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THE MECHANICS OF SOLIDS IN THE PLASTICALLY-
DEFORMABLE STATE

R.v. Mises

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16. Abstract

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The Mechanics of Solids in the Plastically-Deformable State

R. v. Mises[*]

The mechanics of continua, which is based on the general stress model of Cauchy, up to the present has almost exclusively been applied to liquid and solid elastic bodies. Saint-Venant [1] has developed a theory for the plastic or remaining form changes of solids, but it does not give the required number of equations for determining the motion. Other attempts in this direction have not led to any conclusion[#].

The following article leads to a complete set of equations of motion for plastic-deformable bodies, within the framework of Cauchy mechanics and which is supported by certain experimental facts which characterize the range of applications.

1. Notation

The state of stress in a point of a body is assumed to be given by the three normal stresses $\sigma_x, \sigma_y, \sigma_z$ and the tangential stresses τ_x, τ_y, τ_z , assuming a rectangular coordinate system. In the diagram

$$\begin{matrix} \sigma_x & \tau_z & \tau_y \\ \tau_z & \sigma_y & \tau_x \\ \tau_y & \tau_x & \sigma_z \end{matrix} \qquad (1)$$

the quantities in the first line mean the components of the stress vector $\bar{\sigma}_x$ for the surface element, whose other normal has the direction of the positive x axis, etc. The vector complex

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[*]Presented by C. Runge at the meeting of November 1, 1913, with four figures in the text.
[#]Haar and v. Kármán, Göttinger Nachr. 1909, derive equations of motion from a new variational principle, but its relationship to the rest of mechanics has not yet been clarified.

represented by (1), which as is well known can be transformed according to the equation

$$\bar{\sigma}_{x'} = \bar{\sigma}_x \cos{(x, x')} + \bar{\sigma}_y \cos{(y, x')} + \bar{\sigma}_z \cos{(z, x')} \qquad (2)$$

will be called the stress dyad $\bar{\sigma}$.

A similar concept leads to the deformation dyad $\bar{\varepsilon}$ and the dyad of the deformation rate $\bar{\lambda}$. If ξ, η, ζ are the infinitesimally small elastic displacements of a point, then the strains and the angular changes are equal to

$$\varepsilon_x = \frac{\partial \xi}{\partial x}, \ \varepsilon_y = \frac{\partial \eta}{\partial y}, \ \varepsilon_z = \frac{\partial \zeta}{\partial z};$$

$$\gamma_x = \frac{1}{2}\left(\frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y}\right), \ \gamma_y = \frac{1}{2}\left(\frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z}\right), \ \gamma_z = \frac{1}{2}\left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\right); \qquad (3)$$

and the dyad $\bar{\varepsilon}$ has the diagram

$$\begin{matrix} \varepsilon_x & \gamma_z & \gamma_y \\ \gamma_z & \varepsilon_y & \gamma_x \\ \gamma_y & \gamma_x & \varepsilon_z. \end{matrix} \qquad (4)$$

If instead of ξ, η, ζ one uses the components u, v, w of the velocity vector, then one obtains the strain and shear rates

$$\lambda_x = \frac{\partial u_x}{\partial x}, \ \lambda_y = \frac{\partial v}{\partial y}, \ \lambda_z = \frac{\partial w}{\partial z};$$

$$\nu_x = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \ \nu_y = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \ \nu_z = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right); \qquad (5)$$

Figure 1.

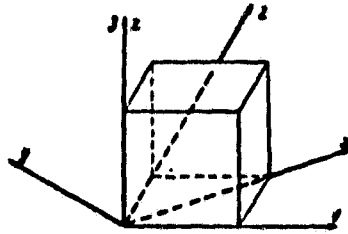and the diagram for the dyad $\overline{\overline{\lambda}}$:

$$
\begin{array}{ccc}
\lambda_x & v_z & v_y \\
v_z & \lambda_y & v_x \\
v_y & v_x & \lambda_z
\end{array}
\qquad (6)
$$

For each dyad there is at least one coordinate system, for which the diagram is reduced to the terms of the principal diagonal, for example for (1) this leads to the form:

$$
\begin{array}{ccc}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{array}
\qquad (7)
$$

Here the "principal stresses" $\sigma_1, \sigma_2, \sigma_3$ are the square roots of the secular equation or determined by the following three

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conditions:

$$\sigma_1 + \sigma_k + \sigma_s = \sigma_x + \sigma_y + \sigma_z,$$

$$\sigma_1\sigma_k + \sigma_1\sigma_s + \sigma_k\sigma_s = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - (\tau_x^2 + \tau_y^2 + \tau_z^2)$$

$$\sigma_1\sigma_k\sigma_s = \begin{vmatrix} \sigma_x & \tau_z & \tau_y \\ \tau_z & \sigma_y & \tau_x \\ \tau_y & \tau_x & \sigma_z \end{vmatrix}. \tag{8}$$

If a coordinate system is placed in such a manner that the
z axis coincides with the third principal axis, whereas the x
and y axes bisect the angles of the first two principal axes
(Figure 1), then the following diagram results because of (2):

$$\begin{array}{ccc} \dfrac{\sigma_1 + \sigma_k}{2}, & \dfrac{\sigma_k - \sigma_1}{2}, & 0 \\[2mm] \dfrac{\sigma_k - \sigma_1}{2}, & \dfrac{\sigma_1 + \sigma_k}{2}, & 0 \\[2mm] 0, & 0, & \sigma_s. \end{array} \tag{9}$$

At the same time one can see that the τ values which occur
here are extremes of the tangential stress, i.e., among the
three quantities

$$\tau_1 = \frac{\sigma_1 - \sigma_s}{2}, \quad \tau_k = \frac{\sigma_1 - \sigma_s}{2}, \quad \tau_s = \frac{\sigma_s - \sigma_1}{2}, \tag{10}$$

one always has the absolutely largest and absolutely smallest
tangential stress. The quantities τ_1, τ_k, τ_s, are called the main   /585

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tangential stresses, and their sum equals zero.

The simplest of all stress dyads is the one for the ideal liquid $-\overline{p}$. In any coordinate system it has the diagram

$$\begin{matrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p. \end{matrix} \tag{11}$$

If one subtracts a stress state of the form (11) from the stresses represented by (1), then the tangential stresses remain unchanged, and we obtain a diagram

$$\begin{matrix} \sigma'_x & \tau_z & \tau_y \\ \tau_z & \sigma'_y & \tau_x \\ \tau_y & \tau_x & \sigma'_z \end{matrix} \tag{12}$$

where

$$\sigma'_x = \sigma_x + p, \quad \sigma'_y = \sigma_y + p, \quad \sigma'_z = \sigma_z + p. \tag{13}$$

The dyad (12) has the same principal directions as (1), and the principal values $\sigma'_1, \sigma'_2, \sigma'_3$ are the principal values of (1) reduced by -p. Then according to (10) the principal tangential stresses for (12) and (1) are identical.

All of these relationships, of course, also apply for the deformation dyad $\overline{\Gamma}$ or for $\overline{\lambda}_i$. We will now give a formula which is used in applications, from which we will derive a

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relationship between (10) and (8): We have

$$
\begin{aligned}
\tau_1^2 + \tau_2^2 + \tau_3^2 &= \tfrac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \tfrac{1}{3}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \\
&= \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)^2 - \tfrac{1}{3}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \\
&= \tfrac{1}{3}(\sigma_x + \sigma_y + \sigma_z)^2 - \tfrac{1}{3}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \tfrac{1}{3}(\tau_x^2 + \tau_y^2 + \tau_z^2) = \\
&= \left(\tfrac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tfrac{\sigma_y - \sigma_z}{2}\right)^2 + \left(\tfrac{\sigma_z - \sigma_x}{2}\right)^2 + \tfrac{1}{3}(\tau_x^2 + \tau_y^2 + \tau_z^2).
\end{aligned}
\tag{14}
$$

2. Fundamentals Due To Experience

We will now introduce those factors from our experience which the equations of motion to follow take into account. We do not attempt to give an axiomatic development, in other words, we are not intent in using a minimum of assumptions.

(a) All solids behave like elastic bodies for sufficiently small stresses: there is a one-to-one correspondence between stress and deformation. /586

With this theorem, we delineate solids with respect to viscous materials. "Solid" is, for example, wax, which will yield even for a small external pressure. Also iron is included, which only reaches the elastic limit under a very high pressure. On the other hand, materials like tar at normal temperature are not plastically deformable, but are liquid.

We will discuss the importance and the form of the elastic limit below.

The connection between the stress dyad and the deformation dyad $\vec{\sigma}$ and $\vec{\epsilon}$, assumes that the mathematical theory of elasticity is linear, as is well-known:

$$
\vec{\sigma} = L(\vec{\epsilon}).
\tag{15}
$$

The most general linear relationship, in which no direction in space is preferred, consists of assuming that the two dyads have the same principal directions and that the principal values are related as follows:

$$\sigma_1 = \alpha \varepsilon_1 + \beta(\varepsilon_1 + \varepsilon_2 + \varepsilon_3), \quad \sigma_2 = \alpha \varepsilon_2 + \beta(\varepsilon_1 + \varepsilon_2 + \varepsilon_3),$$
$$\sigma_3 = \alpha \varepsilon_3 + \beta(\varepsilon_1 + \varepsilon_2 + \varepsilon_3). \tag{16}$$

Here α and β are the elastic constants. As is well-known, (16) can be converted so that relationships between the components referred to arbitrary axes can be created.

(b) If the limit of elasticity is reached, then the solid behaves essentially like a viscous and almost incompressible liquid.

The behavior of the liquid intended here is characterized by the fact that it is not the deformation state, as is the case for the elastic body, but the deformation process which causes stresses. However, one cannot simply assume that the stress dyad $\bar{\sigma}$ is a function of the deformation-rate dyad $\bar{\lambda}$. Instead one has to consider that a volume under a uniform pressure does not experience any finite deformation rate. The volume change which occurs remains always of the order of magnitude of the elastic displacements, as many observations have shown.

Therefore it follows that in the mechanics of viscous fluids, one has to subtract a part $-\bar{p}$ from the stress dyad $\bar{\sigma}$, which corresponds to a uniform pressure in all directions. The remainder $\bar{\sigma}'$ (See (12) in paragraph 1.) can be assumed to be a linear function

$$\bar{\sigma}' = L(\bar{\lambda}) \quad . \tag{17}$$

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If one considers the same symmetry as above, then similar to
(16) we obtain:

$$\sigma'_i = k\lambda_i + k'(\lambda_i + \lambda_i + \lambda_i), \dots \tag{18}$$

However, the expression in parentheses is exactly the divergence
or volume change, which we just discussed and which can be ignored
compared with γ . In this way we obtain:

$$\sigma'_i = k\lambda_i, \quad \sigma'_i = k\lambda_i, \quad \sigma'_i = k\lambda_i. \tag{19}$$

These equations state that σ can be found from λ , if one
multiplies every component of λ by k:

$$\sigma'_s = \sigma_s + p = k\lambda_s, \quad \sigma'_y = \sigma_y + p = k\lambda_y, \quad \sigma'_s = \sigma_s + p = k\lambda_s; \tag{20}$$
$$\tau_s = k\nu_s, \quad \tau_y = k\nu_y, \quad \tau_s = k\nu_s.$$

These are exactly the same equations which the Navier-Stokes
theory of viscous fluids obtains. An important difference will
only occur if we investigate the meaning of the quantity k
more closely. This is done by using the following experimental
theorem.

   c) If one changes the absolute value of the speeds with
which a motion proceeds, while maintaining all ratios, then in
the case of plastically deformable bodies the work does not
change which is required to achieve a certain form change.
       This theorem is derived from the totality of the
observational material, which has been obtained in research on
remaining form changes, that is, technology. For the most part,

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technology uses formulas for the work, which to begin with do not consider the influence of the speed. Wherever this influence has been observed, it was found to be very small[*]. The constant quantities discussed in theorem c) will have to be interpreted similar to the constant nature of the friction coefficients with respect to alternating normal pressure during the sliding friction of solids. In any case, using the assumption c) we have defined

an ideal case, which will then allow a certain theory, and which represents a useful approximation for the actual behavior of bodies.

The work to be expended per volume and second is in general given by:

$$\sigma_x' \lambda_x + \sigma_y' \lambda_y + \sigma_z' \lambda_z + 2\tau_x \nu_x + 2\tau_y \nu_y + 2\tau_z \nu_z =$$
$$= k(\lambda_x^2 + \lambda_y^2 + \lambda_z^2 + 2\nu_x^2 + 2\nu_y^2 + 2\nu_z^2).$$

(21)   /588

If all speeds are multiplied by a factor of c, then this expression changes in proportion to kc^2. At the same time, the duration of the deformation process is reduced by a ratio of 1 : c, and the total work is proportional to kc. That means that the proportionality factor k introduced in (20) has to be inversely proportional to the speed. Or, stated differently: the stress dyad $\vec{\sigma}$ remains the same, when all components of $\bar{\lambda}$ are reduced by the same ratio.

From the last formulation it follows that the stresses in a plastically deformable body must vary in a region of reduced multiplicity, compared with elastic bodies. It is clear that this range can be nothing else than the limit of elasticity. That

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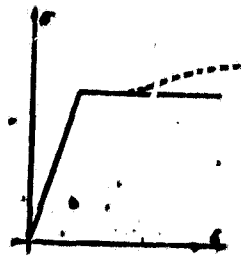[*]Individual proofs, also references, can be found in my Encyclopedia article IV 10, Nr. 5, p. 187.

Figure 2.

means that our theorem c) can also be formulated as follows:

c') In the case of plastic deformations, the stress always remains at the limit of elasticity.

This theorem includes the requirement that the elastic limit must be independent from an additive term of the form (11) (see below).

The theorem c') can be directly confirmed by observations. In the one-dimensional case of a tension load on a rod, the stress-strain diagram according to c') would have to take on the form shown in Figure 2: first an inclined line for the elastic state, then a horizontal limit stress in the plastic region, which accordingly is independent of speed. The observation shows that in the case of iron, steel and similar materials, there is a horizontal segment which comes after the inclined line, but very soon it is transformed into a slightly rising line. This can be attributed to a process which is related to the crystalline nature of the body and which is a highly thermal process, which is called "solidification". This solidification is now not considered by our theory. However, we have to consider the

following: The real range of application of the theory of
plasticity is in the area of pressure loads (positive p).
It has not yet been clarified whether under pressure, iron and
similar materials experience such a solidification. In any
case, it does not seem unlikely that for materials which are
easily deformed, for example, wax and others, "solidification"
is a very unimportant factor.

We will now discuss a last theorem which involves
the nature of the elastic limit:

d) In a coordinate system which has the coordinates of
the principal tangential stresses, the elastic limit is in
the form of a closed curve in the following plane:

$$\tau_1 + \tau_2 + \tau_3 = 0. \tag{22}$$

and this curve includes the origin.

As is well known, O. Mohr gave the first detailed in-
vestigations about the elastic limit and the fracture limit [2].
According to Mohr, this only depends on the largest and smallest
of the three principal stresses, and let us call them σ_1 and
σ_3 . In a coordinate system:

$$x = \frac{\sigma_1 + \sigma_3}{2}, \quad y = \frac{\sigma_1 - \sigma_3}{2} = -\tau_2 \tag{23}$$

the fracture limit has the appearance shown in Figure 3 if
one considers the experiments of von Karman in addition to
those of Mohr [3]. The greatest difference between the
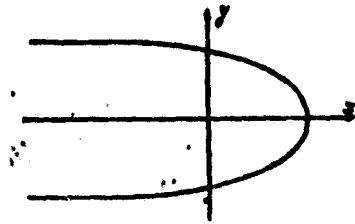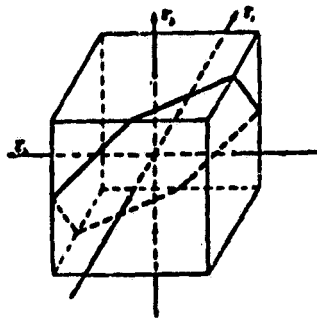behavior for positive x (tension) and negative (pressure) is

Figure 3.

Figure 4.

due to the fact that tearing occurs for tension in all directions,
but there is no compression deformation for uniform pressure.
Therefore, it is not very likely that there is an analog difference
for the limit of the elastic behavior. In addition, since we
are primarily concerned with states with a large average pressure,

it will be allowable to assume that the horizontal asymptotes
of Figure 3 are the important boundaries. This often-
mentioned assumption leads to the elastic limit:

$$|\tau_1| \leq K, \quad |\tau_2| \leq K, \quad |\tau_3| \leq K. \tag{24}$$

The cube (24) is intercepted by the plane (22) in a
regular hexagon (Figure 4.), so that our condition d) is
satisfied.

We will now modify the Mohr theorem in another
direction. Only the corners of the hexagon (22), (24) have
been specified by experiments up to the present. These are
states where one of the τ is zero, and the absolute values of
the other two are the same. A straight line connection is
derived from the assumption that the average principal stress
or the smaller principal tangential stresses are not important
at all. This assumption does not seem to be that plausible
so that one could consider replacing the hexagon by a simpler
curve, that is, a circle which circumscribes it. Instead of the
cube (24), one then has the sphere:

$$\tau_1^2 + \tau_2^2 + \tau_3^2 = 2K^2. \tag{25}$$

In any case, (25) allows a much simpler analytical treatment,
and the difference with respect to (24) is not much greater than
the range of certainty of experiments performed up to the present.

3. Equations of Motion

The quantity ϱ will be called the specific mass of the body,
and X_x, X_y, X_z are the components of the specific volume force

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(gravity, etc.). Then the equations of motion are, in any case:

$$\rho\frac{du}{dt} = x_s - \frac{\partial p}{\partial x} + \frac{\partial \sigma'_s}{\partial x} + \frac{\partial \tau_s}{\partial y} + \frac{\partial \tau_y}{\partial z}$$

$$\rho\frac{dv}{dt} = x_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_s}{\partial x} + \frac{\partial \sigma'_y}{\partial y} + \frac{\partial \tau_s}{\partial z}$$ (I)

$$\rho\frac{dw}{dt} = x_s - \frac{\partial p}{\partial z} + \frac{\partial \tau_y}{\partial x} + \frac{\partial \tau_s}{\partial y} + \frac{\partial \sigma'_s}{\partial z}.$$

The six stress components $\sigma'_s \ldots \tau_s$ are expressed according to
(20) and (5) by the three velocities u, v, w as follows:

$$\sigma'_s = k\frac{\partial u}{\partial x}, \quad \sigma'_y = k\frac{\partial v}{\partial y}, \quad \sigma'_s = k\frac{\partial w}{\partial z};$$ (II)

$$\tau_s = \tfrac{1}{2}k\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad \tau'_y = \tfrac{1}{2}k\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \quad \tau_s = \tfrac{1}{2}k\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

In order to eliminate p, we can use the continuity equation,   /591
just like in hydromechanics

(III)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Here we have assumed incompressibility according to theorem b)
and the related assumptions. However, within the framework of
our theory, we could very easily treat the more general case.
   The assumption (I) to (III) agrees completely with the
one for viscous liquids, but there the quantity k is the given

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viscosity coefficient, and in our case it is a reaction variable, which can only be calculated by knowing the motion itself. For this we use the theorem that the stress remains at the limit of elasticity during plastic deformation.

If one assumes the boundary to be a circle in the form (25), and if we substitute the value (14), then one obtains:

$$(\sigma_x' + \sigma_y' + \sigma_z')^2 - 3(\sigma_x'\sigma_y' + \sigma_y'\sigma_z' + \sigma_z'\sigma_x') + 3(\tau_x^2 + \tau_y^2 + \tau_z^2) = 4K^2. \qquad (26)$$

From the last form of expression (14), it follows that the quantities σ can be replaced by the variable σ' . If one adds the first three of equations (II), and if we consider (III), then we find:

$$\sigma_x' + \sigma_y' + \sigma_z' = 0, \qquad (27)$$

so that (26) is reduced to:

$$(IV)$$

$$\frac{4K^2}{3} = \tau_x^2 + \tau_y^2 + \tau_z^2 - (\sigma_x'\sigma_y' + \sigma_y'\sigma_z' + \sigma_z'\sigma_x').$$

If here one substitutes the values from (II), then one obtains the desired equation for k. The equations (I) through (IV) are the complete system of equations of motion for plastic-deformable bodies.

As a boundary condition we have to add the following: The specification of the velocity components u, v, w for any surface point. This can be replaced over the entire surface or over part of it by specifying the surface stress.

In the case of plane motion, our theorem is reduced to the theorem of Saint-Venant. This is partly due to the fact that in the plane case, the difference between the elastic limit according to (24) (hexagon) or according to (26) (circle) vanishes. This is because one only has two principal tangential stresses τ_1, τ_2 with

$$\tau_1 + \tau_2 = 0, \tag{28}$$

so that $\tau_1^2 + \tau_2^2 \leq 2K'$ says the same as $|\tau_1| \leq K, |\tau_2| \leq K$.

Equations (I) through (IV) can very easily be written in /592 terms of vector notation. If $\bar{v}$ is the velocity vector, $\bar{k}$ the vector of specific force, then we have:

$$\varrho \frac{d\bar{v}}{dt} = \bar{k} - \operatorname{grad} p + \mathscr{P}\bar{\vartheta}', \tag{I'}$$

$$\bar{\vartheta}' = k\bar{\lambda}, \tag{II'}$$

$$\operatorname{div} \bar{v} = 0, \tag{III'}$$

$$-(\bar{\sigma}')_2 = \frac{4K'}{8}. \tag{IV'}$$

Here the quantity $\mathscr{P}$ in (I') is the differentiation to be performed on the dyad, determined by (I). The index 2 in (IV') is intended to indicate that out of the dyad $\bar{\vartheta}'$ one must take the second orthogonal invariant shown in equation (8) of paragraph 1.

From (I') to (IV') one can also easily eliminate $\bar{\vartheta}'$,

and one obtains:

$$\rho \frac{d\overline{v}}{dt} = \overline{\kappa} - \operatorname{grad} p + \mathscr{p}(k\overline{\lambda}), \qquad \text{(a)}$$

$$\operatorname{div} \overline{v} = 0, \qquad \text{(b)}$$

$$k' = -\frac{4K'}{8(\overline{\lambda})_s}. \qquad \text{(c)}$$

If one performs scalar multiplication of (I') and $\overline{v}$ and integrates with respect to volume, then one finds that the dissipation function is represented by (21) after carrying out the corresponding conversion. This therefore proves the agreement of the theorem with our assumption c) in paragraph 2.

Straßburg i. E., October 4, 1913.

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[2] O. Mohr, Abhandl. a. d. Gebiete der techn. Mechanik, Berlin 1906, p. 187.

[3] Zeitschr. d. Vereines deutscher Ingenieure 1911, p. 1749.