

for design values. The degradation in integral quadratic cost due to the

constraint of GSS for the slow subsystem was less than 1.8%. The degra-
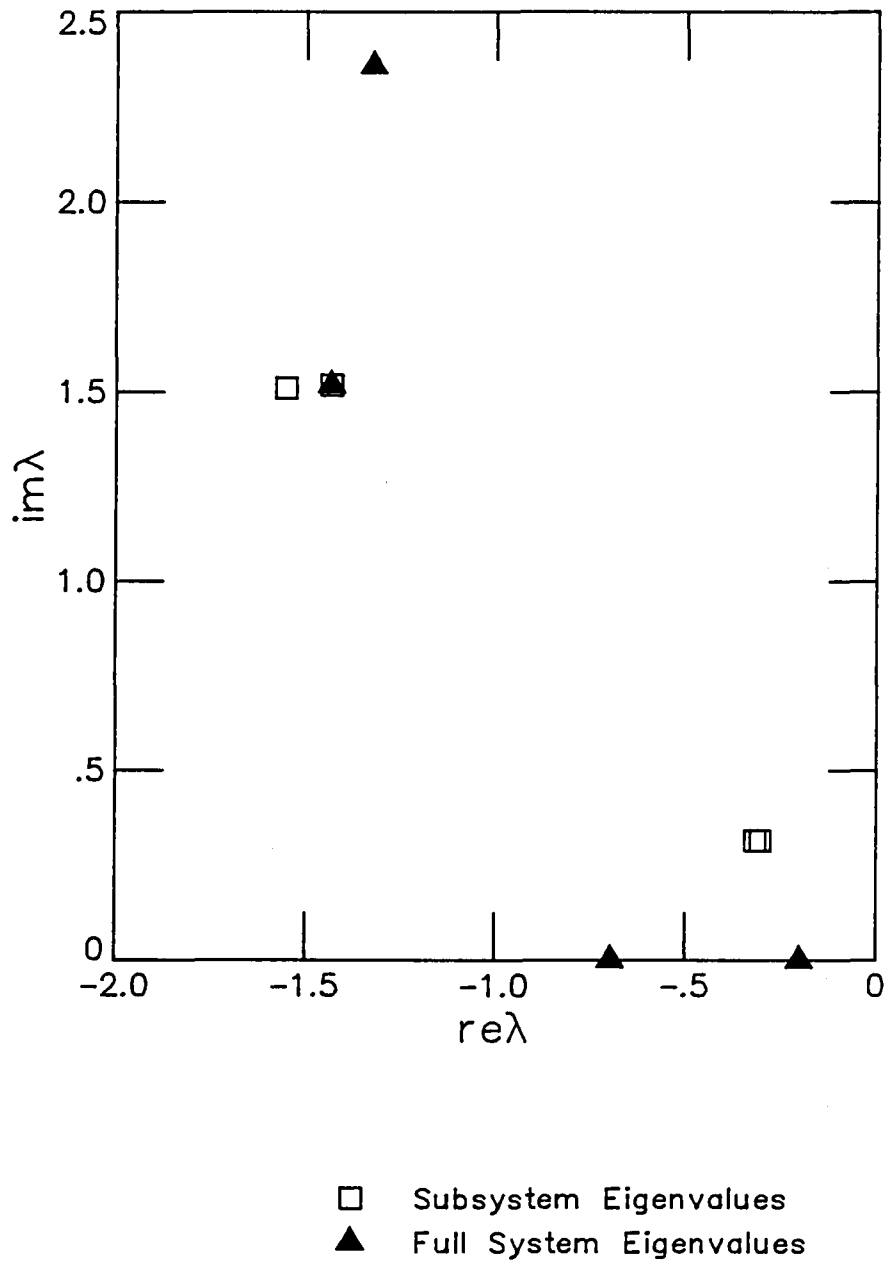
dation in the fast subsystem was negligible.
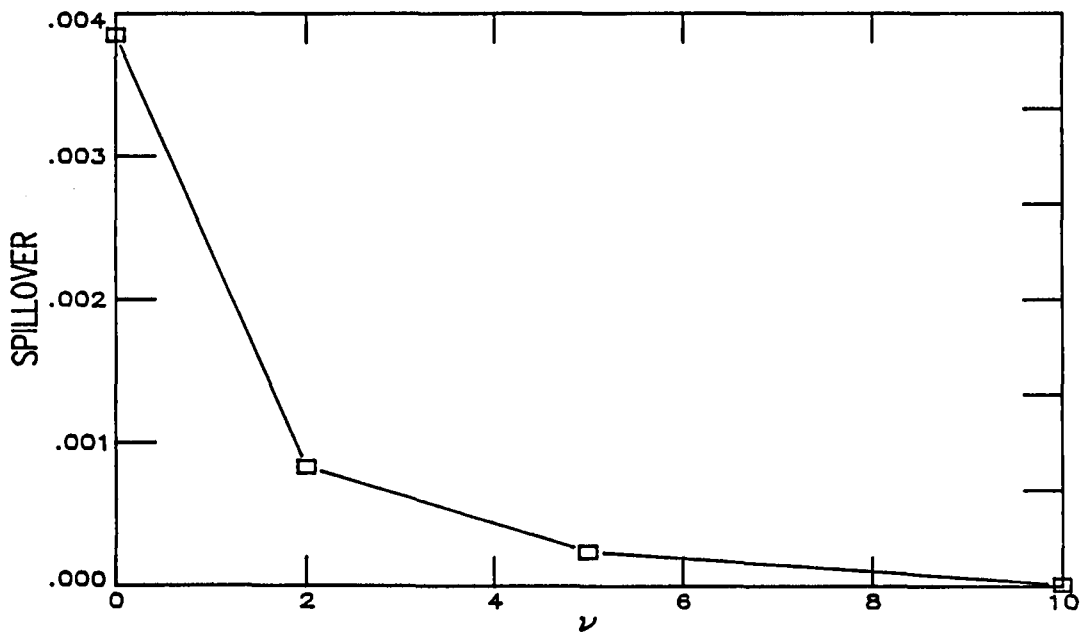
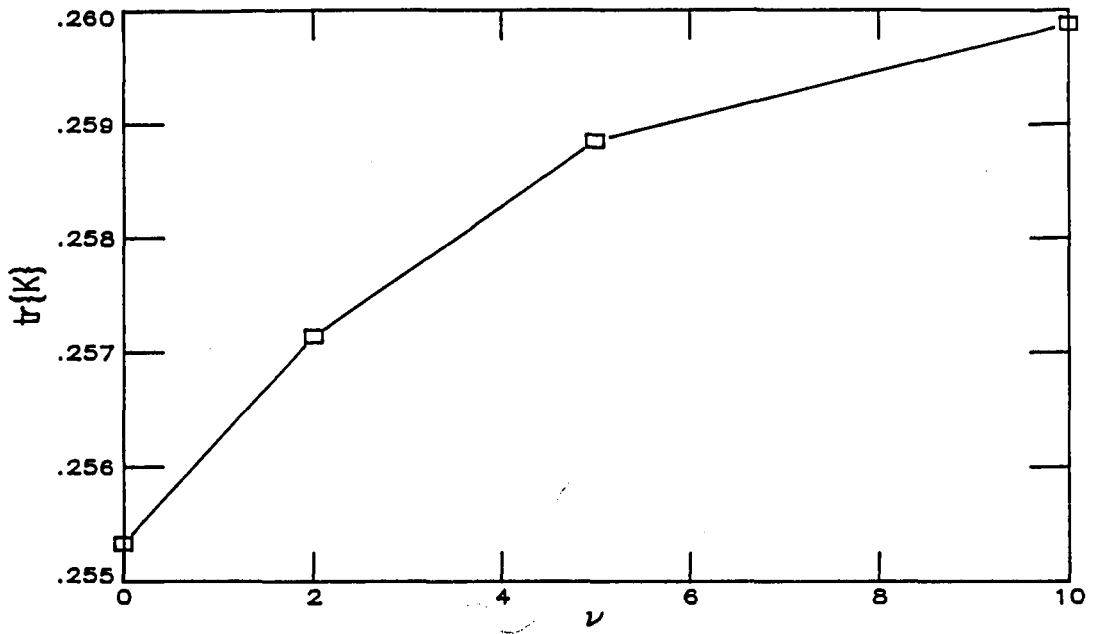FIGURE 3. CLOSED-LOOP EIGENVALUES WITHOUT
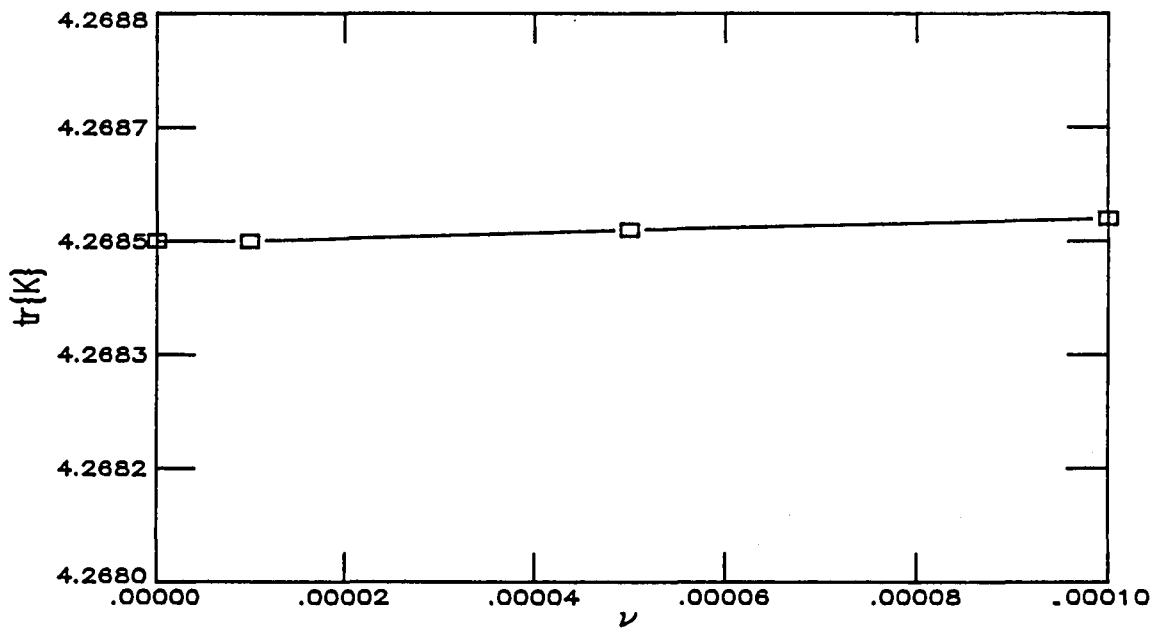SPILLOVER SUPPRESSION.

FIGURE 4.   SLOW SUBSYSTEM DESIGN
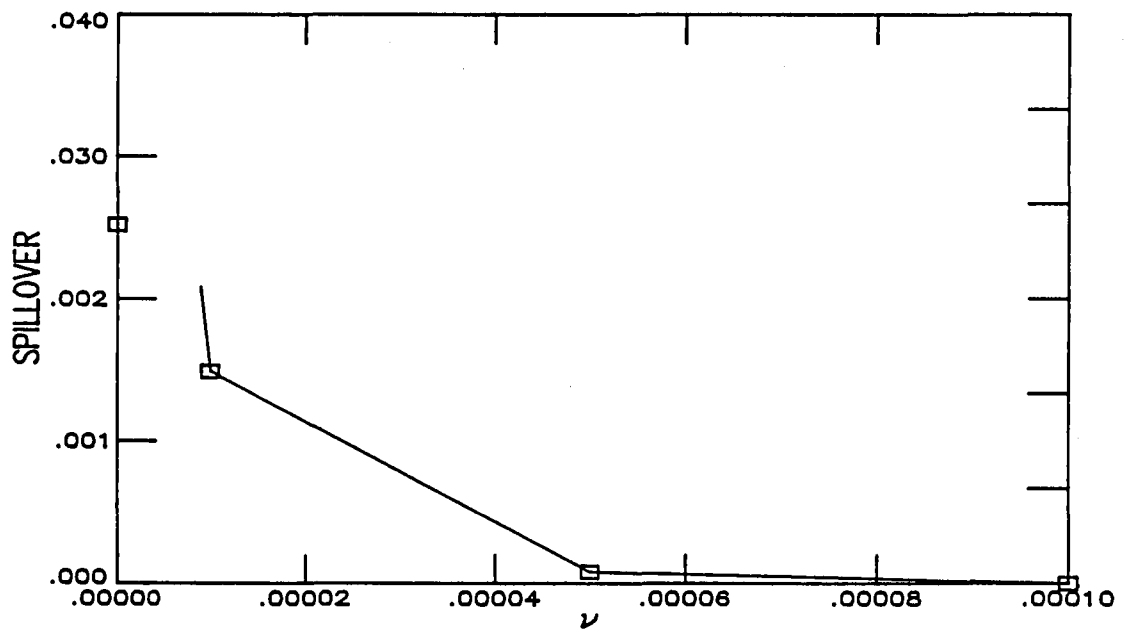
FIGURE 5. FAST SUBSYSTEM DESIGN
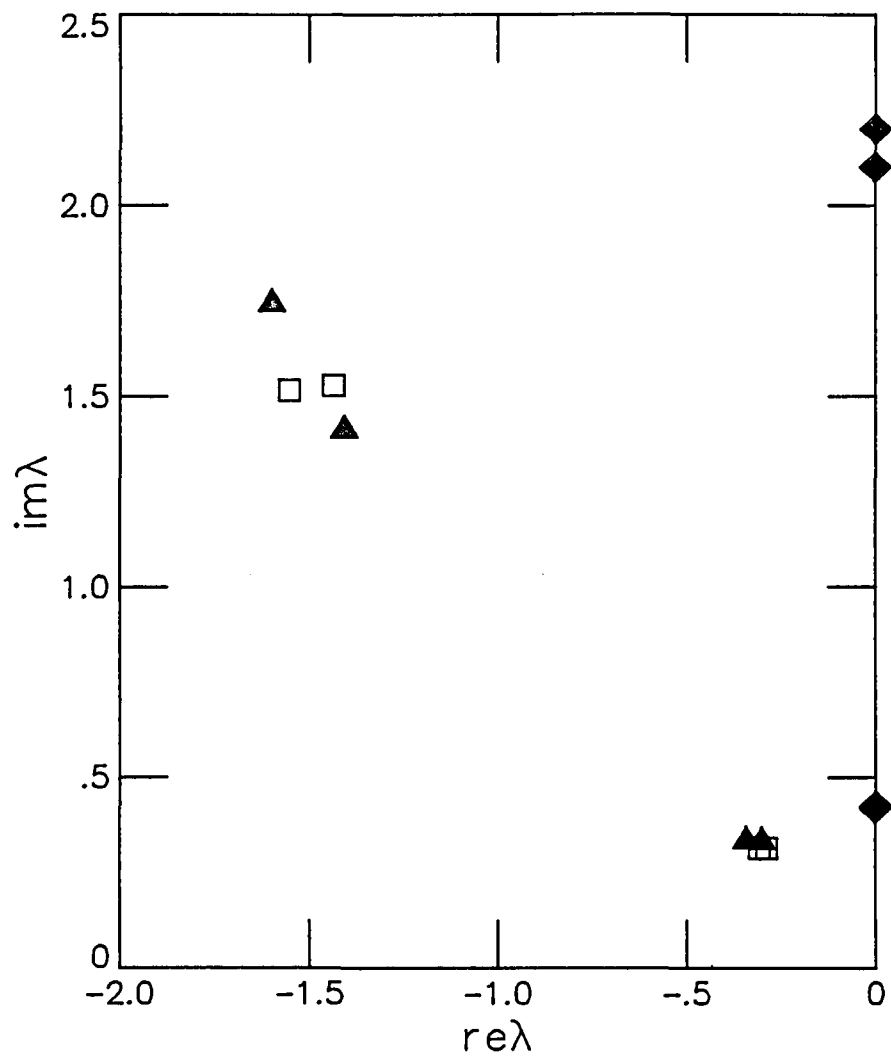
FIGURE 6.   CLOSED-LOOP EIGENVALUES WITH SPILLOVER
SUPPRESSION.

SECTION 6

TWO TIME SCALE DYNAMIC COMPENSATION

In this section, the static gain theory developed in Sections 3-5 is
extended to the case of output feedback regulators which include dynamic
elements in the feedback structure. The approach taken here is a
straightforward adaptation of the standard approach used in dynamic
compensation problems which employ state variable methods [6,7,33,34]:
Slow and fast compensator states are adjoined to the slow and fast plant
states so that, in essence, the compensator becomes a subsystem of the
plant with full state output. The important feature here is that the
compensator has the same two time scale character as the plant, and is
decomposed into two subsystems with it. In [36] an analog of the two
time scale fixed-order compensator is examined - the singularly perturbed
Kalman filter. There, it was shown that, in addition to a complete
separation of the slow and fast subsystem control designs, the filter
dynamics also separate. This results in separate filters for each
subsystem. It will now be seen that, for the fixed order case, "almost"
separate compensators may be designed for each subsystem - separate in
the sense that, although they both use the same static gain feedback
matrix, they do not share any dynamic elements.

The control law takes the form:

$$u_c = -Gy - H_1 z_1 - H_2 z_2 \qquad\qquad (6.1)$$

$$\dot{z}_1 = -P_{11} z_1 - P_{12} z_2 - N_1 y \qquad z_1 \in R^{n z_1} \qquad (6.2)$$

$$\epsilon \dot{z}_2 = -P_{21} z_1 - P_{22} z_2 - N_2 y \qquad z_2 \in R^{n z_1} \qquad (6.3)$$

where y is the system output defined in (3.3). In order to apply the

results of Section 3, the slow and fast plant and compensator dynamics are adjoined by defining

$$v_1^T = [x_1^T, z_1^T] \qquad v_2^T = [x_2^T, z_2^T] \tag{6.4}$$

which gives the following system structure:

$$\begin{bmatrix} \dot{v}_1 \\ \epsilon \dot{v}_2 \end{bmatrix} = \begin{bmatrix} W_{11} & W_{12} \\ W_{12} & W_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} \tilde{u}_c \tag{6.5}$$

with output

$$y_c = \begin{bmatrix} S1 & S2 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} \tag{6.6}$$

where

$$\begin{bmatrix} S1 & S2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 & C_2 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{6.7}$$

$$\begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & I & 0 \\ \hline B_2 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \tag{6.8}$$

$$\begin{bmatrix} W_{11} & W_{12} \\ W_{12} & W_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{12} & 0 \\ 0 & 0 & 0 & 0 \\ \hline A_{21} & 0 & A_{22} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{6.9}$$

The introduction of I in the definition of W_{22} renders it invertible, for invertible A_{22}. The feedback law now becomes

$$\tilde{u}_c = -\tilde{G}_c y_c \tag{6.10}$$

$$\tilde{G}_c = \begin{bmatrix} G & H_1 & H_2 \\ N1 & P_{11} & P_{12} \\ N_2 & P_{21} & P_{22} + I \end{bmatrix} \qquad (6.11)$$

Note that by expressing the compensator in this manner, one can directly apply the static gain theory of Sections 2-5. Decoupling the closed-loop dynamics of (6.4-6.11) by a transformation analogous to (3.13) results in

$$\dot{\xi}_c = [(W_0 - \tilde{\Pi}_0 \tilde{G}_c^0 \tilde{S}_0) + 0(\varepsilon)]\xi_c \qquad (6.12)$$

$$\varepsilon\dot{\eta}_c = [(W_{22} - \tilde{\Pi}_2 \tilde{G}_c \tilde{S}_2) + 0(\varepsilon)]\eta_c \qquad (6.13)$$

where

$$W_0 = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \tilde{\Pi}_0 = \begin{bmatrix} B_0 & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \qquad \tilde{S}_0 = \begin{bmatrix} C_0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \quad (6.14)$$

and

$$\tilde{G}_c^0 = \begin{bmatrix} Go & H_1^0 & H_2^0 \\ N_1^0 & P_{11}^0 & P_{12}^0 \\ N_2^0 & P_{21}^0 & P_{22}^0 \end{bmatrix} \qquad (6.15)$$

The relation analogous to (3.9) is

$$\tilde{G}_c^0 = (I + \tilde{G}_c \tilde{\Psi}_0)^{-1} \tilde{G}_c \qquad (6.16)$$

$$\tilde{\Psi}_0 = \begin{bmatrix} Do & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} \qquad (6.17)$$

At $\varepsilon = 0$, the closed-loop matrices (6.12,6.13) are

$$W_0 - \tilde{\Pi}_0 \tilde{G}_c^o \tilde{S}_0 = \begin{bmatrix} A_0 - B_0 G_0 C_0 & -B_0 H_1^o \\ -N_1^o C_0 & -P_{11}^o \end{bmatrix} \qquad (6.18)$$

$$W_{22} - \tilde{\Pi}_2 \tilde{G}_c \tilde{S}_2 = \begin{bmatrix} A_{22} - B_2 G C_2 & -B_2 H_2 \\ -N_2 C_2 & I - P_{22} \end{bmatrix} \qquad (6.19)$$

Note that in (6.18) and (6.19), the closed-loop subsystem dynamics do not involve any of the cross-coupling terms between the slow and fast compensator states. This system structure is displayed in Figure 8. An examination of this Figure suggests a simplified design problem: Given the input and output matrices

$$\Pi_0 = \begin{bmatrix} B_0 & 0 \\ 0 & I \end{bmatrix} , \qquad S_0 = \begin{bmatrix} C_0 & 0 \\ 0 & I \end{bmatrix} \qquad (6.21)$$

$$\Pi_2 = \begin{bmatrix} B_2 & 0 \\ 0 & I \end{bmatrix} \qquad S_2 = \begin{bmatrix} C_2 & 0 \\ 0 & I \end{bmatrix} \qquad (6.22)$$

design

$$G_c^o = \begin{bmatrix} G^o & H_1^o \\ N_1^o & P_{11}^o \end{bmatrix} \qquad (6.23)$$

to stabilize $W_0 - \Pi_0 G_c^o S_0$, and design

$$G_c = \begin{bmatrix} G & H_2 \\ N_2 & P_{22} \end{bmatrix} \qquad (6.24)$$

to stabilize $W_{22} - \Pi_2 G_c S_2$, where G and G^o are linked by a constraint. After the designs have been completed, reconstruct the implementation compensator matrix $\tilde{G}_c$ in some manner from the elements of G_c^o and G_c.
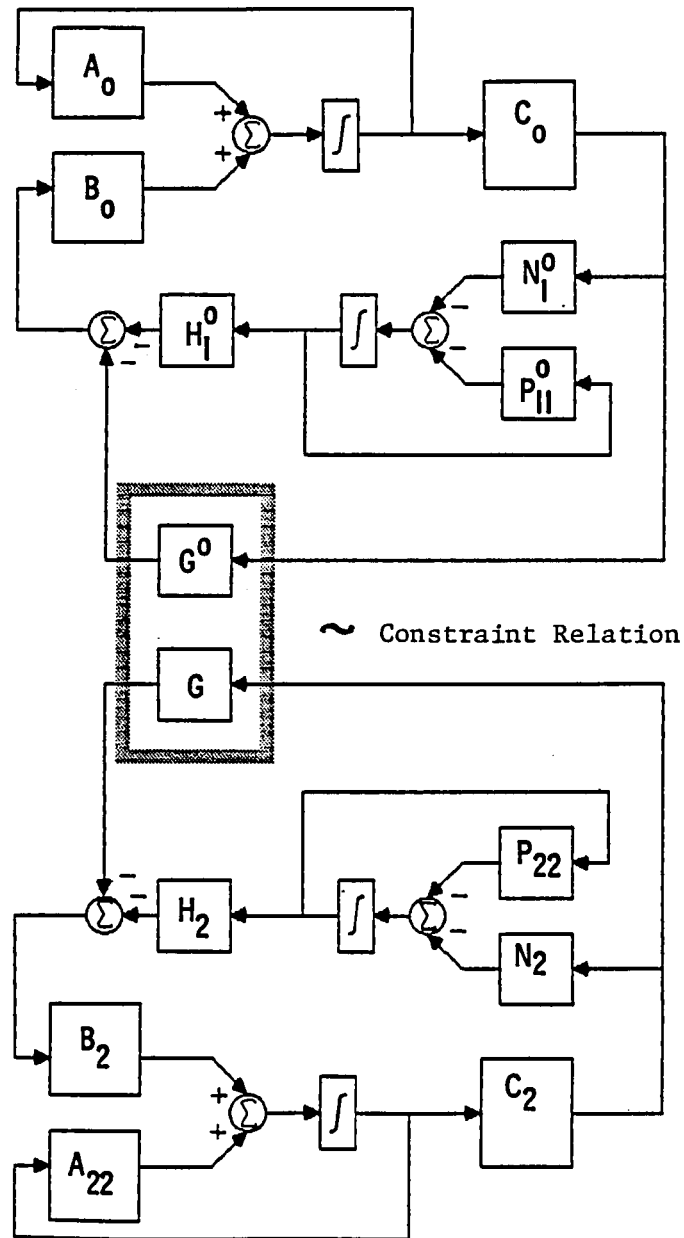
42

FIGURE 7. STRUCTURE OF TWO TIME SCALE
DYNAMIC COMPENSATOR.

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It turns out that if the rank conditions (3.10) and

$$\rho\{P_{22}\} = nz_2 \qquad (6.25)$$

hold, then the design problem can be decomposed in this manner, without imposing any additional constraint on the final solution. From Lemma 3.2, the inverse in (6.16) exists as long as the fast subsystem plant/compensator dynamics are stabilized. Expanding (6.16) as

$$\tilde{G}_c^{\circ} - \tilde{G}_c - \tilde{G}_c \Psi_0 \tilde{G}_c^{\circ} = 0 \qquad (6.26)$$

and partitioning, using (6.11),(6.15) and (6.17) gives a system of nine matrix equations. Among these are the following six, which can be used to derive the remaining compensator blocks in (6.1-6.3):

$$G^{\circ} - G + GD_0G^{\circ} - H_2N_2^{\circ} = 0 \qquad (6.27)$$

$$N_1^{\circ} - N_1 + N_1D_0G^{\circ} - P_{12}N_2^{\circ} = 0 \qquad (6.28)$$

$$-N_2 + N_2D_0G^{\circ} - P_{22}N_2^{\circ} = 0 \qquad (6.29)$$

$$H_1^{\circ} - H_1 + GD_0H_1^{\circ} - H_2P_{21}^{\circ} = 0 \qquad (6.30)$$

$$P_{11}^{\circ} - P_{11} + N_1D_0H_1^{\circ} - P_{12}P_{21}^{\circ} = 0 \qquad (6.31)$$

$$- P_{21} + N_2D_0H_1^{\circ} - P_{22}P_{21}^{\circ} = 0 \qquad (6.32)$$

From (6.27) and (6.29),

$$G = G^{\circ}(I - D_0G^{\circ})^{-1} - H_2P_{22}^{-1}N_2 \qquad (6.33)$$

Combining (6.28) and (6.29) gives

$$N_1 = N_1^{\circ}(I - D_0G^{\circ})^{-1} - P_{12}P_{22}^{-1}N_2 \qquad (6.34)$$

Expressions for H_1, P_{11} and P_{21} follow directly from (6.30-6.32):

$$H_1 = (I + GD_0)H_1^{\circ} - H_2P_{21}^{\circ} \qquad (6.35)$$

$$P_{11} = P_{11}^{\circ} + N_1D_0H_1^{\circ} - P_{12}P_{21} \qquad (6.36)$$

$$P_{21} = N_2D_0H_1^{\circ} - P_{22}P_{21}^{\circ} \qquad (6.37)$$

This formulation raises the following questions:

What is the significance of the off-diagonal compensator blocks $\{P_{12}, P_{21}\}$ and $\{P_{12}^o, P_{21}^o\}$? It is evident from Figure 7 that these elements do not affect closed-loop stability, but what about shaping the system eigenvectors? The constraint (6.26) gives one the option of setting either P_{12} or P_{12}^o to zero and of setting either P_{21} or P_{21}^o to zero. Zeroing $\{P_{12}, P_{21}\}$ seems a reasonable simplification of the compensator dynamics, if its slow and fast states are implemented by separate devices, but this hardly parallels the decomposition seen in the optimal stochastic regulator case [36]. Since they do represent extra degrees of freedom in the design without contributing to dimensionality, zeroing of these elements may be a waste of potential.

SECTION 7

CONCLUSIONS

This research effort has been directed toward easing the difficulties encountered in the calculation of quadratic optimal output feedback gains for linear systems - in particular, linear systems with slow and fast modes. The results of this effort are summarized below.

A fast, simple convergent sequential numerical algorithm has been developed for calculating optimal output feedback gains which minimize a class of performance indices which includes the standard LQ case. In order to demonstrate the theory, a performance index penalizing the magnitude of individual gain elements in an output feedback gain matrix was proposed and demonstrated. Employment of this form of performance index in design allows severing of individual feedback loops by zeroing selected gain elements without sacrificing quadratic optimality in the remaining loops. This dramatically enhances the flexibility of optimal multivariable output feedback design.

It has been shown that two time scale approximation in the output feedback design problem is always well-posed, as long as the fast dynamics are asymptotically stable in the closed loop and as long as there is sufficient time scale separation between the slow and fast dynamics.

A two time scale approximation to the LQ optimal output feedback problem has been derived. When implemented, this approximation provides at least a first-order approximation to optimal quadratic performance. In addition, if the measurement set for the slow subsystem is nonredundant, the performance is a second-order approximation. A computational algorithm for calculating these approximate gains was described and demonstrated. The algorithm is related to and shares the convergence

properties of the general optimal output feedback algorithm previously mentioned.

The two time scale approximation described in the previous paragraph leads to a one-step design procedure. The gain design must balance the performance of the slow and fast dynamics simultaneously. In order to obtain a simpler approach to design, necessary and sufficient conditions were derived for decoupling the slow and fast subsystem designs through mutual suppression of control and observation spillover. A simplified LQ design procedure was developed and demonstrated.

Finally, extension of the two time scale static gain theory to the case of fixed order dynamic compensation was briefly examined. Several questions remain to be resolved, leading to the conclusion that a further investigation of this area should be highly rewarding.

APPENDIX A

USEFUL MATRIX PROPERTIES

From [37], the pseudoinverse of A is defined as X satisfying

$$AXA = A \qquad (A.1)$$

$$XAX = X \qquad (A.2)$$

$$(AX)^* = AX \qquad (A.3)$$

$$(XA)^* = XA \qquad (A.4)$$

where $(.)^*$ is the complex conjugate transpose of $(.)$. Such an X always exists and is unique. If $A \in \mathscr{R}^{m \times n}$ with $m < n$ and $\rho\{A\} = m$ then $AX = I_m$. In this case the pseudoinverse X is a right inverse.

A useful identity for matrix inverses can be found in [38]. If A, C and (A + BCD) are nonsingular

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1} \qquad (A.5)$$

An important eigenvalue property for Kronecker products is supplied supplied by [39]:

Lemma: Given $N \in \mathscr{R}^{s \times s}$ with eigenvalues $\{\lambda_1, \ldots, \lambda_s\}$ and $M \in \mathscr{R}^{t \times t}$ with eigenvalues $\{\mu_1, \ldots, \mu_t\}$, then the eigenvalues of the Kronecker product $N \otimes M \in \mathscr{R}^{st \times st}$ are

$$\sigma_{N \otimes M} = \{\lambda_i \mu_j : i=1, \ldots, s \; ; \; j=1, \ldots, t\} \qquad (A.6)$$

APPENDIX B

PROOF OF THEOREM 2.1

Before proving the theorem, several preliminary results are established.

Lemma B.1: For L satisfying (13) where (i,ii) hold, there exists a Φ such that

$$(CLC^T)^{-1} = \Phi^{+T}\Phi^+ \tag{B.1}$$

$$\Phi\Phi^+ = I_p \tag{B.2}$$

where Φ^+ is the right inverse of Φ, and I_p is the p-dimensional identity matrix.

Proof:

Since $L > 0$ for $G \in \mathscr{G}$, it can be represented as

$$L = L^{1/2}L^{1/2} \tag{B.3}$$

where $L^{1/2} > 0$, so that

$$\Phi = CL^{1/2} \tag{B.4}$$

has full rank p, and therefore, there exists a right inverse of Φ^+.

In order to prove (B.1), first note that

$$CLC^T = \Phi\Phi^T \tag{B.5}$$

Premultiplying by Φ^+ gives

$$\Phi^+CLC^T = \Phi^+\Phi\Phi^T \tag{B.6}$$

Using property (A.1) for pseudoinverses,

$$\Phi^T\Phi^{+T}\Phi^T = \Phi^T \tag{B.7}$$

and property (A.4) gives

$$\Phi^+\Phi\Phi^T = \Phi^T \tag{B.8}$$

Premultiplying (B.8) by Φ^{+T} and using (B.2) results in:

$$\Phi^{+T}\Phi^+CLC^T = (\Phi\Phi^+)^T = I_p \tag{B.9}$$

which demonstrates that $\phi^{+T}\phi^{+}$ is a left inverse of CLC^{T}. Finally, (B.1)

is true by the uniqueness of the inverse of a nonsingular matrix.

Lemma B.2: Let $P_1 > P_2 > 0$, $W > 0$ be Hermitian matrices. Then

$$\text{tr}\{P_1 W\} > \text{tr}\{P_2 W\} > 0 \tag{B.10}$$

The proof is found in [13]. The next lemma establishes the existence of some bounded $G^* \in \mathscr{G}$ which minimizes the performance index.

Lemma B.3: Given the assumptions of the theorem, there exists a $G^* \in \mathscr{G}$ such that

$$J(G^*) < J(G) \text{ for all } G \in \mathscr{G} \tag{B.11}$$

Proof:

In the proof, lower case Roman numerals refer to the conditions of the theorem. $\mathscr{G}$ is an open region in $\mathbb{R}^{m \times p}$ since the characteristic polynomial of $\hat{A}$ is a continuous function of G. Because J takes the form (2.4), it is easy to see that J is $\mathscr{C}^1$ on $\mathscr{G}$, given (iv). Also $J(G)$ has a greatest lower bound, say η, over $\mathscr{G}$. This can be seen by noting that, because of continuity, finite values of $G \in \mathscr{G}$ give rise to finite values of J. If $\gamma(G)$ becomes negatively unbounded for unbounded G, expressing J as

$$J = \text{tr}\{K\}(1 + \gamma(G)/\text{tr}\{K\}) \tag{B.12}$$

and applying (v), shows that $J \to +\infty$ as $\|G\| \to \infty$ for all $\gamma(G)$ satisfying (iv,v). Suppose we specify a $G_0 \in \mathscr{G}$. By the continuity of $J(G)$, the subset of $\mathscr{G}$ defined by the inverse mapping of the closed interval bounded by η and $J(G_0)$,

$$\tilde{\mathscr{G}} = \{G : \eta < J(G) < J(G_0)\} \tag{B.13}$$

is closed.

If G is also bounded, the solution $G^* \in \tilde{\mathcal{G}}$ is guaranteed to exist. To prove the boundedness of $\tilde{\mathcal{G}}$, it is sufficient, from (v), to show that unbounded values of G give rise to infinite integral quadratic cost and, hence, cannot belong to $\tilde{\mathcal{G}}$. Since K satisfies the Lyapunov equation (2.6),

$$2\|K(A-BGC)\| = \|Q+C^TG^TRGC\| \tag{B.14}$$

Using the Cauchy-Schwartz inequality

$$2\|K\| \, \|A-BGC\| > \|Q+C^TG^TRGC\| > \|C^TG^TRGC\| \tag{B.15}$$

or

$$\|K\| > \|C^TG^TRGC\|/2\|A-BGC\| \tag{B.16}$$

We will now specialize our argument to the inner product matrix norm

$$\|\Gamma\| = (\mathrm{tr}\{\Gamma\Gamma^T\})^{1/2} \tag{B.17}$$

since it relates in a direct way to the integral quadratic cost:

$$\|K\| = (\mathrm{tr}\{K^2\})^{1/2} \tag{B.18}$$

It is well known that, for $\Gamma > 0$ and any x of compatible dimension,

$$x^T\Gamma x > (x^Tx)\lambda_{min}(\Gamma) \tag{B.19}$$

where $\lambda_{min}(\Gamma)$ is the smallest eigenvalue of Γ. Suppose that there is some sequence $\{G_i \ i > 0 : \ G_i \in \tilde{\mathcal{G}}\}$ such that some elements in the jth column of G become unbounded. By (B.17) and (B.19),

$$\|G^TRG\| \geq (g_j^Tg_j)\lambda_{min}(R) \tag{B.20}$$

which, since $R > 0$, implies that elements in a column of $R^{1/2}G$ become unbounded, where $R^{1/2} > 0$ is defined by

$$R = R^{1/2}R^{1/2} \tag{B.21}$$

Next, recall that $\rho(C) = p$ implies

$$CC^T > 0 \tag{B.22}$$

Suppose now, that elements in the kth column of $G^T R^{1/2}$ become unbounded.

By (B.17), (B.19) and (B.22) we find that

$$\|R^{1/2}GC\| \to \infty \qquad (B.23)$$

Since, with (B.17,B.18), (B.16) can be expressed

$$(tr\{K^2\})^{1/2} > \frac{\|R^{1/2}GC\|^2}{2\|A-BGC\|} \qquad (B.24)$$

the integral quadratic cost becomes unbounded, in contradiction to

(B.12), completing the proof.

Proof of Theorem:

From (2.12), the gradient of the Lagrangian with respect to G is

given by

$$\mathscr{L}_G = RGCLC^T - B^T KLC^T + \gamma_G(G) \qquad (B.25)$$

The inner product of the search direction (2.16) with the gradient (B.23)

is

$$\beta(G) = tr\{\mathscr{L}_G \Delta G^T\} \qquad (B.26)$$

If it can be shown that

$$\beta(G) < 0 \text{ if } G \in \mathscr{G} \text{ and } \mathscr{L}_G \neq 0 \qquad (B.27)$$

then the proof follows almost immediately. Assume that (B.27) is true.

The continuity of the gradient implies that, for each iteration, there

exists some α^* sufficiently small that (2.18) is satisfied for

$0 < \alpha < \alpha^*$. Under this circumstance, by the definition of G in (34),

$G_i \in \tilde{\mathscr{G}}$ implies that $G_{i+1} \in \tilde{\mathscr{G}}$. Moreover, the sequence $\{J(G_i): i = 0,1,...\}$

with G_i defined by (2.17,2.18) is convergent, since it is monotonic and

bounded. Recall from Lemma 3 that $\tilde{G}$ is closed and bounded. This and

the continuity of J imply that the sequence $\{G_i: i = 0,1,...\}$ is

convergent.

We will now demonstrate (B.29). Expanded, (B.26) is expressed

$$\beta(G) = \text{tr}\{[RGCLC^T - B^T KLC^T + \gamma_G(G)][(CLC^T)^{-1}(CLKB - \gamma_G^T(G)) - G^T R]R^{-1}\} \quad (B.28)$$

Substituting for CLC^T and $(CLC^T)^{-1}$ using Lemma B.1,

$$\beta(G) = \text{tr}\{[RG\Phi\Phi^T - B^T KL^{1/2}\Phi^T + \gamma_G(G)][\Phi^{+T}\Phi^+ + \Phi L^{1/2}KB - \Phi^{+T}\Phi^+ + \gamma_G^T(G) - G^T R]R^{-1}\} \quad (B.29)$$

Expanding, and then factoring (B.28) results in

$$\beta(G) = -\text{tr}\{PP^T R^{-1}\} \quad (B.30)$$

$$P = B^T KL^{1/2}(\Phi^+\Phi)^T - RG\Phi - \gamma_G(G)\Phi^{+T} \quad (B.31)$$

By Lemma B.2, R^{-1} will not affect the sign of $\beta(G)$, which implies that

$\beta < 0$, except at a stationary point in J; thus the sequence must converge

to a stationary point.

APPENDIX C

PROOF OF LEMMAS IN SECTION 3

C.1 Proof of Lemma 3.1

In the proof, matrix calculus operations and Kronecker algebraic

identities are employed, for which [38] is an excellent reference.

Define the function vec (.): $R^{m \times p} \to R^{mp}$ by

$$vec(A_{m \times p}) = \begin{bmatrix} A.1 \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline A.p \end{bmatrix}$$

From (3.9), it follows that

$$F(G_1, G^0) = vec\ G^0 - vec\ G + vec\ (GD_0G^0) \qquad (C.1)$$

$$= vec\ G^0 - [(I_p - D_0G^0)^T\ I_m]\ vec\ G = 0 \qquad (C.2)$$

where I_k denotes the k-dimensional identity matrix. From (A.6), the

Jacobian

$$\frac{\partial F(G,G^0)}{\partial vec^T G} = - [I_{mp} \otimes ((I - D_0G^0)^T \otimes I_m)] \qquad (C.3)$$

is nonsingular iff

$$\rho(I - D_0G^0) = p \qquad (C.4)$$

Assume that this is the case. By the implicit function theorem, (C.4)

implies that G is uniquely defined as a continuous function of G^0 in an

open region around any fixed $\overline{G}^0$ satisfying (C.4); that is, there exists a

continuous function $\phi(G^0)$ such that

$$vec\ G = \phi(G^0) \qquad (C.5)$$

near $\overline{G}^0$ which, by uniqueness is (3.11) and

$$\det\ [\frac{\partial \phi(G^0)}{\partial vec^T G^0}]\ \bigg|_{G^0 = \overline{G}^0} \neq 0 \qquad (C.6)$$

This lets us rewrite (C.3), using the chain rule:

$$[\frac{\partial F(G,\bar{G}^O)}{\partial \text{vec}^T G}] \bigg|_{G\ =\ \phi(\bar{G}^O)} = \frac{\partial F(\phi(G^O),G^O)}{\partial \text{vec}^T G^O} \ [\frac{\partial \phi(G^O)}{\partial \text{vec}^T G^O}]^{-1} \bigg|_{G^O\ =\ \bar{G}^O} \qquad (C.7)$$

Since the LHS of (C.7) has full rank, this implies that

$$\det\ [\frac{\partial F(G,G^O)}{\partial \text{vec}^T G}]\bigg|_{\substack{G\ =\ \phi(\bar{G}^O) \\ G_O\ =\ \bar{G}^O}} \neq 0 \qquad (C.8)$$

where

$$[\frac{\partial F(G,G^O)}{\partial \text{vec}^T G}]\ =\ [I_{mp} \otimes I_p \otimes (I+GD_O) \qquad (C.9)$$

which implies that (3.10) holds if (C.4) does. The converse is proven by reversing the above arguments.

C.2 Proof of Lemma 3.2

Consider the matrix

$$\psi\ =\ I\ +\ GC_2(A_{22}\ -\ B_2GC_2)^{-1}B_2 \qquad (C.10)$$

where the inverse exists by assumption. Suppose that $\rho\{B_2\}\ =\ r < m$. It can be immediately seen from x satisfying

$$[(\lambda\ -\ 1)I\ -\ GC_2(A_{22}\ -\ B_2GC_2)^{-1}B_2]x\ =\ 0 \quad x \neq 0 \qquad (C.11)$$

that ψ has $m-r$ unity eigenvalues, since $\dim \ker\{B_2\}\ =\ m-r$. Now, from property (A.1) for pseudoinverses,

$$B_2^+B_2[I+GC_2(A_{22}-B_2GC_2)^{-1}B_2]B_2^+B_2\ =\ B_2^+[I+B_2GC_2(A_{22}-B_2GC_2)^{-1}]B_2 \qquad (C.12)$$

$$=\ B_2^+[A_{22}(A_{22}-B_2GC_2)^{-1}]B_2 \qquad (C.13)$$

The remaining eigenvalues of ψ are obtained from

$$B_2^+[\lambda I\ -\ A_{22}(A_{22}\ -\ B_2GC_2)^{-1}]B_2x\ =\ 0 \quad x\ \epsilon\ \text{im}\{B_2^T\} \qquad (C.14)$$

Because of the nonsingularity of $A_{22}(A_{22} - B_2GC_2)^{-1}$, ψ is nonsingular.

Using the inverse identity (A.5) and recalling the form of D_0 from (3.7), one obtains

$$\psi^{-1} = I - GC_2A_{22}^{-1}B_2 = I + GD_0 \qquad (C.15)$$

The converse is proven by assuming that $(I + GD_0)$ is invertible, and reversing the logic.

APPENDIX D

PROOF OF THEOREM 4.1

The proof of the theorem requires several preliminary results.

Lemma D.1: Given that G stabilizes the closed-loop system (3.1-3.4), then

$$\bar{K}_1 = \sum_{i=0}^{\infty} \epsilon^i \bar{K}_1^{(i)} \qquad \bar{K}_3 = \sum_{i=0}^{\infty} \epsilon^i \bar{K}_3^{(i)} \qquad (D.1)$$

proof:

This result is well known. From (4.7) and (4.9) at $\epsilon = 0$, $\bar{K}_1$ and $\bar{K}_3$ satisfy

$$F_1(\bar{K}_1, \epsilon) = [(A_0-B_0G^0C_0)^T \otimes I_{n1} + I_{n1} \otimes (A_0-B_0G^0C_0)^T]\text{vec } \bar{K}_1 + \text{vec } \bar{Q}_1 = 0 \quad (D.2)$$

$$F_3(\bar{K}_3, \epsilon) = [(A_{22}-B_2GC_2)^T \otimes I_{n2} + I_{n2} \otimes (A_{22}-B_2GC_2)^T]\text{vec } \bar{K}_3 + \text{vec } \bar{Q}_3 = 0 \quad (D.3)$$

where $\otimes$ denotes the Kronecker product and vec $(.)$ is defined in Appendix C. Since $A_0 - B_0G^0C_0$ and $A_{22} - B_2GC_2$ are both nonsingular, it is easy to verify that

$$\det[\partial F_1/\partial \text{vec}^T \bar{K}_1] \neq 0 \qquad \det[\partial F_3/\partial \text{vec}^T \bar{K}_3] \neq 0 \qquad (D.4)$$

and that these Jacobians are continuous in $\bar{K}_1$ and $\bar{K}_3$, respectively. By the implicit function theorem, this implies that $\bar{K}_1$ and $\bar{K}_3$ are analytic functions of ϵ at $\epsilon = 0$ and, hence, representable by power series expansions.

Lemma D.2: If $J^0(\tilde{G}) < J(G)$ then

$$\partial J^0/\partial G \big|_{G = \tilde{G}} = 0 \qquad (D.5)$$

proof:

It was shown in the proof of Lemma B.3 that, for $\rho\{C\} = p$ and $R > 0$, the control cost in (4.1) for the full-order system becomes unbounded for

$\|G\| \to \infty$. This implies that $\bar{G}$ is bounded. Further, since the characteristic polynomials of $A_0 - B_0 G^0 C_0$ and $A_{22} - B_2 G C_2$ are continuous functions of G^0 and G, the set of asymptotically stabilizing gains is open, implying that $\bar{G}$ is in its interior. This implies that $J^0(\bar{G})$ is a stationary point in $\bar{G}$, so that (D.5) holds.

Lemma D.3: If $\Delta G = O(\varepsilon)$,

$$G^0(G + \Delta G) = G^0(G) + O(\varepsilon) \qquad (D.6)$$

where $G^0(G)$ is defined by (3.9).

<u>proof</u>:

Since the set $\{G: \rho(I + GD_0) = m\}$ is open, there exists some ε^* such that $\rho(I + GD_0 + \varepsilon \Delta G D_0) = m$ for $0 < \varepsilon < \varepsilon^*$ and finite ΔG. From (3.9) and A.5,

$$
\begin{aligned}
G^0(G + \varepsilon \Delta G) &= [(I + GD_0) + \varepsilon \Delta G]^{-1}(G + \varepsilon \Delta G) \\
&= (I + GD_0)^{-1}[I + O(\varepsilon)] [(G + O(\varepsilon)] \\
&= G^0(G) + O(\varepsilon) \qquad (D.7)
\end{aligned}
$$

<u>proof of Theorem</u>:

By Lemma D.1, J can be expanded to first order about $\varepsilon = 0$:

$$J = \text{tr}\{\bar{K}^{(0)}\} + \varepsilon\text{tr}\{\bar{K}_1^{(1)}\} + \varepsilon\text{tr}\{\bar{K}_3^{(0)}\} + O(\varepsilon^2) \qquad (D.8)$$

$$= J^0 + \varepsilon\text{tr}\{\bar{K}_1^{(1)} + O(\varepsilon^2)\} \qquad (D.9)$$

Now, define

$$\delta J = J(G^*) = J(\bar{G}) < 0 \qquad (D.10)$$

$$\delta J^0 = J^0(G^*) - {}^0J(\bar{G}) > 0 \qquad (D.11)$$

$$\delta K_1^{(1)} = K_1^{(1)}(G^*) - K_1^{(1)}(\bar{G}) \qquad (D.12)$$

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Use (D.7 - D.10) to form

$$\delta J = \delta J^0 + \varepsilon tr\{\delta K_1^{(1)}\} + 0 \ (\varepsilon^2) < 0 \qquad\qquad \text{(D.13)}$$

Since $\delta J^0 > 0$, it follows that

$$\delta J^0 = 0(\varepsilon) \qquad\qquad \text{(D.14)}$$

In turn, since C_o has full rank, (D.14) and Lemma 3.1 imply that

$$\|G^* - \tilde{G}\| = 0(\varepsilon) \qquad\qquad \text{(D.15)}$$

Which, with Lemma D.2 implies that

$$\delta J^0 = 0(\varepsilon^2) \qquad\qquad \text{(D.16)}$$

Lemma 3.1 and (D.15) also imply that $\varepsilon tr\{\delta \bar{K}_1^{(1)}\}$ is $0(\varepsilon^2)$; hence it follows

from (D.9) that δJ is $0(\varepsilon^2)$.

APPENDIX E

PROOFS OF LEMMAS IN SECTION 5

Proof of Lemma 5.1: Given that

$$B_2 G_1 C_2 = 0 \tag{E.1}$$

and recalling that

$$D_o = -C_2 A_{22}^{-1} B_2 \tag{E.2}$$

one easily verifies that

$$(I + G_1 D_o)^{-1} = (I - G_1 D_o) \tag{E.3}$$

Now, using (23), (E.3) and (E.1),

$$B_2 G_1^o C_2 = B_2 (I - G_1 D_o) G_1 C_2 = 0 \tag{E.4}$$

The converse is shown by using (5.9):

$$B_2 G_1 C_2 = B_2 G_1^o (I + D_o G_1^o) C_2 \tag{E.5}$$

By (5.6), the RHS of (E.5) is zero, completing the proof.

Proof of Lemma 5.2: From Lemma 3.2, given that A_{22} is invertible, $(I + G_2 D_o)$ is invertible for all G_2 stabilizing the fast dynamics. Employing (A.5),

$$(I + D_o G_2)^{-1} = I - D_o (I + G_2 D_o)^{-1} G_2 \tag{E.6}$$

so that $(I + D_o G_2)^{-1}$ exists. Applying the same identity to the inverse in the LHS of (5.11), along with (E.1-E.3),

$$[I + (G_1 + G_2) D_o]^{-1} = (I - G_1 D_o)[I - G_2 (I + D_o G_2)^{-1} D_o] \tag{E.7}$$

Thus, the LHS of (5.11) can be expressed

$$B_o [I + (G_1 + G_2) D_o]^{-1} (G_1 + G_2) C_o = B_o (I - G_1 D_o)[I - G_2 (I + D_o G_2)^{-1} D_o](G_1 + G_2) C_o \tag{E.8}$$

so that (5.11) becomes

$$B_o (I - G_1 D_o) G_2 [I - (I + D_o G_2)^{-1} D_o (G_1 + G_2)] C_o = 0 \tag{E.9}$$

After some algebra, (E.9) yields

$$B_o (I - G_1 D_o) G_2 (I + D_o G_2)^{-1} (I - D_o G_1) C_o = 0 \tag{E.10}$$

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It is easy to verify that

$$G_2(I + D_oG_2)^{-1} = (I + G_2D_o)^{-1}G_2 \qquad (E.11)$$

This implies that

$$\ker\{G_2(I + D_oG_2)^{-1}\} = \ker\{G_2\} \qquad (E.12)$$

which implies that (E.10) holds iff (5.13) is true.

Proof of Lemma 5.3: A matrix N satisfying

$$MNP = 0 \qquad (E.13)$$

can be written

$$N = N_m + N_p \qquad (E.14)$$

where

$$MN_m = 0 \qquad\qquad N_pP = 0 \qquad (E.15)$$

In (5.13),

$$M = B_o(I - G_1D_o) \qquad (E.16)$$

$$P = (I - D_oG_1)C_o \qquad (E.17)$$

so that

$$G_2 - (I + G_1D_o)N_{B_o} + N_{C_o}(I + D_oG_1) \qquad (E.18)$$

where $(I + D_oG_1) = (I - D_oG_1)^{-1}$. Because of (E.1,E.2), when G_2 from

(E.18) is substituted into $B_2G_2C_2$, G_1 cancels so that

$$B_2G_2C_2 = B_2(N_{B_o} + N_{C_o})C_2 \qquad (E.19)$$

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16. Abstract

Issues pertaining to the well-posedness of a two time scale approach to the output feedback regulator design problem are examined. An approximate quadratic performance index which reflects a two time scale decomposition of the system dynamics is developed. It is shown that, under mild assumptions, minimization of this cost leads to feedback gains providing a second-order approximation of optimal full system performance. A simplified approach to two time scale feedback design is also developed, in which gains are separately calculated to stabilize the slow and fast subsystem models. By exploiting the notion of combined control and observation spillover suppression, conditions are derived assuring that these gains will stabilize the full-order system.

A sequential numerical algorithm is described which obtains output feedback gains minimizing a broad class of performance indices, including the standard LQ case. It is shown that the algorithm converges to a local minimum under nonrestrictive assumptions. This procedure is adopted to and demonstrated for the two time scale design formulations.

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