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Abstract

The steady-state behavior of a particular type of digital phase-locked loop (DPLL) with integrate-and-dump circuit following the phase detector is characterized in terms of the probability density function (pdf) of the phase error in the loop. Although the loop is entirely digital from an implementation standpoint, it operates at two extremely different sampling rates. In particular, the combination of a phase detector and an integrate-and-dump circuit operates at a very high rate whereas the loop update rate is very slow by comparison. Because of this dichotomy, the loop can be analyzed by hybrid analog/digital (s/z domain) techniques. The loop is modeled in such a general fashion that previous analyses of the Real-Time Combiner (RTC), Subcarrier Demodulator Assembly (SDA), and Symbol Synchronization Assembly (SSA) fall out as special cases.

CONTENTS

Figures

Tables

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I. INTRODUCTION

The purpose of this analysis is to derive the steady state probability density function (pdf) of the phase error in a digital phase-locked loop (DPLL). The loop considered in this analysis has an integrate-and-dump device after the phase detector. This device together with the phase detector operates at a much higher rate than the rest of the loop, which makes it possible (or imperative) to model this part of the digital loop as an analog device. In previous analyses, Refs. 1, 2, and 3, each component of the loop was modeled either in the s or the z domain, which gave a hybrid s/z loop model. The resulting open and closed loop transfer functions related integrated phase error to the integrated input phase process.

In the present analysis, difference equations are directly written for the different components of the loop. These difference equations are combined in order to have an overall mathematical model of the loop in the presence of noise. It is shown that essentially the same open/closed loop transfer functions are obtained as in the previous analyses. The main difference is that now the transfer functions relate the phase error to the input phase process. This fact permits the derivation of the steady-state pdf of the phase error. A method is given to transform high-order nonlinear stochastic difference equations. It is shown how this analysis applies to the existing digital phase-locked loops, namely to the Real-Time Combiner (RTC), the Subcarrier Demodulator (SD), and the Symbol Synchronization (SS) digital loops in the Baseband Assembly (BBA).

II. DISCUSSION

The phase-locked loop analyzed in this article, although totally digital, is in a sense a hybrid type because it operates at two extremely different rates. The phase detector and the integrate-and-dump circuit shown in Fig. 1 operate at a very high rate (many millions of samples per second), while the loop update rate is slow (typically, one update period is longer than a second). Because of the very high rate at which the phase detector operates, the input phase, $\theta(t)$, as well as the reference phase $\hat{\theta}(t)$ are modeled as continuous variables. Their difference is the instantaneous loop phase error, namely,
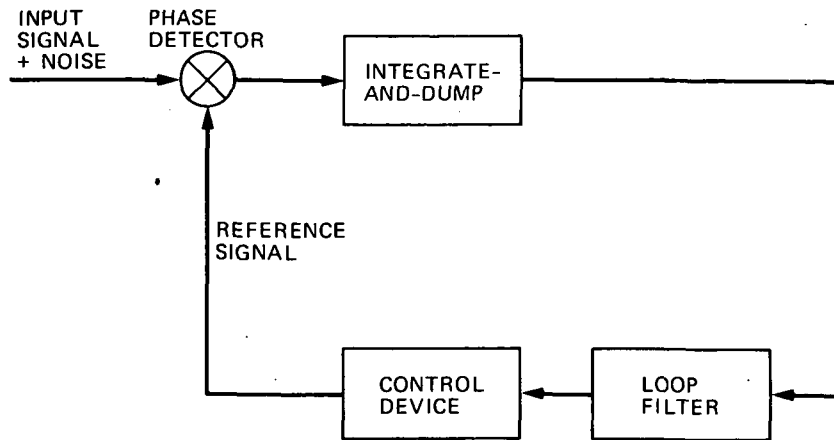
Fig. 1. Main components of a digital phase-locked loop (DPLL)

$$\phi(t) = \theta(t) - \hat{\theta}(t) \qquad (1)$$

In general, the phase detector is a nonlinear device: It transforms $\phi(t)$ to $f(\phi(t))$. Two cases are most frequently considered:

$$f(\phi(t)) = \sin \phi(t) \qquad (2a)$$

when tracking a sinuousoidal signal and

$$f(\phi(t)) = \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{\sin(2i + 1)\phi(t)}{(2i + 1)^2} \qquad (2b)$$

when tracking a square wave.

A typical phase error process is shown in Fig. 2. In the time interval (t_k, t_{k+1}), both $\phi(t)$ and $f(\phi(t))$ are assumed to be monotonic functions of time. Because of this assumption, there is a particular instant, $t_k + \Delta t_k$, in the interval (t_k, t_{k+1}) when the following condition is met:

$$f(\phi(t_k + \Delta t_k)) = \frac{1}{T} \int_{t_k}^{t_{k+1}} f(\phi(t))dt \qquad (3a)$$

$$T = t_{k+1} - t_k$$

2
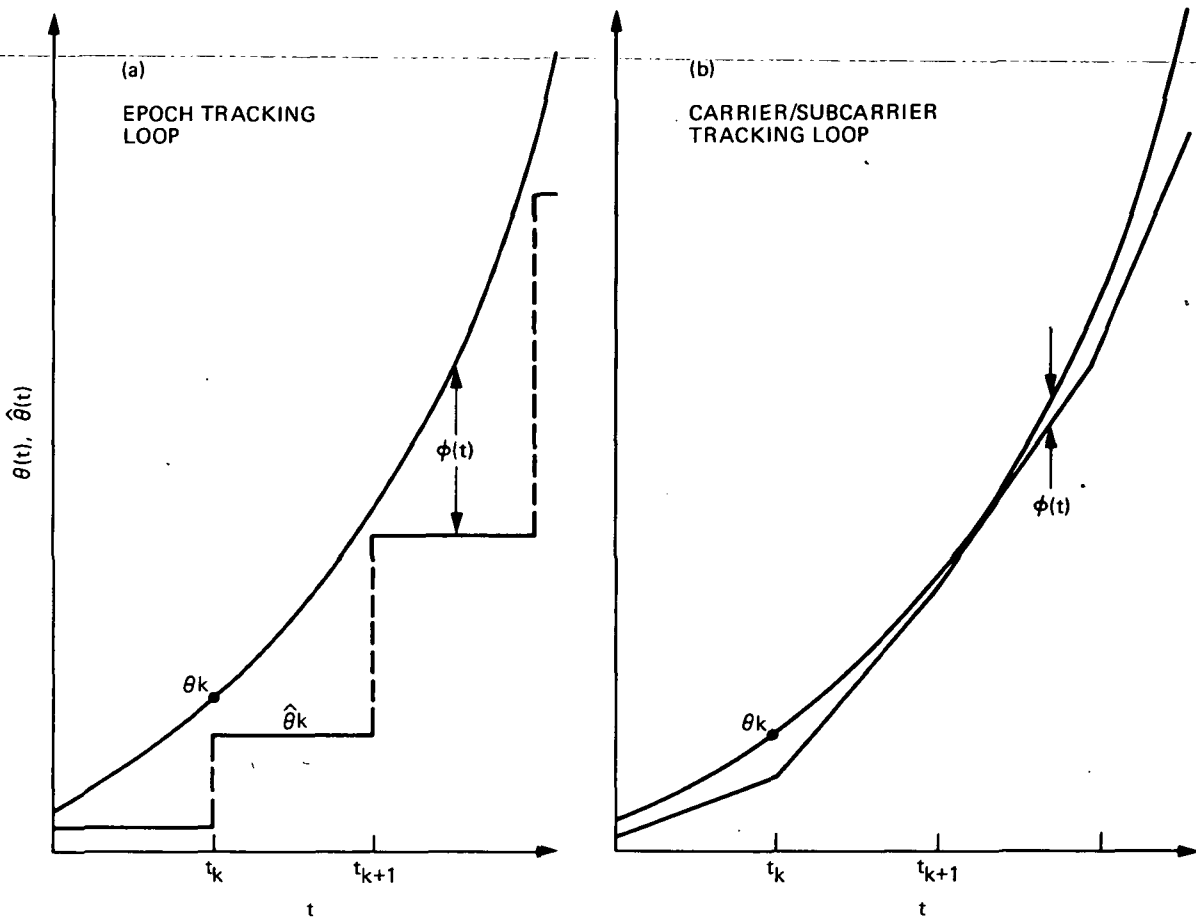
Figure 2.   Input/output phase processes

i.e., when $f(\phi(t))$ passes through its average value. In our analysis, T is the loop update time which is assumed to be a constant. Solving Eq. (3a), we define the new variable

$$\phi_k \triangleq f^{-1}\left[\frac{1}{T} \int_{t_k}^{t_{k+1}} f(\phi(t))dt\right] \qquad (3b)$$

We now replace the actual phase error process, $\phi(t)$, by a staircase process ϕ_k which satisfies Eq. (3b) and which is constant during one loop update interval. With the above substitution, Eq. (1) now becomes

$$\phi_k = \phi(t_k) + \Delta t_k)$$
$$= \theta_k - \hat{\theta}_k \qquad (4)$$

3

where θ_k is the value of $\theta(t)$ at time $t_k + \Delta t_k$ and $\hat{\theta}_k$ is the solution of the difference equation to be developed.

In Fig. 1, the integrate-and-dump circuit integrates the output of the phase detector over an update interval producing ε_k, which we call the error signal. Because of the additive noise in the input signal, ε_k is a random variable. The input-output relation of the integrate-and-dump device can be expressed by the following equation:

$$\varepsilon_k = G_Q f(\phi_k) + n_k \tag{5}$$

where G_Q is the gain of the integrate-and-dump device defined as follows:

$$G_Q \triangleq \frac{E\{\varepsilon_k\}}{f(\phi_k)} \tag{6}$$

The numerator, $E\{\varepsilon_k\}$, is the expected value of ε_k, and ϕ_k is defined by Eq. (3b). The noise sample n_k is modeled as a sequence of independent Gaussian random variables with zero mean and variance σ_n^2. Table 1 gives expressions of G_Q and σ_n^2 for various digital phase-locked loops in use at the Jet Propulsion Laboratory (JPL).

In order to find the pdf of the phase error at the loop update instants, we need the set of difference equations that describes the operation of the loop. In general, between the integrate-and-dump device where the error signal, ε_k, is sampled and the digitally controlled oscillator (DCO) or some other control device which produces the phase estimate $\hat{\theta}(t)$, there is a digital filter which can be described by the following difference equation:

$$y_k D(z) = \varepsilon_k N(z) \tag{7}$$

where

$$D(z) = 1 - a_1 z^{-1} - \ldots - a_n z^{-n}, \tag{8}$$

$$N(z) = b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}, \quad m < n \tag{9}$$

Table 1. Typical gains and transfer functions for some DPLLs

Loop Type, Reference	Loop Gain GQ, Eq. (6)	Loop Filter F(z), Eq. (10)	Control Device C(z) Eqs. (12 and 13)	Open Loop Transfer Function, $G(z) = \hat{\theta}_k/\phi_k = G_Q\, F(z)\, C(z)$	Open Loop Variance, σ_n^2
Type 1 RTC Ref. 1	$\dfrac{K(2P_e - 1)f_p}{m_0}$	$\dfrac{A}{1 - Bz^{-1}}$	FIFO (summer) $\dfrac{z^{-1}}{1 - z^{-1}}$	$G_Q\, \dfrac{z^{-1}A}{(1 - z^{-1})(1 - Bz^{-1})}$	$\dfrac{KN_s}{2m_o}\left[1 - \mathrm{erf}\left(\sqrt{\dfrac{S_1}{2\sigma_{n1}^2}}\right)\mathrm{erf}\left(\sqrt{\dfrac{S_2}{2\sigma_{n2}^2}}\right)\right]$
Type 2 SD and SS Loops Refs. 2 and 3	$\dfrac{KK_p SN_s 2}{T\pi}$	$\dfrac{A + Bz^{-1}}{(1 - z^{-1})(1 - Cz^{-1} - Dz^{-2})}$	DCO $\dfrac{T(1 + z^{-1})z^{-1}}{2(1 - z^{-1})}$	$\dfrac{G_Q Tz^{-1}(1 + z^{-1})(A + Bz^{-1})}{2(1 - z^{-1})^2(1 - Cz^{-1} - Dz^{-2})}$	$\dfrac{KK_p^2 N_s^2 \sigma_n^2}{2}\left[1 - \dfrac{SN_s}{2\sigma_n^2}\right]$

and z^{-1} is the unit delay operator defined as $z^{-1}\varepsilon_k = \varepsilon_{k-1}$. Equation (7) can also be written as

$$y_k = \frac{N(z)}{D(z)} \varepsilon_k \triangleq F(z)\varepsilon_k \qquad (10)$$

where $F(z)$ is defined as the loop filter transfer function.

If $D(z)$ has N integrators, then Eq. (8) can be written as

$$D(z) = (1 - z^{-1})^N (1 - a_1'z^{-1} - \ldots - a_{n-N}'z^{-n+N}) \qquad (11)$$

The mathematical form of the difference equation for the control device in the feedback path will depend on the particular loop we are dealing with. If, for example, the loop tracks the _epoch_ or the relative delay of a signal, as is the case with the RTC, then in the feedback path we have a _summer_, and y_k in Fig. 3 will be the increment in epoch (or in relative time delay). For this case, the input-output relationship of the control device will be described by the equation

$$\theta_k = \sum_{i=-\infty}^{k-1} y_i = \frac{z^{-1}}{(1 - z^{-1})} y_k = \frac{C_1(z)}{(1 - z^{-1})} y_k \qquad (12)$$

On the other hand, if the loop tracks the phase of the incoming signal with a _DCO_ (carrier or subcarrier tracking loop), then y_k of Fig. 2 will be the frequency of the DCO. Since the phase is the integral of frequency, we can approximate the integral operation by the bilinear transformation (or the trapezoidal rule), namely,

$$\hat{\theta}_k = \frac{T (1 + z^{-1})z - 1}{2 (1 - z^{-1})} y_k = \frac{C_2(z)}{(1 - z^{-1})} y_k \qquad (13)$$

Combining Eqs. (5), (10) and (12) or (13) in Eq. (4), the difference equation which describes the operation of the loop at the instants $t_k + \Delta t_k$ is obtained, namely,
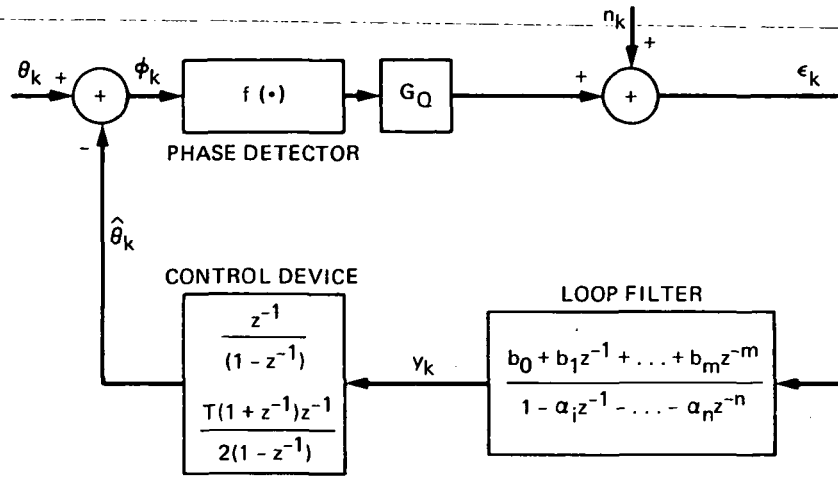
Fig. 3. Equivalent block diagram of a DPLL

$$\phi_k = \theta_k - C_j(z)F(z)(G_Q f(\phi_k) + n_k)/(1 - z^{-1}), \quad j = 1 \text{ or } 2 \qquad (14)$$

The input phase, $\theta(t)$, at discrete time instants $t_k = kT$ can be modeled as being of the form

$$\theta_k = \theta_0 + \theta_1 kT + \theta_2 (kT)^2 + \ldots + \theta_M (kT)^M \qquad (15)$$

where θ_i, $i = 0, 1, 2, \ldots, M$ are constants.

The following fact will be useful in our analysis:

$$\theta_M (kT)^M (1 - z^{-1})^N = \begin{cases} 0 \text{ for } M < N \\ M! \theta_M T^M \text{ for } M = N \\ \text{unbounded otherwise} \end{cases} \qquad (16)$$

After these preliminary steps, we are finally ready to write the overall nonlinear stochastic difference equation for the digital phase-locked loop. Using Eq. (10) in Eq. (14) and rearranging terms, we obtain

$$\phi_k(1 - z^{-1})D(z) = \theta_k(1 - z^{-1})D(z)$$

$$- C_j(z)N(z)(G_Q f(\phi_k) + n_k) \tag{17}$$

with

$$j = \begin{cases} 1 \text{ for an epoch tracking loop and} \\ 2 \text{ for a carrier tracking loop} \end{cases}$$

Assuming that $D(z)$ has N integrators as in Eq. (11) and that θ_k is of the form given by Eq. (15) with $M = N + 1$ (finite steady state error), the above difference equation becomes

$$\sum_{i=0}^{n+1} c(i)\phi_k z^{-1} = u \sum_{i=0}^{n-N} \beta(i)z^{-1} - \sum_{i=1}^{K} \gamma(i)f(\phi_k)z^{-1}$$

$$- \sum_{i=1}^{K} d(i)n_k z^{-i} \tag{18}$$

where

$$K = \begin{cases} m + 1 \text{ for the epoch tracking loop} \\ m + 2 \text{ for the carrier tracking loop} \end{cases} \tag{19}$$

and

$$u \triangleq M!\phi_M T^M \tag{20}$$

Again m and n are the number of delays in the loop filter as defined by Eqs. (8) and (9), and M is the highest component of ϕ_k tracked by the loop with finite steady state error. Note that $c(0) = \beta(0) = 1$.

The above stochastic difference equation is not Markov for n larger than zero. However, it can easily be reduced to a set of first-order stochastic difference equations, resulting in a vector Markov difference equation. This reduction can be accomplished by several methods. Expanding the above difference equation and regrouping terms, we obtain

8

$$\phi_k = u$$

$$+ z^{-1}[\beta(1)u - c(1)\phi_k - \gamma(1)f(\phi_k) - d(1)n_k]$$

$$+ z^{-2}[\beta(2)u - c(2)\phi_k - \gamma(2)f(\phi_k) - d(2)n_k] + \ldots$$

$$+ z^{-n-1}[\beta(n+1)u - c(n+1)\phi_k - \gamma(n+1)f(\phi_k) - d(n+1)n_k] \quad (21)$$

(Note that $\beta(i)$ are zero for i larger than $n - N$, and $\gamma(i)$ and $d(i)$ are zero for i larger than K.)

Representing Eq. (21) in the First Canonical Form, the flow diagram of Fig. 4 is obtained with the following <u>output</u> equation

$$\phi_k = X_k(1) + u \quad (22)$$

and the <u>nonlinear state equation</u>

$$
\begin{bmatrix} X_{k+1}(1) \\ X_{k+1}(2) \\ \cdot \\ \cdot \\ \cdot \\ X_{k+1}(n) \\ X_{k+1}(n+1) \end{bmatrix} =
\begin{bmatrix} -c(1) & 1 & 0 & \ldots & 0 \\ -c(2) & 0 & 1 & \ldots & 0 \\ \cdot & 0 & 0 & \ldots & 0 \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ -c(n) & 0 & 0 & & 1 \\ -c(n+1) & 0 & 0 & & 0 \end{bmatrix}
\begin{bmatrix} X_k(1) \\ X_k(2) \\ \cdot \\ \cdot \\ \cdot \\ X_k(n) \\ X_k(n+1) \end{bmatrix}
$$

$$
-\begin{bmatrix} \gamma(1) \\ \gamma(2) \\ \cdot \\ \cdot \\ \cdot \\ \gamma(n) \\ \gamma(n+1) \end{bmatrix} f(X_k(1) + u)
$$

$$
+\begin{bmatrix} \beta(1) - c(1) \\ \beta(2) - c(2) \\ \cdot \\ \cdot \\ \cdot \\ \beta(n) - c(n) \\ 0 - c(n+1) \end{bmatrix} u
-\begin{bmatrix} d(1) \\ d(2) \\ \cdot \\ \cdot \\ \cdot \\ d(n) \\ 0 \end{bmatrix} n_k \quad (23a)
$$
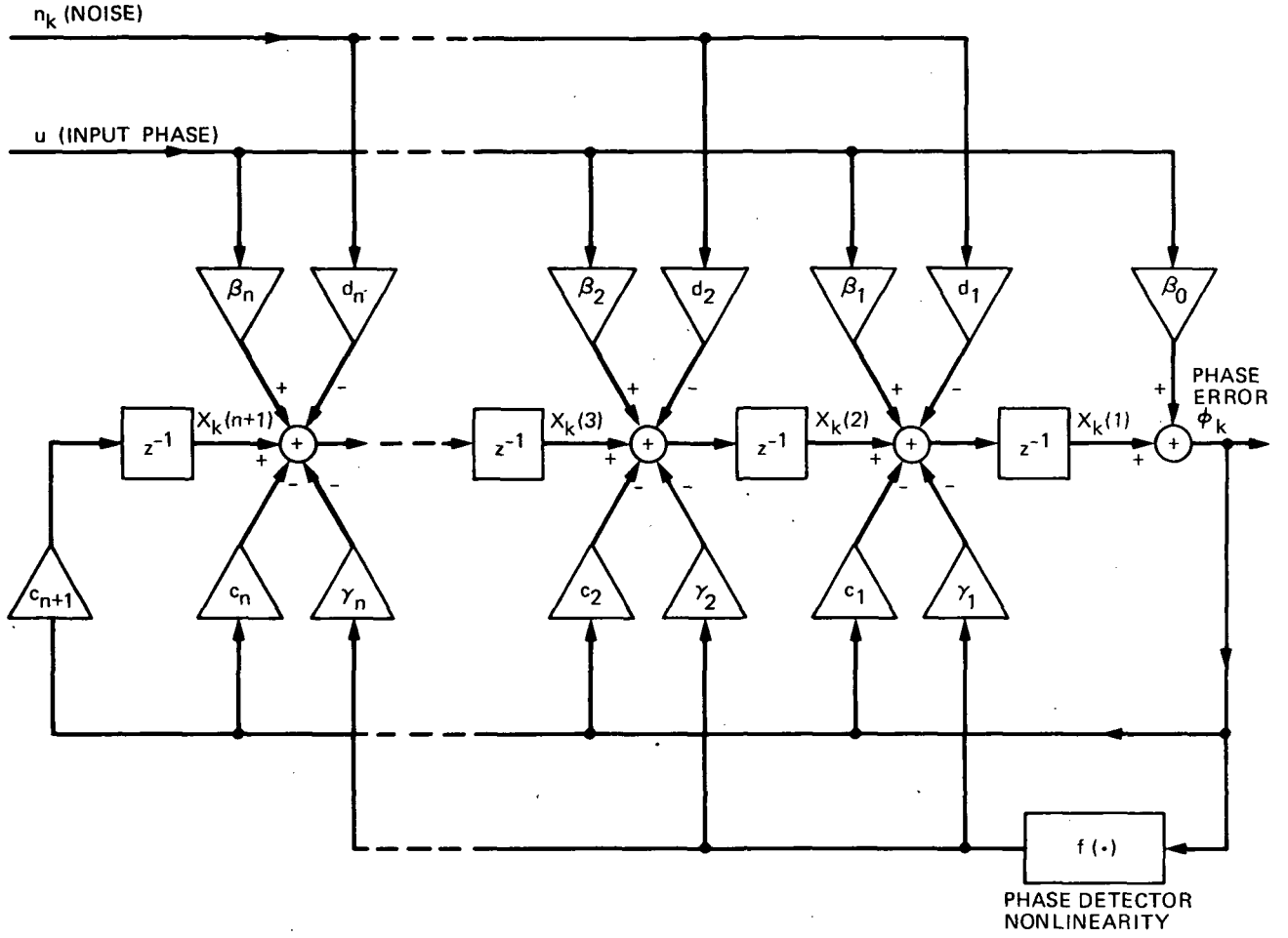
Fig. 4. First canonical form representation of the DPLL difference equation

In vector notation Eq. (23a) can be written as

$$\underline{X}_{k+1} = C\underline{X}_k - \underline{\Gamma} f(X_k(1) + u) + \underline{B}u - \underline{D}n_k \qquad (23b)$$

The above state equation simplifies considerably when the phase detector can be modeled as a linear device with $f(\phi_k) = \phi_k$. In this case, Eq. (18) reduces to

$$\sum_{i=0}^{n+1} \alpha(i)\phi_k z^{-1} = u \sum_{i=0}^{n-N} \beta(i)z^{-1} - \sum_{i=1}^{K} d(i)n_k z^{-1} \qquad (24a)$$

where

$$\alpha(i) = c(i) - \gamma(i) \qquad (24b)$$

10

$$
\begin{bmatrix} X_{k+1}(1) \\ X_{k+1}(2) \\ \cdot \\ \cdot \\ \cdot \\ X_{k+1}(n) \\ X_{k+1}(n+1) \end{bmatrix}
=
\begin{bmatrix} -\alpha(1) & 1 & 0 & \ldots & 0 \\ -\alpha(2) & 0 & 1 & \ldots & 0 \\ \cdot & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ -\alpha(n) & 0 & 0 & & 1 \\ -\alpha(n+1) & 0 & 0 & & 0 \end{bmatrix}
\begin{bmatrix} X_k(1) \\ X_k(2) \\ \cdot \\ \cdot \\ \cdot \\ X_k(n) \\ X_k(n+1) \end{bmatrix}
$$

$$
+
\begin{bmatrix} \beta(1) - \alpha(1) \\ \beta(2) - \alpha(2) \\ \cdot \\ \cdot \\ \cdot \\ \beta(n) - \alpha(n) \\ 0 - \alpha(n+1) \end{bmatrix} u
-
\begin{bmatrix} d(1) \\ d(2) \\ \cdot \\ \cdot \\ \cdot \\ d(n) \\ 0 \end{bmatrix} n_k
\qquad (25a)
$$

Using vector notation we can write Eq. (25a)

$$
\underline{X}_{k+1} = A\underline{X}_k + \underline{B}u - \underline{D}n_k \qquad (25b)
$$

where A is an $(n + 1) \times (n + 1)$ matrix. The values of the constants $\{\alpha_i\}$, $\{\beta_i\}$, and $\{d_i\}$ corresponding to the RTC and the subcarrier loop are given in Table 2.

A.   Chapman-Kolmogorov Equation

The vector Markov process for a nonlinear phase detector is given by Eq. (23) and for a linear phase detector by Eq. (26). The conditional probability density function (pdf) of $\underline{X}_{k+1}$, conditioned on the initial condition $\underline{X}_0$, satisfies the Chapman-Kolmogorov (C-K) equation

$$
p_{k+1}(\underline{X}_{k+1} = \underline{x} \mid \underline{x}_0) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} q_k\,(\underline{X}_{k+1} = \underline{x} \mid \underline{X}_k = \underline{w})
$$

$$
\times\; p_k(\underline{X}_k = \underline{w} \mid \underline{x}_0)dw_1 \cdots dW_L \qquad (26)
$$

where

$$
\underline{x}_0 = \text{initial value of } \underline{X}
$$
$$
p_k(\underline{X}_k = \underline{x} \mid \underline{x}_0) = \text{pdf of } \underline{X}_k \text{ given } \underline{x}_0
$$
$$
q_k(\underline{X}_{k+1} = \underline{x} \mid \underline{X}_k = \underline{w}) = \text{transition pdf of } \underline{X}_{k+1} \text{ given } \underline{X}_k = \underline{w}
$$

Table 2. Value of constants for the linear state Eq. (24)

Constants	RTC	Subcarrier and Symbol Synchronization Loops
α_1	$AG_Q - 1 - B$	$TAG_Q/2 - C - 2$
α_2	B	$2C + 1 + TG_Q(A + B)/2$
α_3	---	$2D - C + TG_QB/2$
α_4	---	$-D$
β_0	1	1
β_1	$-B$	$-C$
β_2	0	$-D$
β_3	---	0
β_4	---	0
d_1	$-A$	$-TA/2$
d_2	0	$-T(A + B)/2$
d_3	---	$-TB/2$
d_4	---	0

$$q_k(\underline{X}_{k+1} = \underline{x} | \underline{X}_k = \underline{w})$$

$$= q_k [X_{k+1}(1) = x(1), X_{k+1}(2) = x(2), \ldots, X_{k+1}(L) = x(L) | \underline{X}_k = \underline{w}]$$

$$= q_k[X_{k+1}(L) = x(L) | X_{k+1}(L - 1) = x(L - 1), \ldots, X_{k+1}(1) = x(1), \underline{X}_k = \underline{w}]$$

$$x\ q_k[X_{k+1}(L - 1) = x(L - 1) | X_{k+1}(L - 2) = x(L - 2), \ldots, X_{k+1}(1) = x(1), \underline{X}_k = w]$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$x\ q_k[X_{k+1}(1) = x(1) | \underline{X}_k = \underline{w}]$$

where $L = n + 1$.

In words, Eq. (27) expresses the total transition pdf as a product of transition pdf's at the input to each delay of Fig. 4. If we refer to that figure and assume Gaussian distribution of the noise samples n_k, these transition pdf's will have Gaussian distributions for $d(i) \neq 0$ and will be delta functions for $d(i) = 0$, $i = 1, \ldots, n + 1$. The pdf's with Gaussian distribution will have the following conditional mean and variance

$$E\{X_{k+1}(i) | X_{k+1}(i - 1) = x(i - 1), \ldots, x_{k+1}(1) = x(1), \underline{X}_k = \underline{w}\} = \mu_i$$

$$= -c(i)w(1) - \gamma(i)f(w(1) + u) + w(i + 1) + (\beta(i) - c(i))u \tag{28}$$

$$\sigma_i^2 = d(i)^2 \sigma_n^2 \tag{29}$$

and the delta functions will be of the form

$$\delta[X_{k+1}(i) - \mu_i] \tag{30}$$

where μ_i is given by Eq. (28).

Note in Eqs. (28), (29), and (30) that the transition pdf $q_k(\bullet|\bullet)$ (Eq. [27]) will be independent of k. This should be expected since we are analyzing a time-invariant system. For simplicity of notation, we shall define

$$q_k(j) \triangleq q_k[X_{k+1}(j)|X_{k+1}(j-1) = x(j-1), \ldots, X_{k+1}(1) = x(1), \underline{X}_k = \underline{w}]$$

$$j = L, L-1, \ldots, 1 \qquad (31)$$

The transition pdf of Eq. (27) now can be written as

$$q_k(\underline{X}_{k+1} = \underline{x}|\underline{X}_k = \underline{w}) = \prod_{j=1}^{L} q_k(j) \qquad (32)$$

The C-K equation for an arbitrary nonlinearity $f(\bullet)$ is given by Eq. (26). When $f(\bullet)$ is periodic, as in Eq. (2), then the pdf of the unrestricted phase error process $p_{k+1}(\underline{X}_{k+1} = \underline{x}|\underline{X} = \underline{w})$ will go to zero as $k \to \infty$, which is not particularly useful for loop analysis. However, the modulo 2π reduction of the random variable $\underline{X}_k$ is of interest in studying the behavior of the loop in the locked condition. We define a new random variable $\underline{X}_k'(j) \in (-\pi, \pi)$ which is obtained from the old random variable $\underline{X}_k(j)$ by taking

$$\underline{X}_k'(j) = \underline{X}_k(j) \bmod 2\pi, \quad j = 1,2,\ldots, L \qquad (33)$$

Using Eqs. (26) and (32), the corresponding C-K equation that applies to the mod 2π process $\{\underline{X}'\}$ is modified to be

$$p_{k+1}(\underline{X}_{k+1}' = \underline{x}'|\underline{x}_0) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{j=1}^{L} q_k(j)p_k(\underline{X}_k' = \underline{w}|\underline{x}_0)dw_1\ldots dw_L \qquad (34)$$

where all the variables are now $\in (-\pi, \pi)$. The transition pdf's, $q_k(j)$ of Eq. (31), will have to be modified now as follows:

$$q_k(j) = \sum_{m=-\infty}^{\infty} q_k[X_{k+1}'(j) + 2\pi m) \,|\, X_{k+1}'(j-1)$$

$$= x(j-1), \ldots, X_{k+1}' = x(1), \underline{X}_k' = \underline{w}] \tag{35}$$

In solving the C-K equation as expressed by Eq. (34) in the steady state, i.e., $k \to \infty$, we know that $p_{k+1}(\cdot)$ on the left hand side will be equal to $p_k(\cdot)$ on the right hand side. The dependence of distribution $p_k(\cdot)$ on $\underline{x}_0$, the initial value of the state vector, can be dropped for large k.

In order to conclude our discussion, we will show a method of solving the C-K equation for the first-order loop, i.e., when L = (n + 1) = 1 (no loop filter). In this case the transition pdf will have only one term with Gaussian distribution, namely,

$$q(X_{k+1}' \,|\, X_k' = w)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{m=-\infty}^{\infty} \exp\left[-\frac{\{(X'+2\pi m) - (-cw - \gamma f(w+u) - cu)\}^2}{2\sigma^2} \right] \tag{36}$$

Note that we have dropped the index 1 from all the variables and constants in order to simplify our notation.

The C-K equation for this first-order loop in the steady state will be

$$p(X' = x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} \exp\left[-\frac{\{(X'+2\pi m) - (-cw - \gamma f(w+u) - cu)\}^2}{2\sigma^2} \right] p(X = w) \, dw,$$

$$X' \in (-\pi, \pi) \tag{37}$$

We now approximate integration with summation. The continuous variables X' and w are converted to discrete variables taking values x_i and w_i in the interval $(-\pi, \pi)$, namely,

$$x_i = i(\pi/N), \quad w_i = i(\pi/N), \quad i = -N, -N+1, \ldots, N \tag{39}$$

where N is an integer large enough to make the above approximation valid.

15

With these changes, the discrete C-K equation becomes

$$p(X' = x_i) = \sum_{j=-N}^{N} q_{ij} p(X' = w_j) \qquad (40)$$

where

$$q_{ij} = \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{m=-\infty}^{\infty} \exp\left[-\frac{\{(x_i + 2\pi m) - (-cw_j - \gamma f(w_j + u) - cu)\}^2}{2\sigma^2} \right] \qquad (41)$$

Let Q be the $(2N + 1) \times (2N + 1)$ matrix with elements q_{ij} defined by Eq. (41), and let $\underline{P}$ be the $(2N + 1)$ vector with elements x_i and w_i. Then the discrete C-K equation in vector notation will be

$$\underline{P} = Q\underline{P} \qquad (42)$$

Using the definition of eigenvalues, we recognize that Eq. (42) can be written as

$$(\lambda I - Q)\underline{P} = \underline{0} \qquad (43)$$

where λ is an eigenvalue of Q and I is a $(2N + 1) \times (2N + 1)$ identity matrix. The C-K discrete equation will have a solution if one of the eigenvalues of Q will be 1. In this case, the desired discrete pdf of the phase error $\underline{P}$ will be the eigenvector corresponding to $\lambda = 1$.

B.   A Numerical Example

The modulo 2π steady-state pdf of the phase error process was numerically computed for a first order loop having a summer as the control device and a filter with $F(z) = A$. It was assumed that the loop tracks a square wave which results in the sawtooth nonlinearity given by Eq. (2). These assumptions result in the following nonlinear stochastic difference equation:

$$\phi_k = \theta_k - z^{-1} A(G_Q f(\phi_k) - n_k)/(1 - z^{-1}) \qquad (44)$$

16

The elements q_{ij} of the matrix Q defined by Eq. (42) are normalized as follows

$$q_{ij}' = \begin{cases} q_{ij}\pi/(2N + 1) & \text{for } i = 1 \text{ and } (2N + 1) \\ q_{ij}2\pi/(2N + 1) & \text{otherwise} \end{cases} \qquad (45)$$

where q_{ij} are computed using Eq. (41). This normalization makes the sum of the elements q_{ij}' along each column of Q' equal to 1.0 which in turn makes the largest eigenvalue of Q', λ_{max}, equal to 1.0 and the desired, discrete pdf, $\underline{P}$, is obtained. In our numerical evaluation of $\underline{P}$, the following assumptions were made:

$$G_Q = 1.0$$
$$0.01 \leq A \leq 1.0$$
$$u = 0.0 \text{ (no loop stress)}$$
$$0.01 \leq \sigma^2 \leq 1.0$$

Figures 5, 6, and 7 summarize the numerical results. In these figures the pdfs of the modulo 2π process of the phase error are compared with a Gaussian pdf of the same mean and variance values.

III. Conclusions

Using a method based on transforming high-order nonlinear stochastic difference equations to a set of first-order nonlinear difference equations, the steady-state probability density function of a digital phase-locked loop with integrate-and-dump circuit following the phase detector has been derived. This work ties together in a unified fashion, the previous analyses of specific DSN Baseband Assembly (BBA) loops such as the Real-Time Combiner (RTC), Subcarrier Demodulation Assembly (SDA), and Symbol Synchronization Assembly (SSA). A numerical example for a first-order loop having a summer as a control device and a constant loop filter was given to illustrate the simplicity of the application of the theory. It is believed that the results derived here are sufficiently general so as to apply to a host of other loops of this generic type.
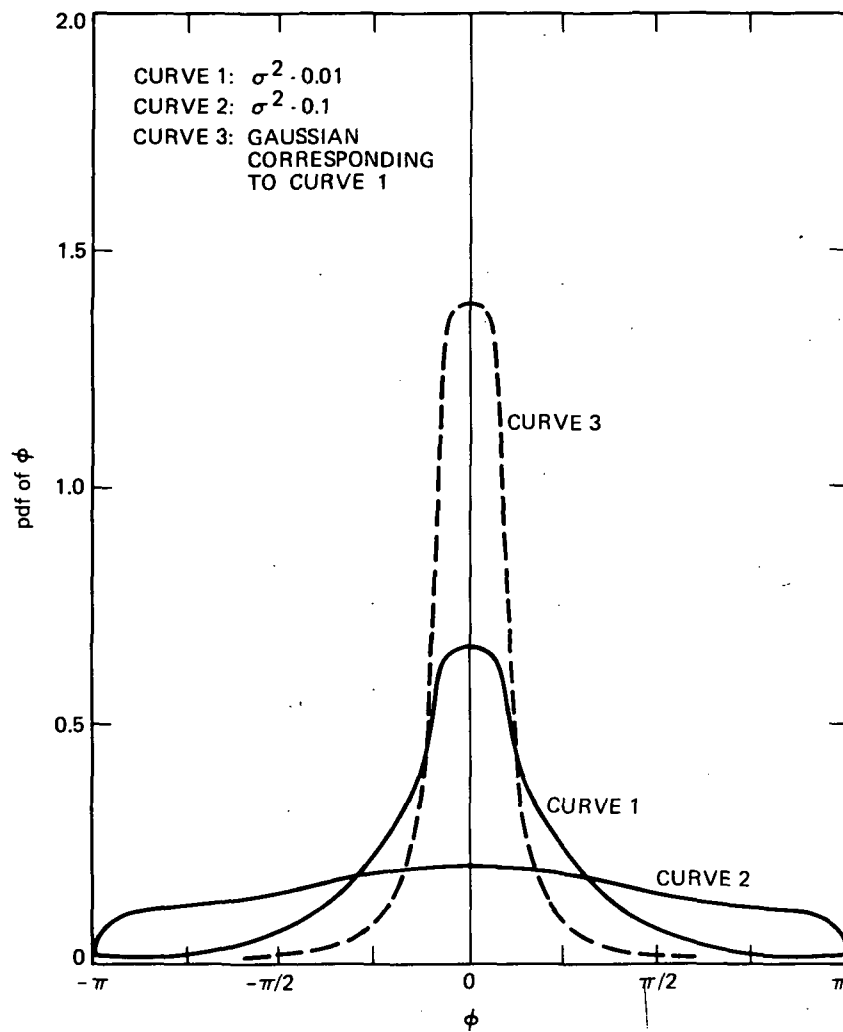
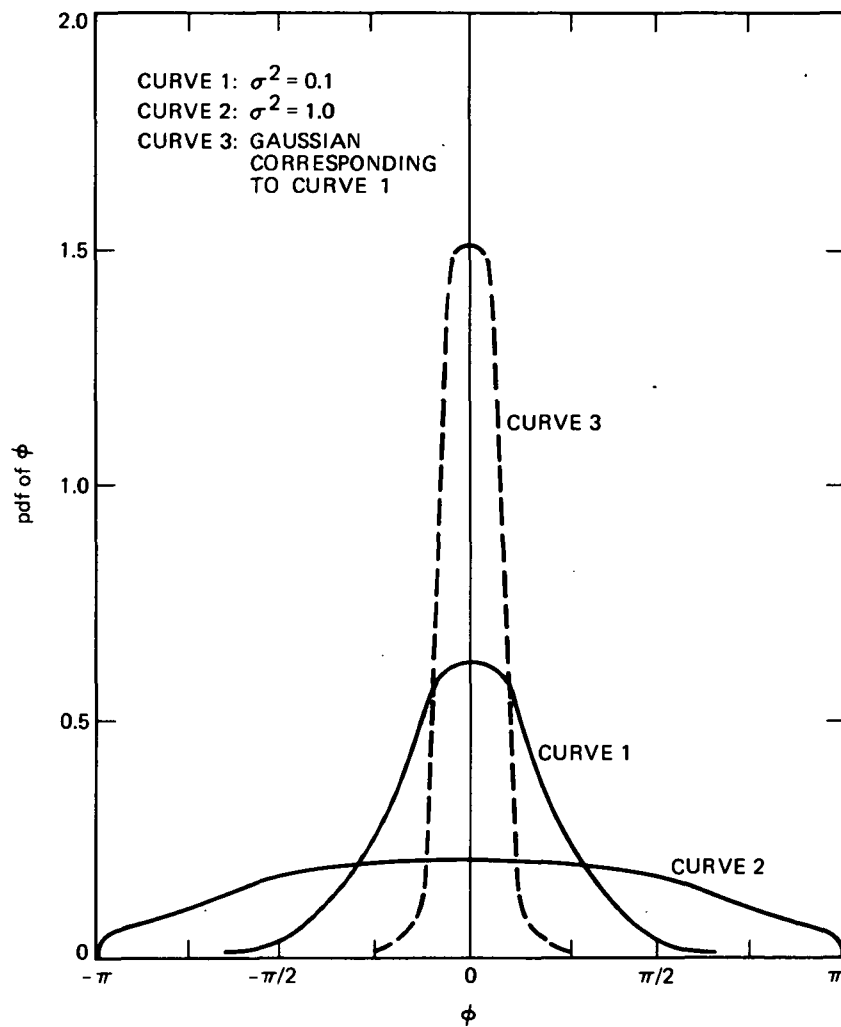Fig. 5. Modulo 2π steady-state pdf of the phase error. First order loop. Loop gain $G_QA = 0.01$.

Fig. 6. Modulo 2π steady-state pdf of the phase error. First order loop.
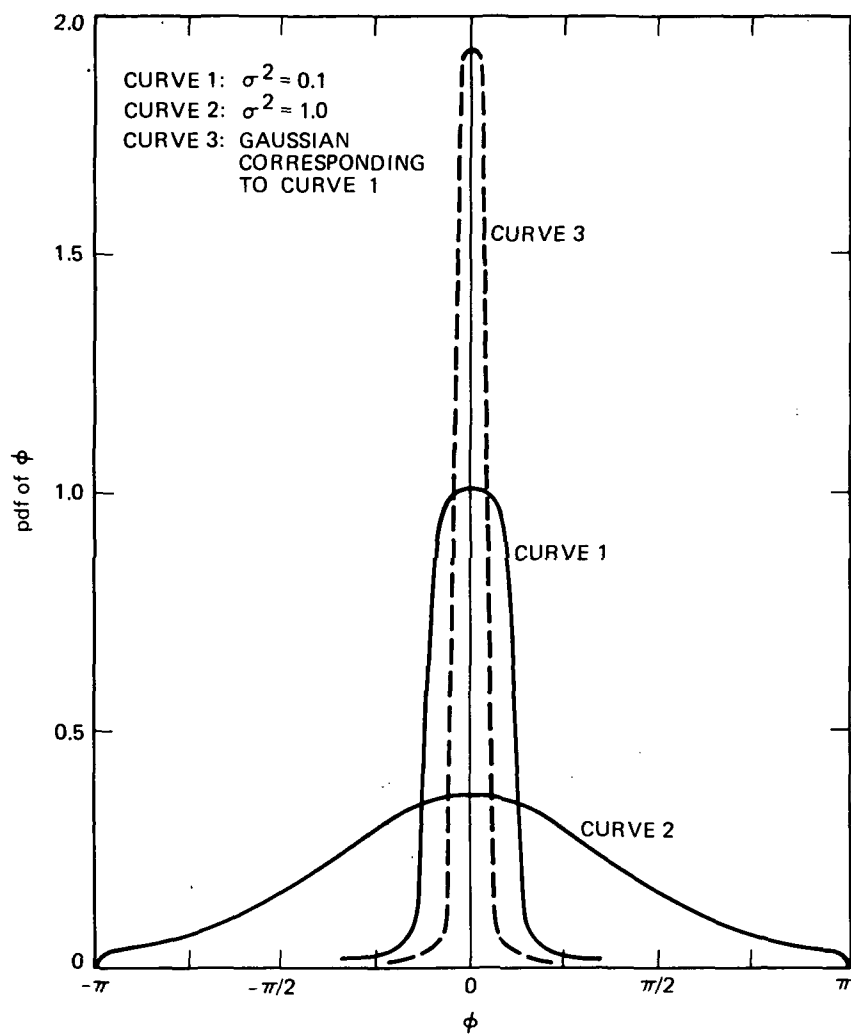Loop gain $G_Q A$ = 0.1.

Fig. 7. Modulo 2π steady-state pdf of the phase error. First order loop. Loop gain $G_Q A = 1.0$.

IV. REFERENCES

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15. Supplementary Notes

16. Abstract

The Steady-state behavior of a particular type of digital phase-locked loop (DPLL) with an integrate-and-dump circuit following the phase detector is characterized in terms of the probability density function (pdf) of the phase error in the loop. Although the loop is entirely digital from an implementation standpoint, it operates at two extremely different sampling rates. In particular, the combination of a phase detector and an integrate-and-dump circuit operates at a very high rate whereas the loop update rate is very slow by comparison. Because of this dichotomy, the loop can be analyzed by hybrid analog/digital (s/z domain) techniques. The loop is modeled in such a general fashion that previous analyses of the Real-Time Combiner (RTC), Subcarrier Demodulator Assembly (SDA), and Symbol Synchronization Assembly (SSA) fall out as special cases.

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