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FOR ELLIPTIC SYSTEMS OF DIFFERENCE EQUATIONS

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Abstract

We prove regularity estimates up to the boundary for solutions of elliptic systems of finite difference equations. The regularity estimates, obtained for boundary–fitted coordinate systems on domains with smooth boundary, involve discrete Sobolev norms and are proved using pseudo–difference operators to treat systems with variable coefficients. The elliptic systems of difference equations and the boundary conditions which are considered are very general in form. We prove that regularity of a regular elliptic system of difference equations is equivalent to the nonexistence of "eigensolutions". The regularity estimates obtained are analogous to those in the theory of elliptic systems of partial differential equations, and to the results of Gustafsson, Kreiss, and Sundström [1972] and others for hyperbolic difference equations.

1. Introduction

In this paper we prove boundary regularity estimates for finite difference schemes for elliptic systems of partial differential equations. These estimates express the smoothness, or regularity, of the solution of the difference scheme up to the boundary in terms of the smoothness of the data in the interior and on the boundary of the domain on which the equations hold. For those difference schemes which require more boundary conditions than the corresponding differential equations, the estimates show the effect of the additional boundary data on the smoothness of the solution.

Our results are analogous to the Sobolev norm estimates of Agmon, Douglis and Nirenberg [1] for solutions of elliptic systems of differential equations. In previous work (Bube and Strikwerda [3]) interior regularity estimates for systems of elliptic difference schemes were derived and a theory of pseudo–difference operators was developed (see also Frank [4]). It is assumed that the reader is familiar with this paper. That theory is used here to prove the boundary regularity estimates.

The approach used in this paper is similar to that used by Gustafsson, Kreiss, and Sundström [6] on the initial boundary value problem for difference schemes for hyperbolic equations. The essential ideas are first to transform the system to a one–step scheme in the direction normal to the boundary, and then construct a Hermitian pseudo–difference operator, the Gustafsson–Kreiss–Sundström symmetrizer, for this one–step scheme which enables one to obtain the appropriate estimates for the solution. Michelson [9] has extended these ideas to multidimensional initial–boundary value problems. Because the principal symbol of an elliptic system of partial differential equations can involve different orders of

differentiation on different variables, the construction used here is more general than that of Gustafsson, Kreiss, and Sundström.

We consider only boundary–fitted grids, i.e., those in which the boundary is a coordinate surface. This excludes the grid systems used by Bramble and Hubbard [2] and others in which the boundary curve is not parallel to a coordinate line. Boundary–fitted grids are in common use in computational fluid dynamics (e.g. Thompson et al. [12]). A scheme for the Stokes equations, an elliptic system, which is second–order accurate on boundary–fitted grids has been presented by Strikwerda [10].

The significance of the regularity estimates proven here is that they show the effect of the discretizations of the boundary conditions on the smoothness of the solution near the boundary. Regular elliptic finite difference schemes satisfy interior regularity estimates and thus have smooth solutions away from the boundary. In this paper the conditions under which the solutions have optimal smoothness at the boundary are given.

In a subsequent work we intend to extend the results given here to estimates of the accuracy of the finite difference solution as an approximation to the solution of an elliptic system of partial differential equations. In this paper we give only the estimate in Theorem 3.2 which gives the accuracy of the divided differences of the solution in terms of the accuracy of the solution itself.

The outline of the paper is as follows. We begin by considering elliptic systems of difference schemes defined on a half–space with a rectangular grid. Section 2 is concerned with the basic definitions and assumptions. The main result of this paper, Theorem 3.1, is stated in Section 3 after the Complementing Conditions are defined. Section 4 is

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concerned with the reduction of general schemes to a one–step scheme, and in Section 5 the Gustafsson–Kreiss–Sundström symmetrizer is constructed. The boundary regularity estimates are obtained in Section 6 which proves the main theorem, Theorem 3.1. Section 7 is a discussion of the more general situation of difference schemes on domains with curvilinear grids, and Section 8 is a summary of the paper which is intended to be a user's guide, enabling one to apply the results of the paper to particular problems.

2. Definitions and Assumptions

We begin by considering the regular elliptic system of difference equations

$$\sum_{j=1}^{n} L_{ij} u_j(x_\nu, y_\mu) = F_i(x_\nu, y_\mu), \qquad i = 1, ..., n, \quad \mu \in \mathbf{Z}^{d-1}, \tag{2.1}$$

on the half-space $\mathbf{R}_+^d = \{(x, y) : x \geq 0, y \in \mathbf{R}^{d-1}, d \geq 2\}$, where the grid points are $x_\nu = \nu h, \quad y_\mu = \mu h, \quad \nu \geq 0$. The boundary conditions are given by

$$\sum_{j=1}^{n} B_{kj} u_j(x_0, y_\mu) = \phi_k(y_\mu), \qquad k = 1, ..., q, \quad \mu \in \mathbf{Z}^{d-1}, \tag{2.2}$$

where the B_{kj} are difference operators which involve only forward differences and translations. The number of boundary conditions which must be specified will be determined later, (see Assumption 2.1). Without loss of generality (see Section 7), we assume that $F(x, y)$ and $\phi_k(y)$ are 2π–periodic in each y_j. The system (2.1) is assumed to be regular elliptic of order (σ, τ), (Bube and Strikwerda [3]). We also assume that there is a $\rho \in \mathbf{Z}^q$ such that

$$b_{kj}(h, y, \varsigma) \in \mathbf{S}^{\rho_k + \tau_j}, \tag{2.3}$$

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where the b_{kj} are the symbols of the B_{kj} when considered as difference operators on all of $\mathbf{R}^d$.

Following Bube and Strikwerda [3], we define the tangential grid

$$G := \{y_\mu : 0 \le \mu_j \le 2N, j = 1, ..., d-1\},$$

where $y_\mu := h\mu \in [0, 2\pi]^{d-1}$ and $h := 2\pi/(2N+1)$. We also define

$$(u(x,\cdot), v(x,\cdot))_G := (\frac{h}{2\pi})^{d-1} \sum_{y \in G} u(x,y)^* v(x,y),$$

and

$$\hat{u}(x,\xi) := (\frac{h}{2\pi})^{d-1} \sum_{y \in G} u(x,y) e^{-iy\cdot\xi},$$

where $\Gamma := \{\xi \in \mathbf{Z}^{d-1} : |\xi_j| \le N \quad \text{for} \quad j = 1, ..., d-1\}$. By the Fourier inversion formula,

$$u(x,y) = \sum_{\xi \in \Gamma} e^{iy\cdot\xi} \hat{u}(x,\xi).$$

The norms used in the regularity estimate are as follows. Let $\tau \in \mathbf{Z}^{d-1}$ and $s \in \mathbf{R}$. Define

$$\Lambda(h,\omega)^2 := 1 + \sum_{j=1}^{d-1} \frac{4}{h^2} \sin^2(\frac{1}{2}\omega_j), \qquad \omega = h\xi \in h\Gamma,$$

and

$$\|u_j\|_s^2 := \frac{h}{2\pi} \sum_{\xi \in \Gamma} \sum_{\nu=0}^{\infty} \sum_{m=0}^{[s]} |\Lambda(h,h\xi)^{s-m} \delta_{x+}^m \hat{u}_j(x_\nu,\xi)|^2,$$

$$\|u\|_{s+\tau}^2 := \sum_{j=1}^{n} \|u_j\|_{s+\tau_j}^2.$$

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At the boundary we define

$$|u_j|_s^2 := \sum_{\xi \in \Gamma} \sum_{m=0}^{[s]} |\Lambda(h,\omega)^{s-m} \delta_{x+}^m \hat{u}_j(x_0, \xi)|^2,$$

$$|u|_{s+\tau}^2 := \sum_{j=1}^{n} |u_j|_{s+\tau_j}^2.$$

Note that $|u_j|_s$ is the sum of norms of u_j and its forward divided differences in x of order

at most s evaluated at x_0, and considered as discrete functions of y.

We will assume that the operators L_{ij} and B_{kj} contain only differences of order $\sigma_i + \tau_j$

and $\rho_k + \tau_j$, respectively, with coefficients which are independent of h. That is, l_{ij}, the

symbol of L_{ij}, is in $S^{\sigma_i + \tau_j}$ and

$$l_{ij}(h, x, y, \varsigma) = l'_{ij}(x, y, \varsigma)h^{-(\sigma_i + \tau_j)}, \tag{2.4}$$

$$\varsigma = h\xi, \quad \xi \in \mathbf{Z}^d, \quad |\xi| \leq N, \quad h = 2\pi/(2N+1).$$

for some function l'_{ij} independent of h. Similarly for $b_{kj}(h, y, \varsigma)$. The inclusion of lower

order terms does not affect the form of the final regularity estimates, as will be discussed

in Section 7. For simplicity of exposition we will assume that the symbols $l_{ij}(h, x, y, \varsigma)$ are

independent of x.

We will also assume that $\tau_j \geq 1$ for each j and that $\max_i \sigma_i = 0$. Note that if, for

some j, $\tau_j = 0$ then u_j may be expressed as a linear combination of differences of the other

components of u, therefore u_j may be eliminated from the system without affecting the

ellipticity.

To apply the theory of pseudo–difference operators to the boundary value problem (2.1–2.2) we introduce the reduced system of difference equations as follows. Each of the difference operators L_{ij} is a difference operator in both x and y. The reduced operator $\tilde{L}_{ij}(h, y, \omega)$ is the symbol of L_{ij} with respect to y only, and thus remains as a difference operator in x. Here $\omega = h\xi$, where $\xi \in \mathbf{Z}^{d-1}$ is the dual variable to y. Boundary operators $\tilde{B}_{kj}(h, y, \omega)$ are defined in a similar way. The reduced system is

$$\sum_{j=1}^{n} \tilde{L}_{ij}(h, y, \omega)\tilde{u}_j(x_\nu, \omega) = \tilde{F}_i(x_\nu, \omega), \qquad i = 1, ..., n, \tag{2.5}$$

with boundary conditions

$$\sum_{j=1}^{n} \tilde{B}_{kj}(h, y, \omega)\tilde{u}_j(x_0, \omega) = \tilde{\phi}_k(\omega), \qquad k = 1, ..., q. \tag{2.6}$$

The reduced system (2.5) will be transformed to a one–step difference scheme in x using a new vector of variables $W(x_\nu, \omega)$ so that the reduced system is equivalent to one of the form

$$W(x_{\nu+1}, \omega) = M(y, \omega)W(x_\nu, \omega) + h\mathcal{F}(x_\nu, \omega), \quad \nu \geq 0, \tag{2.7}$$

with boundary conditions

$$\mathcal{B}(y, \omega)W(x_0, \omega) = \Phi(\omega). \tag{2.8}$$

The regularity estimates will be derived for the one–step scheme (2.7) using the function $W(x_\nu, \omega)$. The final form of the estimate will follow by transferring back to the original function $u(x_\nu, \omega)$.

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The estimates will be derived through the use of a matrix symbol $H(y,\omega)$, the symmetrizer, which will be constructed to satisfy the two matrix inequalities

$$M^* H M - H \geq c_0 h \Lambda_0 \qquad (2.9)$$

$$H + c_1 B^* B \geq c_2 \qquad (2.10)$$

for some positive constants c_0, c_1, c_2 and $\Lambda_0 = \Lambda_0(h,\omega) := \sqrt{4h^{-2} \sum_{i=1}^{d-1} \sin^2(\frac{1}{2}\omega_i)}$. The construction of the matrix H is analogous to that used by Gustafsson, Kreiss, and Sundström [6] for hyperbolic difference schemes.

Before transforming the reduced equation to the one–step scheme we require some additional definitions and assumptions. Related to the reduced equation (2.5) is the resolvent equation

$$\sum_{j=1}^{n} \tilde{l}_{ij}(h,y,\omega,z) v_j = 0, \qquad i = 1,...,n \qquad (2.11)$$

where $\tilde{l}_{ij}(h,y,\omega,z) = l_{ij}(h,y,\varsigma)$, the symbol of L_{ij}, and $\varsigma = (\frac{1}{i}\log(z),\omega), v_j \in \mathbf{C}, j = 1,...,n$. Here $\log(z)$ is any logarithm of $z \in \mathbf{C}$.

Definition 2.1

The values of $z = z(y,\omega)$ for which the resolvent equation (2.11) has nontrivial solutions are called eigenvalues of the resolvent equation.

Let

$$R(y,\omega,z) := \det\{\tilde{l}_{ij}(h,y,\omega,z)\} h^{2p}, \qquad 2p := \sum_{i=1}^{n}(\tau_i + \sigma_i).$$

Note that $R(y,\omega,z)$ is independent of h by equation (2.4). The eigenvalues $z(y,\omega)$ are the roots of $R(y,\omega,z) = 0$. We will need the following lemma.

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Lemma 2.1

The eigenvalues $z(y, \omega)$ satisfy

$$|z(y, \omega)| \neq 1 \qquad if \quad \omega \neq 0.$$

Moreover, if $\omega = 0$ and $|z| = 1$ then $z = 1$.

Proof:

Suppose $z(y, \omega) = \exp(i\varsigma_1)$ with ς_1 real and $|\varsigma_1| \leq \pi$. Then

$$0 = R(y, \omega, \exp(i\varsigma_1)) = \det\{\tilde{l}_{ij}(h, y, \omega, \exp(i\varsigma_1))\}$$

implies

$$\det\{l_{ij}(h, y, \varsigma)\} = 0,$$

where $\varsigma = (\varsigma_1, \omega)$. Since the matrix of the l_{ij} is a regular elliptic symbol with only highest order terms, the determinant can not vanish unless $\varsigma = 0$. This proves the lemma.

To transform the reduced equation (2.5) to the one–step scheme (2.7) we must place restrictions on the size of the stencil of the difference equations.

Definition 2.2

If Q is a difference operator in x written in the form

$$Qf(x_\nu) = \sum_{\mu=a}^{b} Q_\mu(h, x) T^\mu f(x_\nu),$$

where neither Q_a nor Q_b are identically zero, then the extent of Q is the ordered pair (a, b). If Q_1 is a difference operator with extent (a_1, b_1) we say the extent of Q_1 is less than the extent of Q if $a \leq a_1 \leq b_1 \leq b$.

The condition we place on the extent of the reduced system is:

Resolvent Condition

If the operators $\tilde{L}_{ij}$ in the reduced equation (2.5) have extent (a_{ij}, b_{ij}), we assume there are $\alpha^-, \alpha^+, \beta^-, \beta^+$ in $\mathbf{Z}^n$ such that

$$\alpha_i^- + \beta_j^- \le a_{ij} \le b_{ij} \le \alpha_i^+ + \beta_j^+, \qquad i, j = 1, ..., n, \tag{2.12}$$

and such that the number of roots $z(y, \omega)$, counting multiplicity, of the equation

$$R(y, \omega, z) = 0$$

is precisely

$$\sum_{k=1}^n (\alpha_k^+ + \beta_k^+) - (\alpha_k^- + \beta_k^-)$$

for any value of ω. If $\tilde{L}_{ij}$ is identically zero for some values of (i, j) then we place no restriction of the form (2.12) for that value of (i, j).

We now show that without loss of generality we may take $\alpha^- = \beta^- = 0$ in the resolvent condition. By subtracting some positive integer from all the α_i^- and adding it to all the β_j^- we can have $\beta^- \ge 0$, with $\min \beta_j^- = 0$, without altering the resolvent condition.

By operating on the i–th equation with $T_h^{-\alpha_i^-}$, where T_h is the translation operator in the x direction, we obtain an equivalent system with $\alpha^- = 0$. (The new value of α^+ is $\alpha^+ - \alpha^-$.) Then by defining new dependent variables

$$u_j'(x_\nu, \omega) = u_j(x_\nu + \beta_j^- h, \omega), \qquad \nu \ge 0$$

a new system is obtained with $\beta^- = 0$. (The new value of β^+ is $\beta^+ - \beta^-$.) Note that the (old) variables $u_j(x_0, \omega), ..., u_j(x_{\beta_j^- - 1}, \omega)$ do not appear in the difference equations and

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thus are superfluous, appearing only in the boundary conditions. We will assume that these boundary variables can be expressed as linear combinations of the other (nonsuperfluous) variables and thus be eliminated. If these variables can not be so eliminated then the system does not have sufficient boundary conditions to determine the solution.

We now make an assumption on the number of boundary conditions.

Assumption 2.1

The resolvent condition holds with $\alpha^- = \beta^- = 0$ and the number of boundary conditions q is equal to the number of roots in z of

$$R(y, \omega, z) = 0 \tag{2.13}$$

which satisfy

$$0 < |z(y, \omega)| < 1 \qquad \text{for} \quad |\omega| \neq 0.$$

Recall that $|z(y, \omega)| \neq 1$ for $\omega \neq 0$ by Lemma 2.1. We also know that

$$\sigma_i + \tau_j \leq \alpha_i^+ + \beta_j^+. \tag{2.14}$$

since a consistent difference operator approximating a differential operator of order s must involve at least $s + 1$ points. In the case that the number of boundary conditions q is larger than p as defined in Definition 2.1, we need an additional assumption. We assume that the boundary conditions are ordered so that the ρ_k are in increasing order and we then define two important quantities. Let

$$\bar{\rho} := \max_{1 \leq k \leq p} (\rho_k + 1, 0). \tag{2.15}$$

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and

$$\rho^* := \begin{cases} \min_{k \geq q}(\rho_k) + 1, & \text{if } q > p; \\ \infty, & \text{if } q = p. \end{cases} \tag{2.16}$$

The number ρ^* is only used to limit the order of divided differences in the regularity estimates, when ρ^* is infinite there is no restriction on the order of the differences, e.g. Theorem 3.1.

Assumption 2.2

If $q > p$, then we assume that

$$\rho_k \geq \bar{\rho}, \qquad \text{for} \quad p < k \leq q.$$

That is, the last $q - p$ boundary conditions have weights ρ_k which are nonnegative and larger than the weights of the first p boundary conditions.

This assumption is needed to obtain the regularity estimates in the appropriate norms, and can be motivated as follows. If the difference scheme (2.1) is an approximation to a system of differential equations then p of the boundary conditions (2.2) would correspond to the boundary conditions of the differential equations. The remaining $q - p$ boundary conditions which are required by the difference equations should be intrinsically distinct from the first p boundary conditions, and this distinction is maintained by having the weights ρ_k of the last $q - p$ boundary conditions sufficiently large and indeed larger than the weights of the first p boundary conditions. Note that $\bar{\rho}$ is zero for the classical Dirichlet and Neumann boundary value problems for elliptic equations of order $2p$.

Definition 2.3

Corresponding to the system of difference equations (2.1) is the associated system of differential equations

$$\sum_{j=1}^{n} \hat{L}_{ij}(y, \partial_x, \partial_y)\hat{w}_j(x,y) = F_i(x,y), \qquad i = 1,...,n, \qquad (2.17)$$

where $\hat{L}_{ij}$ is the differential operator whose symbol $\hat{l}_{ij}(y,\xi)$ is the limit of $l_{ij}(y,h\xi)$ as h tends to zero. The associated boundary conditions are

$$\sum_{j=1}^{n} \hat{B}_{kj}(y, \partial_x|_{x=0}, \partial_y)\hat{w}_j(0,y) = \phi_k(y) \qquad k = 1,...,p, \qquad (2.18)$$

These are obtained by the same limiting procedure from the first p boundary conditions (2.2), (see Assumption 2.2).

The above limits exist by equation (2.4) since the L_{ij} and b_{kj} are difference operators in $S^{\sigma_i+\tau_j}$ and $S^{\rho_k+\tau_j}$ respectively; see also equation (2.8) of Bube and Strikwerda [3].

As in Agmon et al. [1, p.39], we require the associated differential equation to satisfy the following condition.

Assumption 2.3   *Supplementary condition on $\hat{L}$.*

$\hat{L}(y,\xi)$ is of even degree $2p$ with respect to ξ. For every pair of vectors ξ and ξ' in $\mathbf{R}^d$ the polynomial $\hat{L}(y,\xi + \tau\xi')$ in the complex variable τ has exactly p roots with positive imaginary part.

3. Complementing Condition

In order for the regularity estimates to hold for the system (2.1) certain conditions must be satisfied by the boundary conditions (2.2). Before stating these conditions, called Complementing Conditions, it will be helpful to introduce some notation. We will write the reduced equation (2.5) as

$$\tilde{L}(h,y,\omega)\tilde{u}(x_\nu,\omega) = \tilde{F}(x_\nu,\omega), \qquad \nu \geq 0, \tag{3.1}$$

and the boundary conditions (2.6) as

$$\tilde{B}_1(h,y,\omega)\tilde{u}(x_0,\omega) = \tilde{\phi}_1(\omega),$$
$$\tilde{B}_2(h,y,\omega)\tilde{u}(x_0,\omega) = \tilde{\phi}_2(\omega), \tag{3.2}$$

where $\tilde{B}_1$ is composed of the first p boundary operators and $\tilde{B}_2$ the last $q - p$ boundary operators (see assumption 2.2).

The reduced equation (3.1) can be replaced by an equivalent equation independent of h by the following scaling procedure. Multiply the i-th equation by h^{σ_i} and replace the variables u_j by $u'_j h^{\tau_j}$. This gives a system of difference equations in the u'_j which is equivalent to the original system. The boundary conditions can be scaled in a similar manner.

The Complementing Conditions will be stated in terms of eigensolutions, of which there are three types. Because of the above scaling procedure, we need only consider h equal to 1.

Definition 3.1

An eigensolution of type I is a nontrivial solution to the difference equation

$$\tilde{L}(1, y, \omega) u(x_\nu, \omega) = 0 \qquad \nu \geq 0, \quad for \quad some \quad \omega \neq 0,$$

satisfying

$$
\begin{aligned}
a) \quad & \tilde{B}_1(1, y, \omega) u(x_0, \omega) = 0 \\
& \tilde{B}_2(1, y, \omega) u(x_0, \omega) = 0,
\end{aligned}
$$

$$b) \quad u(x_\nu, \omega) \to 0 \qquad as \quad \nu \to \infty.$$

Definition 3.2

An eigensolution of type II is a nontrivial solution to the associated differential equation

$$\hat{L}(y, \partial_x, \theta) \hat{w}(x, \theta) = 0 \qquad for \quad \theta \in \mathbf{R}^{d-1}, \quad |\theta| = 1,$$

satisfying

$$a) \quad B_1(y, \partial_x, \theta) \hat{w}(x, \theta)|_{x=0} = 0,$$

$$b) \quad \hat{w}(x, \theta) \to 0 \qquad as \quad x \to \infty.$$

In the case $q > p$ we formulate:

Definition 3.3

An eigensolution of type III is a nontrivial solution to the difference equation

$$\tilde{L}(1, y, \omega) \tilde{u}(x_\nu, \omega)|_{\omega=0} = 0,$$

satisfying

$$a) \quad \tilde{B}_2(1, y, x_0)\tilde{u}(x_0, \omega)|_{\omega=0} = 0,$$

$$b) \quad \tilde{u}(x_\nu, 0) \to 0 \quad \text{as} \quad \nu \to \infty.$$

We now state the Complementing Condition and the main theorem of this paper.

Complementing Condition

The system (2.1) with boundary conditions (2.2) satisfies the Complementing Condition if there are no eigensolutions of type I,II, or III.

Theorem 3.1

If $u(x_\nu, y_\mu)$ is a solution to the system (2.1) with the boundary conditions (2.2) and Assumptions 2.1, 2.2, and 2.3 are satisfied, then the following regularity estimate holds for each s with $\bar{\rho} \leq s < \rho^*$ and h sufficiently small if, and only if, the Complementing Condition holds.

$$\|u\|_{r+s}^2 + |u|_{r+s-\frac{1}{2}}^2 \leq C_s(|\phi_1|_{s-\rho-\frac{1}{2}}^2 + |h^{\rho-t+\frac{1}{2}}\phi_2|_{s-t}^2 + \|F\|_{s-\sigma}^2 + \|u\|_0^2). \tag{3.3}$$

where $t = \bar{\rho} + \frac{1}{2}\lfloor 2(s - \bar{\rho})\rfloor$.

The boundary data ϕ_1 and ϕ_2 are defined by equation (3.2) according to Assumption 2.2. The subscript on the norm of ϕ_1 uses only the first p components of the multi–index ρ, and $h^{\rho-s+\frac{1}{2}}\phi_2$ is understood as multiplication of the $(k - p + 1)$–st component of ϕ_2 by $h^{\rho_k - s + \frac{1}{2}}$ for $k > p$.

Note that the nonexistence of eigensolutions of type II is equivalent to a regularity estimate analogous to (3.3) (with $\phi_2 = 0$) holding for the associated differential equation

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(2.14). Theorem 3.1 is proved in Section 6 as Theorems 6.1 and 6.2.

An immediate corollary of Theorem 3.1 is Theorem 3.2.

Theorem 3.2

Suppose v is a solution to the finite difference equations (2.1) and u is a solution to the corresponding partial differential equation (2.17) such that

$$1) \quad Lv = F, \quad Bv = \phi,$$

$$2) \quad Lu = F + O(h^{r_1}),$$

$$3) \quad B_1 u = \phi_1 + O(h^{r_2}), \quad B_2 u = \phi_2 + O(h^{r_3 - \rho}),$$

$$4) \quad \|u - v\|_0 = O(h^{r_4}).$$

Then, for $\bar{\rho} \le s < \rho^*$,

$$\|u - v\|_{\tau+s} + |u - v|_{\tau+s-\frac{1}{2}} = O(h^r),$$

where $r := \min\{r_1, r_2, r_3 + \frac{1}{2} - s, r_4\}$.

4. The One–Step Scheme

We now describe the transformation of the reduced equation (2.5) to the one–step scheme (2.7). We first operate on the i–th equation of the reduced system (2.5) with

$$(\delta_+ - \Lambda_0)^{-\sigma_i + \bar{\rho}}, \tag{4.1}$$

where Λ_0 is the symbol $\Lambda_0(h, \omega)$ defined in equation (2.9). Recall that $\sigma_i \le 0$ and $\bar{\rho} \ge 0$. Let

$$\tilde{L}'_{ij} := (\delta_+ - \Lambda_0)^{-\sigma_i + \bar{\rho}} \tilde{L}_{ij}.$$

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The resolvent equation for this new system has the additional root $z = 1 + h\Lambda_0$ with multiplicity $|\sigma| + n\bar{p}$. Since $1 + h\Lambda_0 \geq 1$, no additional boundary conditions are required.

The first p boundary conditions are modified by multiplying the k-th boundary condition by $\Lambda_0^{\bar{p}-1-\rho_k}$ and, when q is greater than p, the last $q - p$ boundary conditions are multiplied by $h^{\rho_k - \bar{p}+1}$. Note that

$$\bar{p} - 1 - \rho_k \geq 0, \qquad \text{for} \quad 0 \leq k \leq p,$$

$$\bar{p} - 1 - \rho_k < 0, \qquad \text{for} \quad p < k \leq q.$$

The resulting system of difference equations is elliptic of order $(0, \tau + \bar{p})$ and the resulting boundary conditions are all of weight greater than or equal to -1. We obtain

$$\sum_{j=1}^{n} \tilde{L}'_{ij}(h, y, \omega) u_j(x_\nu, \omega) = \tilde{F}'_i(x_\nu, \omega), \qquad \nu \geq 0, \quad i = 1, ..., n, \qquad (4.2)$$

$$\sum_{j=1}^{n} \tilde{B}'(h, y, \omega) u_j(x_0, \omega) = \tilde{\phi}'_k(\omega), \qquad k = 1, ..., n. \qquad (4.3)$$

This system of difference equations will now be written in a more canonical form. The left side of each equation of the reduced system (4.2) can be written as a sum of terms of the form

$$p(h, y, \omega) \delta_+^a T_h^\gamma u_j(x_\nu, \omega),$$

where $p(h, y, \omega) \in \mathbf{S}^{\tau_j + \bar{p} - a}, a \leq \tau_j + \bar{p}$ and, by Assumption 2.1, $\gamma \geq 0$. If $\gamma \geq 1$ and $a \leq \tau_j + \bar{p} - 1$ we can rewrite this term as

$$p(h, y, \omega) \delta_+^a T_h^{\gamma-1} u_j(x_\nu, \omega) + hp(h, y, \omega) \delta_+^{a+1} T_h^{\gamma-1} u(x_\nu, \omega).$$

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In this way the order of translation γ is reduced by one and the order of differencing is increased by one. Note that $hp(h,y,\omega) \in \mathbf{S}^{\tau_j + \bar{p} - 1 - a}$. By continuing in this way, we obtain a system where the nontrivial translation operators, i.e., $\gamma > 0$, operate only on the highest order differences.

The resulting system may be written as

$$\sum_{j=1}^{n} g_{ij}(h,\omega,T)\delta_+^{\tau_j + \bar{p}} u_j(x_\nu,\omega) + \sum_{j=1}^{n} \sum_{a=0}^{\tau_j + \bar{p} - 1} \tilde{l}_{ija}(h,y,\omega)\delta_+^a u_j(x_\nu,\omega) = \tilde{F}_i'(x_\nu,\omega), \quad (4.4)$$

$$\nu \geq 0, \quad i = 1, ..., n.$$

where g_{ij} is a polynomial in T (the translation operator in the x–direction) with coefficients in $\mathbf{S}^0$, and $\tilde{l}_{ija} \in \mathbf{S}^{\tau_j + \bar{p} - a}$.

Before transforming the system (4.4) to a one–step scheme, it is necessary to examine more closely the matrix of translation operators $G(T) := (g_{ij}(T))$. The extent of $g_{ij}(T)\delta_+^{\tau_j + \bar{p}}$ is at most $(0, \alpha_i^+ + \beta_j^+ - \sigma_i + \bar{p})$, by Assumption 2.1. Decompose $G(T)$ as a sum of $G_1(T)$ and $G_0(T)$, where $G_1(T)$ is composed of elements $g_{ij}'(T)$ which, if they are nonzero, are of degree $\gamma_{ij} = \alpha_i^+ + \beta_j^+ - \sigma_i - \tau_j$, and $G_0(T)$ has elements $g_{ij}^0(T)$ of degree less than γ_{ij}.

Lemma 4.1

$G_1(T)$ *is equivalent through elementary row operations to a diagonal matrix, i.e., we can assume that*

$$g_{ij}'(T) = \begin{cases} 0, & \text{if } i \neq j; \\ T^{\mu_i}, & \text{if } i = j, \end{cases} \quad \mu_i := \gamma_{ii}. \tag{4.5}$$

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Moreover, (4.4) is equivalent to

$$T^{\mu_i}\delta_+^{\tau_i+\bar{p}}u_i(x_\nu,\omega) = -\sum_{j=1}^{n}\sum_{b=0}^{\mu_j-1} g_{ijb}^0(y,\omega)T^b\delta_+^{\tau_j+\bar{p}}u_j(x_\nu,\omega)$$

$$\quad (4.6)$$

$$-\sum_{j=1}^{n}\sum_{a=0}^{\tau_j+\bar{p}-1} \tilde{l}_{ij}''(h,y,\omega)\delta_+^a u_j(x_\nu,\omega) + \tilde{F}_i''(x_\nu,\omega).$$

$$\nu \geq 0, \quad i = 1,...,n.$$

By an elementary row operation we mean permutation of the rows, multiplication of a row by a nonvanishing function of (y,ω) which is in $\mathbf{S}^0$, or addition of a translate of one row to another row.

Proof:

$$G_1(T) = (\tilde{g}_{ij}'(h,y,\omega)T^{\gamma_{ij}}),$$

and since $\det(\tilde{g}_{ij}')$ is the coefficient of the highest power of z in $R(y,\omega,z)$, $\det(\tilde{g}_{ij}')$ does not vanish for any (y,ω). Also, in adding to row l a translate of row m we can restrict ourselves to the translation operator $T^{(\alpha_l^+ - \sigma_l)-(\alpha_m^+ - \sigma_m)}$. It is then easy to see that the elements of $G_1(T)$ will remain monomials in T of degree γ_{ij}, although the rows, and thus the γ_{ij}, will be permuted.

It is a standard result that by means of these elementary row operations $G_1(T)$ can be transformed to upper triangular form (see Gantmacher [5,p. 135], and since $G_1(T)$ remains a matrix of monomials in T it can be transformed to a diagonal matrix. This

matrix has nonvanishing diagonal elements; by dividing through by the coefficients of $T^{\gamma_{ii}}$ we achieve the form (4.5).

The lemma will be proven if we can show that the elementary row operations keep the degree of the (i,j)-th element of $G_0(T)$ less than γ_{ij} and do not create new elements for $G_1(T)$ from the second sum of equation (4.4). It is easy to see that by operating on the i-th row of (4.4) with a translation of degree $(\alpha_l^+ - \sigma_l) - (\alpha_i^+ - \sigma_i)$, and adding it to the l-th row, the degree of the (l,j)-th polynomial in G_0 remains less than γ_{lj}. The effect of this operation on the second sum in (4.4) results in terms of the form

$$\tilde{l}_{ija} T^{\gamma} \delta_+^a u_j(x_\nu, \omega),$$

in the l-th row, where $\gamma = (\alpha_l^+ - \sigma_l) - (\alpha_i^+ - \sigma_i)$. If $\gamma + a < \tau_j + \bar{\rho}$, this term can be written as

$$\tilde{l}_{ija} (1 + h\delta_+)^{\gamma} \delta_+^a u_j(x_\nu, \omega),$$

which, when expanded, again has the form of the terms in the second sum of (4.4). If $\gamma + a \geq \tau_j + \bar{\rho}$ then by repeated substitution of $1 + h\delta_+$ for T one obtains terms as in (4.4) with the term of highest power in T being

$$T^{\gamma + a - \tau_j - \bar{\rho}} \delta_+^{\tau_j + \bar{\rho}} u_j(x_\nu, \omega).$$

Now by inequality (2.14) $a < \tau_j + \bar{\rho} < \alpha_i^+ + \beta_j^+ - \sigma_i + \bar{\rho}$, so

$$\gamma + a - \tau_j - \bar{\rho} < \gamma + \alpha_i^+ + \beta_j^+ - \sigma_i - \tau_j \leq (\alpha_l^+ - \sigma_l) + (\beta_j^+ - \tau_j),$$

and hence this term is also of order less than γ_{lj}. This proves the lemma.

We now define the variables $W_{j,a}(x_\nu, \omega)$ which are the components of the vector W in the one–step scheme (2.7). We set

$$W_{j,a}(x_\nu, \omega) := \Lambda_0^{\tau_j + \bar{p} - 1 - a} \delta_+^a u_j(x_\nu, \omega) \qquad \text{for} \quad a = 0, ..., \tau_j + \bar{p} - 1,$$

If $\mu_j > 0$, we put

$$W_{j,a}(x_\nu, \omega) := T^{a - (\tau_j + \bar{p} - 1)} \delta_+^{\tau_j + \bar{p} - 1} u_j(x_\nu, \omega), \qquad \text{for} \quad a = \tau_j + \bar{p}, ..., \tau_j + \bar{p} - 1 + \mu_j.$$

Thus there are a total of $|\mu| + |\tau| + n\bar{p}$ components in W.

The equations comprising the one–step scheme are: first, if $\tau_j + \bar{p} - 1 > 0$,

$$W_{j,a}(x_{\nu+1}, \omega) = W_{j,a}(x_\nu, \omega) + h \Lambda_0 W_{j,a+1}(x_\nu, \omega) \qquad \text{for} \quad a = 0, ..., \tau_j + \bar{p} - 2, \qquad (4.7)$$

and, if $\mu_j > 0$,

$$W_{j,a}(x_{\nu+1}, \omega) = W_{j,a+1}(x_\nu, \omega), \qquad \text{for} \quad a = \tau_j + \bar{p} - 1, ..., \tau_j + \bar{p} + \mu_j - 2. \qquad (4.8)$$

For $a = \tau_j + \bar{p} + \mu_j - 1$ we use equation (4.6) to obtain an expression for $W_{j,a}(x_\nu, \omega)$. Multiplying equation (4.6) by h we have

$$T^{\mu_i} \delta_+^{\tau_i + \bar{p} - 1} u_i(x_{\nu+1}, \omega) =$$

$$T^{\mu_i} \delta_+^{\tau_i + \bar{p} - 1} u_i(x_\nu, \omega) - \sum_{j=1}^{n} \sum_{b=0}^{\mu_j - 1} g_{ijb}^0 [T^{b+1} \delta_+^{\tau_j + \bar{p} - 1} u_j(x_\nu, \omega) - T^b \delta_+^{\tau_j + \bar{p} - 1} u_j(x_\nu, \omega)]$$

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$$+ \sum_{j=1}^{n} \sum_{b=0}^{\tau_j + \bar{p} - 2} \tilde{m}_{ijb} W_{j,b}(x_\nu, \omega) + h \tilde{F}_i''(x_\nu, \omega),$$

where $\tilde{m}_{ijb} = -h \hat{l}_{ijb}'' \Lambda_0^{-(\tau_j + \bar{p} - 1 - b)} \in \mathbf{S}^0$. Replacing $T^b \delta_+^{\tau_j + \bar{p} - 1} u_j(x_\nu, \omega)$ by $W_{j,b+\tau_j+\bar{p}-1}$ in the above, we have (with $a := \tau_i + \bar{p} - 1 + \mu_i$)

$$W_{i,a}(x_{\nu+1}, \omega) = \qquad (4.9)$$

$$W_{i,a}(x_\nu, \omega) - \sum_{j=1}^{n} \sum_{b=0}^{\mu_j - 1} g_{ijb}^0 [W_{j,\tau_j+\bar{p}+b}(x_\nu, \omega) - W_{j,\tau_j+\bar{p}-1+b}(x_\nu, \omega)]$$

$$+ \sum_{j=1}^{n} \sum_{b=0}^{\tau_j + \bar{p} - 2} \tilde{m}_{ijb} W_{j,b}(x_\nu, \omega) + h \tilde{F}_i''(x_\nu, \omega).$$

The equations (4.7), (4.8), and (4.9) together give the one-step scheme (2.7). We also will write equation (2.7) as

$$W_{i,a}(x_{\nu+1}, \omega) = \sum_{j,b} M_{ijab}(y, \omega) W_{j,b}(x_\nu, \omega) + h \mathcal{F}_i(x_\nu, \omega), \qquad (4.10)$$

$$i = 1, ..., n, \quad a = 0, ..., \tau_i + \bar{p} - 1 + \mu_i.$$

Note that the matrix $M(y, \omega)$ as constructed is independent of h, depending only on y and ω. This follows from equation (2.4) and the means used to define the $W_{i,a}(x_\nu, \omega)$.

Note that $h \Lambda_0(h, \omega) = \sqrt{4 \sum_{i=1}^{d-1} \sin^2(\frac{1}{2}\omega_i)}$ is independent of h, depending only on ω.

The boundary conditions (4.3) can be written in terms of the variables $W_{i,a}$ as

$$\mathcal{B}_1(y, \omega) W(x_0, \omega) = \Phi_1(\omega)$$

$$\qquad (4.11)$$

$$\mathcal{B}_2(y, \omega) W(x_0, \omega) = \Phi_2(\omega)$$

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where $\mathcal{B}_1$ and $\mathcal{B}_2$ consist of the first p and last $q - p$ boundary operators, respectively, as follows. Each of the terms in the sum (2.6), describing the boundary conditions for the reduced problem, can be written as

$$\tilde{B}_{kj} u_j = \sum_{a=0}^{\tau_j + \rho_k} b_{k,j,a}(h, y, \omega) \Lambda_0^{\tau_j + \rho_k - a} \delta_+^a u_j,$$

where the $b_{k,j,a}$ are symbols of order 0 in ω. The first p boundary conditions are scaled by $\Lambda_0^{\bar{p} - 1 - \rho_k}$, as mentioned earlier in this section, and thus

$$\Lambda_0^{\bar{p} - 1 - \rho_k} \tilde{B}_{kj} u_j = \sum_{a=0}^{\tau_j + \rho_k} b_{k,j,a}(h, y, \omega) \Lambda_0^{\tau_j + \bar{p} - 1 - a} \delta_+^a u_j = \sum_{a=0}^{\tau_j + \rho_k} b_{k,j,a}(h, y, \omega) W_{j,a}.$$

This defines the elements of $\mathcal{B}_1(y, \omega)$. The components of $\Phi_1(\omega)$ are $\Lambda_0^{\bar{p} - 1 - \rho_k} \tilde{\phi}_k(\omega)$.

The elements of $\mathcal{B}_2(y, \omega)$ are obtained by scaling with $h^{\rho_k - \bar{p} + 1}$. Thus each term in the sum in (2.6) for k from $p + 1$ to q becomes

$$h^{\rho_k - \bar{p} + 1} \sum_{a=0}^{\tau_j + \rho_k} b_{j,k,a} \Lambda_0^{\tau_j + \rho_k - a} \delta_+^a u_j = \sum_{a=0}^{\tau_j + \bar{p} - 1} b_{j,k,a} (h\Lambda_0)^{\rho_k - \bar{p} + 1} \Lambda_0^{\tau_j + \bar{p} - 1 - a} \delta_+^a u_j$$

$$+ \sum_{a=\tau_j + \bar{p}}^{\tau_j + \rho_k} b_{j,k,a} (T - 1)^{a - \bar{p} + 1 - \tau_j} (h\Lambda_0)^{\tau_j + \rho_k - a} \delta_+^{\tau_j + \bar{p} - 1} u_j.$$

The first summation can be expressed as

$$\sum_{a=0}^{\tau_j + \bar{p} - 1} b_{j,k,a} (h\Lambda_0)^{\rho_k - \bar{p} + 1} W_{j,a} \tag{4.12}$$

and the second as

$$\sum_{a=1}^{\rho_k - \bar{p} + 1} b_{j,k,a+\tau_j+\bar{p}-1} (h\Lambda_0)^{\rho_k - \bar{p} + 1 - a} \sum_{c=0}^{a} \binom{a}{c} (-1)^{a-c} T^c \delta_+^{\tau_j + \bar{p} - 1} u_j.$$

This second summation is further expressed as the sum of several terms,

$$\sum_{a=1}^{\rho_k - \bar{p} + 1} b_{j,k,a+\tau_j+\bar{p}-1} (h\Lambda_0)^{\rho_k - \bar{p} + 1 - a} \sum_{c=0}^{\min(a,\mu_j)} \binom{a}{c} (-1)^{a-c} W_{j,c+\tau_j+\bar{p}-1}, \tag{4.13}$$

and

$$\sum_{a=\mu_j+1}^{\rho_k - \bar{p} + 1} b_{j,k,a+\tau_j+\bar{p}-1} (h\Lambda_0)^{\rho_k - \bar{p} + 1 - a} \sum_{c=\mu_j+1}^{a} \binom{a}{c} (-1)^{a-c} T^{c-\mu_j} W_{j,\tau_j+\bar{p}-1+\mu_j}, \tag{4.14}$$

where this last expression is taken to be zero if μ_j is greater than $\rho_k - \bar{p} + 1$. The expressions $T^{c-\mu_j} W_{j,\tau_j+\bar{p}-1+\mu_j}$ are replaced by repeated use of the equation $TW = MW + h\mathcal{F}$. This defines the elements of $\mathcal{B}_2(y,\omega)$ and the components of $\Phi_2(\omega)$ are $h^{\rho_k - \bar{p} + 1} \tilde{\phi}_k(\omega)$ plus terms from $h\mathcal{F}$.

We now state and prove two lemmas about the matrix M.

Lemma 4.2

$$det(\frac{M - I}{h}) = h^{-|\mu|} \Lambda_0^{|\sigma| + n\bar{p}} L(h,y,\omega) c(y,\omega),$$

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where $L(h,y,\omega) = det(l_{ij}(h,y,\varsigma))$ *for* $\varsigma = (0,\omega)$, $c(y,\omega) \neq 0$ *for all* ω, *and* $c(y,\omega)$ *is independent of* h.

Lemma 4.3 .

$$\|(\frac{M-I}{h})^{-1}\| \leq C\Lambda_0^{-1},$$

for some constant C.

Proof of Lemma 4.2:

There are three types of rows in the matrix $(M-I)/h$. Thinking of W as a doubly subscripted vector, the first type is obtained from equation (4.7) and contributes one nonzero element, Λ_0, in row (i,a) and column $(i,a+1)$ for $a = 0, ..., \tau_i + \bar{p} - 2, 1 \leq i \leq n$. The second type, obtained from equation (4.8), contributes two nonzero elements $-h^{-1}$ and h^{-1} in row (i,a) and columns (i,a) and $(i,a+1)$ for $a = \tau_i + \bar{p} - 1, ..., \tau_i + \bar{p} + \mu_i - 2, \quad 1 \leq i \leq n$.

We now evaluate the determinant. First, each row corresponding to an equation of the type (4.7) has only one element, Λ_0, in the position $((j,a),(j,a+1)), 1 \leq j \leq n, a = 0, ..., \tau_j + \bar{p} - 2$. Evaluating about these rows gives $\Lambda_0^{|\tau|+n(\bar{p}-1)}$ times the determinant of the remaining rows and columns. The second class of rows corresponds to equation (4.8). These have the elements h^{-1} in the position $((j,a),(j,a))$ and $-h^{-1}$ in $((j,a),(j,a+1))$, $1 \leq j \leq n, a = \tau_j + \bar{p} - 1, ..., \tau_j + \bar{p} - 1 + \mu_j - 1$. However, in evaluating about the first class of rows the column $(j, \tau_j + \bar{p} - 1)$ is eliminated, so by evaluating the determinant remaining after the first reduction there results a factor h^{-1} times $\tilde{m}_{ij0}(\omega)$ in the $(i, \tau_i + \bar{p} - 1 + \mu_i)$ row and the $(j,0)$ column which is, by definition, equal to $\tilde{l}_{ij0}(h,\omega)\Lambda_0^{-\tau_j-\bar{p}+1}$. Thus, from the definition of the $\tilde{l}_{ij0}$ resulting from the operators (4.1), the determinant of $(M-I)/h$

is

$$\pm h^{-|\mu|} \Lambda_0^{|\sigma|+n\bar{p}} \det(l_{ij}(h,y,\omega))/\det(g'_{ij}(\omega)),$$

where $g'_{ij}(\omega)$ are the coefficients of $G_1(T)$. Since $\det(\tilde{g}'_{ij}(\omega))$ does not vanish for any ω, this proves the lemma.

Proof of Lemma 4.3:

Using Cramer's rule to evaluate the inverse matrix of $(M - I)/h$ one sees that each minor is bounded by either Λ_0^{-1} or h times the determinant of $(M - I)/h$. Since $h \le c/\Lambda_0$ the estimate follows easily.

Lemma 4.4

For $\omega = 0$, 1 is a semi-simple eigenvalue of M. That is, there are exactly $|\tau| + n\bar{p}$ linearly independent vectors $V^{j,b}, j = 1, ..., n, b = 0, ..., \tau_j + \bar{p} - 1$ such that $MV^{j,b} = V^{j,b}$. Moreover, these eigenvectors are given by

$$V_{i,a}^{j,b} = \begin{cases} \delta_{i,j}\delta_{a,b}, & \text{if } 0 \le b \le \tau_j + \bar{p} - 2; \\ \delta_{i,j}\delta_{\tau_i+\bar{p}-1,b}, & \text{if } a \ge \tau_i + \bar{p} - 1, \ b = \tau_j + \bar{p} - 1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof:

We consider the equation $MV = V$. From the (4.7), we see that $(MV)_{i,a} = V_{i,a}$ for $0 \le a \le \tau_j + \bar{p} - 2$ and from (4.8) $(MV)_{i,a} = V_{i,a+1}$ for $\tau_j + \bar{p} - 1 \le a \le \tau_j + \bar{p} + \mu_j - 2$. Thus if V is an eigenvector of M with eigenvalue 1 then $V_{i,a} = V_{i,\tau_i+\bar{p}-1}$ for these μ_i components. In (4.9) note that $h^{-1}\tilde{m}_{i,j,b} \in S^1$ and hence vanishes at $\omega = 0$. Thus for an eigenvector, (4.9) reduces to $(MV)_{i,a} = V_{i,a}$ for $a = \tau_i + \bar{p} - 1 + \mu_i$. It is now easy to check

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that the $|\tau| + n\bar{p}$ independent vectors $V^{j,b}$ given in the lemma are indeed eigenvectors. Moreover, by Theorem 5.3 the multiplicity of the eigenvalue 1 is $|\tau| + n\bar{p}$, hence these are all the eigenvectors and 1 is a semi–simple eigenvalue.

Lemma 4.5

If $MW = W$ for $\omega = 0$, then $\mathcal{B}_2 W = 0$.

Proof:

If $\omega = 0$ then $\Lambda_0 = 0$ and all the coefficients in (4.12) vanish since $\rho_k \geq \bar{p}$ for $k > p$ by Assumption 2.2. Since $MW = W$ the sum of (4.13) and (4.14) gives the coefficient of $b_{j,k,a+\tau_j+\bar{p}-1}$ as

$$\sum_{c=0}^{\min(a,\mu_j)} \binom{a}{c} (-1)^{a-c} W_{j,c+\tau_j+\bar{p}-1} + \sum_{c=\mu_j+1}^{a} \binom{a}{c} (-1)^{a-c} W_{j,\tau_j+\bar{p}-1+\mu_j}. \qquad (4.15)$$

Again since $MW = W$ the equation (4.8) implies that

$$W_{j,a} = W_{j,a+1}, \qquad \text{for} \qquad a = \tau_j + \bar{p} - 1, ..., \tau_j + \bar{p} + \mu_j - 2,$$

and hence (4.15) is equal to the quantity

$$\left\{ \sum_{c=0}^{a} \binom{a}{c} (-1)^{a-c} \right\} W_{j,\tau_j+\bar{p}-1},$$

which vanishes identically. This shows that $\mathcal{B}_2 W = 0$, as asserted.

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5. Construction of $H(y,\omega)$

We now construct the matrix $H(y,\omega)$ to satisfy inequalities (2.9) and (2.10).

Theorem 5.1

There exists a bounded Hermitian matrix symbol $H(y,\omega) \in \mathbf{S}^0$ which is a C^∞ function

of (y,ω) for $y \in \mathbf{R}^{d-1}$ and $\omega \in \mathbf{R}^{d-1} \setminus \{0\}$ such that

$$a) \quad M^* H M - H \geq c_0 h \Lambda_0$$

$$(5.1)$$

$$b) \quad W^* H W \geq c_1 (\eta |W_+|^2 - |W_-|^2),$$

where $W_+ = W_+(\omega)$ is the projection of W into the span of the generalized eigenvectors of

M whose eigenvalues are of modulus greater than unity, and $W_- = W_-(\omega)$ is the projection

into the span of the generalized eigenvectors of M whose eigenvalues are of modulus less

than unity. The constants c_0, c_1 and η are all positive and, moreover, η can be chosen

arbitrarily large with c_0, c_1, and H depending on η.

The proof of Theorem 5.1 will be postponed until after we prove Theorems 5.2–5.4.

We begin by constructing H in a neighborhood of $\omega_0 \neq 0$.

Theorem 5.2

There exists a smooth, bounded matrix function $P(\omega)$ defined in a neighborhood U

of $\omega_0 \neq 0$ such that

$$P(\omega)M(\omega)P(\omega)^{-1} = \begin{pmatrix} M_+(\omega) & O \\ O & M_-(\omega) \end{pmatrix}, \tag{5.2}$$

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where $M_+(\omega)$ is a $m_+ \times m_+$ matrix and $M_-(\omega)$ is a $m_- \times m_-$ matrix. Moreover, for $\omega \in U$

$$M_+^*(\omega)M_+(\omega) \geq 1 + \tilde{c}_0,$$

$$\text{and} \quad M_-^*(\omega)M_-(\omega) \leq 1 - \tilde{c}_0.$$

(5.3)

m_+ is the number of eigenvalues of $M(\omega)$ of modulus greater than unity and m_- is the number with modulus less than unity.

Proof:

By lemma 4.2, $M(\omega)$ has no eigenvalues of modulus one for $\omega \neq 0$. Therefore it follows by standard linear algebra that $M(\omega)$ can be transformed to the form (5.2). That the inequalities (5.3) can be satisfied is also a standard result, see Gustafsson et al. [6].

For $\omega_0 \neq 0$, $H(\omega)$ can be constructed in the neighborhood U of ω_0 as

$$H(\omega) = P^*(\omega) \begin{pmatrix} \eta I_{m_+} & O \\ O & -I_{m_-} \end{pmatrix} P(\omega),$$

where I_{m_+} and I_{m_-} are the identity matrices of order m_+ and m_-, respectively. It is easily checked that $H(\omega)$ satisfies (5.1) in U.

We now consider $M(\omega)$ in a neighborhood of $\omega_0 = 0$.

Theorem 5.3

For $|\omega|$ near zero the eigenvalues of $M(\omega)$ separate into three distinct classes. These are:

1) There are $|\sigma| + n\bar{p}$ eigenvalues with $\kappa = 1 + h\Lambda_0$.

2) There are $|\tau| - |\sigma| = 2p$ eigenvalues which are the roots of the equation

$$det(l_{ij}(h,\varsigma)) = 0,$$

with

$$\varsigma = (\varsigma_1, \omega) \quad \text{and} \quad \kappa = \exp(i\varsigma_1),$$

where

$$\kappa = 1 \pm O(h\Lambda_0).$$

Moreover, p of these eigenvalues satisfy

$$|\kappa| \leq 1 - ch\Lambda_0,$$

and p of them satisfy

$$|\kappa| \geq 1 + ch\Lambda_0,$$

for some positive constant c.

3) There are $|\mu|$ eigenvalues which satisfy

$$||\kappa| - 1| \geq \delta > 0.$$

Proof:

The proof depends on the equivalence of the one–step scheme to the original reduced equations (2.5). The first class of $|\sigma| + n\bar{p}$ eigenvalues with $\kappa = 1 + h\Lambda_0$ are due to operating on the reduced system with $(\delta_+ - \Lambda_0)^{-\sigma_i + \bar{p}}$ (see 4.1). There are a total of

$$\sum_{i=1}^{n}(-\sigma_i + \bar{p}) = |\sigma| + n\bar{p},$$

such eigenvalues. The third class of eigenvalues is determined as follows. At $\omega_0 = 0$ the eigenvalues of M are the roots of

$$\det(g_{ij}(\kappa))(\kappa - 1)^{|\tau| + n\bar{\rho}} = 0,$$

by equation (4.4). Notice that the $\tilde{l}_{ija}$ in equation (4.4) vanish at $\omega_0 = 0$ since they are

symbols in $S^{\tau_j + \bar{\rho} - 1 - a}$ with only highest order terms. Since the system (4.4) is elliptic of

order $(0, \tau + \bar{\rho})$ the root $\kappa = 1$ cannot have multiplicity greater than $|\tau| + n\bar{\rho}$. These are

the roots of class (1) and (2). Thus the roots at $\omega_0 = 0$ which are distinct from 1 are all

the roots of

$$\det(g_{ij}(\kappa)) = 0,$$

of which there are $|\mu|$. By Lemma 2.1, $|\kappa|$ is not equal to 1. By continuity there exists

some neighborhood of $\omega_0 = 0$ and constant for which the inequality in part (3) holds.

The remaining $|\tau| - |\sigma|$ eigenvalues are easily seen to be related to the associated

system of partial differential equations (2.15) . If we define $\tilde{\kappa}(\theta)$ by

$$\det(\hat{l}_{ij}(y, \tilde{\kappa}, \theta)) = 0,$$

then

$$\kappa(\omega) = 1 + |\omega|\tilde{\kappa}(\frac{\omega}{|\omega|}) + O(|\omega|^2).$$

By the supplementary condition (Assumption 2.3) we easily obtain the inequalities in

part (2) for the p eigenvalues less than and greater than one in modulus.

Theorem 5.4

For ω in a neighborhood of zero there is a continuous nonsingular matrix $Q(\omega)$ such

that $\|Q(\omega)\|$ and $\|Q^{-1}(\omega)\|$ are bounded and

$$\tilde{M}(\omega) := Q(\omega)M(\omega)Q^{-1}(\omega) = \begin{pmatrix} L_0(\omega) & O & O & O \\ O & N_0(\omega) & O & O \\ O & O & L_1(\omega) & O \\ O & O & O & N_1(\omega) \end{pmatrix}, \qquad (5.4)$$

where $L_0, L_1, N_0,$ and N_1 are square matrices. Moreover,

$$a) \quad L_0^* L_0 \leq 1 - \delta,$$

$$b) \quad N_0^* N_0 \geq 1 + \delta,$$

$$\qquad (5.5)$$

$$c) \quad L_1^* L_1 \leq 1 - c_1 h \Lambda_0,$$

$$d) \quad N_1^* N_1 \geq 1 + c_1 h \Lambda_0,$$

with $L_1(0)$ and $N_1(0)$ being identity matrices. The dimension of L_1 is $\frac{1}{2}(|\tau| - |\sigma|) = p$ and that of N_1 is $\frac{1}{2}(|\tau| + |\sigma|) + n\bar{p}$.

Proof of Theorem 5.4:

For ω in a neighborhood of zero the eigenvalues of $M(\omega)$ separate into three classes: those which are strictly less than one in modulus, those which are strictly greater than one in modulus, and those which are not bounded away from one in modulus. Thus $Q(\omega)$ can be constructed so that

$$Q(\omega)M(\omega)Q^{-1}(\omega) = \begin{pmatrix} L_0 & O & O \\ O & N_0 & O \\ O & O & M_1 \end{pmatrix},$$

where the three matrices on the diagonal correspond to these three classes of eigenvalues. Moreover, $Q(\omega)$ can be constructed so that L_0 and N_0 satisfy the inequalities of the theorem.

By Schur's theorem we can assume that $M_1(\omega)$ is in upper triangular form with the first $\frac{1}{2}(|\tau| - |\sigma|)$ diagonal elements being those eigenvalues which are less than one in modulus for ω nonzero. There are $\frac{1}{2}(|\tau| - |\sigma|) = p$ such eigenvalues by Theorem 5.3. Thus

$$M_1(\omega) = \begin{pmatrix} L_2(\omega) & M_2(\omega) \\ O & N_2(\omega) \end{pmatrix}.$$

We now show that there is a matrix $D(\omega)$ such that

$$\begin{pmatrix} I & D \\ O & I \end{pmatrix} \begin{pmatrix} L_2 & M_2 \\ O & N_2 \end{pmatrix} \begin{pmatrix} I & -D \\ O & I \end{pmatrix} = \begin{pmatrix} L_2 & O \\ O & N_2 \end{pmatrix},$$

with $\|D(\omega)\|$ bounded for ω near zero.

The matrix $D(\omega)$ is the solution to

$$L_2(\omega)D(\omega) - D(\omega)N_2(\omega) = M_2(\omega).$$

In order to prove the existence of $D(\omega)$ we give the following

Lemma 5.1

There is a unique solution to

$$L_2 D - D N_2 = M_2,$$

with

$$\|D\| \leq K\|M_2\|/h\Lambda_0,$$

where K is independent of h and ω.

Proof:

By Schur's theorem there are orthogonal matrices O_1 and O_2 such that $\tilde{L} := O_1^* L_2 O_1$

and $\tilde{N} := O_2^* N_2 O_2$ are lower and upper triangular, respectively. With $\tilde{M} := O_1^* M_2 O_2$ and

$\tilde{D} := O_1^* D O_2$, the equation becomes

$$(\tilde{L}_{ii} - \tilde{N}_{kk})\tilde{D}_{ik} = \tilde{M}_{ik} - \sum_{j<i} \tilde{L}_{ij}\tilde{D}_{jk} + \sum_{j<K} \tilde{D}_{ij}\tilde{N}_{jk}.$$

This is a recursive formula for the elements of $\tilde{D}$ in the order $\tilde{D}_{11}, \tilde{D}_{12}, ..., \tilde{D}_{21}, ..., \tilde{D}_{ab}$.

By Theorem 5.3 $(\tilde{L}_{ii} - \tilde{N}_{kk}) = O(h\lambda_0)$ and the estimate follows easily. This proves the

lemma.

It remains to show that $\|M_2\| \leq ch\lambda_0$. We will need the following lemma, whose proof

will be given after the completion of the proof of Theorem 5.4.

Lemma 5.2

Let $x(\varepsilon)$ be an upper triangular matrix depending on ε for $0 < \varepsilon \leq \varepsilon_0$. If

$$\|x(\varepsilon)^{-1}\| \leq c/\varepsilon,$$

and

$$c_1\varepsilon \leq |x_{ii}| \leq C_1\varepsilon \qquad for \quad 0 < \varepsilon \leq \varepsilon_0,$$

then

$$|x_{ij}(\varepsilon)| \leq C\varepsilon,$$

for some constant C.

Now with $\varepsilon = h\Lambda_0$ and $x(\varepsilon) = M_1(\omega) - I$ it follows from Lemmas 4.3 and 4.4 that $\|M_2\| \leq ch\Lambda_0$. Hence $D(\omega)$ is bounded in norm for ω near zero.

An argument similar to the above shows that the off–diagonal elements of $L_2(\omega)$ and $N_2(\omega)$ are all of order $h\Lambda_0$. Thus there are constant diagonal matrices $D_1(\omega)$ and $D_2(\omega)$ such that for

$$L_1(\omega) := D_1 L_2(\omega) D_1^{-1},$$

$$N_1(\omega) := D_2 N_2(\omega) D_2^{-1},$$

we have

$$L_1^* L_1 \leq 1 - ch\Lambda_0,$$

$$N_1^* N_1 \leq 1 + ch\Lambda_0.$$

This completes the proof of Theorem 5.4.

Proof of Lemma 5.2:

The proof is by induction on j. Suppose $(x_{ij}) \leq C(l)\varepsilon$ for $j = i + l$, $0 \leq l \leq m - 1$. For $j = i$, $|x_{ij}| \leq C\varepsilon$ by hypothesis. Let $z_{ij} := (x^{-1}(\varepsilon))_{ij}$, then

$$0 = x_{i,i} z_{i,i+m} + x_{i,i+1} z_{i+1,i+m} + \ldots + x_{i,i+m} z_{i+m,i+m},$$

$$x_{i,i+m} = \frac{-1}{z_{i+m,i+m}} \left(\sum_{j=i}^{i+m-1} x_{ij} z_{j,i+m} \right),$$

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and hence

$$|x_{i,i+m}| \le C_1 \varepsilon \sum_{j=i}^{i+m-1} (C(j)\varepsilon)(c/\varepsilon) \le C(m)\varepsilon,$$

which proves the lemma for C equal to the maximum of the $C(l)$.

For $\tilde{M}(\omega)$ in the form (5.4) satsfying (5.5), we construct $\tilde{H}$ as

$$\tilde{H} := \begin{pmatrix} -I & O & O & O \\ O & \eta I & O & O \\ O & O & -I & O \\ O & O & O & \eta I \end{pmatrix},$$

and then

$$H(\omega) := Q^*(\omega)\tilde{H}Q(\omega).$$

It follows that

$$M^*HM - H = Q^*(\tilde{M}^*\tilde{H}\tilde{M} - \tilde{H})Q$$

$$\ge c_0 h \Lambda_0 Q^* Q$$

$$\ge c_0' h \Lambda_0.$$

This satisfies (5.1a) locally; similarly (5.1b) is satisfied.

We have constructed $H(y,\omega)$ satisfying (5.1) in a neighborhood of each ω. Since the set of ω with $|\omega_i| \le \pi$ is compact we can choose a finite set of these neighborhoods which cover $\mathbf{R}^{d-1}$, and through the use of a partition of unity we can construct $H(\omega)$ for all ω. This proves Theorem 5.1.

We now show that the relation (2.10) is satisfied for some positive constants c_1 and c_2. By equation (5.1) we need only show that

$$|\beta W_-|^2 \geq \tilde{c}|W_-|^2,$$

for some positive constant $\tilde{c}$ where W_- is defined as in Theorem 5.2. For then

$$W^*HW + c_1|\beta W|^2 \geq \tilde{c}_1\eta|W_+| - c_1|\beta W_+|^2 + (c_1\tilde{c} - \tilde{c}_1)|W_-|^2$$

$$\geq c_2(|W_+|^2 + |W_-|^2)$$

$$\geq c_2|W|^2,$$

for c_1 and η chosen large enough. Thus we need to prove

Theorem 5.5

If the Complementing Condition is satisfied then there exists a positive constant $\tilde{c}$ such that

$$|\beta(\omega)W_-|^2 \geq \tilde{c}|W_-|^2, \tag{5.6}$$

for all $\omega \in \mathbf{R}^{d-1}$, and all W_- defined as in Theorem 5.1.

Proof of Theorem 5.5:

Suppose for some ω_0 with $\omega_0 \neq 0$

$$\beta(\omega_0)W_- = 0 \qquad \text{for} \quad |W_-| = 1, \tag{5.7}$$

where W_- is in the generalized eigenspace of $M(\omega)$ whose eigenvalues are of modulus less than unity. Then by setting

$$W_0 := W_-$$

$$W_{\nu+1} := M(\omega_0)W_\nu \qquad \nu \geq 0,$$

W_ν is an eigensolution of type I. W_ν tends to zero as ν increases since W_- is in the generalized eigenspace whose eigenvalues of modulus less than unity. Thus (5.7) violates the Complementing Condition.

Next suppose that (5.7) holds with $\omega_0 = 0$. It follows that

$$B_2(0)W_- = 0,$$

and we see that W_ν, generated by $M(0)$ as above, corresponds to an eigensolution of type III. Therefore (5.7) violates the Complementing Condition when $\omega_0 = 0$. Now consider the possibility of the existence of sequences $\{\omega_k\}_{k=1}^\infty$ and $\{W_-^k\}_{k=1}^\infty$ such that

$$\omega_k \to 0 \quad \text{as} \quad k \to \infty, \qquad |W_-^k| = 1,$$

and

$$B(\omega_k)W_-^k \to 0 \qquad \text{as} \quad k \to \infty.$$

We may assume that there exists a W_-^0 such that

$$W_-^k \to W_-^0 \quad \text{and} \quad \omega_k/|\omega_k| \to \theta \quad \text{as} \quad k \to \infty.$$

Set $W_-^0 := W_{II}^0 + W_{III}^0$ where W_{II}^0 is in the linear space spanned by the eigenvectors of M with eigenvalue 1 and W_{III}^0 is in the span of the eigenvectors with eigenvalues strictly less than 1 in magnitude. By lemma 4.5, $B_2 W_{II}^0 = 0$, and, since $B_2 W_-^0 = B_2 W_{III}^0 = 0$, the nonexistence of eigensolutions of type III implies W_{III}^0 is zero. Therefore, without loss of generality, we can assume

$$W_\nu^k = \sum_{b=1}^{p} f_b^k \kappa_b(\omega_j)^\nu,$$

where the $\kappa_b(\omega_j)$ are the eigenvalues of $M(\omega_j)$ with $\kappa_b(\omega_j) \to 1$ as $\omega_j \to 0$, $f_b^k \to f_b^0$, and $|\kappa_b(\omega_j)| < 1$ for $\omega_j \neq 0$. Define the vector–valued function $Z(x)$ by

$$Z(x) := \sum_{b=1}^{p} f_b^0 e^{-\eta_b x}$$

where

$$\eta_b := \lim_{k \to \infty} \frac{1 - \kappa_b(\omega_j)}{|\omega_j|}.$$

By the structure of the eigenvalues of M at $\omega = 0$ (Lemma 4.4), we have that $Z_{j,a}(x) = Z_{j,\tau_j+\bar{p}-1}(x)$ for $\tau_j + \bar{p} - 1 \leq a \leq \tau_j + \bar{p} - 1 + \mu_j$. From (4.7), for $0 \leq a \leq \tau_j + \bar{p} - 1$ we have

$$W_{j,a,\nu+1} - W_{j,a,\nu} = h\Lambda_0 W_{j,a+1,\nu},$$

or

$$\sum_{b=0}^{p} f_{b,j,a}^k \kappa_b^\nu (\kappa_b - 1) = h\Lambda_0 W_{j,a+1,\nu}.$$

Dividing by $|\omega_j|$ and taking the limit we have

$$\sum_{b=0}^{p} f_{b,j,a}^0 (-\eta_b) e^{-\eta_b x} = \sum_{b=0}^{p} f_{b,j,a+1} e^{-\eta_b x},$$

hence

$$\frac{\partial}{\partial x} Z_{j,a}(x) = Z_{j,a+1}(x)$$

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for $0 \le a \le \tau_j + \bar{\rho} - 1$. Thus

$$Z_{j,a}(x) = (\frac{\partial}{\partial x})^a Z_{j,0}(x) \qquad \text{for} \quad a = 0, ..., \tau_j + \bar{\rho} - 1.$$

Let $Z_j(x)$ be $Z_{j,0}(x)$. Finally, from (4.9), we obtain

$$(\frac{\partial}{\partial x})^{\tau_j + \bar{\rho}} Z_i(x) = -\sum_{j=1}^{n} \sum_{b=1}^{\mu_j - 1} g_{i,j,b}(\frac{\partial}{\partial x})^{\tau_j + \bar{\rho}} Z_j(x)$$

$$+ \sum_{j=1}^{n} \sum_{b=0}^{\tau_j + \bar{\rho} - 2} \hat{l}''_{ijb}(\theta)(\frac{\partial}{\partial x})^b Z_j(x).$$

By reversing the transformations done after equation (4.4) (see Lemma 4.1) we have that

this system is equivalent to one analogous to (4.4), namely,

$$\sum_{j=1}^{n} g_{ij}(0, \theta, 1)(\frac{\partial}{\partial x})^{\tau_j + \bar{\rho}} Z_j(x) + \sum_{j=1}^{n} \sum_{a=0}^{\tau_j + \bar{\rho} - 1} \tilde{l}_{ij}(\theta)(\frac{\partial}{\partial x})^a Z_j(x) = 0.$$

By eliminating the factors $(\frac{\partial}{\partial x} - 1)^{\sigma_i + \bar{\rho}}$ corresponding to (4.1) we obtain the associated

system of differential equations (2.17) and the vector function $\tilde{Z}(x) = (Z_{j,0}(x))$ is a solution

to (2.17).

By a similar reduction one can show that $\tilde{Z}(x)$ satisfies $\mathcal{B}_1 \tilde{Z}(0) = 0$. The nonexistence

of eigensolutions of type II implies that $\tilde{Z}(x)$, hence $Z(x)$, is zero. Thus we have, for

$|\omega_-| = 1$, that $|\mathcal{B}(\omega)W_-| \ge c$ which proves Theorem 5.5.

6. The Regularity Estimate

In this section we prove the regularity estimate (3.3) using the inequalities (2.9) and (2.10) and Gårding's inequality (Bube and Strikwerda [3]). We state the final estimate as:

Theorem 6.1

If the elliptic difference scheme (2.1) with boundary conditions (2.2) satisfies the Complementing Condition and Assumptions 2.1, 2.2, and 2.3, then the following regularity estimate holds for $\bar{\rho} \leq s < \rho^*$ and h sufficiently small,

$$\|u\|_{\tau+s}^2 + |u|_{\tau+s-\frac{1}{2}}^2 \leq C_s(|\phi_1|_{s-\rho-\frac{1}{2}}^2 + |h^{\rho-t+\frac{1}{2}}\phi_2|_{s-t}^2 + \|F\|_{s-\sigma}^2 + \|u\|_0^2). \tag{6.1}$$

where $t = \bar{\rho} + \frac{1}{2}\lfloor 2(s-\bar{\rho})\rfloor$.

The first step to proving Theorem 6.1 is to prove estimates on the tangential differences. We first define the norms

$$\|u\|_{\tau+t,\tau+s}^2 = \sum_{j=0}^{n} \sum_{a=0}^{\tau_j+t} \|\Lambda^{\tau_j+s-a}\delta_+^a u_j\|_0^2,$$

and

$$|u|_{\tau+t,\tau+r}^2 = \sum_{j=0}^{n} \sum_{a=0}^{\tau_j+t} |\Lambda^{\tau_j+r-a}\delta_+^a u_j|_0^2,$$

which limit the normal differences, i.e. those with repect to x, which are included in the norm.

Our first result is

Theorem 6.2

If the elliptic difference scheme (2.1) with boundary conditions (2.2) satisfies the Complementing Condition and Assumptions 2.1, 2.2, and 2.3, then the following estimate holds for any real s.

$$\|u\|^2_{\tau+\bar{p}-1,\tau+s} + |u|^2_{\tau+\bar{p}-1,\tau+s-\frac{1}{2}} \leq C_s(|\phi_1|^2_{s-p-\frac{1}{2}} + |h^{p-\bar{p}+1}\phi_2|^2_{s-\bar{p}+\frac{1}{2}} + \|F\|^2_{\bar{p}-\sigma,s-\sigma}$$

$$+ \|u\|^2_{\tau+\bar{p}-1,0} + |u|^2_{\tau+\bar{p}-1,-\frac{1}{2}}).$$

$$(6.2)$$

Proof:

We have that

$$\|f\|^2_{0,s} := h \sum_{\nu=0}^{\infty} |f_\nu|^2_{0,s},$$

where $|f_\nu|_{0,s}$ is the Sobolev norm of order s on the tangential variables only at x_ν. Using equation (2.9) and Gårding's inequality on the tangential variables, we obtain

$$\|W\|^2_{0,s} \leq C \sum_{\nu=0}^{\infty} ((\tilde{M}^*\tilde{H}\tilde{M} - \tilde{H})W_\nu, W_\nu)_{0,s-\frac{1}{2}} + (W_\nu, W_\nu)_{0,s-\frac{1}{2}},$$

where the tilde indicates the pseudo–difference operator corresponding to the given symbol. For solutions of (2.7) this gives

$$\|W\|^2_{0,s} \leq C(-(W_0, \tilde{H}W_0)_{0,s-\frac{1}{2}} + \varepsilon\|W\|^2_{0,s} + \frac{1}{\varepsilon}\|\mathcal{F}\|^2_{0,s-1} + \|W\|^2_{0,s-\frac{1}{2}}).$$

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Equation (2.10) and Gårding's inequality again imply

$$\|W\|_{0,s}^2 + |W_0|_{0,s-\frac{1}{2}}^2 \le C(|\mathcal{B}W_0|_{0,s-\frac{1}{2}}^2 + \varepsilon\|W\|_{0,s}^2 + \frac{1}{\varepsilon}\|\mathcal{F}\|_{0,s-1}^2 + \|W\|_{0,s-\frac{1}{2}}^2 + |W_0|_{0,s-1}^2).$$

Taking ε sufficiently small we obtain

$$\|W\|_{0,s}^2 + |W_0|_{0,s-\frac{1}{2}}^2 \le C(|\mathcal{B}W_0|_{0,s-\frac{1}{2}}^2 + \|\mathcal{F}\|_{0,s-1}^2 + \|W\|_{0,s-\frac{1}{2}}^2 + |W_0|_{0,s-1}^2).$$

Also, for $\varepsilon > 0$ there is a $c_{s,\varepsilon}$ such that

$$\|W\|_{0,s-\frac{1}{2}}^2 \le \varepsilon\|W\|_{0,s}^2 + c_{s,\varepsilon}\|W\|_{0,0}^2,$$

and similarly for the boundary norm, e.g. Thomée and Westergren [11], or Bube and Strikwerda [3]. Thus we obtain

$$\|W\|_{0,s}^2 + |W_0|_{0,s-\frac{1}{2}}^2 \le C(|\mathcal{B}W_0|_{0,s-\frac{1}{2}}^2 + \|\mathcal{F}\|_{0,s-1}^2 + \|W\|_{0,0}^2 + |W_0|_{0,-\frac{1}{2}}^2). \qquad (6.3)$$

The estimate (6.2) for the original dependent variables is obtained from (6.3) by replacing the $W_{j,a}$ by the equivalent expression in terms of the u_j. The norms and scaling of the boundary data ϕ_1 and ϕ_2 in (6.2) are the result of the modifications to obtain the one–step scheme as described at the beginning of Section 4.

Estimates on the Normal Differences

We now consider estimates on the normal finite differences, and begin by stating interpolation results for the normal differences.

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Lemma 6.1

For integers t, s with $t \leq s$ and $\varepsilon > 0$, there exists a constant C_ε such that for any discrete function $v(x_\nu, y_\mu)$

$$|\delta_+^s v|_{0,r}^2 \leq \varepsilon \|\delta_+^{s+1} v\|_{0,r-\frac{1}{2}}^2 + C_\varepsilon \|\delta_+^t v\|_{0,r+\frac{1}{2}}^2$$

and

$$\|\delta_+^s v\|_{0,r} \leq \varepsilon \|\delta_+^{s+1} v\|_{0,r-1} + C_\varepsilon \|\delta_+^t v\|_{0,r+1}.$$

Proof:

By summation by parts,

$$\left(\Lambda^r \delta_+^s v(x_0, \omega)\right)^2 = -h\Lambda^r \sum_{\nu=0}^{\infty} \delta_+ \left(\delta_+^s v(x_\nu, \omega)\right)^2$$

$$= -h \sum_{\nu=0}^{\infty} \left(\Lambda^{\frac{1}{2}(r-\frac{1}{2})} \delta_+^{s+1} v(x_\nu, \omega)\right) \Lambda^{\frac{1}{2}(r+\frac{1}{2})} \left(\delta_+^s v(x_{\nu+1}, \omega) + \delta_+^s v(x_\nu, \omega)\right).$$

Hence

$$|\delta_+^s v|_{0,r}^2 \leq \varepsilon \|\delta_+^{s+1} v\|_{0,r-\frac{1}{2}}^2 + C_\varepsilon \|\delta_+^s v\|_{0,r+\frac{1}{2}}^2.$$

Then by using an interpolation inequality (e.g Thomée and Westergren [11]) the first inequality of the lemma is easily proven. The second estimate is proved in a similar manner; the proof will be omitted.

For the case $\mu = 0$ the restriction on the differences in x in Theorem 6.2 is removed in precisely the same way as it is for the partial differential equation. From equation (4.6) we have that $\delta_+^{r_j + \bar{p}} u_j(x_\nu, \omega)$ is equal to a sum of lower order differences of u_j and $\tilde{F}_j''(x_\nu, \omega)$.

We easily obtain

$$\|\delta_+^{\tau+\bar{p}} u\|_{0,s-\tau-\bar{p}+1} \leq C \big(\|u\|_{\tau+\bar{p}-1,\tau+s+1} + \|F\|_{s+1-\sigma} \big). \tag{6.4}$$

The use of the interpolation estimates in Lemma 6.1 can then be used on the norms of u on the right–hand side of (6.2) to give (6.1) for $s = \bar{p}$ in the case that $\mu = 0$, which implies that $q = p$. By operating on equation (4.6) with $\delta_+^{s-\bar{p}}$ and using an estimate similar to (6.4) along with Lemma 6.1 we obtain (6.1).

The case $\mu \neq 0$ is more difficult because one must solve for $\delta_+^{\tau_j+\bar{p}} u_j(x_\nu, \omega)$ from equation (4.6). We now show how this is to be done. It essentially involves solving an elliptic system of order $(0,0)$ for the $\delta_+^{\tau_j+\bar{p}} u_j(x_\nu, \omega)$.

Consider the system of equations (4.4) and write it as

$$G(\omega, T)\delta_+^{\tau+\bar{p}} u(x_\nu, \omega) + \tilde{K}(\omega)u(x_\nu, \omega) = \tilde{F}'(x_\nu, \omega),$$

where $\delta_+^{\tau+\bar{p}} u(x_\nu, \omega)$ denotes the vector with components $\delta_+^{\tau_j+\bar{p}} u_j(x_\nu, \omega)$, $j = 1, ..., n$, and $G(\omega, T)$ is the matrix of translation operators $g_{ij}(\omega, T)$. The term $\tilde{K}(\omega)u(x_\nu, \omega)$ contains all the differences of the $u_j(x_\nu, \omega)$ with respect to x of order less than $\tau_j + \bar{p}$. The operator G is an elliptic operator of type $(0,0)$ by the Resolvent Condition.

As in the proof of Lemma 4.1 the matrix $G(\omega, T)$ is upper triangular and without loss of generality we can assume that the rows are ordered so that the degree of $g_{i,i}(\omega, T)$ is greater than the degree of $g_{j,j}(\omega, T)$ if $i < j$, i.e. $\mu_i \geq \mu_j$ if $i < j$. Let n' be the integer such that $\mu_i = 0$ for i greater than n', with $n' = n$ if $\mu_n > 0$. For $i > n'$, $\delta_+^s u_i$ can be expressed in terms of lower order differences of the u_j as in the case $\mu = 0$.

As in section 4 we construct a one-step scheme for $\mathcal{W}_\nu$, the vector whose components are $T^l \delta_+^{\tau_j + \bar{p}} u_j(x_\nu, \omega)$, for $0 \le l \le \mu_j - 1$ and $1 \le j \le n'$. This scheme can be written as

$$\mathcal{W}_{\nu+1} = \mathcal{M}\mathcal{W}_\nu + \mathcal{G}_\nu \tag{6.5}$$

where $\mathcal{G}_\nu$ contains both the function $\tilde{F}'$ and difference of u_j with respect to x of order less than $\tau_j + \bar{p}$. The boundary conditions for this scheme can be written as

$$\mathcal{C}\mathcal{W}_0 = \Phi \tag{6.6}$$

where Φ contains both the data Φ_2 from (4.11) and differences in $u_j(x_0, \omega)$ of order less than $\tau + \bar{p}$, and $\mathcal{C}$ has been obtained from (3.2) by scaling with $h^{p-\bar{p}}$ analogous to the scaling at the beginning of section 4.

The matrix $\mathcal{M}$ is of order $|\mu|$ and it is easily shown that for ω near 0 its eigenvalues are those of class 3 as given in Theorem 5.3. Thus we can construct a symmetrizer $\mathcal{H}$ such that

$$\mathcal{M}^*\mathcal{H}\mathcal{M} - \mathcal{H} \ge c_0 \tag{6.7}$$

as in section 5. By the nonexistence of eigenvalues of type III, there is a neighborhood for ω near 0, such that

$$\mathcal{H} + c_1 \mathcal{C}^* \mathcal{C} \ge c_2 \tag{6.8}$$

for some positive constants c_1 and c_2. Note that if $q = p$ then $\mathcal{C}$ is taken to be zero, and $\mathcal{H}$ is a positive definite matrix. The relation $q = p$ results from all the eigenvalues of $\mathcal{M}$ being larger than 1 in magnitude. The inequalities (6.7) and (6.8) are analogous to (2.9) and (2.10).

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We now extend the estimate (6.2) to include normal differences of $\tau + \bar{p}$. Since (6.8) only holds for small values of ω we employ a cut–off function $\psi(\omega)$ with $\psi(\omega) = 0$ for $\delta_0 \leq |\omega|_\infty \leq \pi$, $\psi(\omega) = 1$ for $|\omega|_\infty \leq \delta_0/2$ and extend $\psi(\omega)$ periodically for all $\omega \in \mathbf{R}^{d-1}$. Let $\mathcal{V}_\nu$ be $\psi(\omega)\mathcal{W}_\nu$. Analogous to the proof of Theorem 6.2, but using the Gårding inequality proved by Lax and Nirenberg [7], we have

$$\|\mathcal{V}\|_{0,r}^2 \leq Ch \sum_{\nu=0}^{\infty} \{((\tilde{M}^*\tilde{\mathcal{H}}\tilde{M} - \tilde{\mathcal{H}})\mathcal{V}_\nu, \mathcal{V}_\nu)_{0,r} + ch(\mathcal{V}_\nu, \mathcal{V}_\nu)_{0,r}\},$$

$$\leq C\{-h(\mathcal{V}_0, \mathcal{H}\mathcal{V}_0)_{0,r} + \varepsilon\|\mathcal{V}\|_{0,r}^2 + C_\varepsilon\|\mathcal{G}\|_{0,r}^2 + ch\|\mathcal{V}\|_{0,r}^2\}. \tag{6.9}$$

By (6.8) and the definition of $\mathcal{C}$ and $\mathcal{G}$ we have for small h

$$\|\mathcal{V}\|_{0,r}^2 + h|\mathcal{V}_0|_{0,r}^2 \leq C\{h|\mathcal{C}\mathcal{V}_0|_{0,r}^2 + \|\mathcal{G}\|_{0,r}^2\} \tag{6.10}$$

$$\leq C\{h|u|_{\tau+\bar{p}-1,r+1}^2 + |h^{\rho-\bar{p}+\frac{1}{2}}\phi_2|_r^2 + \|F\|_{\bar{p}+r-\sigma}^2 + \|u\|_{\tau+\bar{p}-1,r+1}^2\}.$$

Since $h\Lambda$ is bounded, we have

$$h^{\frac{1}{2}}|u|_{\tau+\bar{p}-1,r+1} \leq c|u|_{\tau+\bar{p}-1,r+\frac{1}{2}}.$$

For $|\omega|_\infty \geq \delta_0/2$ we use equation (2.7) as follows. We have

$$\frac{1}{h}(W(x_{\nu+1},\omega) - W(x_\nu,\omega)) = \frac{1}{h}(M(\omega) - I)W(x_\nu,\omega) + \mathcal{F}(x_\nu,\omega),$$

and estimating only the differences of the components $W_{i,\tau_i+\bar{p}-1}$ of W we obtain

$$|\delta_+^{\tau+\bar{p}}u(x_\nu,\omega)|^2 \leq \frac{c}{h^2}|W(x_\nu,\omega)|^2 + |\mathcal{F}(x_\nu,\omega)|^2 \leq C\Lambda_0^2|W(x_\nu,\omega)|^2 + |\mathcal{F}(x_\nu,\omega)|^2, \tag{6.11}$$

since $h^{-1} \leq C(\delta_0)\Lambda_0$ for $|\omega|_\infty \geq \delta_0/2$.

Combining the estimates (6.10) and (6.11) we obtain

$$\sum_{j=1}^{n} \|\delta_+^{\tau_j+\bar{\rho}} u_j\|_r^2 \leq \sum_{j=1}^{n} \|\psi * \delta_+^{\tau_j+\bar{\rho}} u_j\|_r^2 + \sum_{j=1}^{n} \|(1 - \psi) * \delta_+^{\tau_j+\bar{\rho}} u_j\|_r^2$$

$$\leq C\left\{|h^{\rho-\bar{\rho}+\frac{1}{2}}\phi_2|_r^2 + \|F\|_{\bar{\rho}+r-\sigma}^2 + \|u\|_{\tau+\bar{\rho}-1,\tau+\bar{\rho}+r}^2 + h|u|_{\tau+\bar{\rho}-1,\tau+\bar{\rho}+r}^2\right\}.$$

Then, using estimate (6.2) and Lemma 6.1, we obtain

$$\|u\|_{\tau+\bar{\rho},\tau+s}^2 + |u|_{\tau+\bar{\rho}-1,\tau+s-\frac{1}{2}}^2 \leq C\left\{|\phi_1|_{s-\rho-\frac{1}{2}}^2 + |h^{\rho-\bar{\rho}+\frac{1}{2}}\phi_2|_{s-\bar{\rho}}^2 + \|F\|_{s-\sigma}^2 + \|u\|_0^2\right\}, \quad (6.12)$$

which proves (6.1) for $\bar{\rho} \leq s < \bar{\rho} + \frac{1}{2}$.

To prove (6.1) for larger values of s we obtain a sharper estimate than (6.10) for the boundary terms, i.e. $|\mathcal{V}_0|_{0,r}$, as follows. Since the eigenvalues of $\mathcal{M}$ are bounded away from 1 independent of h, there is a number β with $0 < \beta < 1$ such that for any eigenvalue λ of $\mathcal{M}$

$$c \leq ||\beta\lambda|^2 - 1|,$$

analogous to the estimate for the third class of eigenvalues of M in Theorem 5.3, and moreover, $|\beta\lambda| > 1$ if $|\lambda| > 1$. Thus the matrix $\mathcal{H}$ can be constructed to satisfy

$$c_0 \leq \beta\mathcal{M}^*\mathcal{H}\mathcal{M}\beta - \mathcal{H}$$

analogous to (6.7). The variables $\beta^\nu \mathcal{V}_\nu$ satisfy

$$\beta^{\nu+1}\mathcal{W}_{\nu+1} = (\beta\mathcal{M})\beta^\nu\mathcal{W}_\nu + \beta^{\nu+1}\mathcal{G}_\nu$$

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by (6.5). Similar to (6.10) we have

$$\|\beta \mathcal{V}\|_{0,r}^2 + h|\mathcal{V}_0|_{0,r}^2 \le C\{h|\mathcal{C}\mathcal{V}_0|_{0,r}^2 + \|\beta \mathcal{G}\|_{0,r}^2\},\tag{6.13}$$

and we estimate the last term as follows. We have

$$\|\beta \mathcal{G}\|_{0,r}^2 = h \sum_{\nu=0}^{\infty} \beta^{2\nu}|\mathcal{G}|_{0,r}^2 \qquad = \frac{h}{1-\beta^2}\sum_{\nu=0}^{\infty}(\beta^{2\nu} - \beta^{2(\nu+1)}|\mathcal{G}_\nu|_{0,r}^2$$

$$= \frac{h^2}{1-\beta^2}\sum_{\nu=0}^{\infty}(\delta_+\beta^{2\nu})|\mathcal{G}_\nu|_{0,r}^2 = -\frac{h^2}{1-\beta^2}\left(|\mathcal{G}_0|^2 + \beta^2\sum_{\nu=0}^{\infty}\beta^{2\nu}\delta_+|\mathcal{G}_\nu|_{0,r}^2\right)$$

$$\le \frac{h^2\beta^2}{1-\beta^2}\sum_{\nu=0}^{\infty}|\delta_+|\mathcal{G}_\nu|_{0,r}^2| \qquad \le Ch\|\mathcal{G}\|_{0,r+\frac{1}{2}} \|\delta_+\mathcal{G}\|_{0,r-\frac{1}{2}}.$$

Thus we have from (6.13)

$$|\mathcal{V}_0|_{0,r}^2 \le C\left(|\mathcal{C}\mathcal{V}_0|_{0,r}^2 + \varepsilon\|\delta_+\mathcal{G}\|_{0,r-\frac{1}{2}}^2 + C_\varepsilon\|\mathcal{G}\|_{0,r+\frac{1}{2}}^2\right)$$

and thus, with $r = s - \frac{1}{2}$,

$$|u|_{\tau+\bar\rho,\tau+s-\frac{1}{2}}^2 \le C\left(|u|_{\tau+\bar\rho-1,\tau+s-\frac{1}{2}}^2 + \varepsilon\|u\|_{\tau+\bar\rho,\tau+s}^2 + C_\varepsilon\|u\|_{\tau+\bar\rho-1,\tau+s}^2\right.$$

$$\left. + |h^{\rho-\bar\rho}\phi_2|_{s-\bar\rho-\frac{1}{2}}^2 + |\phi_1|_{s-\rho-\frac{1}{2}}^2 + \|F\|_{s-\sigma}^2\right).$$

This inequality with (6.10) and (6.12) gives

$$\|u\|_{\tau+\bar\rho,\tau+s}^2 + |u|_{\tau+\bar\rho,\tau+s-\frac{1}{2}}^2 \le C\big(|\phi_1|_{s-\rho-\frac{1}{2}}^2 + |h^{\rho-\bar\rho}\phi_2|_{s-\bar\rho-\frac{1}{2}}^2$$

$$+ \|F\|_{s-\sigma}^2 + \|u\|_0^2\big),\tag{6.14}$$

which proves (6.1) for $\bar{p} \leq s < \bar{p} + 1$.

Estimates for higher normal differences are obtained in the following way. The equation for $\delta_+^r \mathcal{W}_\nu$ is

$$\delta_+^r \mathcal{W}_{\nu+1} = \mathcal{M} \delta_+^r \mathcal{W}_\nu + \delta_+^r \mathcal{G}_\nu, \tag{6.15}$$

where for simplicity we have assumed that $\mathcal{M}$ is independent of x. If $\mathcal{M}$ is not independent of x then the right-hand side of (6.15) would contain lower order differences of $\mathcal{W}$, which would not effect the final estimate, but would complicate the following formulas.

For integer values of r with $0 < r < \rho^* - \bar{p}$ we have boundary conditions

$$C^{(r)} \delta_+^r \mathcal{W}_0 = \Phi^{(r)} \tag{6.16}$$

analogous to (6.6) where now $C^{(r)}$ is scaled by $h^{\rho - \bar{p} - r}$. In the same way that equations (6.12) and (6.14) were obtained we obtain

$$\|u\|_{r+\bar{p}+r, r+s}^2 + |u|_{r+\bar{p}+r-1, r+s-\frac{1}{2}}^2 \leq C(|\phi_1|_{s-\rho-\frac{1}{2}}^2 + |h^{\rho-\bar{p}-r+\frac{1}{2}} \phi_2|_{s-\bar{p}-r}^2$$
$$\tag{6.17}$$
$$+ \|F\|_{s-\sigma}^2 + \|u\|_0^2),$$

and

$$\|u\|_{r+\bar{p}+r, r+s}^2 + |u|_{r+\bar{p}+r, r+s-\frac{1}{2}}^2 \leq C(|\phi_1|_{s-\rho-\frac{1}{2}}^2 + |h^{\rho-\bar{p}-r} \phi_2|_{s-\bar{p}-r-\frac{1}{2}}^2$$
$$\tag{6.18}$$
$$+ \|F\|_{s-\sigma}^2 + \|u\|_0^2).$$

These estimates, with $r + \bar{p} \leq s < r + \bar{p} + \frac{1}{2}$ and $r + \bar{p} + \frac{1}{2} \leq s < r + \bar{p} + 1$, respectively, prove Theorem 6.1.

We now consider estimates on differences of order ρ^* or higher in the case that $q > p$.

Theorem 6.3

If the elliptic difference scheme (2.1) with boundary conditions (2.2) satisfies the Complementing Condition and Assumptions 2.1, 2.2, and 2.3, then the following estimate holds for $s \geq \rho^*$

$$\|u\|_{r+s}^2 + |u|_{r+s-\frac{1}{2}}^2 \leq Ch^{-2(s-\rho^*+\frac{1}{2})}(|\phi_1|_{s-\rho-\frac{1}{2}}^2 + |h^{\rho-s-\frac{1}{2}}\phi_2|_0^2 + \|F\|_{s-\sigma}^2 + \|u\|_0^2). \quad (6.19)$$

Proof:

As in equation (6.9) we obtain, for h sufficiently small

$$\|\delta_+^r \mathcal{W}\|_{0,r}^2 \leq C\{-h(\delta_+^r \mathcal{W}_0, \mathcal{H}\delta_+^r \mathcal{W}_0) + \|\delta_+^r \mathcal{G}\|_{0,r}^2\}. \quad (6.20)$$

Let $\rho' = \rho^* - \bar{\rho} - 1$, then we then use equation (6.5) to obtain

$$\delta_+^r \mathcal{W}_0 = \left(h^{-1}(\mathcal{M} - I)\right)^{r-\rho'}\delta_+^{\rho'}\mathcal{W}_0 + h^{-(r-\rho')}\sum_{\nu=0}^{r-\rho'-1} p_\nu(\mathcal{M})\delta_+^\nu \mathcal{G}_0$$

where $p_\nu(\mathcal{M})$ is a polynomial in $\mathcal{M}$. We then have, using Lemma 6.1 on $\mathcal{G}$,

$$h|\delta_+^r \mathcal{W}_0|^2 \leq Ch^{-2(r-\rho'-\frac{1}{2})}\left(|\delta_+^{\rho'}\mathcal{W}_0|^2 + \sum_{\nu=0}^{r-\rho'-1}|\delta_+^\nu \mathcal{G}_0|^2\right)$$

$$\leq Ch^{-2(r-\rho'-\frac{1}{2})}\left(|\delta_+^\nu \mathcal{W}_0|^2 + \|\mathcal{G}\|_{r-\rho'}^2\right),$$

$$\leq Ch^{-2(r-\rho'-\frac{1}{2})}\left(|u|_{r+\rho^*-1}^2 + \|u\|_{r+\rho^*-1,r+\rho^*+r}^2\right),$$

from which the estimate (6.19) follows easily from (6.20) with $s = r + \bar{\rho}$.

To complete the proof of Theorem 3.1, we state

Theorem 6.4

If $u(x_\nu, y_\mu)$ is a solution to the system (2.1) with boundary conditions (2.2) and Assumptions 2.1, 2.2, and 2.3, are satisfied, then the regularity estimate (6.1) holds for each s, with $\bar{\rho} \le s < \rho^*$ and h sufficiently small, only if the Complementing Condition holds.

Proof:

Assume the Complementing Condition does not hold. If there is an eigensolution of type I or type III, then as in the proof of Theorem 5.5 we can construct a solution of (2.1) and (2.2) with homogeneous data for any $h > 0$. Since the eigenvalues, $\kappa_b(\omega)$, are bounded away from 1, the norms $\|u\|_{\tau+s}$ and $|u|_{\tau+s-\frac{1}{2}}$ will be $O(h^{-s})$ as h tends to zero even though $\|u\|_0$ is $O(1)$. Thus the estimates (6.1) and (3.3) can not hold.

If there is an eigensolution of type II, then the regularity estimate analogous to (3.3) fails to hold for the solutions of the associated system of differential equations, (Agmon et al. [1]). The eigensolution is a solution to the difference equations with inhomogeneous data, i.e. the "truncation error", which tends to zero as h tends to zero. It is easily seen that the estimate (3.3) can not hold for sufficiently small h. This proves Theorem 6.4.

7. General Domains and Lower Order Terms

We now discuss the modifications required to handle the cases when we have domains which are not a half–space or the equations have lower order terms. For a bounded domain Ω with a smooth boundary we assume the grid is boundary–fitted. This means that for each

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boundary point there is a neighborhood which can be smoothly mapped onto a portion of a half–space with the grid being mapped onto an orthogonal grid on the half–space. Through the use of such mappings and a partition of unity one can obtain regularity estimates up to the boundary on the boundary–fitted grid on Ω.

Lower order terms cause no problems unless their extent is greater than the extent of the highest order portion of the system. If their extent is greater, they may require additional boundary conditions and this could adversely affect the regularity at the boundary. In most problems of interest lower order terms would have an extent no larger than that of the highest order terms and the regularity estimates would hold true in the same form as (3.3). This is proved in exactly the same fashion as for systems of differential equations. That is, the lower order terms can be considered as part of a right–hand side of (2.1) and then the estimate (3.3) with this modified data follows. Then using the interpolation estimates in Lemma 6.1 the estimate with the original data follows.

8. Summary of Results and Examples

In this section we summarize the results of this paper and apply the theory to several examples. These examples are chosen to illustrate the theory; the boundary conditions we consider are not representative of those used in practice. Unfortunately, more realistic boundary conditions lead to a great deal of algebraic manipulation. To apply Theorem 3.1 to determine the regularity of a boundary value problem for a regular elliptic system of difference equations on a boundary–fitted coordinate system one must only consider the "frozen coefficient problem" for the system at each boundary grid point. The frozen coefficient problem at a point on the boundary is the constant coefficient problem obtained

by fixing the coefficients of both the system and the boundary conditions at their values at that boundary point. This frozen coefficient problem is considered on the half-space determined by the tangent space to the boundary and the inward unit normal at the point. If the frozen coefficient problem is regular for each point on the boundary then the original variable coefficient problem is regular. Thus we need only consider constant coefficient boundary value problems on a half-space.

The steps one must take to check the regularity of the boundary value problem are as follows. The n-tuples σ and τ defining the order of the elliptic system of difference equations must be determined and lower order terms can then be neglected. The regularity of the scheme must also be checked (Bube and Strikwerda [3]). The reduced equation, obtained by Fourier transforming in the tangential variables, must satisfy the resolvent condition of Section 2 and must be adjusted so that Assumption 2.1 is satisfied. With the reduced equation in this form the number of boundary conditions can be determined (Assumption 2.1) and they should be ordered so that Assumption 2.2 is satisfied. Assumption 2.3 will be satisfied for most systems arising in practical application. The final step is to check for eigensolutions of types I, II and III. Theorem 3.1 states that regularity up to the boundary is equivalent to the nonexistence of eigensolutions.

The regularity estimate (3.3) shows that for those schemes which require as many boundary conditions as does the associated differential equation the solution and its finite differences are bounded independently of the grid spacing. For those schemes which require more boundary conditions, i.e. numerical boundary conditions, the estimate (3.3) shows that these should be of high order to achieve smooth solutions. These observations seem

to justify the use of compact difference schemes wherever possible. (A compact scheme is one which has the smallest extent possible, for a given accuracy.) For those schemes that do require numerical boundary conditions, Theorem 3.2 shows that these extra conditions should be of sufficiently high order so as not to affect the accuracy of the solution and its finite differences near the boundary. The interior regularity estimates of Bramble and Hubbard [2] and others show that for second–order elliptic equations the finite differences of the solution are approximations of the corresponding derivatives with the same order of accuracy as the solution itself away from the boundary. Theorem 3.2 shows that this can also be true up to the boundary under certain circumstances. In particular, it can be true if no numerical boundary conditions are required and the boundary conditions are of the same order of accuracy as the scheme.

To illustrate the theory consider several examples. We begin with the Cauchy–Riemann equations

$$u_x - v_y = f_1$$

$$u_y + v_x = f_2,$$

on the half–space $\{(x,y) : x \geq 0, \quad y \in \mathbf{R}\}$ with boundary condition

$$u(0,y) = g(y).$$

Define $a(T) := (\frac{-1}{2h})(T^2 - 4T + 3)$ and $\tilde{a}(T) := -a(T^{-1})$. We approximate the elliptic system with the second order accurate scheme given by

$$\begin{pmatrix} a(T_x) & -a(T_y) \\ \tilde{a}(T_y) & \tilde{a}(T_x) \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (x_\nu, y_\mu) = F(x_\nu, y_\mu),, \qquad \mu \in \mathbf{Z}. \tag{8.1}$$

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One can easily check that this is a regular elliptic system of difference equations. Let $\varsigma(\omega) := a(e^{i\omega})$ be the symbol of a. Fourier transforming with respect to the tangential variable yields the reduced system

$$\begin{pmatrix} a(T_x) & -\varsigma(\omega) \\ -\bar{\varsigma}(\omega) & \tilde{a}(T_x) \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (x_\nu, \omega) = \tilde{F}(x_\nu, \omega),$$

$$\nu \geq 2, \quad \omega := h\xi.$$

To determine how many boundary conditions are needed by the system (8.1), we must consider the eigenvalues of the resolvent equation, (see Assumption 2.1), given by

$$\det \begin{pmatrix} a(z) & -\varsigma(\omega) \\ -\bar{\varsigma}(\omega) & \tilde{a}(z) \end{pmatrix} = a(z)\tilde{a}(z) - |\varsigma|^2 = 0. \tag{8.2}$$

If $\omega \neq 0$, Lemma 2.1 implies that none of the four roots of this equation is on the unit circle. Since $a(z) = -\tilde{a}(z^{-1})$ we conclude that there are exactly two roots inside the unit circle and two outside. Denote the two which are inside by z_1 and z_2. Because there are two roots inside the unit circle, it is necessary to specify two boundary conditions in order to have a well–posed system of difference equations. The conditions which we impose are

$$a) \quad \frac{\varepsilon}{2}(\tilde{u}_0 + \tilde{u}_1) + (1 - \varepsilon)\tilde{u}_0 = \tilde{g}_0,$$

$$b) \quad h^{-r}(T - 1)^r \tilde{v}_0 = 0, \tag{8.3}$$

for ε real and r a positive integer. The first boundary condition is the operator B_1 corresponding to specifying u and the second is B_2; we have $\rho_1 = -1$, $\rho_2 = r - 1$.

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Now we consider the resolvent condition. One could take $\alpha_1^+ = \alpha_2^+ = \beta_2^+ = 0, \beta_1^+ = 2, \alpha_1^- = \alpha_2^- = \beta_1^- = 0, \beta_2^- = -2$, or one could take $\alpha_1^+ = 2, \alpha_2^+ = \beta_1^+ = \beta_2^+ = 0, \alpha_1^- = \beta_1^- = \beta_2^- = 0, \alpha_2^- = -2$. Following the procedure for reducing to a canonical form, i.e., all differences forward, we obtain two equivalent reduced systems which are to be analyzed for eigensolutions. We will work with the one which comes from the second set of α and β. This system is

$$\begin{pmatrix} a(T) & -\varsigma(\theta) \\ -\bar{\varsigma}(\theta)T^2 & \tilde{a}(T)T^2 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (x_\nu, \omega) = \begin{pmatrix} \tilde{f}_1 \\ T^2 \tilde{f}_2 \end{pmatrix} (x_\nu, \omega),$$

$$\nu \geq 0, \quad \theta := h\omega.$$

If we would have chosen the first possibility then the two variables $\tilde{u}_0$ and $\tilde{u}_1$ would have been declared superfluous. We could then eliminate these by applying the first equation at $\nu = 0$ and $\nu = 1$.

The general decaying solution to the resolvent equation is

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (x_\nu, \omega) = c_1 \begin{pmatrix} \varsigma \\ a(z_1) \end{pmatrix} z_1^\nu + c_2 \begin{pmatrix} \varsigma \\ a(z_2) \end{pmatrix} z_2^\nu. \tag{8.4}$$

First we check for eigensolutions of type III. Since one of the roots goes to 1 as $\omega \to 0$ the general decaying solution at $\omega = 0$ becomes

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (x_\nu, \omega) = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} (\frac{1}{3})^\nu, \quad \nu \geq 0.$$

This shows that using any extrapolation with $r \geq 1$ for a numerical boundary condition gives no eigensolutions of type III. Note that if we would have taken $(3T-1)^{r_1}(T-1)^{r_2}\tilde{v}_0 =$

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0 for a numerical boundary condition instead of (8.3b) then there would be eigensolutions of type III. For example, take $r_1 = 1$ and $r_2 = 2$, then this condition becomes $\tilde{v}_0 = 3\tilde{v}_3 - 7\tilde{v}_2 + 5\tilde{v}_1$ and this boundary condition would not be regular.

Since (8.1) with the indicated boundary condition is regular, there are no eigensolutions of type II, so it only remains to check for the existence of type I eigensolutions. Applying the boundary conditions to (8.4), we obtain the condition for the existence of a nontrivial solution as

$$\det \begin{pmatrix} \varepsilon(z_1 + 1)/2 + (1 - \varepsilon) & \varepsilon(z_2 + 1)/2 + (1 - \varepsilon) \\ (z_1 - 1)^r a(z_1) & (z_2 - 1)^r a(z_2) \end{pmatrix} = 0. \qquad (8.5)$$

Using the two equations $a(z_j)\tilde{a}(z_j) - |\varsigma|^2 = 0$, for $j = 1, 2$, we obtain the equation

$$a(z_1)\tilde{a}(z_1) - a(z_2)\tilde{a}(z_2) = 0. \qquad (8.6)$$

We now have to determine whether there are any roots z_1, z_2 of equations (8.5) and (8.6) which are both of modulus less than unity. Using the symbolic manipulation language MACSYMA [8] to determine the solutions to this equation, we observe that there are no solution with z_1 and z_2 both less than or equal to 1 in magnitude. Thus we conclude that there are no eigensolutions of type I for the cases $\varepsilon \in \{0, \frac{1}{2}, 1\}, r \in \{1, 2, 3\}$. Therefore the system is regular up to the boundary for these values of ε and r.

As the next example, consider the Stokes equations

$$\Delta u \quad + p_x = f_1$$

$$\Delta v + p_y = f_2$$

$$u_x + v_y \quad = f_3,$$

on the half–space $\{(x,y) : x \geq 0, y \in \mathbf{R}\}$ with boundary conditions $u(0,y) = g_1(y)$ and $v(0,y) = g_2(y)$. The scheme we consider is

$$\begin{pmatrix} (\delta_{x+}\delta_{x-} + \delta_{y+}\delta_{y-}) & O & \delta_{x-} \\ O & (\delta_{x+}\delta_{x-} + \delta_{y+}\delta_{y-}) & \delta_{y-} \\ \delta_{x+} & \delta_{y+} & O \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} (x_\nu, y_\mu) = F(x_\nu, y_\mu),$$

$$\nu \geq 1, \quad \mu \in \mathbf{Z}.$$

One can easily check that this is a regular elliptic system of difference equations. Let $b(T) := (T + T^{-1} - 2)/h^2$ and $\varsigma(\phi) := (e^{i\phi} - 1)/h$. After Fourier transforming with respect to the tangential variable, we obtain the reduced system

$$\begin{pmatrix} b(T_x) - |\varsigma|^2 & O & (1 - T_x^{-1})/h \\ O & b(T_x) - |\varsigma|^2 & -\bar{\varsigma} \\ (T_x - 1)/h & \varsigma & O \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{p} \end{pmatrix} (x_\nu, \omega) = \tilde{F}(x_\nu, \omega), \qquad (8.7)$$

$$\nu \geq 1.$$

The determinant of the resolvent equation is

$$\det \begin{pmatrix} b(z) - |\varsigma|^2 & O & (1 - z^{-1})/h \\ O & b(z) - |\varsigma|^2 & -\bar{\varsigma} \\ (z - 1)/h & \varsigma & O \end{pmatrix} = 0,$$

and the eigenvalues are the roots of this equation,

$$z_\pm = (2 + h^2|\varsigma|^2 \pm \sqrt{(2 + h^2|\varsigma|^2)^2 - 4})/2, \tag{8.8}$$

where each root is double. We conclude that there must be two boundary conditions since z_- is a double root with modulus less than unity (Assumption 2.1). Consider the following boundary conditions:

$$a) \quad \frac{\varepsilon}{2}(\tilde{u}_0 + \tilde{u}_1) + (1 - \varepsilon)\tilde{u}_0 = \tilde{g}_1,$$

$$\tag{8.9}$$

$$b) \quad \frac{\delta}{2}(\tilde{v}_0 + \tilde{v}_1) + (1 - \delta)\tilde{v}_0 = \tilde{g}_2.$$

Note that, as $\omega \to 0$, $z_\pm \to 1$ so we have no type III eigensolutions. The general decaying solution to the homogeneous difference equation (8.7) is

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{p} \end{pmatrix} (x_\nu, \omega) = c_1 \begin{pmatrix} -\varsigma \\ (z_- - 1)/h \\ 0 \end{pmatrix} z_-^\nu + c_2 \begin{pmatrix} -\nu\varsigma \\ ((\nu + 1)z_- - \nu)/h \\ 0 \end{pmatrix} z_-^{\nu-1}. \tag{8.10}$$

Applying the boundary conditions (8.9), the condition for a nontrivial solution is

$$\det \begin{pmatrix} \varepsilon(z_- + 1)/2 + (1 - \varepsilon) & \frac{\varepsilon}{2} \\ (\delta(z_- + 1)/2 + (1 - \delta))(z_- - 1) & \delta z_- + (1 - \delta) \end{pmatrix} = 0.$$

After simplification this equation reduces to

$$\varepsilon\delta z_-^2 + 2\delta(2 - \varepsilon)z_- + 4(1 - \delta) + \varepsilon\delta = 0,$$

and since $z_-^2 = (2 + h^2|\varsigma|^2)z_- - 1$ we obtain

$$z_- = -\frac{2(1-\delta)/\delta}{2 - \varepsilon + \varepsilon(2 + h^2|\varsigma|^2)}.$$

Noting that z_-, as given by (8.8), is strictly positive, we conclude that this equation is not satisfied if $0 \le \varepsilon \le 1$ and $0 < \delta \le 1$. When $\delta = 0$ it is easy to see that the determinant never vanishes. Therefore there are no eigensolutions of type I. This implies regularity up to the boundary for this system of difference equations. If $\varepsilon = 0$ and δ is taken larger than $(2(\sqrt{2} - 1))^{-1}$ then there exist type I eigensolutions, so the difference equations are not regular up to the boundary in this case.

Finally, consider the same difference equations but with the boundary conditions for the differential system given by

$$a) \quad u_x(0,y) = 0,$$

$$b) \quad v(0,y) = g(y).$$

Approximate these equations by

$$a) \quad \varepsilon(u(x_1,y_\mu) - u(x_0,y_\mu)) + (1-\varepsilon)(u(x_2,y_\mu) - u(x_0,y_\mu)) = 0,$$

$$b) \quad \frac{\delta}{2}(v(x_0,y_\mu) + v(x_1,y_\mu)) + (1-\delta)v(x_0,y_\mu) = g(x_0,y_\mu).$$

(8.11)

After applying these boundary conditions to the solution (8.10) we obtain

$$\det \begin{pmatrix} \varepsilon(z_- - 1) + (1-\varepsilon)(z_-^2 - 1) & \varepsilon + 2(1-\varepsilon)z_- \\ (\delta(z_- + 1)/2 + (1-\delta))(z_- - 1) & \delta z_- + (1-\delta) \end{pmatrix} = 0.$$

(8.12)

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Taking $\varepsilon = 0$ and $\delta = 0$ gives no eigensolutions of type I. However, if $\varepsilon = 1$ and $\delta = 0$ then the determinant vanishes for any value of $\omega \in \mathbf{R}$. In this case the constants from equation (8.10) can be taken to be $c_1 = 1$ and $c_2 = 1 - z_-$. Thus, since there exist type I eigensolutions, the scheme is not regular up to the boundary.

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16. Abstract

We prove regularity estimates up to the boundary for solutions of elliptic systems of finite difference equations. The regularity estimates, obtained for boundary-fitted coordinate systems on domains with smooth boundary, involve discrete Sobolev norms and are proved using pseudo-difference operators to treat systems with variable coefficients. The elliptic systems of difference equations and the boundary conditions which are considered are very general in form. We prove that regularity of a regular elliptic system of difference equations is equivalent to the nonexistence of "eigensolutions". The regularity estimates obtained are analogous to those in the theory of elliptic systems of partial differential equations, and to the results of Gustafsson, Kreiss, and Sundstrom [1972] and others for hyperbolic difference equations.

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