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LEAST SQUARES FINITE ELEMENT SIMULATION OF TRANSONIC FLOWS

T. F. Chen

G. J. Fix

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NASA Langley Research Center, Hampton, Virginia 23665

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T. F. Chen
Carnegie-Mellon University

G. J. Fix
Carnegie-Mellon University

Dedicated to Milton E. Rose
on Occasion of his 60th Birthday

ABSTRACT

Finite difference approximation of transonic flow problems is a well-developed and largely successful approach. Nevertheless, there is still a real need to develop finite element methods for applications arising from fluid-structure interactions and problems with complicated boundaries. In this paper we introduce a least squares based finite element scheme. It is shown that, if suitably formulated, such an approach can lead to physically meaningful results. Bottlenecks that arise from such schemes are also discussed.

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1. INTRODUCTION

In this paper we consider the approximation of transonic flows by finite element methods based on a variational method of the least squares type. The objective here is purely computational. In particular, we have sought to fully exploit the ideas arising from mathematical analysis of such methods (see, for example, [1] - [6]) and directly apply them to a nontrivial transonic flow problem. The major conclusion drawn from this work is that finite element methods--suitably formulated--can give physically meaningful results.

There is a significant and largely successful array of finite difference techniques for transonic flows (e.g., [17]). Nevertheless, an assumption implicit in this work is that there is still a need for stable and accurate finite element approaches. First, there are applications from fluid-structure interactions that would benefit from the availability of a finite element flow model. Second, there is the issue of complicated boundaries in the flow field. The importance of the finite element ideas in such a context--while largely untested--is still promising.

Variational principles of the least squares types have a number of valuable computational properties. For example, the algebraic system generated is always Hermitian semidefinite. In addition, such schemes, if properly formulated, are insensitive to equation type, be it hyperbolic (supersonic flows) or elliptic (subsonic flows). In fact, the majority of the finite element ideas that have been used for hyperbolic problems to date tend to be either implicitly or explicitly of the least squares type.

Least squares based schemes do have, however, some major computational defects. First, they tend to be sensitive to singularities and discontinuities in the flow variables. Moreover, mesh refinement alone does not overcome these defects [7]. Based on the work in [7] we introduce weighted least squares variational principles, which in combination with mesh refinement is capable of dealing with shocks in the flow field.

In Section 2 we describe the basic numerical formulation, and outline the essential computational properties associated with the approach. A key feature is the proper choice of weighting functions to use in the least squares functional. A closely allied issue is the density modifications needed to rule out nonphysical expansion shocks.

In Section 3 we present sample numerical results. As a model problem we select the planar potential flow over a cylinder.

Other authors have considered finite element approximation of transonic flows. Selected references are [18] - [21].

2. THE LEAST SQUARES FORMULATION

We consider the potential flow over a body $\hat{\Omega}$. Let $\underline{u}$ denote the velocity and ρ the density. Then a mass balance yields

$$\mathrm{div}[\rho \underline{u}] = 0. \quad (2.1)$$

In addition, we have

$$\underline{u} = \mathrm{grad}\ \phi \quad (2.2)$$

for the velocity potential ϕ. The density ρ is given as a function of $\underline{u}$ by the Bernoulli equation. The system is closed by specifying the normal velocity

$$\underline{u}\cdot\underline{n} = v \tag{2.3}$$

at the boundaries of the flow region. On the body $\hat{\Omega}$ the no flow condition

$$\underline{u}\cdot\underline{n} = 0$$

applies. We assume that the flow region is contained in a box B and that (2.3) is specified on the boundary of B. Thus

$$\Omega = B/\hat{\Omega} \tag{2.4}$$

defines the flow region, and (2.1) - (2.2) hold in Ω with (2.3) holding on the boundary Γ and Ω.

Since the flow is assumed to be irrotational, (2.1) - (2.2) can be replaced with

$$\mathrm{div}(\rho\underline{u}) = 0 \quad \text{in} \quad \Omega \tag{2.5}$$

$$\mathrm{curl}(\underline{u}) = 0 \quad \text{in} \quad \Omega \tag{2.6}$$

$$u\cdot n = v \quad \text{on} \quad \Gamma. \tag{2.7}$$

A least squares scheme based on this system takes the form

$$\int_\Omega \{|\mathrm{div}(\rho\underline{u})|^2 + |\mathrm{curl}(\underline{u})|^2\} = \min, \tag{2.8}$$

where the variation is taken $\underline{u}$ in some finite element space satisfying the boundary conditions (2.7). Such a div - curl system has proven to be very effective for elliptic systems (subsonic flows) in cases where the density $\rho = \rho(\underline{u})$ and the velocity field $\underline{u}$ are smooth [8].

Preliminary results indicate that with appropriate weighting functions on the terms in (2.8), the nonsmooth cases can be treated as well. Nevertheless, in this paper we shall focus attention on (2.1) - (2.2) and least squares schemes of the form

$$\int_{\Omega} \left\{ \left| \frac{\underline{v}}{\rho} - \text{grad } \phi \right|^2 + w \left| \text{div } \underline{v} \right|^2 \right\} = \min, \tag{2.9}$$

where $\underline{v} = \rho\underline{u}$ is the mass flow and w is a weighting function to be chosen. In this setup the variables are the potential ϕ and the mass flow $\underline{v}$.

The density in (2.9)

$$\rho = \rho(|\text{grad } \phi|)$$

is obtained from Bernoulli's equation, i.e.,

$$\rho^{\gamma-1} = \left\{1 - \left(\frac{\gamma-1}{2}\right) M_\infty^2 \left(|\text{grad } \phi|^2 - 1\right)\right\}.$$

Thus, (2.9) is a nonlinear least squares formulation, which is appropriate since it reflects the nonlinear character of transonic flow. Once a grid is selected (specific examples are given in the next section), the minimization of (2.9) over the associated finite element space leads to a nonlinear system

$$K(\underline{\Phi})\underline{\phi} = \underline{F}. \tag{2.10}$$

In all of the numerical examples reported in the next section, (2.10) was solved by a combination of Newton's method and elimination. Issues related to this choice for the equation solver will be discussed in the next section.

There are three main cases that are considered in this paper:

Case 1: smooth subsonic flows,

Case 2: smooth transonic flows,

Case 3: transonic flows with shocks.

In the first case (2.9) can be used without modification, and in particular no weighting function is needed (i.e., $w \equiv 1$ can be used). One does need special grids to obtain optimal accuracy (see [1]), and the criss-cross grid pattern which satisfies the grid decomposition property of [1] is used.

In the second case a hyperbolic region appears but the flow field remains smooth. In this case there is a loss of accuracy in the hyperbolic region. In particular, with linear elements the pointwise accuracy in the mass flow $\underline{v}$ drops from $O(h^2)$--in a generic mesh spacing--to $O(h)$. This can be corrected with a suitable choice of weighting function w, and details are given in [8]. This modification was not used in the results reported in this paper since the hyperbolic regions in question were too small for the suboptimal accuracy to have a major effect on the qualitative features of the flow.

The third case is, by a wide margin, the most important as well as the most challenging. Here we use a weight w so that the term

$$\int_\Omega \left\{ w|\mathrm{div}\ \underline{u}|^2 + \left|\frac{\underline{u}}{\rho} - \mathrm{grad}\ \phi\right|^2 \right\} \tag{2.11}$$

remains meaningful. In addition, modification to the density $\rho = \rho(|\text{grad } \phi|)$ must be introduced so that nonphysical expansion shocks are eliminated.

For the choice of the weight w, we follow the developments introduced in [7]. For most flows, $\underline{v} = \rho\underline{u}$ is continuous across the shock [10]. Nevertheless, it does not follow that div $\underline{v}$ is square integrable, and the primary rule derived from [7] is that w be chosen so that

$$\int_\Omega w|\text{div } \underline{v}|^2 < \infty. \tag{2.12}$$

This requires that w vanishes appropriately on the shock, which in turn means that (2.11) is a least squares principle in a degenerate L^2 norm. A point of significance, on the other hand, is the fact that if w vanishes to minimal order on the shock (in that (2.12) still holds), then optimal $O(h^2)$ can be achieved in unweighted L^2 norms provided appropriate mesh refinement is introduced. This has been proved rigorously only in special cases (see [7]), yet the numerical results in the next section seem to indicate that the principle is general.

These modifications alone do not yield an accurate simulation of the flow problem. To do this one must deal with the presence of nonphysical expansion shocks. In effect, (2.9) does not have a unique minimum, neither over infinite-dimensional function spaces nor over the finite-dimensional finite-element spaces. One can have expansion shocks, compression shocks, or both. What is interesting is the results in the next section tend to indicate that the case where both type of shocks appear tends to be the stable mode for (2.10). That is, an arbitrary choice of starting vector for Newton's methods applied to (2.10) tends to converge to this solution.

To eliminate expansion shocks we consider density biasing which in effect introduces streamwise diffusion into (2.1) - (2.2). Following [11] (see also [12] - [14]) the modified density takes the form

$$\overline{\rho} = \rho - \mu \rho_s \, \Delta s, \tag{2.13}$$

where ρ_s is the derivative of the density ρ along the streamwise direction. Since the density has the form

$$\rho = \rho(|\text{grad } \phi|),$$

the derivative ρ_s formally involves second derivatives of ϕ. Since ϕ is expanded in terms of linear elements, it is necessary to replace ρ_s with a streamwise difference quotient; i.e.,

$$\overline{\rho} = \rho - \mu \, \Delta\rho \, \Delta s, \qquad (2.13')$$

in the least squares formulation.

3. NUMERICAL RESULTS

To illustrate the above ideas we selected the classic problem of a planar flow past a cylinder. The flow region plus boundary conditions are given in Figure 3.1. The configuration shown in this figure assumes that both the outflow and inflow remain subsonic. Figure 3.2 contains a typical grid. For economy only the top part of the flow region is shown, and the special refinement needed for the shocks is not shown.

The first set of results shows a typical subsonic flow pattern. The results are given in Figure 3.3 for a free stream Mach number of

$$M_\infty = 0.1.$$

Convergence studies at such Mach numbers are reported in [5] - [6]. These results indicate, with the type of grid shown in Figure 3.2, one can readily achieve L^2 error of 1% or less for the velocity field.

The next set of results deal with the smooth transonic case. Of special interest here is the ability of the scheme to detect the onset of supersonic flow. Analytical techniques (see [15] and [16]) have given accurate values for the critical free stream Mach number M_* as a function of d/D, where d is the diameter of the cylinder and D is the width of the channel. These results are reproduced in Figure 3.4. Numerical results from the least squares scheme are given in Figures 3.5 - 3.7 for M_∞ = .42, .45, and .50, respectively. The d/D ratio used for this case is 1/6. Extrapolation based on these results indicates that the critical Mach number is approximately .41, which is good agreement with Figure 3.4.

The next set of results show what least squares based schemes produce when diffusion via density modification is not used. These are shown in Figure 3.8 which contains plots of the velocity $q = |\underline{u}|$ versus angle θ along the cylinder and at a radius slightly above the cylinder. The free stream Mach number is $M_\infty = .5$. The shock at the front of the cylinder is an expansion shock and is nonphysical. The one at the rear is a compression shock. A remarkable feature of this approximation is that the physically relevant compression shock is approximately in its correct position and is apparently unaffected by the spurious shock. (Compare Figures 3.8 and 3.9.)

The solution shown in Figure 3.8 is apparently a stable mode for the nonlinear system (2.10). Indeed, Newton's method converged to this solution rather rapidly for a wide variety of initial conditions.

In this regard, it is interesting to note that for the least squares formulation the Jacobian is not singular near the solution shown in Figure 3.8. Density modifications are needed to remove the spurious shock shown at the front of the cylinder. However, they are not needed to obtain nonsingular Jacobians.

The final results deal with the complete least squares system with the density modification discussed in the previous section. Figures 3.9 - 3.11 show the velocity field over the cylinder, at a radius slightly larger that that of the cylinder, and at a radius in the free stream. Note that the spurious expansion shock has been totally eliminated. Moreover, the shock location and strength as well as the velocity profile appear to be correct as is the supersonic bubble shown in Figure 3.12.

While we regard these numerical experiments as successful, there are a number of areas where the approach could be improved. The first issue concerns the equation solver. Once the density modification were introduced, the number of iterations increased by a factor of 2 to 3. Moreover, the solution shown in Figure 3.9 tended to be less "attractive" to the Newton iterations than that shown in Figure 3.8 (without density modifications). In fact, it was not difficult to find starting vectors where nonconvergence was seen, in the former case, although the starting state of a uniform flow always leads to convergence. This suggests that an alternative equation solver (e.g., preconditioned conjugate gradient) might be a more efficient choice for the equation solver.

A second issue concerns post-shock oscillations. These are seen in Figure 3.10, which is the radius where the oscillations were found to be the most significant. These oscillations were not seen on the body of the cylinder (Figure 3.9) and disappeared rather rapidly away from the cylinder (Figure 3.11). This is clearly a grid effect due to the slight misalignment of shock and grid.

4. CONCLUSIONS

Finite difference approximations to transonic flow problems are well-developed and have been successfully used for a wide range of problems. Nevertheless, there is still a need to develop finite element approaches for such problems for a variety of applications. We feel that the results presented here do show that such schemes can give physically meaningful simulations.

On the other hand, our experience has tended to indicate that straightforward application of the basic finite element idea may not always be successful. Key computational issues are as follows:

(i) There is a need to carefully develop the spaces in which the approximations are formulated. Classical L^2 spaces are generally inappropriate.

(ii) Some form of diffusion (via density modifications or otherwise) appears to be needed. Moreover, care is needed in the way this diffusion is introduced.

(iii) The geometrical pattern of the grid selected is of importance. Some patterns are definitely superior to others.

Finally, there are some important "bottlenecks" associated with the scheme employed in this paper, which, if properly addressed, could lead to an even more efficient approach. These include the following:

(i) There is a need for an equation solver that is more efficient than the Newton method used in this paper.

(ii) There is a need for adaptive grid refinement techniques that would lead to a better shock grid alignment than that achieved in this paper.

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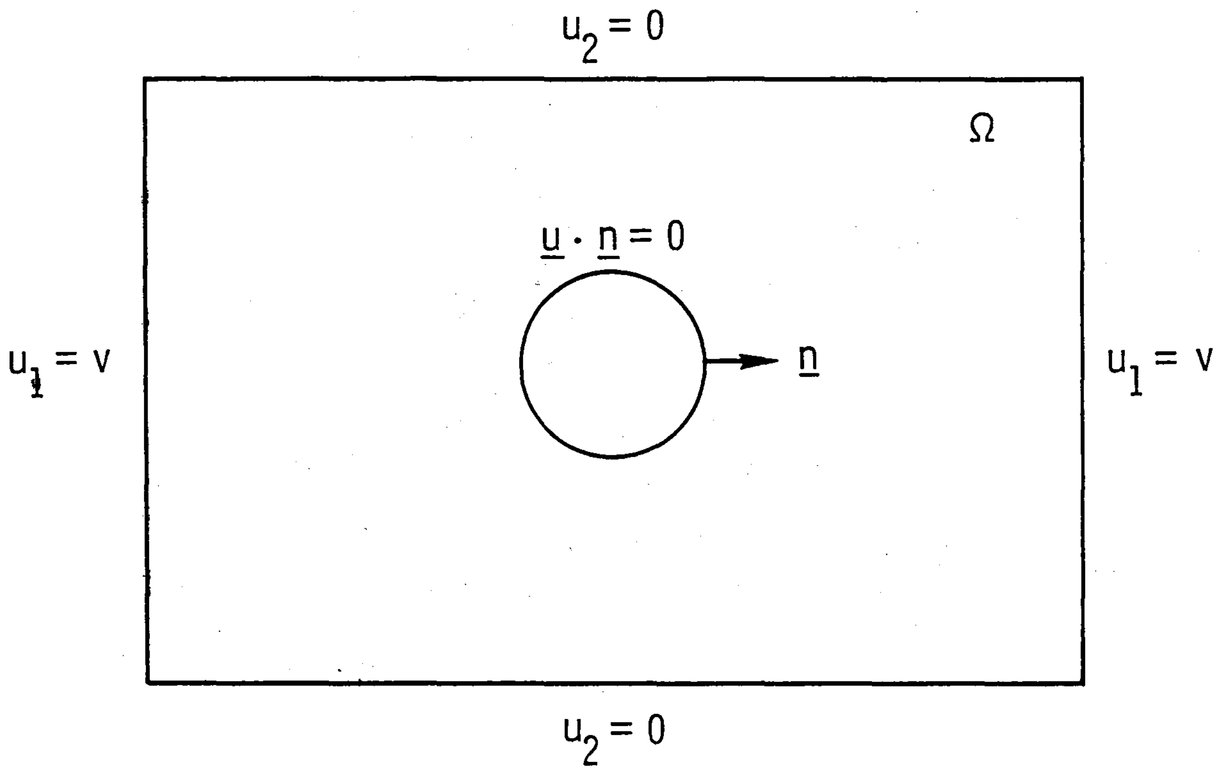

Figure 3.1. The flow region Ω.

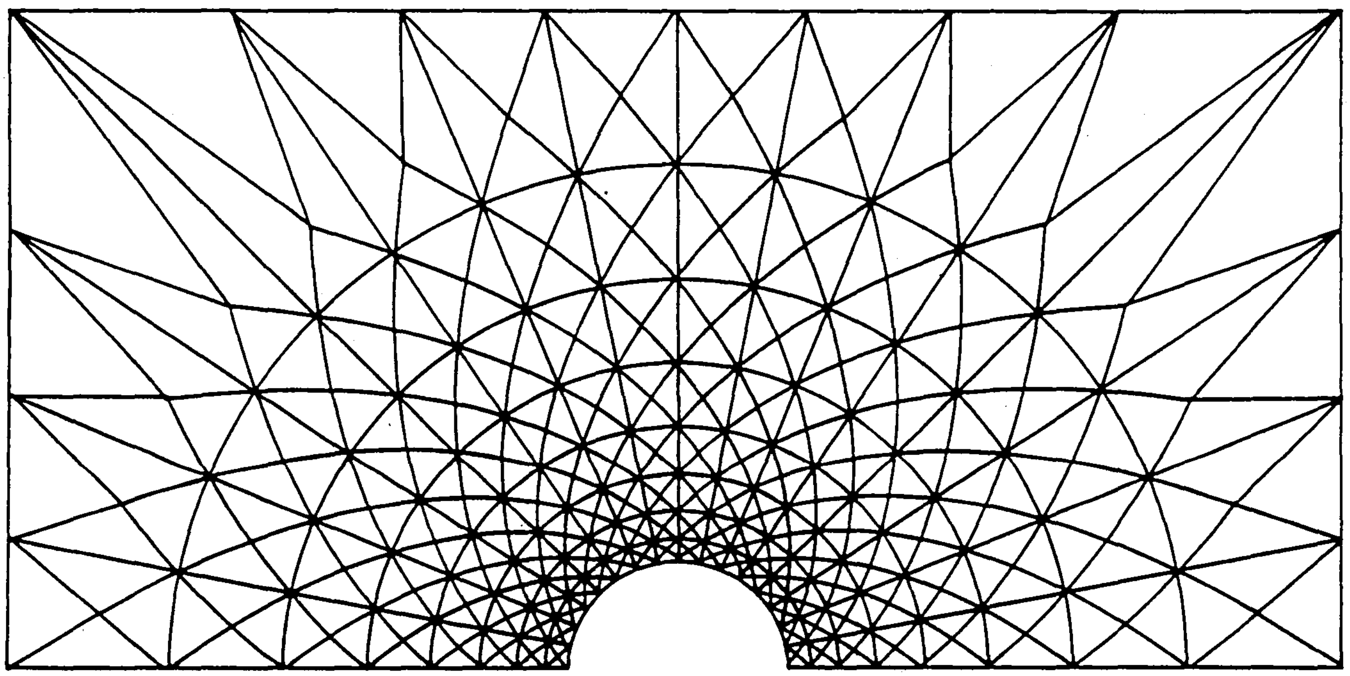

Figure 3.2. 512 elements, 281 nodes, $h = 0.30907 \times 10^{-1}$.

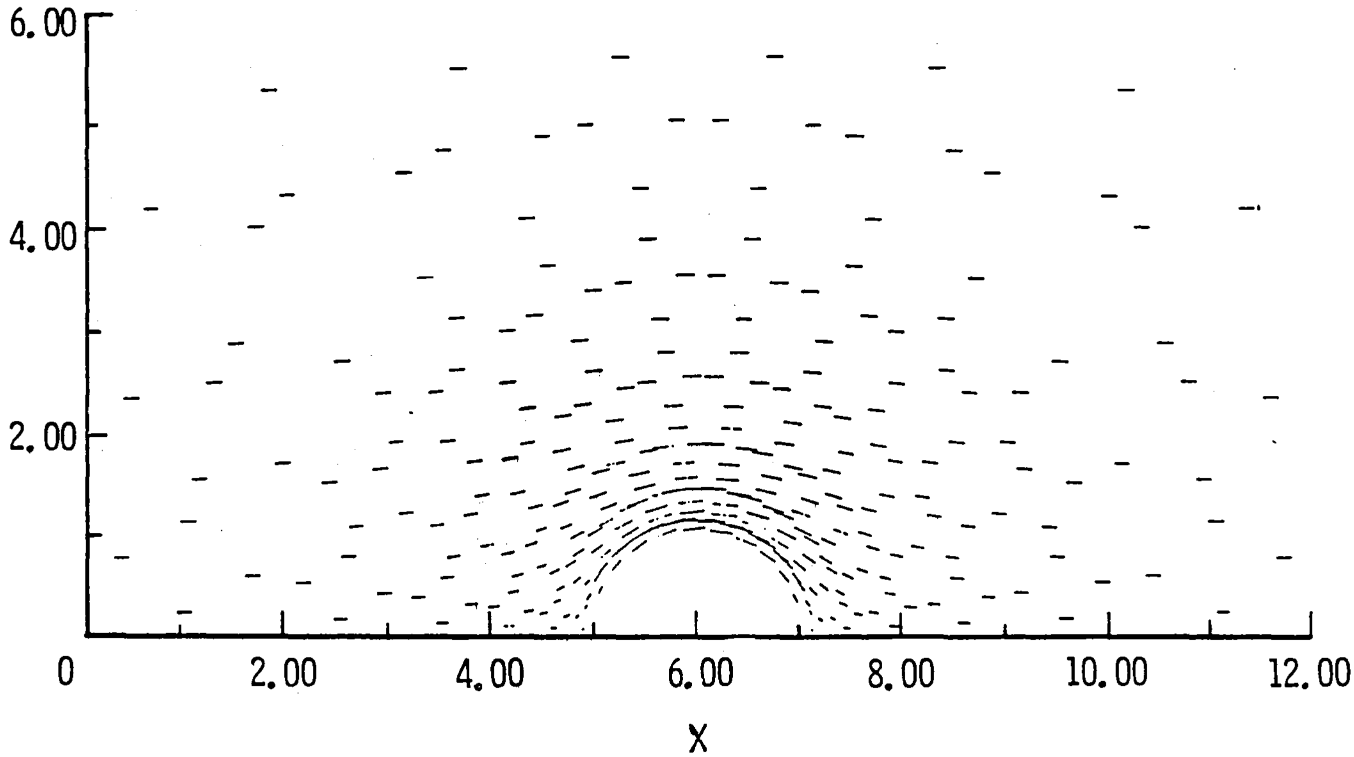

Figure 3.3. Flow pattern for the free stream Mach number $M_\infty = 0.1$.

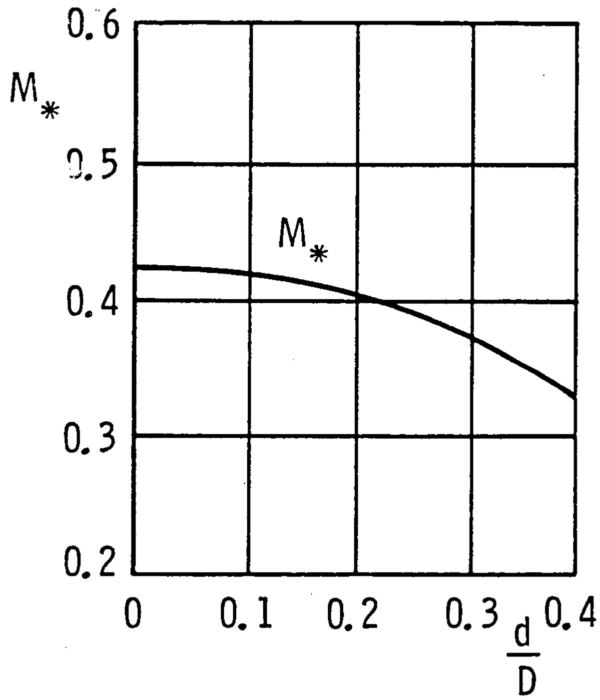

Figure 3.4. Critical Mach number versus d/D.

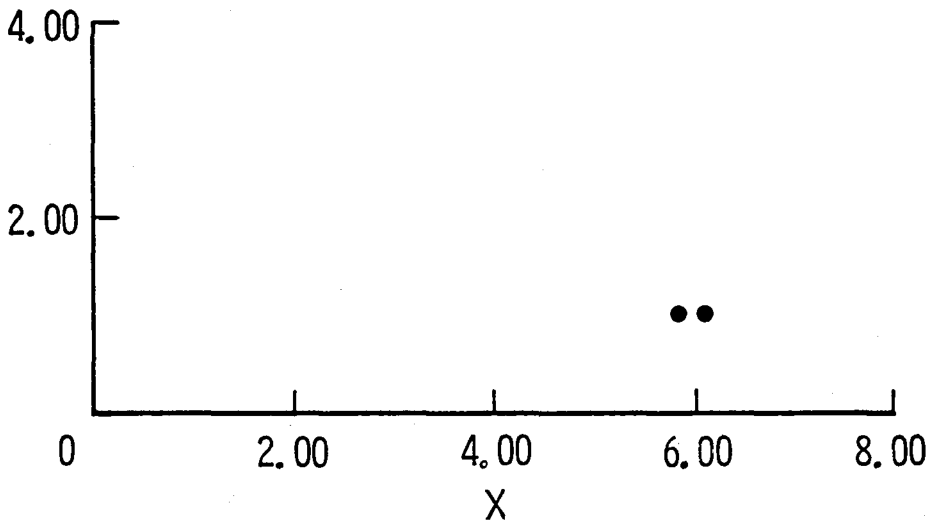

Figure 3.5. Plots of the supersonic pocket for $M_\infty = 0.42$.

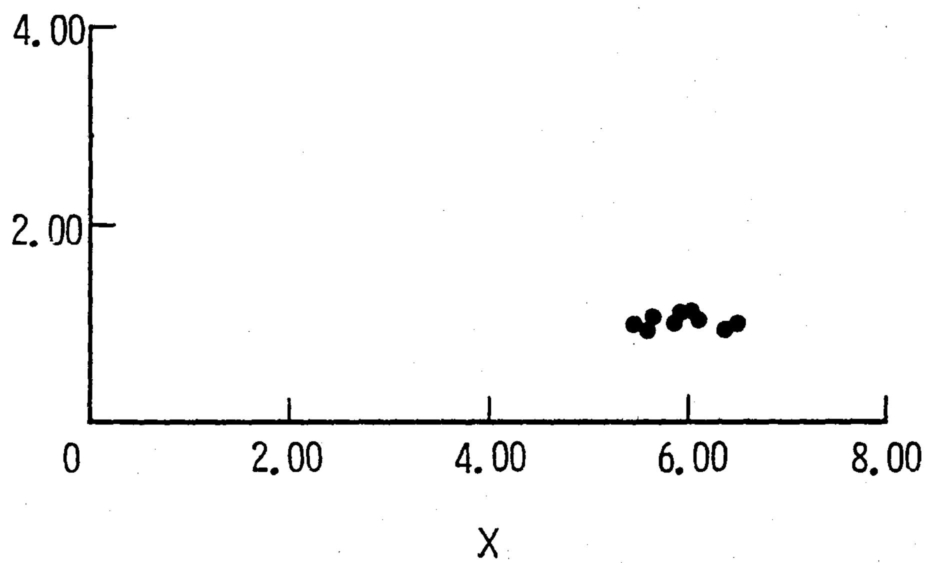

Figure 3.6. Plots of the supersonic pocket for $M_{\infty} = 0.45$.

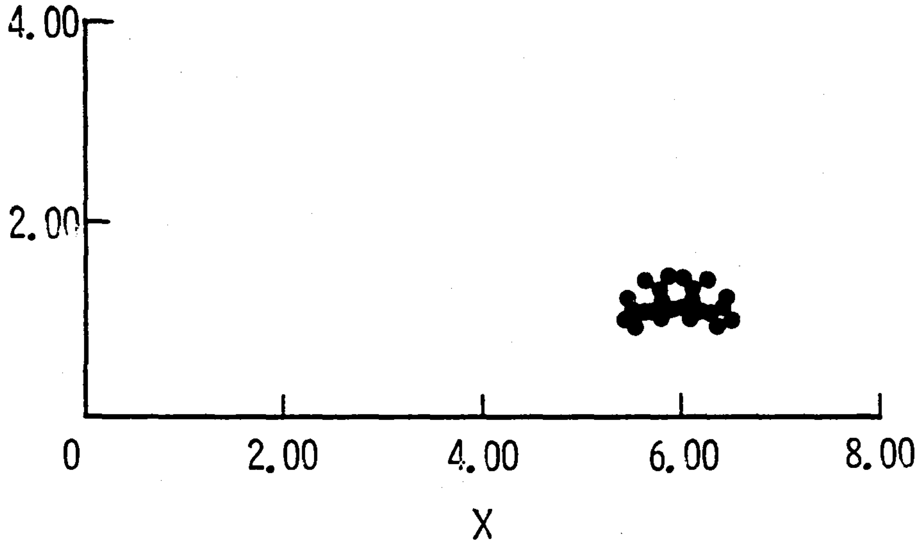

Figure 3.7. Plots of the supersonic pocket $M_\infty = 0.50$.

(a)

(b)

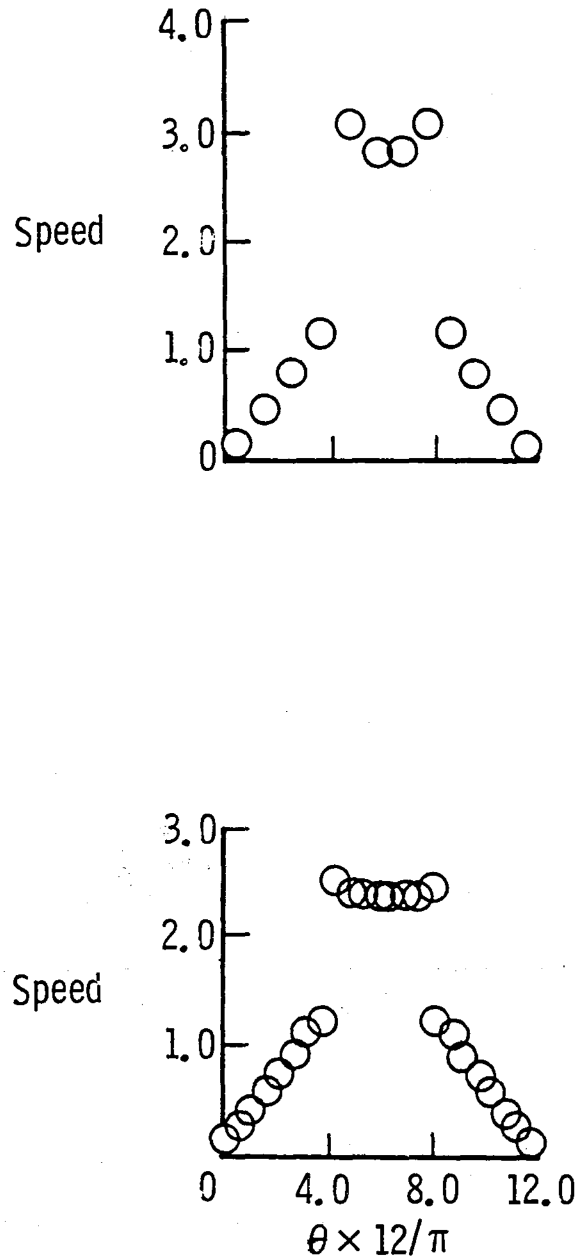

Figure 3.8. Velocity as a function of angle: (a) on cylinder, (b) slightly off cylinder — M_β = .51.

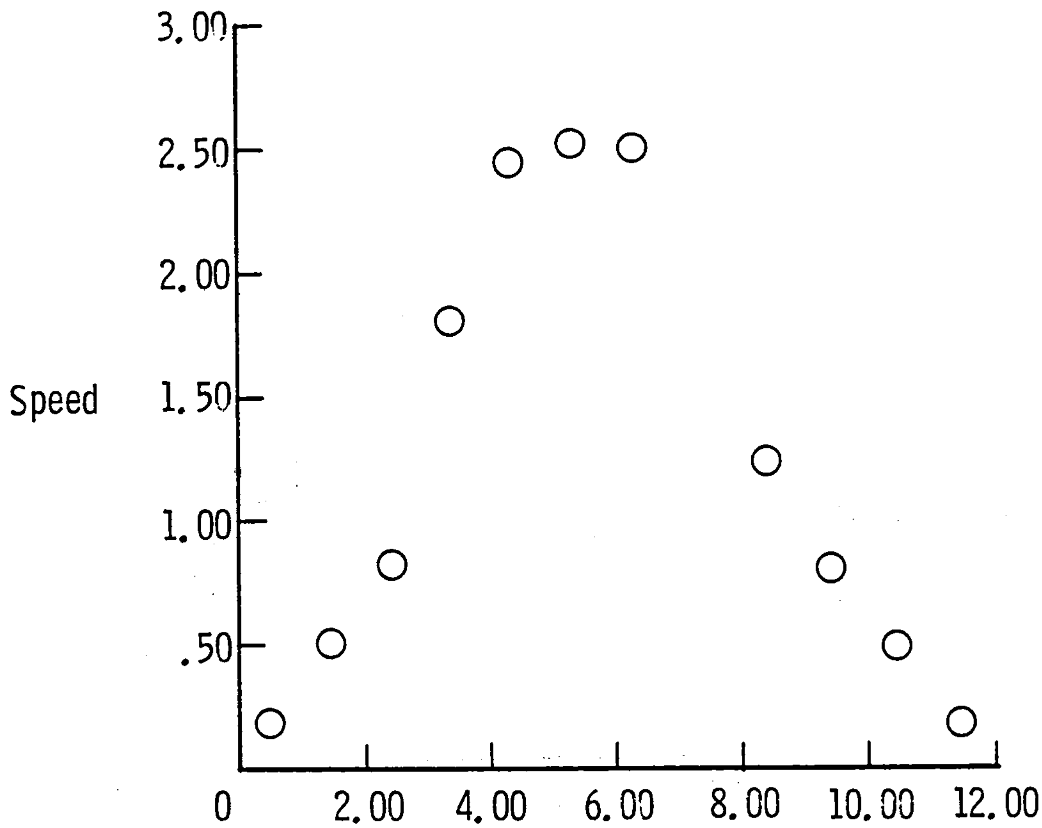

Figure 3.9. Velocity as a function of angle on the cylinder -- full least squares scheme with density modification -- $M_\infty = .5$.

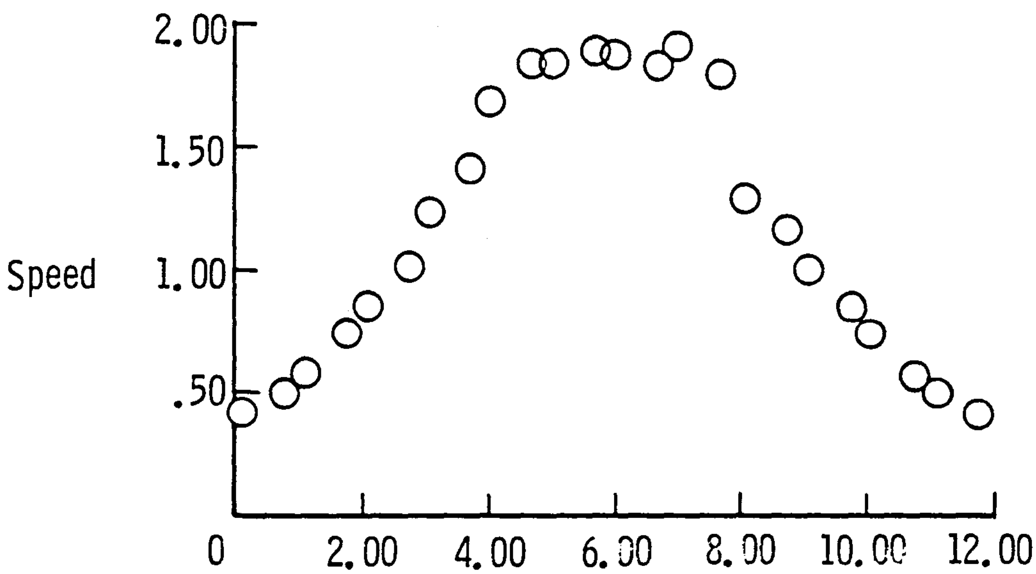

Figure 3.10. Velocity as a function of angle slightly off cylinder — full least squares scheme with density modification — $M_{\infty} = .5$.

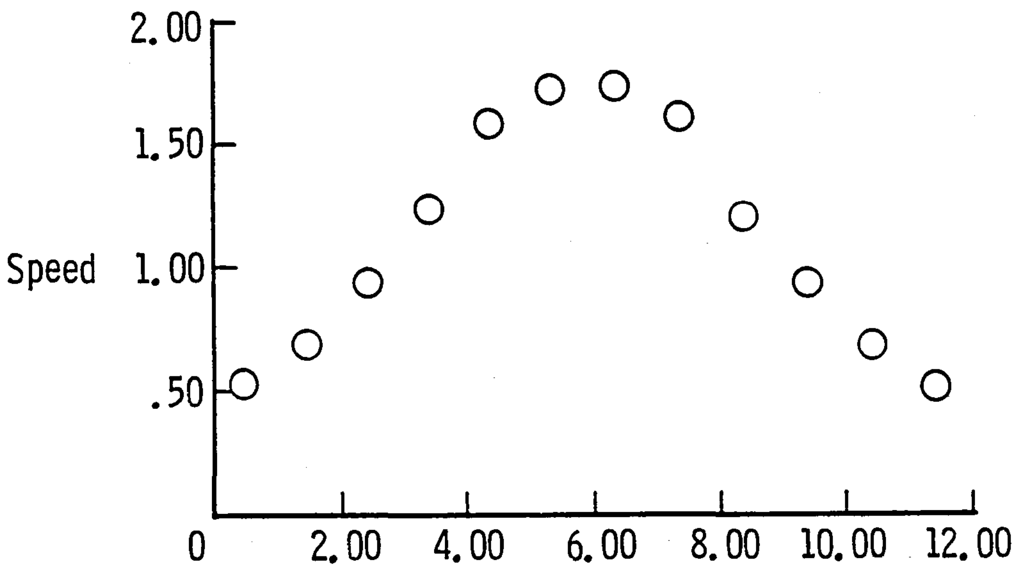

Figure 3.11. Velocity as a function of angle half radius above cylinder — full least squares scheme with density modification -- $M_\infty = .5$.

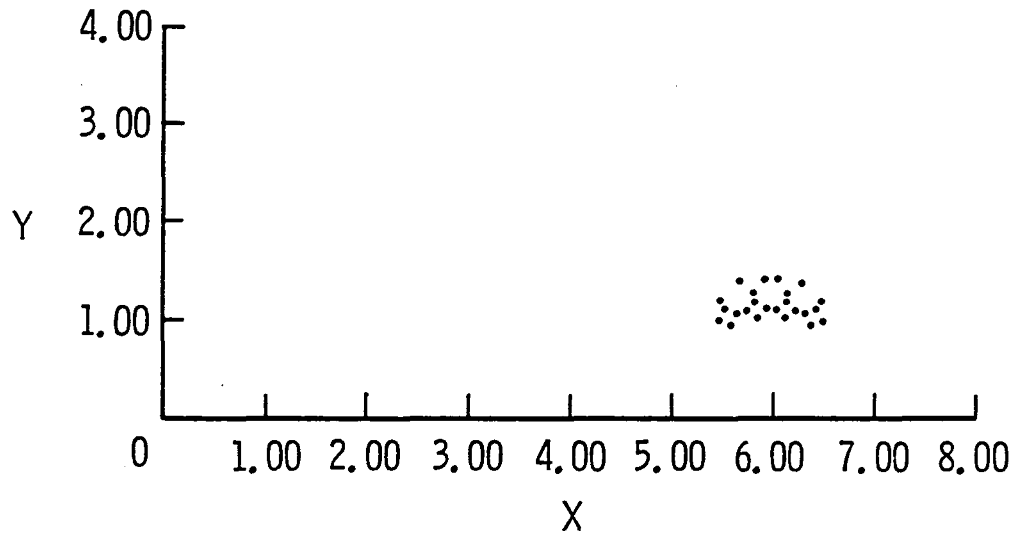

Figure 3.12. Supersonic bubble -- full least squares scheme with density modification -- $M_\infty = .5$.

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