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COMPARATIVE EFFICIENCY OF FINITE, BOUNDARY, AND HYBRID

ELEMENT METHODS IN ELASTOSTATICS

by

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ABSTRACT

The comparative computational efficiencies of the finite
element (FEM), boundary element (BEM), and hybrid boundary
element-finite element (HBFEM) analysis techniques are evaluated
for representative bounded domain "interior" and unbounded domain
"exterior" problems in elastostatics. Computational efficiency
is carefully defined in this study as the computer time required
to attain a specified level of solution accuracy. The study
found the FEM superior to the BEM for the interior problem, while
the reverse was true for the exterior problem. The hybrid
analysis technique was found to be comparable or superior to both
the FEM and BEM for both the interior and exterior problems.

INTRODUCTION

The overall purpose of the research described in this report is the investigation of the feasibility and advantages of incorporating boundary element analysis techniques into a finite element environment and, ultimately, into the NASTRAN finite element code. The expected benefits from combining boundary element and finite element analysis techniques include: (a) more effective analyses, both in terms of computational efficiency and numerical accuracy; and (b) simpler analysis preparation, especially in an interactive computer graphics modeling environment where boundary elements can conveniently be used to discretize just the surfaces of complex three-dimensional geometries.

The past year's effort has focused on two areas: (a) the definition of the modifications required to incorporate a boundary element formulation into a conventional finite element context; and (b) the quantitative evaluation of the relative computational efficiencies of the pure boundary element, finite element, and hybrid boundary element-finite element analysis methods.

BACKGROUND

All numerical methods for stress analysis problems are based upon approximations that transform the underlying differential equations for equilibrium and strain compatibility into a set of simultaneous algebraic equations amenable to solution on a computer. The nature of this approximation forms the principal distinction between the two common numerical techniques used today, the finite element and boundary element methods. In the finite element method (FEM), the assumed displacement fields within each subregion or "element" of the problem domain form the basis of the approximation; the entire problem domain (including the boundaries) must be discretized into elements to obtain a solution. In the boundary element method (BEM), on the other hand, the approximation is based on simplified assumed variations of the prescribed conditions along each segment of the boundary contour; therefore, only the problem boundaries need be discretized. Discretizing only the boundaries instead of the entire problem domain reduces the effective dimension of the analysis problem by one order. For example, a three-dimensional geometry, which would be analyzed using three-dimensional volume elements in the FEM, can be analyzed using two-dimensional surface elements in the BEM.

This difference in the underlying approximation "philosophy" gives each method an inherent set of advantages and limitations. The major advantages of the finite element method are:

(1)   a long history of development and application and an associated accumulation of knowledge and experience;

(2)   the ability to model complex geometries, material behavior,
       geometric nonlinearities, and loading conditions;

(3)   a "convenient" set of equations to solve; the equations are
       usually narrowly banded and symmetric and consequently have
       relatively small machine storage and computation
       requirements.

The countervailing limitations include:

(1)   a very large number of equations to solve (particularly for
       three-dimensional problems) since the entire problem domain
       must be discretized;

(2)   the inability to model infinite domain problems without
       resorting to artificial truncation of the problem geometry;

(3)   difficulty in obtaining accurate solutions to problems
       involving singularities.

In contrast, the principal advantages of the boundary element
are:

(1)   a relatively small system of equations to solve, since only
       the problem boundary is discretized;

(2)   improved accuracy within the domain at points away from the
       boundary approximations;

(3)   the direct incorporation of infinite domain boundary
       conditions;

(4)   the ability to obtain accurate solutions to problems
       involving singularities.

The principal limitations of the boundary element method include:

(1)   difficulty in incorporating nonlinear material behavior,
       material inhomogeneity, and/or dynamic effects;

(2)   an "inconvenient" set of equations to solve (the coefficient
       matrix is usually unsymmetric and fully populated);

(3)   a large computational effort for assembling the system
       coefficient matrices.

       Which numerical method is best for any specific problem is
often unclear.   Conventional wisdom based on the general
qualitative merits listed above suggests that the finite element
method is more suited to finite domains and/or problems involving
nonlinear behavior while the boundary element method is most
effective for infinite domains and linear behavior.   Often the
best approach is of a mixed or hybrid nature that takes optimal
advantage of the particular characteristics of each (Zienkiewicz,
Kelly, and Bettess, 1977; Kelly, Mustoe, and Zienkiewicz, 1979;

Brady and Wassyng, 1981). Figure 1 illustrates typical problems that may profit from a hybrid boundary element-finite element discretization.

PREVIOUS STUDIES

Many investigations into the relative efficiencies of finite element and boundary element methods have been conducted over the past several years. Most of these studies have been largely qualitative, although a few have provided comparative numerical results and associated costs.

Bettess (1980) considered the simple problem of a square plate discretized using boundary element or finite element meshes having the same node densities along the boundaries. Bettess counted the number of calculations required to solve the systems of·equations for each method and concluded that the "dimensionality " advantage of boundary elements over finite elements is "more apparent than real" and that the BEM is computationally cheaper only for problems with a very large number of degrees of freedom. Bettess' arguments are inconclusive, however, in that the computation counts in his study include only the equation solution step; the computations required to formulate the coefficient matrices were not considered. The more serious limitation of Bettess' study, however, is that his comparisons of the two methods were not made at the same level of solution accuracy.

Beer (1983) performed a more realistic comparison of boundary element, finite element, and hybrid analyses by considering the problem of a two-dimensional mine opening--i.e., an "exterior" problem in boundary element terminology. Although solution accuracy was not controlled explicitly, the results for selected stresses and displacements varied by less than 2% among all analyses. Beer found that the boundary element, finite element, and hybrid analyses for this problem all required about the same magnitude of computer time (within $\pm 10\%$). He also noted that the finite element analysis required over three times the input data needed for the boundary element case.

Radaj, Mohrmann, and Schilberth performed a study similar to Beer's for two-dimensionsal "interior" problems typical of those encountered in industry. They found that a finite element analysis may require several times the CPU time needed for a boundary element analysis of comparable accuracy. They also commented on the large expenditure of time for meshing and input preparation in the finite element method as compared to the boundary element analysis: "It is exactly this difference in expenditure, which is quite obvious although it is hard to measure accurately, that is responsible for the boundary element method being more economical for practical application in industry."

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Mukherjee and Morjaria (1984) investigated the twin issues of computational efficiency and accuracy for the solution of Laplace's equation with mixed boundary conditions in a two-dimensional square domain. Solution accuracy was defined as the mean square error of the computed solution variable and its derivatives evaluated at the node points in the finite element analysis and at the corresponding domain locations in the boundary element solution. They observed that "for the same level of discretization, the BEM results are more accurate than the FEM". If the solution is required only on the boundaries and at a few interior points, they concluded that the BEM requires less computational effort than the FEM; if the solution is required throughout the domain, the FEM is most efficient. Mukherjee and Morjaria also note that problem symmetry favors the BEM because the symmetry boundaries need not be discretized.

Hume, Brown, and Deen (1985) performed a study similar to Mukherjee and Morjaria's for the case of Laplace's equation in a domain with a moving boundary (a formulation encountered in surface wave/solidification/capillary front problems). There comparisons were all made at the same solution accuracy, defined as the RMS error in the solution variable along the moving boundary. They concluded that the FEM was computationally more efficient for bounded interior problems except when very high solution accuracy is desired. For unbounded problems, the boundary element formulation incorporating the problem symmetry conditions was found to be the most efficient.

In reviewing these previous investigations, it is clear that there is little consensus regarding which analysis method is most efficient for any given problem category. For interior problems, the Radaj, Mohrmann, and Schilberth and the Mukherjee and Morjaria studies found the BEM more effective than the FEM while the Hume, Brown and Deen and the Bettess studies concluded just the opposite. For exterior problems (which conventional wisdom claims should be most advantageous for the BEM), Hume, Brown, and Deen found that the BEM was more effective than the FEM while Beer found that the two analysis methods were roughly comparable. There is general agreement that the BEM requires considerably less effort for input preparation than does the FEM.

PURPOSE OF PRESENT STUDY

The failure of previous comparisons of finite element and boundary element analyses to yield a consensus regarding their relative merits is due in part to differences in the problems analyzed (e.g., elastostatics vs. potential problems). To a larger degree, however, these comparisons have been flawed by the inadequate consideration and control of solution accuracy. The BEM is generally more accurate than the FEM at the same level of discretization for a given problem (Mukherjee, 1982). This higher accuracy compensates to some extent for the greater computational effort per equation required in the boundary element method.

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The purpose of the present study is to compare the computational efficiency of the finite element and boundary element methods at equivalent solution accuracy. This comparison is performed for interior and exterior two-dimensional elastostatic problems. In addition, the relative efficiency of the hybrid boundary element-finite element method (HBFEM) will be evaluated. At this point, very little is known regarding the performance of the hybrid analysis method.

MODEL PROBLEMS

The model problems selected for the comparison studies satisfy two major criteria: (a) existence of an analytical solution for evaluating the accuracy of the numerical solutions; and (b) geometry suited to simple discretization without the introduction of complications such as mesh gradation, zoning, etc.

The model problem for the bounded domain or "interior problem" case is a solid cylinder subjected to compressive normal pressures applied over finite arcs at opposite ends of the cylinder diameter (Figure 2a) This problem corresponds to the standard split cylinder tension test. The cylinder is assumed to be homogeneous, isotropic, and linear elastic. The analytic solution to this problem is expressed in series form (see Jaeger and Cook, 1976):

$$\sigma_r = \frac{2\alpha P}{\pi} + \frac{2P}{\pi} \sum_{m=1}^{\infty} (\frac{r}{R})^{2m-2} \{1 - (1 - \frac{1}{m})(\frac{r}{R})^2 \} \sin 2m\alpha \cos 2m\theta \qquad (1a)$$

$$\sigma_\theta = \frac{2\alpha P}{\pi} - \frac{2P}{\pi} \sum_{m=1}^{\infty} (\frac{r}{R})^{2m-2} \{1 - (1 + \frac{1}{m})(\frac{r}{R})^2 \} \sin 2m\alpha \cos 2m\theta \qquad (1b)$$

$$\tau_{r\theta} = \frac{2P}{\pi} \sum_{m=1}^{\infty} \{ (\frac{r}{R})^{2m} - (\frac{r}{R})^{2m-2} \} \sin 2m\alpha \sin 2m\theta \qquad (1c)$$

in which P is the radial pressure, α is the angle defining the arc segment along which the pressure is applied, R is the radius of the cylinder, and (r,θ) are the polar coordinates of any point.

The radial displacements u at at the boundary of the cylinder r=R for plane strain conditions are given as:

$$U = \frac{PR}{2\pi Gr} \left\{ \alpha(\kappa-1) + \sum_{m=1}^{\infty} \frac{1}{m} \left[\frac{\kappa}{2m+1} + \frac{1}{2m-1} \right] \right\} \sin 2m\alpha \cos 2m\theta \qquad (2)$$

in which $\kappa = (3 - 4\nu)$ and λ and G are Lame's constants.

The model problem for the unbounded domain or "exterior problem" case is a circular cavity in an infinite domain subjected to a uniform internal pressure (see Figure 2b). The material surrounding the cavity is assumed to be homogeneous, isotropic, and linearly elastic material. The analytical solution to this problem can be expressed as follows (see, e.g., Poulos and Davis, 1974):

$$\sigma_r = P \left(\frac{R}{r}\right)^2 \qquad (3a)$$

$$\sigma_\theta = -P \left(\frac{R}{r}\right)^2 \qquad (3b)$$

$$\tau_{r\theta} = 0 \qquad (3c)$$

in which P is the applied internal pressure, R is the radius of the cavity, and r is the radial distance to the point of interest.

The radial displacements for plane strain conditions are given by:

$$u = \frac{P(1+\nu)}{E} \left(\frac{R}{r}\right)^2 \qquad (4)$$

in which E and ν are the elastic constants for the material surrounding the cavity.

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Typical boundary element, finite element, and hybrid FE-BE meshes for the interior and exterior problems are illustrated in Figure 3. Note that the hybrid mesh is the same as the boundary element mesh--i.e., it contains no finite elements. Nevertheless, the hybrid analysis is expected to give results different from the pure boundary element analysis because of differences in the underlying formulation, as described in more detail in the next section.

A common difficulty in finite element analyses of infinite domain "exterior" problems is that the mesh must be finite in size. The finite element mesh for the exterior problem (Figure 3c) is truncated at a distance of six radii from the center of the cylindrical cavity consistent with the modeling guidelines suggested by Kulhawy (1974). This mesh truncation introduces "modeling" errors in addition to the inherent finite element approximation errors. For a fair evaluation of the finite element analysis for the exterior problem, the numerical results should therefore be compared not to the infinite domain analytical solution (Eqs. 3 and 4) but to the "constrained thick-walled cylinder" problem illustrated in Figure 4. The analytical solution for these conditions can be easily derived:

$$\sigma_r = P\ [\frac{A}{r^2} + B] \tag{5a}$$

$$\sigma_\theta = P\ [-\frac{A}{r^2} + B] \tag{5b}$$

$$\tau_{r\theta} = 0 \tag{5c}$$

in which a is the internal radius, b is the external radius, and:

$$A = \frac{a^2 b^2 (1 - 2\nu)}{a^2 + B^2 (1 - 2\nu)} \tag{5d}$$

$$B = (1 - \frac{A}{a^2}) \tag{5e}$$

The corresponding radial displacements for plane strain conditions are given as:

$$u = \frac{P}{Er(1-\nu)}\left[-A + Br^2(1-2\nu)\right] \qquad (6)$$

NUMERICAL FORMULATIONS

The finite element algorithm is based on the standard virtual displacement formulation (see, e.g., Bathe and Wilson, 1976). The well-known matrix form of finite element equations is:

$$[K]\{U\} = \{F\} \qquad (7)$$

in which [K] is the global stiffness matrix, {U} is the nodal displacements vector, and {F} is the nodal force vector. Four-node quadrilateral isoparametric elements with linear interpolation were used throughout this study.

The boundary element formulation employed in this study is based on the weighted residual procedure described by Brebbia (Brebbia, 1978; Brebbia and Walker, 1980; Brebbia, Telles, and Worbel, 1984; this formulation is often referred to as the "direct BEM formulation" in the literature). The matrix form of the BEM for elastostatics problems in the absence of body forces is given as:

$$[H]\{U\} = [G]\{P\} \qquad (8)$$

in which [H] and [G] are the boundary element influence coefficient matrices, {U} is the nodal displacements vector, and {P} is the nodal tractions vector. For a general mixed boundary value problem, some elements of {U} and {P} will be prescribed boundary conditions. Eq. (8) can then be rearranged such that all unknown boundary quantities appear on the left side of the equation and all prescribed boundary conditions on the right:

$$[A]\{X\} = \{Y\} \qquad (9)$$

in which {X} is the vector of unknown boundary traction and displacement quantities, [A] is the matrix of influence coefficients corresponding to the entries in {X}, and {Y} is the product of the vector of prescribed boundary quantities premultiplied by the matrix of corresponding influence coefficients. Two-noded boundary elements with linear interpolation functions for boundary displacements and tractions were used throughout this study.

In the hybrid finite element-boundary element formulation, the boundary element region is treated as a "super" finite element that can be incorporated into the standard finite element global stiffness matrix. Consider a problem discretized in part by finite elements and in part by boundary elements as shown in Figure 5. Rearranging Eq. (8) for the boundary element domain gives:

$$[G]^{-1} \, [H] \, \{U\} = \{P\} \tag{10}$$

The nodal traction vector $\{P\}$ can be converted to an equivalent nodal force vector $\{F\}$ through use of the transformation matrix $[M]$ (Brebbia, Telles, and Worbel, 1984):

$$\{F\} = [M] \, \{P\} \tag{11}$$

Multiplying both sides of Eq. (10) by $[M]$:

$$[M] \, [G]^{-1} \, [H] \, \{U\} = [M] \, \{P\} = \{F\} \tag{12}$$

Defining:

$$[K_B] = [M] \, [G]^{-1} \, [H] \tag{13}$$

Eq. (12) can then be expressed as:

$$[K_B] \, \{U\} = \{F\} \tag{14}$$

Eq. (14) is in a form similar to the standard finite element formulation given by Eq. (7). However, $[K_B]$ is not in general symmetric and thus cannot be solved using the symmetric equation solvers found in standard finite element codes. Symmetry can be imposed on Eq. (14) through the procedures suggested by Brebbia, Telles, and Worbel (1984):

$$[K_{BS}] = (\, [K_B] + [K_B^T] \,) \, / \, 2 \tag{15}$$

or

$$[K \quad] = (\, [M] \, [G]^{-1} \, [H] + (\, [M] \, [G]^{-1} \, [H] \,)^T \,) \, / \, 2 \tag{16}$$

in which $[K_{BS}]$ is the symmetric equivalent stiffness matrix for the boundary element region. This equivalent stiffness matrix can be assembled into the FEM stiffness matrix using standard procedures. Assuming that similar interpolation functions are used for both the boundary elements and finite elements along the interface, displacement compatibility is ensured.

All boundary element analyses of the model problems were performed using the computer program listed in Brebbia, Telles, and Worbel (1984). Standard Gauss elimination procedures were used to solve the full, unsymmetric equations in the boundary element analyses. The computer program code CBFE (Coupled Boundary-Finite Element method) was developed to perform all

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finite element and hybrid analyses for the model problems. CBFE is based on the simple finite element program STAP described in Bathe and Wilson (1976). For the banded and symmetric equations encountered in the finite element and hybrid analyses, the specialized Gauss elimination "skyline" equation solver COLSOL (Bathe and Wilson, 1976) was employed. All computations were performed on the Sperry 1100/92 mainframe computer system at the University of Maryland Computer Science Center.

SOLUTION ERROR DEFINITION

Numerical solution errors for stresses and displacements are defined in terms of Euclidean error norms as follows:

$$\varepsilon_\sigma = [\ \sum_{i=1}^{M} (\sigma_i - \overline{\sigma}_i)^2 / \sum_{i=1}^{M} \overline{\sigma}_i^2\ ]^{1/2} \qquad (17a)$$

$$\varepsilon_u = [\ \sum_{i=1}^{M} (u_i - \overline{u}_i)^2 / \sum_{i=1}^{M} \overline{u}_i^2\ ]^{1/2} \qquad (17b)$$

in which ε_σ and ε_u are the error norms for stresses and displacements, respectively; σ_i and u_i are the numerical stress and displacement solution at a specific point i; and $\overline{\sigma}_i$ and $\overline{u}_i$ are the corresponding analytical values for the stress and displacement at point i. The summations are carried out over a representative sample of M solution points.

For the interior problem, the M sample points for the stress error norm calculations were distributed at equally spaced intervals along the horizontal and vertical diameters of the cross-section (Figure 6a). In the finite element analyses, these sample point locations were adjusted to coincide with the location of the nearest element integration point (Figure 6b). Since the analytical solution for the interior problem gives displacements only at r=R, all M sample points for the displacement error norm calculation were equally spaced around the circumferential boundary.

For the exterior problem, the M sample points were distributed at equally spaced intervals along a radial line for both the stress and displacement error norm computations (Figure 7a). As in the interior problem, these sample point locations were adjusted for the finite element analyses to coincide with the nearest element integration point (Figure 7b).

As mentioned earlier, the finite element mesh for the exterior problem is truncated at a distance of six times of

cavity radius. This mesh truncation introduces an additional "modeling" error in the analysis; in other words, the finite element model does not correspond to the actual problem of a cavity in an infinite domain. In order to eliminate this modeling error, the analytical solution for the constrained thick-walled cylinder was used in the computations of the error norms for the finite element analyses.

The cost of a numerical analysis in its broadest sense includes data preparation (preprocessing) time, execution CPU time, and postprocessing time. Human labor time and machine storage demands must also be considered. Many of these cost components are difficult to quantify precisely. In this study, only the CPU time required for forming and solving the system of equations is considered. Note that the CPU time required for calculation of stresses is not included in these comparisons.

RESULTS

The model interior and exterior problems were analyzed using the finite element, boundary element, and hybrid analysis methods for several meshes representing a range of discretization refinement. Mesh refinement is defined as the total number of elements, N. For the boundary element and hybrid analysis meshes, all elements lie along the boundary and are of equal length. For the finite element meshes, the elements are all approximately equidimensional. Symmetry conditions were not incorporated in any of the meshes (e.g., the full cylinder was analyzed in the interior problem).

Solution Convergence

Solution accuracy as a function of number of elements for the interior model problem is shown in Figure 8. As would be expected, all solutions converge toward the exact result with increasing N. However, the rates of convergence vary among the different numerical methods. For the same level of discretization (i.e., the same number of elements), the hybrid analysis method is the most accurate, followed by the BEM and FEM analyses, respectively.

Solution accuracy for the exterior problem is plotted in Figure 9. Both the BEM and hybrid analyses converge with increasing N toward the analytical values. The FEM results do not converge toward the analytical values for a cavity in an infinite domain as the number of elements increases. Instead, the numerical results slowly and asymptotically approach an error of approximately 8%, which represents is the result of the truncated mesh required for the exterior problem. The comparison of the FEM results with the constrained thick-walled cylinder analytical values gives better agreement, although the rate of convergence is still extremely slow. The hybrid analysis method is again the most accurate at any given degree of discretization.

Computational Efficiency

The CPU time required to obtain a given level of solution accuracy for the interior problem is given in Figures 10 and 11 for stresses and displacements, respectively. Defining computational efficiency as the CPU time required to obtain a specified level of solution accuracy, it is clear from the results in Figures 10 and 11 that the FEM is more efficient than the BEM for calculation of both stresses and displacements for the interior problem. For example, at the five percent stress error level the CPU time required for the BEM is 4 to 5 times greater than for the FEM. This trend is even more clear for the displacement calculations (Figure 11). The hybrid analysis is more computationally efficient than both the BEM and FEM for stress calculations. It is also significantly better than the BEM for displacement calculations, although less effective than the FEM.

Solution accuracy as a function of CPU time for the exterior problem is plotted in Figure 12 for stress computations and Figure 13 for displacements. The errors attributable to the truncated mesh in the FEM model are again quite evident. When the FEM results are compared to the constrained thick-walled cylinder solution the numerical results converge to the analytical values. Nevertheless, the overall performance of the FEM is still poor compared to the BEM. The hybrid analysis is slightly more efficient than the BEM except in the range of very small solution errors.

From an overall viewpoint, the hybrid analysis seems to be competitive with or superior to the FEM and BEM formulations for both interior and exterior problems. This result is somewhat surprising, since the hybrid method requires the most calculations of all three methods: it includes all of the calculation steps in the BEM as well as the additional and substantial computations needed to compute the inverse of [G] and to enforce symmetry of the stiffness matrix. Apparently, the process of enforcing symmetry of the equivalent stiffness matrix eliminates much of the approximation error in the pure BEM. Specifically, the equivalent stiffness matrix in the hybrid formulation is not symmetric because different classes of "trial" functions and "test" functions (to use Brebbia's weighted residual terminology) are used to form the underlying BEM influence coefficients. However, lack of symmetry in the stiffness matrix violates energy conservation principles and thus represents a component of error in the approximation. The procedure for enforcing symmetry of the hybrid method stiffness matrix appears to negate much of this error, producing a more accurate solution.

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Solution Time Breakdown

The CPU time logs for various calculation phases in the FEM, BEM, and HBFEM analyses are given in Table 1. The HBFEM requires a large computational effort to obtain the equivalent stiffness matrix because of the need to invert the [G] matrix. However, this computational expenditure is partially compensated by the smaller amount of CPU time required for the solution of equations in the HBFEM as compared to the BEM and FEM analyses.

CONCLUSIONS

The performance of finite element, boundary element, and hybrid boundary element-finite element solution algorithms has been evaluated by comparing computation times at comparable levels of solution accuracy. Typical but simple interior and exterior problems were analyzed at varying levels of discretization refinement. For the interior problem, the FEM was found to be superior to the BEM; that is, the FEM required less computation time to achieve solutions of comparable accuracy. For the exterior problem, the BEM was more efficient than the FEM. The hybrid boundary element-finite element method was comparable or superior to both the BEM and FEM analyses for both exterior and interior problems. This exceptional performance of the hybrid method is attributed to a reduction in the approximation error resulting from the enforcement of symmetry in the equivalent stiffness matrix.

Even though input preparation requirements and results calculation times were not considered in this study, our experience confirms observations in the literature that the boundary element method (and hybrid method) requires significantly less time and effort for data preparation than does the finite element method. Stress and displacement calculation times for the boundary element and hybrid analysis methods were also less than those in the finite element method as long as results are needed only at a few selected points; the computational effort required by the boundary element and hybrid methods increases proportionally with the number of points at which the solution is sought.

ACKNOWLEDGEMENTS

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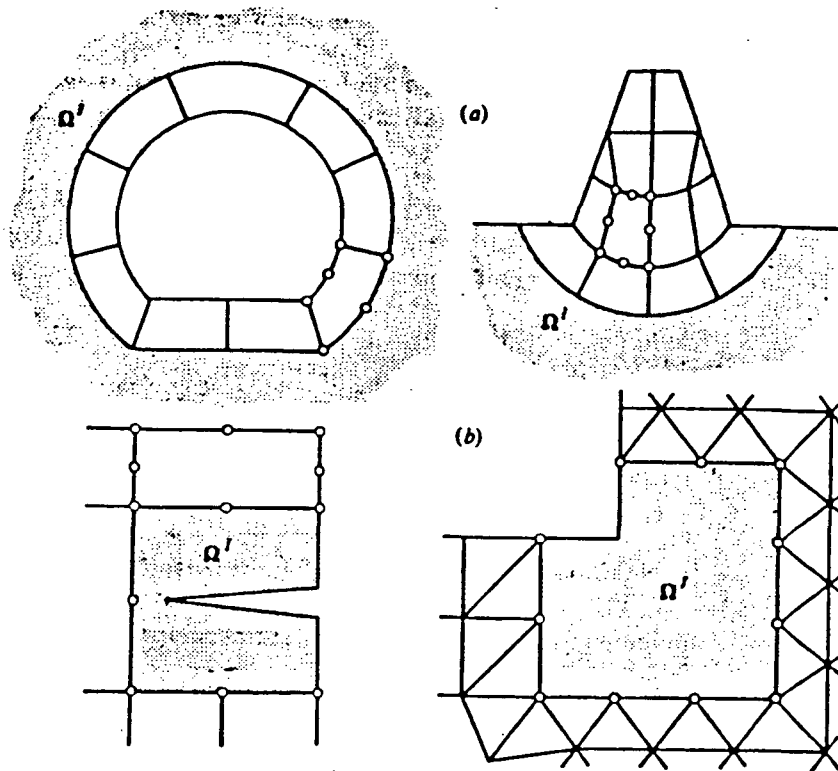
Figure 1.   Potential uses of hybrid
boundary element-finite
element analysis
(Zienkiewicz, 1977)
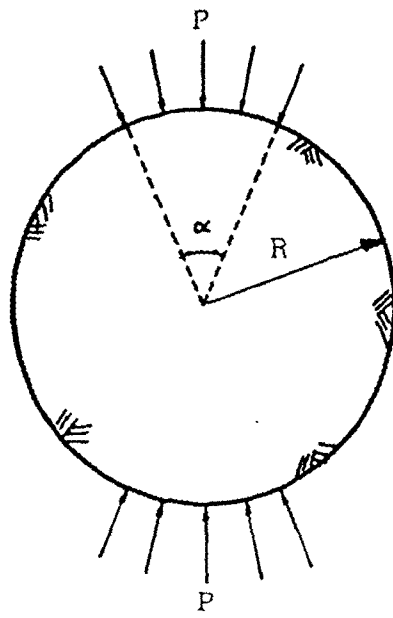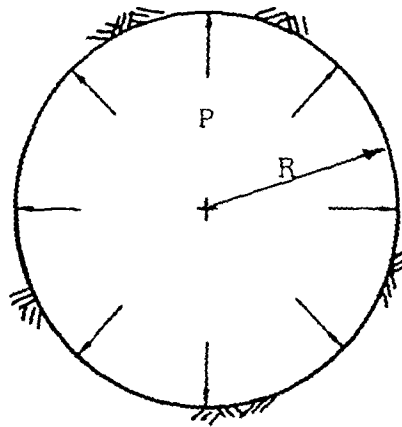
Figure 2. Model problems: (a) interior problem
                         (b) exterior problem

(b)

(a)
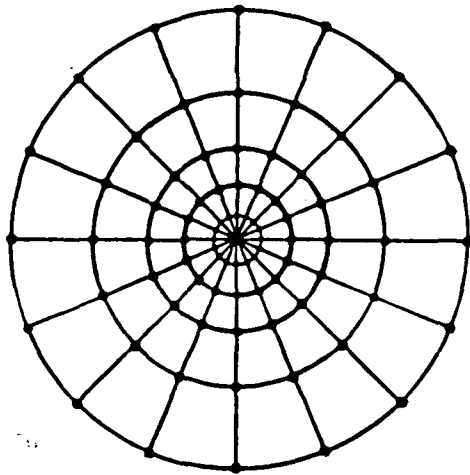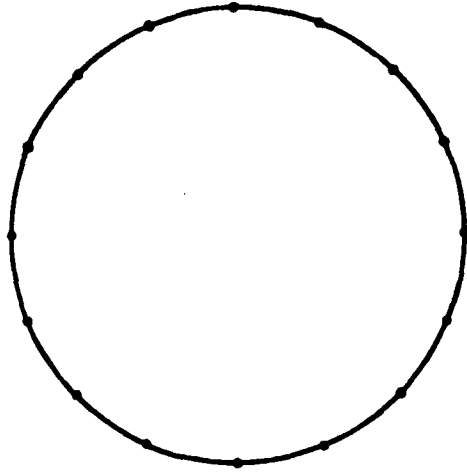
Figure 3. Typical meshes for Model problems
(a) FEM mesh for interior problem
(b) BEM and HBFEM mesh for both interior
and exterior problems
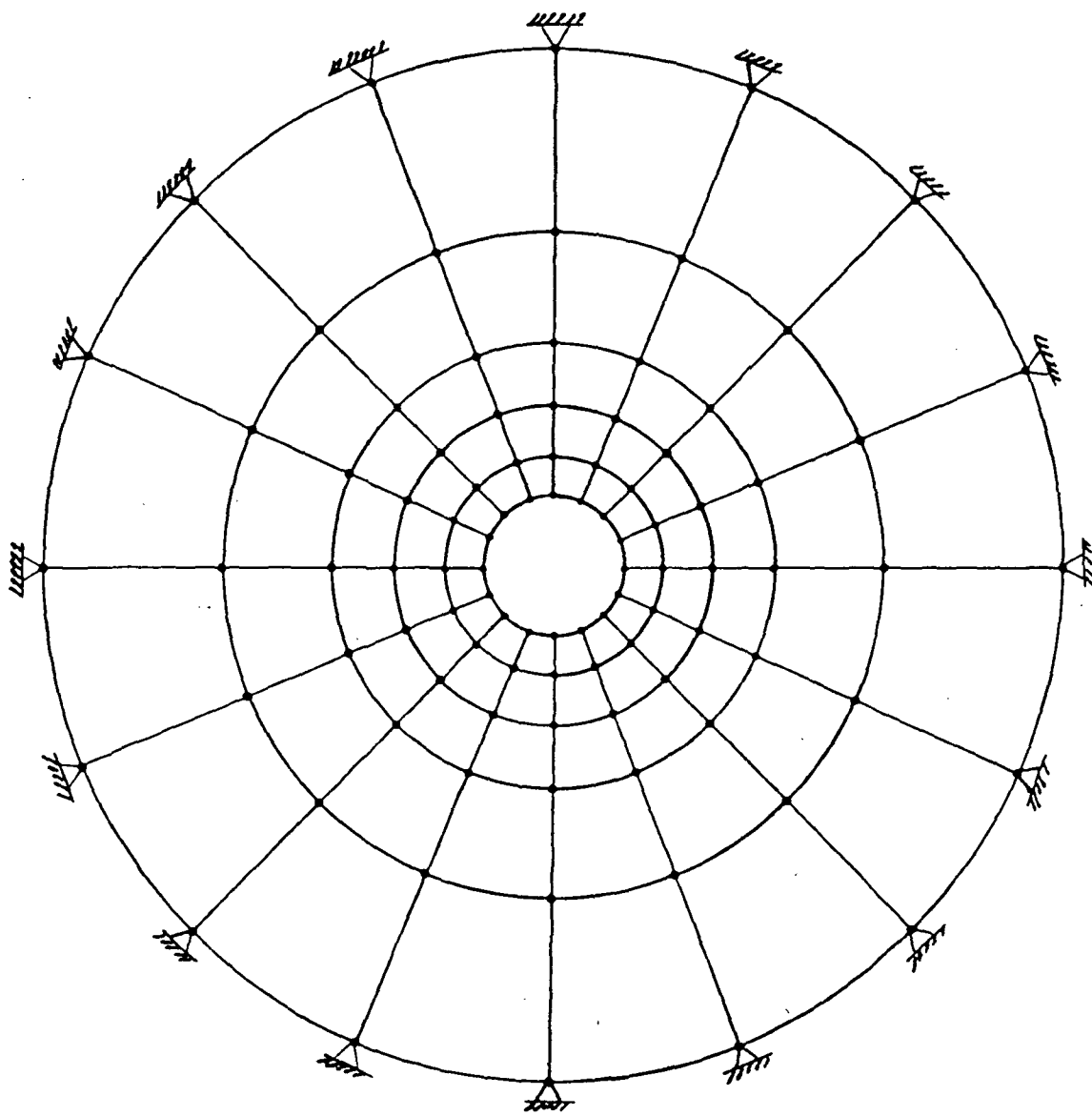
(c)

Figure 3(c). FEM mesh for exterior problem

$\bar{u} = 0$
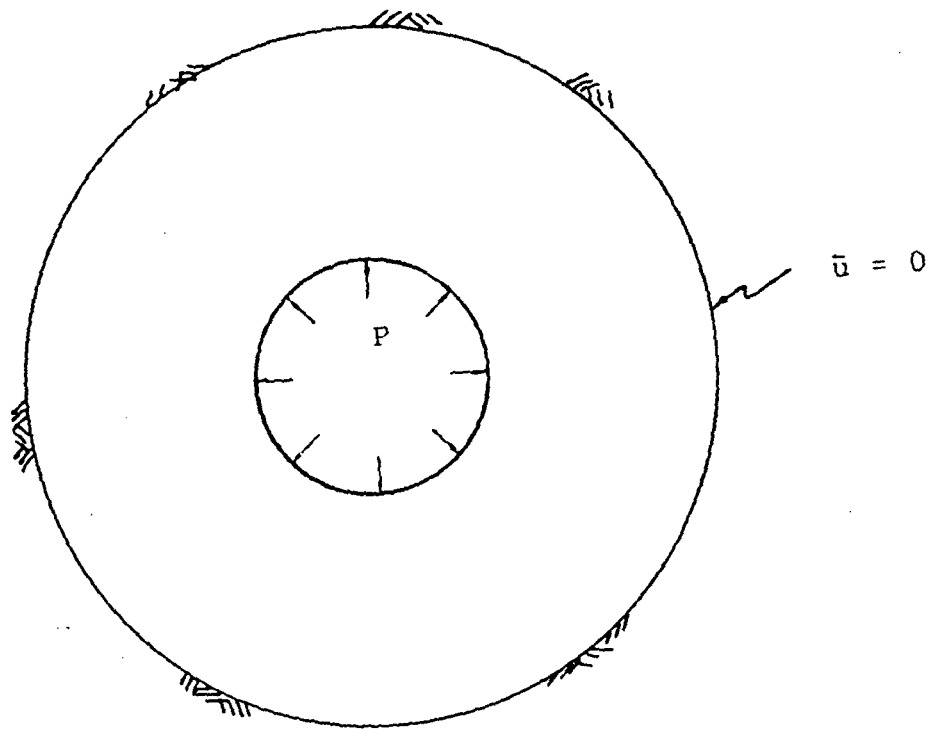
Figure 4. Constrained thick-walled cylinder

Finite element   Boundary element
domain   domain
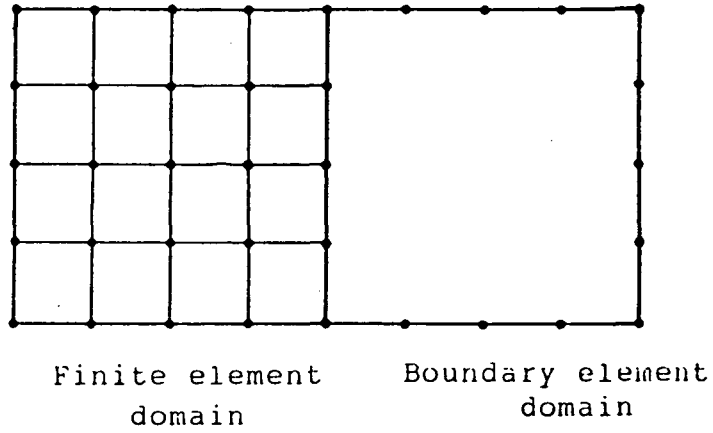
Figure 5.   Hybrid boundary element-finite
element discretization
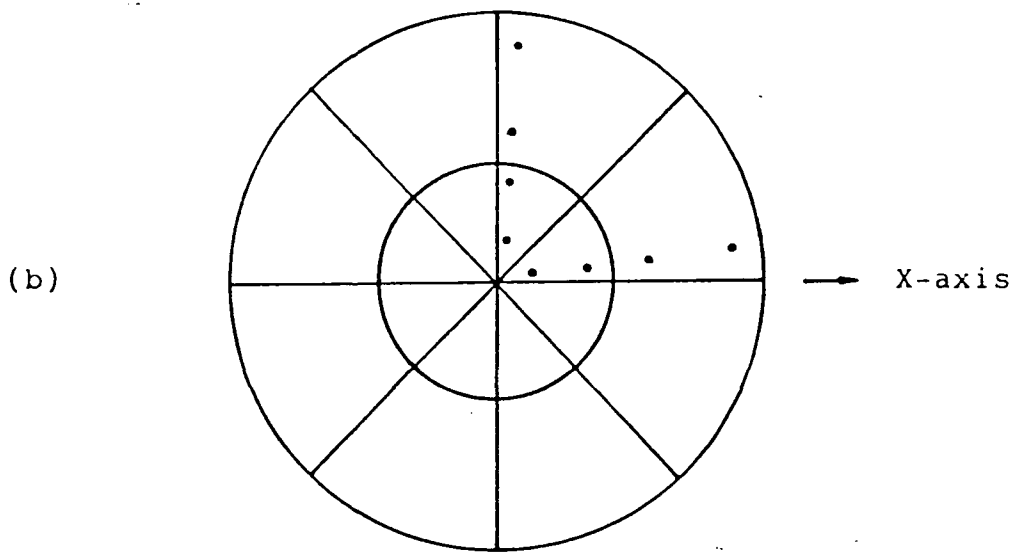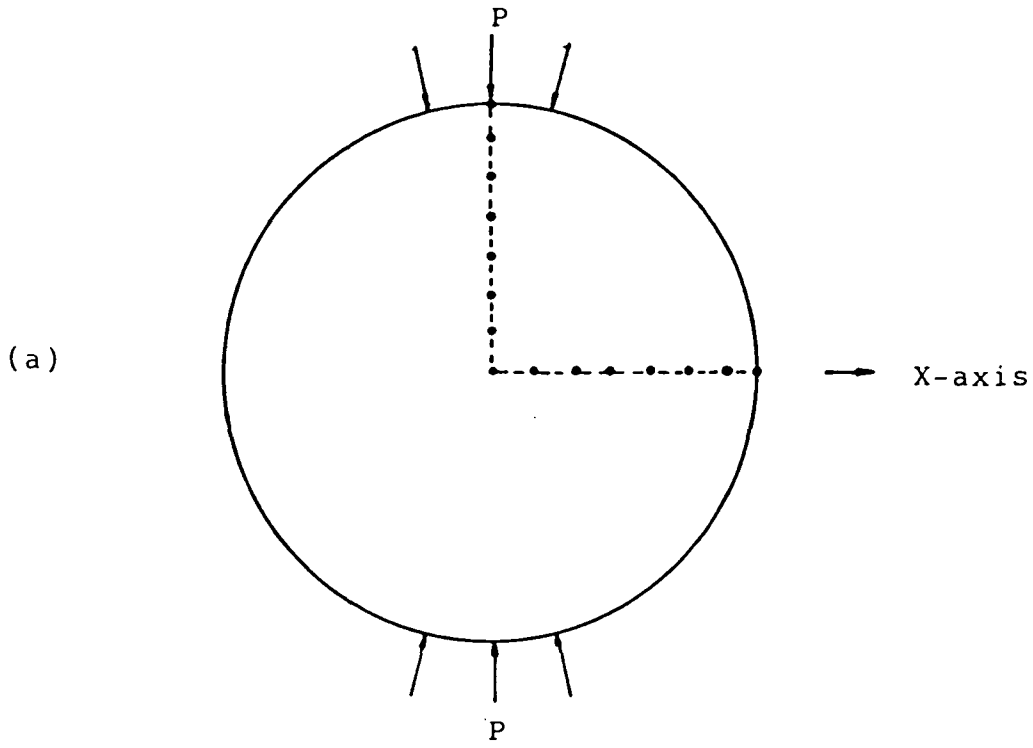
Figure 6. Stress calculation locations
         for interior problem:
         (a) boundary/hybrid element analysis
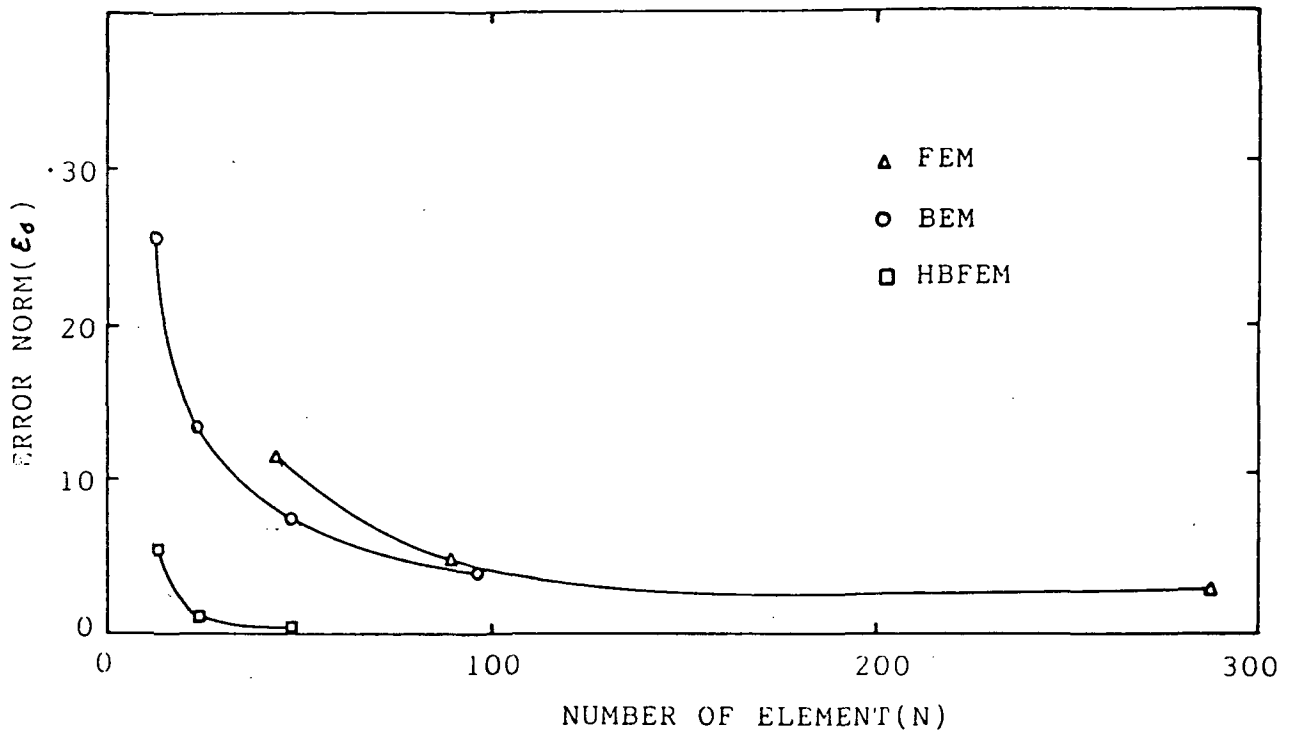   (b) finite element analysis

(a)

P

X

(b)

P

X
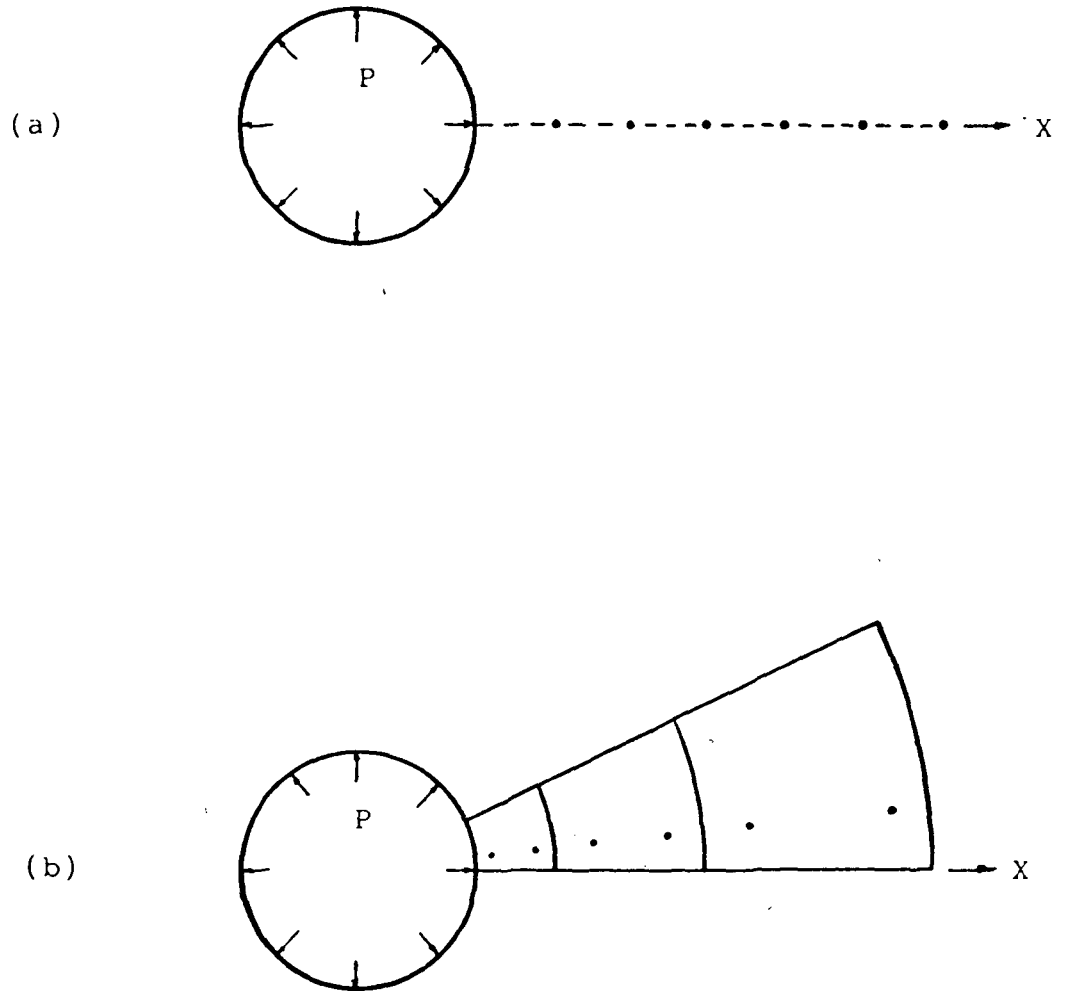
Figure 7. Stress calculation locations for
exterior problem:
(a) boundary/hybrid element analysis
(b) finite element analysis

Figure 8.   Stress error vs. number of elements for the interior
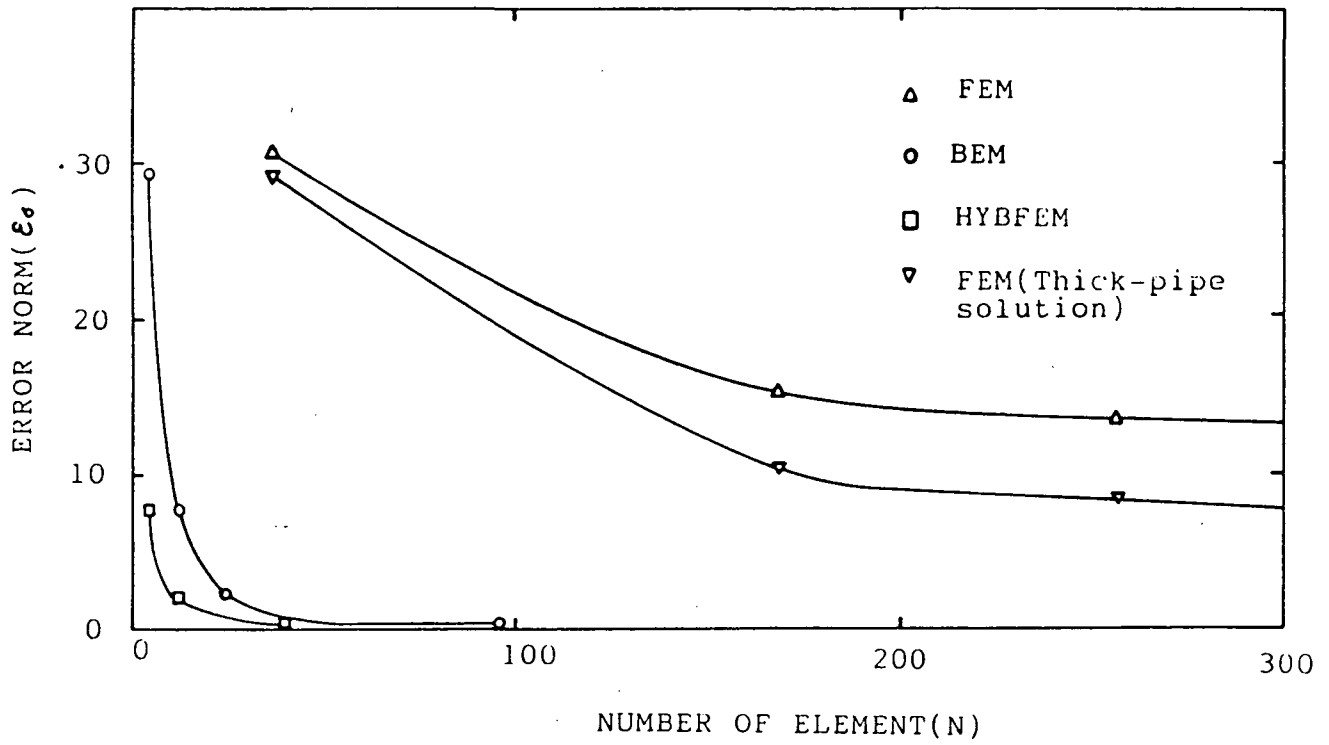problem

Figure 9.   Stress error norm vs. number of elements for exterior
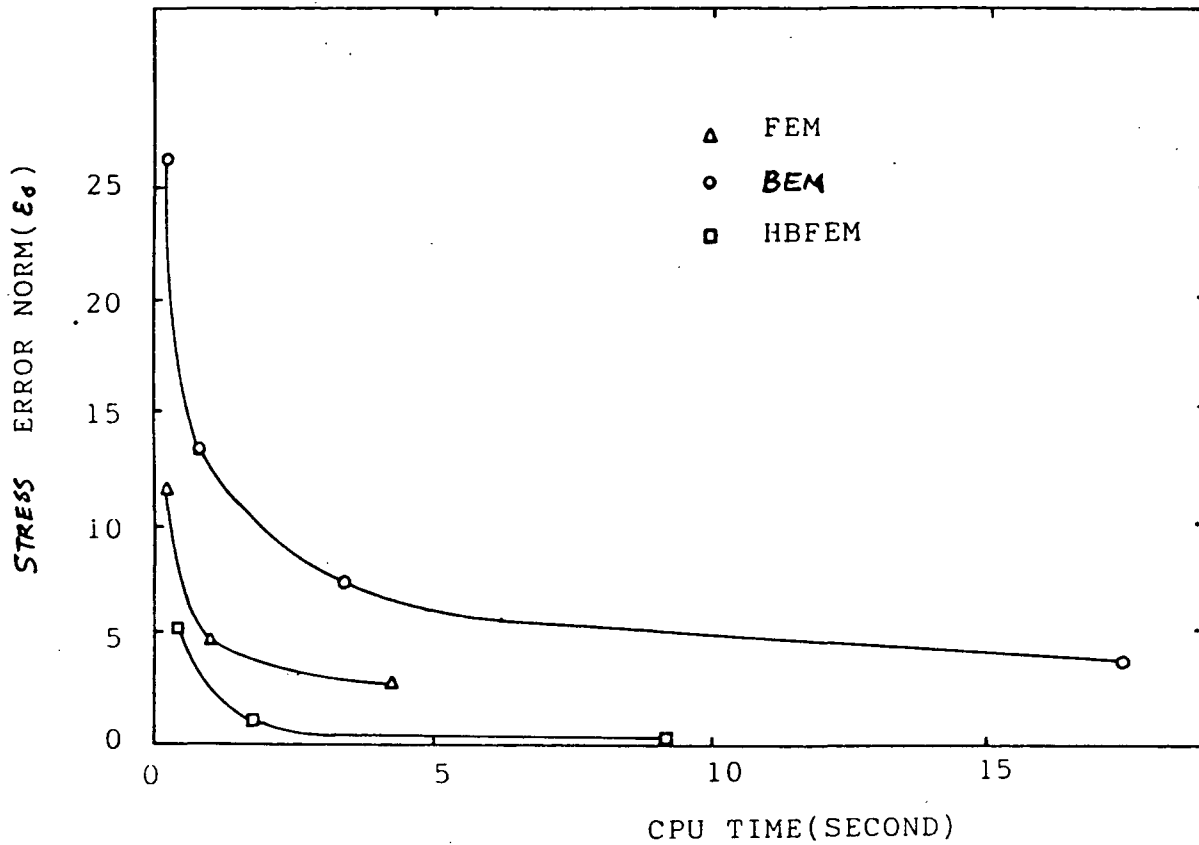problem.

Figure 10.   Computational efficiency for interior problem:
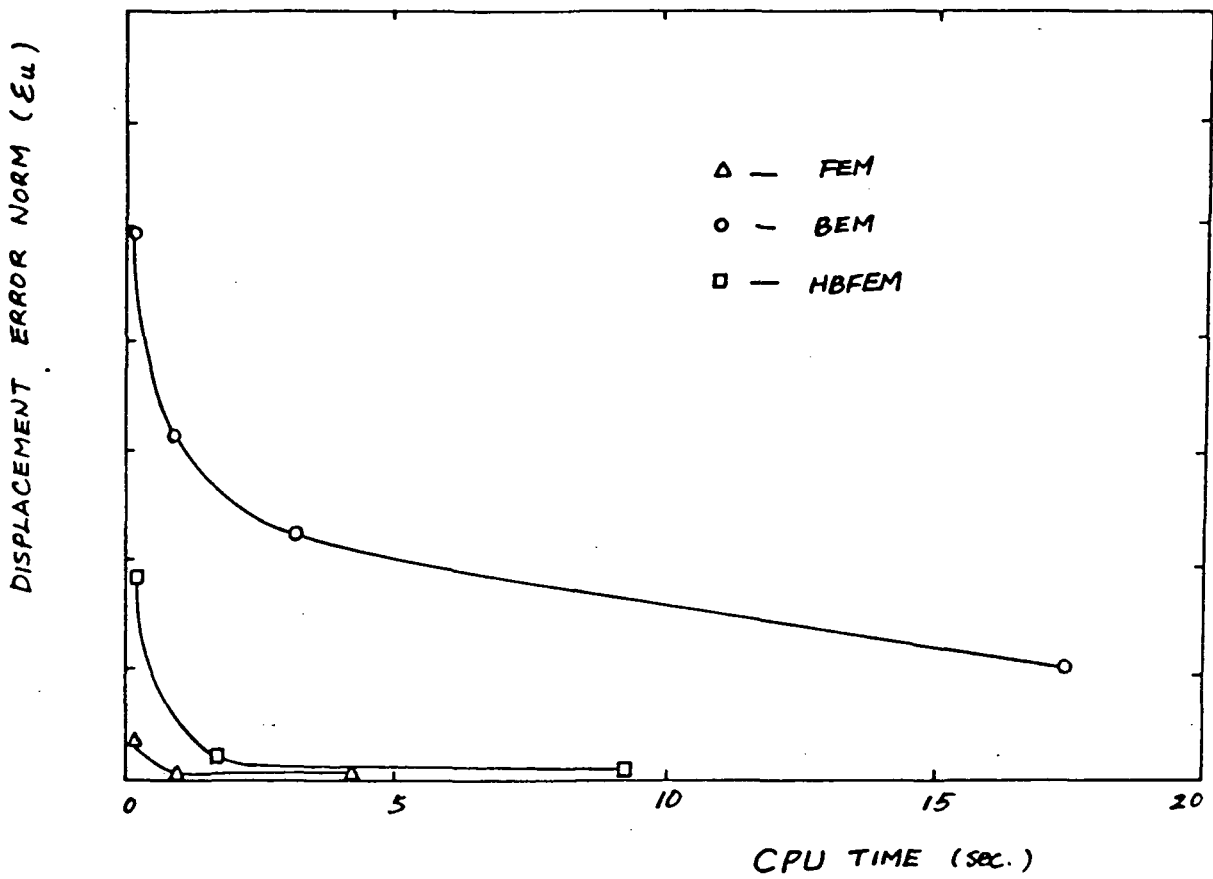   stress error norm.

Figure 11. Computational efficiency for interior
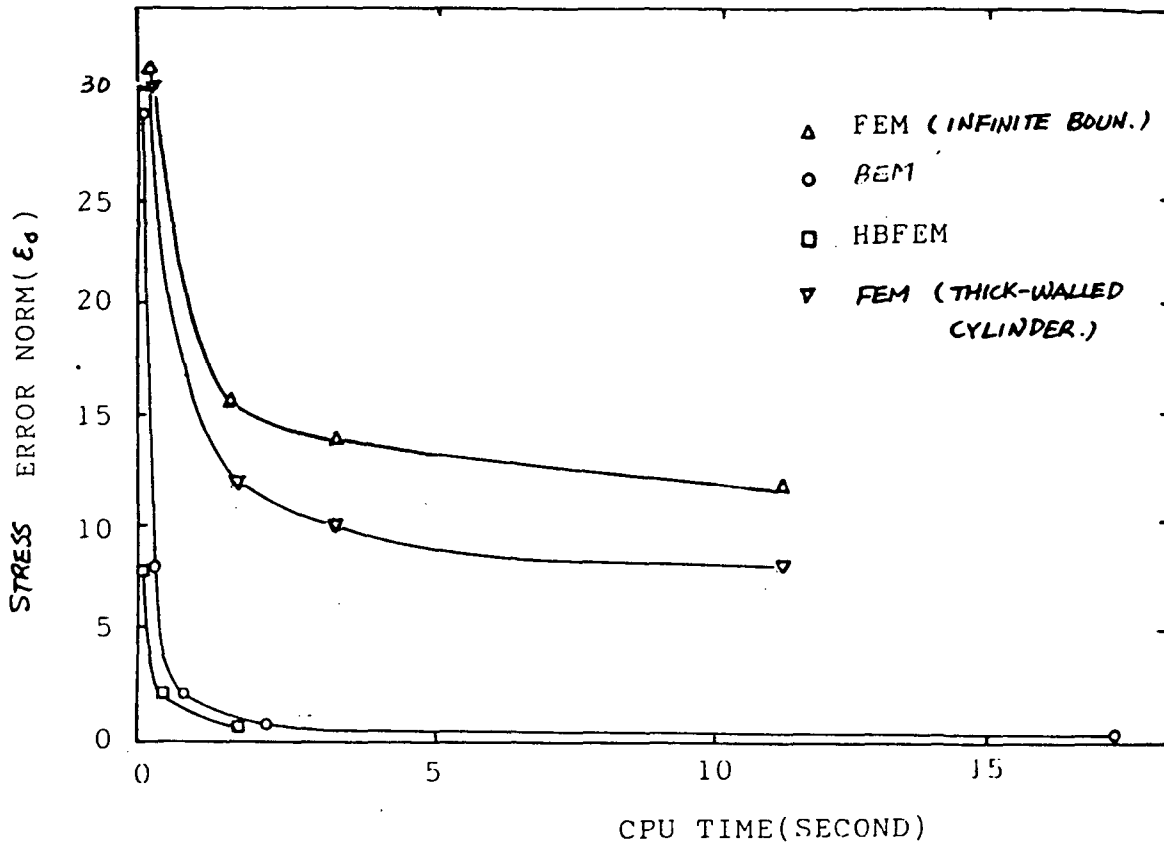problem: displacement error norm.

Figure 12. Computational efficiency for exterior problem:
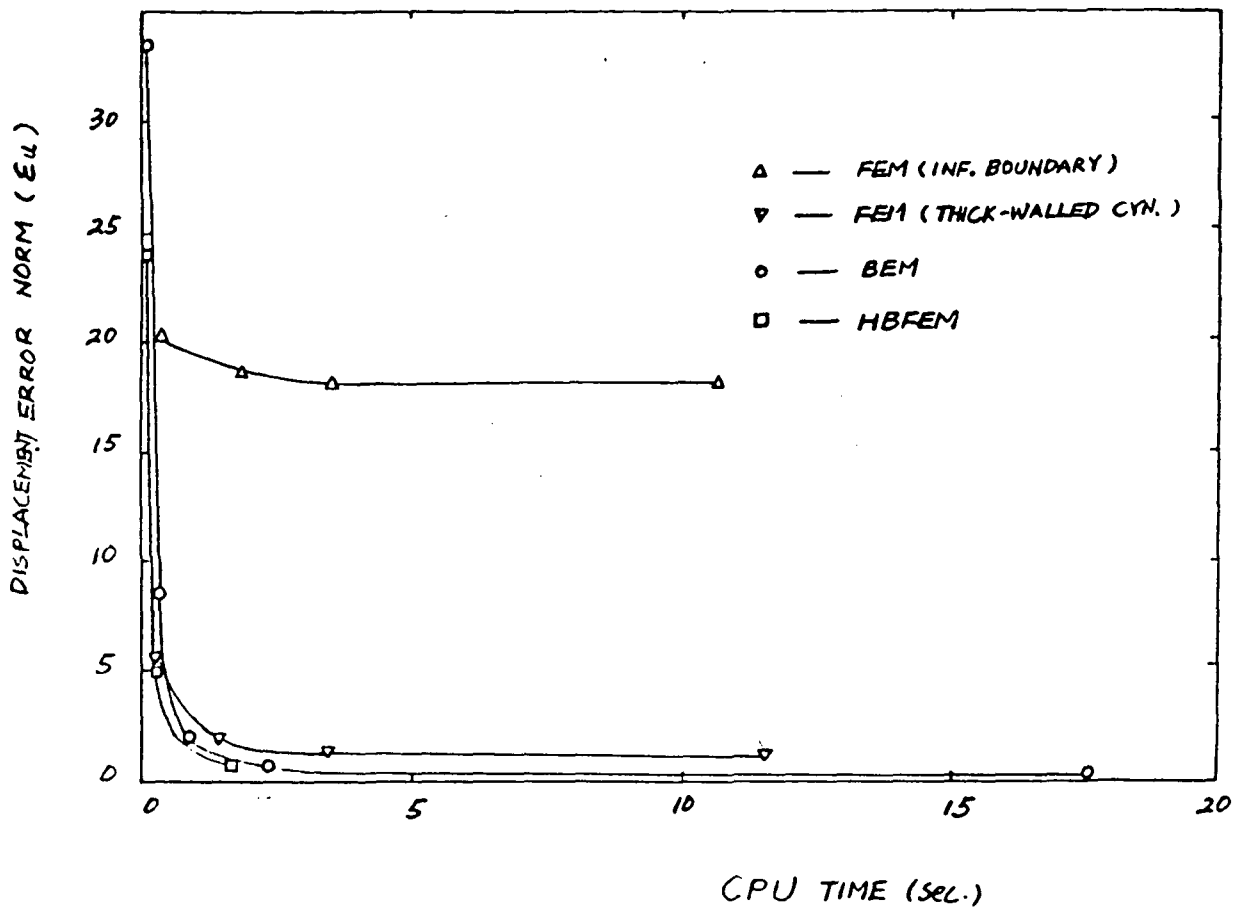stress error norm.

Figure 13. Computational efficiency for exterior problem:
displacement error norm.

cal. phase \ no. of ele.	12			24			48		
	FEM	BEM	HBFEM	FEM	BEM	HBFEM	FEM	BEM	HBFEM
Input phase	.06	.02	.10	.20	.03	.30	.58	.06	1.06
Cal.&assemb. of G & H			.19			.62			2.18
$G^{-1}H$			.079			.64			5.09
Cal.&assemb. of M			.002			.006			.017
$MG^{-1}H$			.006			.025			.101
Total Time to assemb. K or A	.10	.19	.311	.39	.61	1.41	1.17	2.18	7.88
Time to solv. equations	.02	.02	.00	.34	.15	.03	2.55	1.17	0.24
stress cal.	.07	.92	.91	.31	1.54	1.53	1.02	2.07	1.75
Total sol.	.25	1.15	1.32	1.24	2.33	3.27	5.32	5.48	10.93

Table 1. Time log for solution phases.