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# A Constitutive Law for Finite Element Contact Problems With Unclassical Friction

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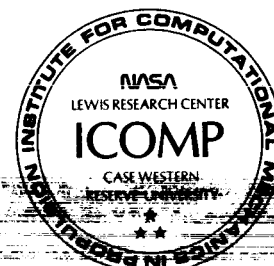
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A CONSTITUTIVE LAW FOR FINITE ELEMENT CONTACT PROBLEMS WITH  
UNCLASSICAL FRICTION

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SUMMARY

This report addresses techniques for modeling complex, unclassical contact-friction problems arising in solid and structural mechanics. A constitutive modeling concept is employed whereby analytic relations between increments of contact surface stress (i.e., traction) and contact surface deformation (i.e., relative displacement) are developed. Because of the incremental form of these relations, they are valid for arbitrary load-deformation histories. The motivation for the development of such a constitutive law is that more realistic friction idealizations can be implemented in finite element analysis software in a consistent, straightforward manner. Of particular interest in this report is modeling of two-body (i.e., unlubricated) metal-metal, ceramic-ceramic, and metal-ceramic contact. Interfaces involving ceramics are of engineering importance and are being considered for advanced turbine engines in which higher temperature materials offer potential for higher engine fuel efficiency.

1. INTRODUCTION

Material interfaces are common in mechanical systems and often have a substantial influence on structural response. The behavior of a material discontinuity is complex and involves frictional sliding, possible contact surface separation and various types of surface damage. Because of the non-linearity of such behavior, numerical methods such as the finite element method (FEM) are popular for the analysis of problems involving discontinuities. However, solution methodologies are still not advanced to the point where contact-friction capabilities are included in general purpose FEM programs and in most analyses, special purpose analysis programs are used.

Because of the difficulty in analyzing contact-friction problems and our rather poor quantitative understanding of frictional phenomena, virtually all of the analyses reported in the literature have employed a very simple and idealized form of friction law which can be attributed to Coulomb's original

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work published in 1781. In its "modern" form, this law postulates that the magnitude of the tangential component of the surface traction, or simply the tangential stress, will not exceed the product of the normal component of the traction, or normal stress, and a constant called the coefficient of friction. For states of tangential stress below the friction limit, various types of behavior such as stick-impenetrability or penalized deformability are common. Remarkably, for many problems, Coulomb's law is accurate and is sufficiently faithful to reality for engineering analysis. In fact, much of the research in tribology since Coulomb's work has been directed at obtaining a qualitative understanding of why and under what circumstances this law is accurate. However, it has been established by experimentation that true friction behavior is considerably more complicated than Coulomb's idealization, although for engineering analyses, it is still an open question when such detail is warranted.

In this paper, we employ a constitutive modeling concept to develop analytic relations between increments of contact surface stress (i.e., traction) and deformation (i.e., relative surface displacement). The development begins by distinguishing between macrostructural and microstructural features of a contact surface. Through macrostructural considerations and the assumption that contact surface deformations consist of reversible (elastic) plus irreversible (plastic) parts, the general form of the constitutive law is obtained. Microstructural considerations permit the specialization of the constitutive law for various applications by allowing the inclusion of special frictional phenomena such as adhesion, microsliding effects on coefficient of friction and surface damage. In particular, the behavior of unlubricated metal-metal, ceramic-ceramic, and metal-ceramic contacts will be discussed.

Because of the simplicity of Coulomb's law, most analysis programs employ ad hoc implementations of the contact-friction conditions which usually are not amenable to implementation of more general friction laws. Our objective is the development of a general, explicit, incremental constitutive law that leads to straightforward implementation in finite element software. In Section 4 of this report, a simple finite element and solution procedure for plane contact problems is discussed.

#### NOMENCLATURE

|             |  |
|-------------|--|
| A           | actual contact area  |
| $A_0$       | available or macroscopic contact area                                  |
| $B_I$       | denotes body number I where $I = 1, 2$                                 |
| $E_{ij}$    | interface stiffnesses where $i = t, n$ and $j = t, n$                  |
| $\hat{e}_i$ | unit vector in coordinate direction $i = t, n$                         |
| F           | slip function  |
| G           | slip potential   |
| $g_i$       | relative discontinuity displacement in coordinate direction $i = t, n$ |

H hardening or softening parameter  
 L length of contact finite element  
 $N_I$  finite element shape function for node  $I = 1, 2, 3, 4$   
 $n$  surface index for finite element  $n = \begin{cases} -1 & \text{surface 1-2} \\ +1 & \text{surface 3-4} \end{cases}$   
 $p$  arbitrary contact point between two bodies  
 $\text{sgn}(\sigma_t)$  denotes the sign of  $\sigma_t$ ;  $\text{sgn}(\sigma_t) = \begin{cases} +1 & \text{if } \sigma_t \geq 0 \\ -1 & \text{if } \sigma_t < 0 \end{cases}$   
 $t$  variable denoting tangential distance along finite element  
 $\tilde{u}_I$  displacement vector of a point associated with body  $B_I$ ;  $I = 1, 2$   
 $W^p$  plastic work  
 $\alpha_K$  asperity angle  
 $\Delta t$  time step for dynamic analyses  
 $n$  exponent used in friction law given by equation (3.7)  
 $\lambda$  scalar giving magnitude of plastic slip  
 $\mu$  friction coefficient  
 $\xi$  ratio of actual contact area to available contact area  
 $\sigma_i$  interface stress;  $i = t, n$

#### Matrix Symbols

$\tilde{B}$  matrix relating relative displacements to finite element node displacements  
 $\tilde{f}$  vector of nodal forces  
 $\tilde{k}^{ep}$  element matrix including nonlinearities  
 $\tilde{M}, \tilde{N}$  shape function matrices  
 $\tilde{T}_1, \tilde{T}_2$  transformation matrices  
 $\tilde{u}$  vector of nodal displacements

#### Subscripts

$i, j$  denotes coordinate direction  $t$  or  $n$   
 $I$  denotes body number  $B_1$  and  $B_2$  or finite element node numbers 1 through 4

- K denotes sliding direction to right or to left
- n,t denotes directions normal and tangential, respectively, to the macroscopic contact surfaces
- y used in equation (3.8) denoting contact yield stress with a certain amount of constrained plastic deformation
- ~ denotes a vector or matrix

### Superscripts

- e,p denotes elastic and plastic, respectively
- j denotes state of stress and deformation at iteration j for incremental static analysis or time  $j\Delta t$  for dynamic analysis
- ( $\dot{\cdot}$ ) denotes the time rate of change

## 2. CONSTITUTIVE LAW - MACROSTRUCTURAL CONSIDERATIONS

Shown in figure 1 is a macroscopically smooth contact surface with local tangential and normal coordinate directions t and n, respectively, with origin at point  $p^-$  which is affixed to body  $B_1$ . Any roughness that may be present on the contact surface is not shown in figure 1 and will be considered as microstructural features which are discussed in Section 3 of this paper. The only requirement that the macroscopic contact surface must possess is that it be sufficiently smooth so that the surface tangent and normal are not ill-defined.

The kinematic variables to be used in the constitutive law are the relative surface displacements in the tangential and normal directions which are defined as

$$g_i = (\underline{u}_2 - \underline{u}_1) \cdot \hat{e}_i \quad (2.1)$$

where  $u_I$  is the displacement of the point p associated with body I and  $\hat{e}_i$  is a unit vector in coordinate direction  $i = t, n$ . The stresses (in proper terminology, these are tractions, but we adopt the more conventional nomenclature of stresses) the interface supports at point p are denoted by  $\sigma_i$  and the convention that compressive stresses are negative is used. Our objective is the development of an explicit relation between increments of stress and deformation. Because the relation is incremental, it is valid for arbitrary load and deformation histories that can involve unloading and subsequent reloading (i.e., changes in the direction of frictional sliding).

A basic assumption is that the deformation can be additively decomposed into

$$g_i = g_i^e + g_i^p \quad (2.2)$$

where superscripts e and p denote the elastic (recoverable) and plastic (irrecoverable) parts of the deformation. There exists experimental evidence for almost every class of friction problem that has been carefully studied

supporting this decomposition. Furthermore, it has the added advantage of leading to a more convenient numerical implementation compared to friction idealizations in which a "stick" condition precedes frictional sliding.

Because the stress that an interface supports must be reversible (i.e., stresses must go to zero upon unloading), it can be related to the elastic part of equation (2.2) only. The simplest relation possible is the linear law

$$\dot{\sigma}_i = E_{ij} \dot{g}_j^e \quad (2.3)$$

where the  $E_{ij}$  are constant interface stiffnesses, superposed dots denote time differentiation and the summation convention is applied to repeated indices. Although nonlinear  $E_{ij}$  are possible, it is not known if such detail is warranted. Based upon physical considerations, it is appropriate to take  $E_{tn} = E_{nt} = 0$  so that changes of stress in the tangential and normal directions are unrelated to changes of deformation in the normal and tangential directions, respectively. For example, expanding the second of equations (2.3) gives  $\dot{\sigma}_n = E_{nt} \dot{g}_t + E_{nn} \dot{g}_n$ . If changes in tangential elastic displacement are not to give rise to changes in normal stress, then  $E_{nt} = 0$ . A similar argument can be made showing  $E_{tn} = 0$ .

It is necessary to postulate a rule for the plastic deformation. By assuming that

(1) A slip potential (scalar valued) function,  $F$ , can be defined such that negative  $F$  corresponds to nonsliding states of stress, zero  $F$  corresponds to sliding and positive  $F$  is not possible, and

(2) Increments of stress are linearly related to increments of deformation,

it can be shown (see ref. 1) that a slip rule, which is analogous to the flow rule employed in continuum plasticity, is the appropriate law for the plastic deformations

$$\dot{g}_i^p = \begin{cases} 0 & \text{if } F < 0 \text{ or } \dot{F} < 0 \text{ (unloading)} \\ \dot{\lambda} \frac{\partial G}{\partial \sigma_i} & \text{if } F = \dot{F} = 0 \text{ (loading)} \end{cases} \quad (2.4)$$

where  $G$  is the slip potential whose gradient gives the direction of the slip and  $\dot{\lambda}$  is a nonnegative scalar that gives the magnitude of the slip. When  $F = G$ , the friction is "associated" and when  $F \neq G$  the friction is "non-associated". Only under unusual circumstances is friction associated. Unfortunately, the more typical case of nonassociated friction results in a nonsymmetric tangent material matrix which poses difficulties for many computer solution schemes; methods of reducing these difficulties will be mentioned in Section 4 of this report.

Combining equations (2.1) through (2.4) provides the general form of the contact-friction constitutive law

$$\dot{\sigma}_i = E_{ij}^{ep} \dot{g}_j \quad (2.5)$$

where for purely elastic response or unloading (i.e.,  $F < 0$  or  $\dot{F} < 0$ ), respectively),

$$E_{ij}^{ep} = E_{ij} \quad (2.6(a))$$

and for imminent slip (i.e.,  $F = \dot{F} = 0$ )

$$E_{ij}^{ep} = E_{ij} - \frac{\frac{\partial F}{\partial \sigma_p} E_{iq} E_{pj} \frac{\partial G}{\partial \sigma_q}}{\frac{\partial F}{\partial \sigma_p} E_{pq} \frac{\partial G}{\partial \sigma_q} - H} \quad (2.6(b))$$

where  $H$  is a hardening or softening parameter given by

$$H = \frac{\partial F}{\partial W^p} \sigma_l \frac{\partial G}{\partial \sigma_l} \quad (2.6(c))$$

in which  $\dot{W}^p = \sigma_l \dot{g}_l^p$  is the plastic work. If no hardening or softening behavior is permitted, then  $H = 0$ .

#### Remarks

1. Completion of the constitutive law requires the specification of the slip function  $F$  and slip potential  $G$ . These are determined by considering the microstructure of the material interface and will be considered in Section 3 of this report.

2. Equation (2.5) is an explicit, incremental relation that is in a form convenient for numerical implementation and the analysis of problems with arbitrary load and deformation histories.

3. This constitutive law enforces compatibility (i.e., impenetrability constraint preventing contacting bodies from penetrating one another) through the use of a "penalty" stress that is proportional to the violation of compatibility. Thus, when bodies are in contact,  $g_{\bar{n}} \leq 0$  (bodies are allowed to interpenetrate slightly) and the penalty stress is  $\sigma_n = E_{nn} g_{\bar{n}}$ .

### 3. CONSTITUTIVE LAW - MICROSTRUCTURAL CONSIDERATIONS

The general framework of the constitutive law given by equations (2.5) and (2.6) is applicable to a variety of contact problems. These varied applications are obtained by formulating the slip function  $F$  and slip potential  $G$  such that important physical features of the contact problem are accounted for. In this section we demonstrate several examples.

The first example to be considered is illustrative, simple, and identical to the penalized form of the classic Coulomb friction law (ref. 2). We idealize the material discontinuity as smooth with no roughness whatsoever. Coulomb's law of friction requires

$$|\sigma_t| \leq -\mu\sigma_n \quad (3.1)$$

The slip function and slip potential for this case are

$$\begin{aligned} F &= |\sigma_t| + \mu\sigma_n \\ G &= |\sigma_t| \end{aligned} \quad (3.2)$$

These surfaces are shown in figure 2. Equations (2.5) and (2.6(b,c)) become

$$\dot{\sigma}_t = -(\text{sgn}(\sigma_t)\mu E_n) \dot{g}_n \quad (3.3)$$

$$\dot{\sigma}_n = E_n \dot{g}_n \quad (3.4)$$

where  $\text{sgn}(\sigma_t)$  denotes the sign of  $\sigma_t$  and  $\mu$  is the coefficient of friction. Equation (3.4) is seen to be the correct relation between rates of normal stress and normal relative displacement and equation (3.3) provides the relation between the tangential stress and normal deformation rates where  $E_n \dot{g}_n$  is recognized as the rate of change of normal stress and  $\text{sgn}(\sigma_t)$  insures that the tangential stress rate has the appropriate sign.

Rough, dilatant material discontinuities. - Prototypical material interfaces are microscopically very rough; two extreme surface profiles are shown in figure 3. The situation depicted in figure 3(a) has a rough surface with close initial mating and is characteristic of interfaces, or cracks, that propagate through an initially continuous medium. An important feature of these types of discontinuities is dilatancy: contact surface separation during relative tangential motion due to the asperity surfaces of one body riding up on those of the other. Such a discontinuity is typical of crack surfaces in polycrystalline and aggregate materials and the effects of dilatancy may be important. For example, consider a plane tortuous discontinuity with a sharp crack front in a material that is loaded in plane shear or antiplane shear. If the crack was not tortuous then these loading conditions would give rise to mode II and III crack growth, respectively. However, because of the rough, closely mated discontinuity, mixed mode crack growth (addition of mode I) may result due to dilatancy. A simplified model for the surface shown in figure 3(a) was developed in references 3 and 4 with the following slip function and potential

$$F = |\sigma_n \sin \alpha_K + \sigma_t \cos \alpha_K| + \mu(\sigma_n \cos \alpha_K - \sigma_t \sin \alpha_K) \quad (3.5)$$

$$G = |\sigma_n \sin \alpha_K + \sigma_t \cos \alpha_K| \quad (3.6)$$

where  $\alpha_K$  is an average asperity orientation with respect to the macroscopic interface tangent and K is either R or L to denote if sliding is in the right hand or left hand direction, respectively. Equations (3.5) and



(3.6) were obtained by assuming that simple Coulomb friction as given by equations (3.3) and (3.4) is active on each asperity surface. The surface  $F = 0$  is shown in figure 4; the step discontinuity in  $F$  occurs when the asperities have overridden one another and the discontinuity behavior becomes nondilatant and is assumed to be governed by the simple Coulomb friction law given by equations (3.3) and (3.4). More realistic friction laws, such as equation (3.7) discussed in the following subsection, are possible for dilatant surfaces, however, this might correspond to the modeling of second order effects since for rough, closely mated contact problems, dilatancy is probably responsible for the behavior that is observed.

Rough, nondilatant material discontinuities. - The situation depicted in figure 3(b) is a rough interface with unmatched surfaces so that contact takes place at only a small number of asperity peaks. Such a discontinuity is characteristic of man-prepared surfaces that come into contact. For engineering purposes, these interfaces are nondilating with sometimes complex frictional behavior. The friction of such contacting solids is extremely sensitive to a number of factors including environment, roughness, presence of contaminating film layers, sliding history, and temperature. Thus, it is very difficult to make generalities about frictional behavior, although in this paper we will attempt to do this with the understanding that there are numerous exceptions. In postulating models for this behavior, we draw heavily upon the work of Courtney-Pratt and Eisner (ref. 5), Bowden and Tabor (ref. 6), Kragelskii (ref. 7), and Buckley (ref. 8). Experimental methods for the determination of necessary friction law parameters are not discussed and may pose some open questions requiring additional research.

In many respects, metals, glasses and ceramics have similar friction and wear behavior. In this paper, we will attribute two-body (unlubricated) friction to three primary sources: (1) plasticity of the surface film layer, (2) adhesion due to the molecular attraction of bodies in very close proximity, and (3) plowing due to the penetration of prominent asperities through a surface film layer, if any, into the adjacent material. Usually only one of these sources of friction will be active at any given time although sometimes sources (1) and (3) or (2) and (3) can exist simultaneously. To represent this behavior the following friction law is considered

$$\mu = \mu_{\text{film}} + \mu_{\text{attr}} \left( \frac{\xi}{1 - \xi} \right) + \mu_{\text{plow}} \left( \frac{\sigma_n}{\sigma_{\text{seize}} - \sigma_n} \right)^n \quad (3.7)$$

The first term of equation (3.7) represents the basic frictional resistance of the film (contaminant) layer to plastic deformations. Surface films are almost always present in noninert environments such as air. The frictional properties and wear resistance of most materials is extremely sensitive to environment and with the notable exception of glass, most materials exhibit reduced friction and wear in film-producing environments compared to their behavior in a vacuum in which surface films are not present or are rapidly destroyed without regeneration. Compared to their frictional resistance in a film-producing environment, ceramics and glasses show up to a 30 percent increase in a vacuum, while the increase for metals is significantly greater at up to two orders of magnitude (complete seizure). The reason for this marked behavior is a change in the source of friction. In the presence of a

surface film, the second and often times third terms of equation (3.7) are negligible while the opposite is true in an inert environment (ref. 8).

The second term of equation (3.7) represents frictional resistance due to molecular attraction between bodies in very close contact. Thus, the larger the number of asperities in intimate contact, i.e., the larger the actual area of contact, the greater the adhesion. In equation (3.7),  $\xi$  is the ratio of the actual contact area,  $A$ , to the available, or macroscopic area,  $A_0$ . To quantify this, we consider the model of asperity contact shown in figure 5 in which the state of stress at the asperity is assumed to be sufficiently high to cause local yielding. Assuming a maximum shear stress yield criterion provides

$$\xi^2 = \frac{(\sigma_n^2 + 4\sigma_t^2)}{\sigma_y^2} \quad (3.8)$$

where  $\sigma_y$  is the yield stress, although it must be interpreted in a slightly different sense than its usual definition as the uniaxial yield stress since behavior of asperities involves local large deformation yielding. Rather, it can be interpreted as the stress necessary to produce a certain amount of constrained plastic deformation which is significantly higher than the initial yield stress (ref. 9, pg. 335). This constant and  $\mu_{attr}$  are determined from experiments in which conditions are selected to eliminate the other sources of friction appearing in equation (3.7).

Where surfaces are atomically clean, this source of friction can be very substantial and particularly with metals, the frictional resistance from adhesion and plowing can approach full seizure. Anything that tends to disrupt atomically clean surfaces such as surface films or contaminants resulting from an air environment decreases  $\mu_{attr}$  and makes the affects of  $\mu_{film}$  more significant.

The last term of equation (3.7) represents friction due to the plowing of asperities of one body through those of the other. This source of friction becomes more prominent at higher compressive loads but is diminished by the presence of film layers which add a zone of softer material that must be penetrated. For ceramics, frictional resistance is generally load independent except when a transition is made when asperities penetrate a surface layer and begin plowing. The normal stress at which seizure occurs,  $\sigma_{seize}$ , along with  $n$  and  $\mu_{plow}$  must be determined from experiment. This source of friction is prominent when contact involves materials with disparate hardnesses. For example, ceramic-ceramic and glass-glass contact is controlled by film layer and adhesive friction with no plowing. However, when metal contacts ceramic, the metal is much softer and the ceramic will plow offering additional resistance.

Although the constitutive law presented in this paper does not model wear, it is an important phenomenon that can influence frictional behavior. Because of adhesion, there often is a transfer of one material to another thus affecting a change in surface profile and characteristics. This type of wear is most pronounced for metals in contact with ceramic and glass (ref. 8). With metal-ceramic contact systems, metal transfers to ceramic and eventually the contact behaves as a metal-metal system. With metal-glass in air, metal also transfers

to glass although when in a vacuum, the opposite is generally the case; the reason for this anomalous behavior is that the fracturing resistance of glass is strongly water moisture dependent.

Anisotropy of frictional sliding and wear has been experimentally observed for many single crystal metal and ceramic contact systems. For this case, a version of equation (3.7) can be employed for each of the two principal tangential directions (directions displaying maximum and minimum values of friction). Slip functions and slip potentials for anisotropic friction are presented in references 10 and 11.

#### 4. FINITE ELEMENT IMPLEMENTATION

In this section we review the development of a simple two-dimensional linear displacement finite element for contact-friction problems (ref. 2). The element is suitable for modeling contact between regions that are discretized by, for example, constant strain triangles or bilinear displacement quadrilaterals. The macroscopic contact surface shown in figure 1(b) is discretized into elements. One such element is shown in figure 6 with a natural (t,n) coordinate system having origin at the center. The element has zero thickness in the undeformed state but is shown separated for clarity.

Kinematics. - The element has four nodes and each node has two degrees of freedom corresponding to displacements  $u_t$  and  $u_n$ , tangent and normal to the plane of the surface, respectively. The length of the element is  $L$ . The displacement of any point on surface 1-2 or surface 3-4 can be expressed in terms of nodal displacements by

$$u_i(t,n) = N_I u_{iI} \quad \begin{matrix} i = t,n \\ I = 1,2,3,4 \end{matrix} \quad (4.1)$$

where a lower case subscript denotes a vector component, upper case subscript  $I$  denotes a node number, and the shape functions are

$$\begin{aligned} N_1 &= \frac{1}{4} \left( 1 - \frac{2t}{L} \right) (1 - n) \\ N_2 &= \frac{1}{4} \left( 1 + \frac{2t}{L} \right) (1 - n) \\ N_3 &= \frac{1}{4} \left( 1 + \frac{2t}{L} \right) (1 + n) \\ N_4 &= \frac{1}{4} \left( 1 - \frac{2t}{L} \right) (1 + n) \end{aligned} \quad (4.2)$$

where  $-L/2 \leq t \leq +L/2$ ,  $n = -1$  for surface 1-2 and  $n = +1$  for surface 3-4; values of  $n$  such that  $-1 < n < +1$  have no meaning since the interface is assumed to have infinitesimal thickness when in contact. In matrix notation, equation (4.1) becomes

$$\underline{u}(t,n) = \underline{N} \underline{u} \quad (4.3)$$

where

$$\begin{aligned} \underline{u}(t,n)^T &= [u_t(t,n), u_n(t,n)] \\ \underline{u}^T &= [u_{t1}, u_{t2}, u_{t3}, u_{t4}, u_{n1}, u_{n2}, u_{n3}, u_{n4}] \quad (4.4) \\ \underline{N} &= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \end{aligned}$$

Superscript T denotes vector or matrix transposition.

The relative tangent and normal contact displacements are given by

$$g_i(t) = u_i(t, +1) - u_i(t, -1) \quad (4.5)$$

The distribution of the relative displacements,  $g_i(t)$ , can be related to the node relative displacements by

$$\underline{g}(t) = \underline{M} \underline{u} \quad (4.6)$$

where

$$\begin{aligned} \underline{g}^T(t) &= [g_t(t), g_n(t)] \\ \underline{g}^T &= [g_{t1}, g_{t2}, g_{n1}, g_{n2}] \quad (4.7) \\ \underline{M} &= \begin{bmatrix} M_1 & M_2 & 0 & 0 \\ 0 & 0 & M_1 & M_2 \end{bmatrix} \end{aligned}$$

The shape functions  $\underline{M}$  are related to the shape functions  $\underline{N}$  by

$$\underline{M} = \underline{N} \underline{T}_1 \quad (4.8)$$

where

$$\underline{T}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (4.9)$$

Furthermore, the relative node displacements are related to the total node displacements by

$$\underline{g} = \underline{I}_2 \underline{u} \quad (4.10)$$

where

$$\underline{I}_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \quad (4.11)$$

Combining equations (4.6), (4.8), and (4.10), the distribution of the relative displacements are related to the total node displacements by

$$\underline{g}(t) = \underline{B} \underline{u} \quad (4.12)$$

where

$$\underline{B} = \underline{N} \underline{I}_1 \underline{I}_2 \quad (4.13(a))$$

If the matrix product in equation (4.13(a)) is expanded, we obtain

$$\underline{B} = \frac{1}{2} \begin{bmatrix} -(1-t) & -(1+t) & (1+t) & (1-t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(1-t) & -(1+t) & (1+t) & (1-t) \end{bmatrix} \quad (4.13(b))$$

Equilibrium. - To equilibrate the distribution of the contact stresses,  $\sigma(t)$ , with the slave nodal (concentrated) forces,  $\underline{f}$ , we will require the work of the distributed stresses to be equal to the work of the nodal forces. Thus,

$$\int_{-L/2}^{L/2} d\underline{g}^T(t) [\underline{\sigma}^j(t) + d\underline{\sigma}(t)] dt = d\underline{u}^T \underline{f}^{j+1} \quad (4.14)$$

where

$$\underline{f}^T = [f_{t1}, f_{t2}, f_{t3}, f_{t4}, f_{n1}, f_{n2}, f_{n3}, f_{n4}] \quad (4.15)$$

and superscript  $j$  denotes, in the case of a dynamic analysis, time  $j\Delta t$  where  $\Delta t$  is the time step and in the case of an incremental static analysis,  $j$  denotes the state of stress and deformation at iteration number  $j$ .

Combining (4.14) with (4.12) and noting the arbitrariness of  $d\underline{u}$  results in

$$\underline{f}^{j+1} = \underline{f}^j + \int_{-L/2}^{L/2} \underline{B}^T d\underline{\sigma}(t) dt \quad (4.16)$$

where

$$\underline{f}^j = \int_{-L/2}^{L/2} \underline{B}^T \underline{\sigma}^j(t) dt \quad (4.17)$$

or in other words, the increment of  $\underline{df}$ ;  $\underline{df} = \underline{f}^{j+1} - \underline{f}^j$ , is related to the increment of the stresses by

$$\underline{df} = \int_{-L/2}^{L/2} \underline{B}^T d\underline{\sigma}(t) dt \quad (4.18)$$

where  $\underline{B}$  and  $d\underline{\sigma}$  are given by equations (4.13) and (2.5), respectively.

**Solution Schemes.** - For transient finite element analysis, the implementation of the constitutive law presented in this paper is particularly straightforward if the time integration scheme is explicit. Two popular explicit integrators are the central difference and the explicit Newmark methods (ref. 13). The advantage of explicit methods is that at each time step the new displacement configuration is determined completely in terms of historical information. This enables equation (4.18) to be evaluated using stresses based upon historical displacements. Furthermore, if a lumped (diagonal) mass matrix is employed, time stepping can be accomplished without solving systems of nonlinear simultaneous equations. Explicit methods are easy to program and show good convergence for nonlinear problems but are conditionally stable and often permit only small time step sizes. Nevertheless, for short and moderate duration transients, these methods can be very good. Implicit methods, such as trapezoidal rule and the implicit Newmark family, compute a new displacement configuration using historical information and derivatives of the new (unknown) displacement. Thus, these methods require the solution of a nonlinear system of simultaneous equations at each step which can entail considerable difficulty. The attractive feature of these methods is that they have good stability properties that permit the use of time steps that are larger than those for explicit methods. However, these methods are more difficult to program and convergence for large time steps with the constitutive law presented in this paper is an open question.

For static analysis, an incremental solution scheme can be obtained by combining equation (4.18) with equations (4.12) and (2.5) to obtain

$$\underline{df} = \underline{k}^{ep} d\underline{u} \quad (4.19)$$

where the instantaneous (tangent) stiffness matrix is

$$\underline{k}^{ep} = \int_{-L/2}^{L/2} \underline{B}^T \underline{\underline{E}}^{ep} \underline{B} dt \quad (4.20)$$

As previously mentioned,  $\underline{\underline{E}}^{ep}$  and hence,  $\underline{k}^{ep}$  is unsymmetric for frictional response. In addition, it is possible that it may occasionally be singular. Thus, approximate tangent stiffness matrix methods (refs. 14 and 15) (i.e., approximate Newton-Raphson methods) that use symmetric matrices may be more robust and economical.

## SUMMARY AND CONCLUSIONS

A constitutive model for complex, unclassical contact-friction problems has been presented for applications to dilatant and nondilatant surfaces. Surface deformation in the normal and tangential directions are decomposed into elastic (recoverable) and plastic (irrecoverable) components. Resultant normal and tangential stresses are determined from the deformations through a general elasto-plastic material matrix that accounts for several important physical features of the contact problem, namely surface roughness, relative mating of contact surfaces, and an appropriate surface friction law. For the unlubricated two body contact problem considered here, a three term general friction law was formulated including plasticity of a contact surface film layer, adhesion due to molecular attraction, and plowing due to penetration of contacting surface asperities. Application of this law to metal-metal, ceramic-ceramic, and metal-ceramic contact systems was discussed. Under general conditions, usually only one or two terms of this model would actually be required to adequately describe the contact.

Finally, a linear displacement contact interface finite element that is compatible with constant strain triangle or bilinear displacement quadrilateral finite elements was formulated. Implementation of this element into computer codes combined with the solution schemes discussed should afford sufficient convergence rates and solution stability. Based on this study, the following specific results were obtained:

1. The constitutive law presented in this paper results in an elasto-plastic material matrix for general frictional behavior and leads to straightforward finite element implementation and solution using existing methodologies.

2. The incremental form of the constitutive relation allows modelling of complex load-deformation histories and can account for reversals of sliding direction at any time.

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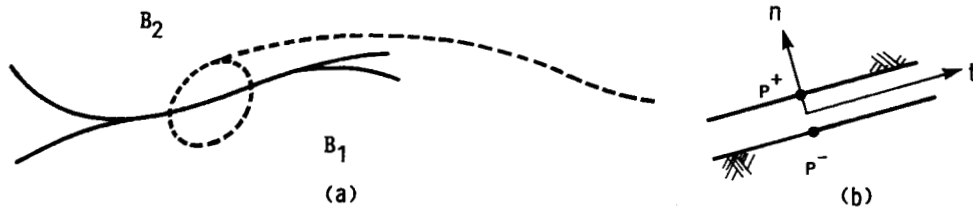


FIGURE 1.- (a) TWO BODY CONTACT IN 2-DIMENSIONS, (b) MACROSTRUCTURAL CONTACT SURFACE WITH COORDINATE SYSTEM AND SURFACES SHOWN SEPARATED FOR CLARITY.

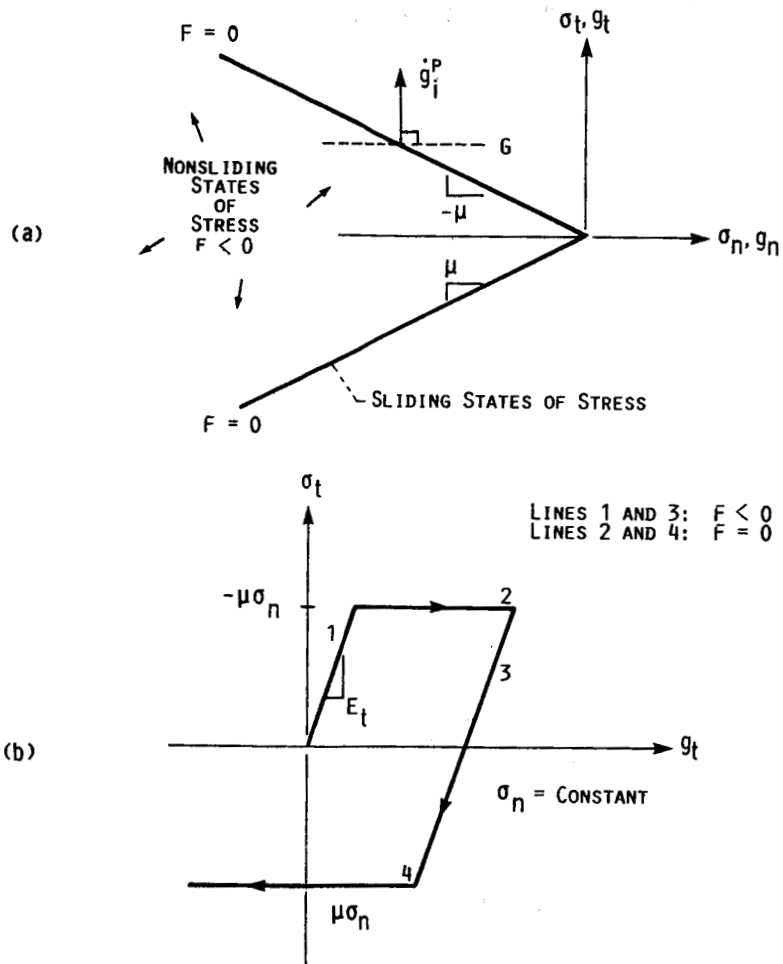


FIGURE 2. - (a) SLIP SURFACE  $F = 0$ . SLIP SURFACE AND SLIP POTENTIAL FOR NONDILATANT COULOMB FRICTION. THE SURFACE  $G$  WHOSE NORMAL GIVES THE DIRECTION OF SLIDING,  $\hat{g}_i^P$ , IS ALSO SHOWN, (b) POSSIBLE STRESS-DEFORMATION RESPONSE SHOWING UNLOADING AND SLIDING IN THE OPPOSITE DIRECTION.

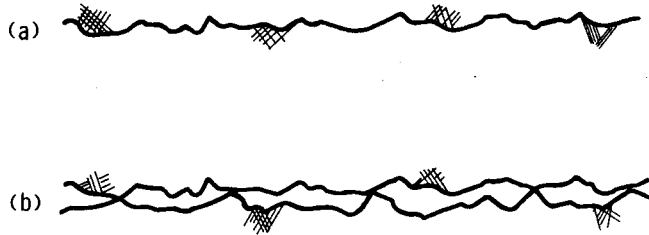


FIGURE 3.- EXAMPLES OF ROUGH SURFACE PROFILES; (a) DILATANT SURFACE PROFILE WITH VERY CLOSE MATING, (b) NONDILATANT SURFACE PROFILE WITH CONTACT AT ISOLATED ASPERITY PEAKS ONLY.

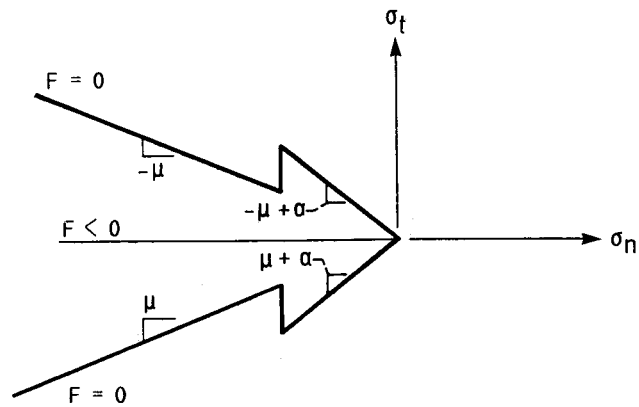


FIGURE 4.- SLIP FUNCTION  $F = 0$  FOR DILATANT COULOMB FRICTION.

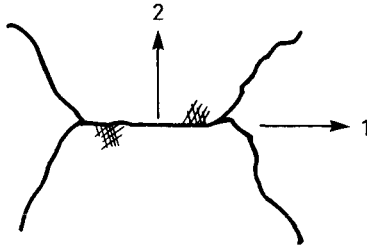


FIGURE 5.- IDEALIZATION OF PLASTIC DEFORMATION BETWEEN ASPERITIES IN CONTACT.

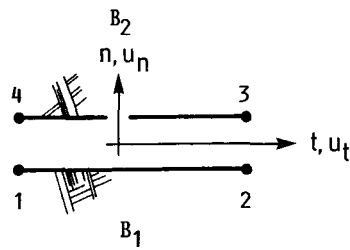


FIGURE 6.- 2-DIMENSIONAL LINEAR DISPLACEMENT CONTACT FINITE ELEMENT.

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| 16. Abstract<br><p>This report addresses techniques for modeling complex, unclassical contact-friction problems arising in solid and structural mechanics. A constitutive modeling concept is employed whereby analytic relations between increments of contact surface stress (i.e., traction) and contact surface deformation (i.e., relative displacement) are developed. Because of the incremental form of these relations, they are valid for arbitrary load-deformation histories. The motivation for the development of such a constitutive law is that more realistic friction idealizations can be implemented in finite element analysis software in a consistent, straightforward manner. Of particular interest in this report is modeling of two-body (i.e., unlubricated) metal-metal, ceramic-ceramic and metal-ceramic contact. Interfaces involving ceramics are of engineering importance, ceramics being considered for advanced turbine engines in which higher temperature materials offer potential for higher engine fuel efficiency.</p> |  |   |  |  |            |
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