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MATHEMATICAL ASPECTS OF FINITE ELEMENT METHODS FOR INCOMPRESSIBLE VISCOUS FLOWS

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We survey some mathematical aspects of finite element methods for incompressible viscous flows, concentrating on the steady primitive variable formulation. We address the discretization of a weak formulation of the Navier-Stokes equations; we then consider the div-stability condition, whose satisfaction insures the stability of the approximation. Specific choices of finite element spaces for the velocity and pressure are then discussed. Finally, the connection between different weak formulations and a variety of boundary conditions is explored.

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MATHEMATICAL ASPECTS OF FINITE ELEMENT METHODS FOR INCOMPRESSIBLE VISCOUS FLOWS

One of the most successful and well developed mathematical theories concerning finite element methods is that connected with incompressible flow problems. The success of this theory lies not only in the accumulated elegant mathematical results, but also in its impact on practical computations. The outstanding monographs by Girault and Raviart [GR1,GR2] give a rigorous account of this theory and to this day remain the definitive sources.

In this survey we examine certain mathematical aspects of finite element methods for the approximate solution of incompressible flow problems. Our principal goal is to present some of the important mathematical results which are relevant to practical computations. In so doing we also discuss useful algorithms. Due to space limitations we focus on the steady primitive variable formulation. Moreover, even within this narrow context, we will concentrate on only one of the very many different known approaches. Some other approaches are discussed in, e.g., [GR1,GR2,To].

We state at the outset that we make no attempt at being comprehensive in our coverage or in our attributions. To anyone who takes offense, we sincerely apologize.

I - The Primitive Variable Formulation

Let Ω denote a bounded, possibly multiply connected, domain in $\mathbf{R}^d$, d=2 or

3, and let Γ denote its boundary. As a prototype for incompressible flow problems we consider the Navier-Stokes equations

$$\mathbf{u} \cdot \operatorname{grad} \mathbf{u} + \operatorname{grad} p = \nu \Delta \mathbf{u} + \mathbf{f} \quad \text{in } \Omega \tag{1.1}$$

together with the incompressibility constraint

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \tag{1.2}$$

and the boundary condition

$$\mathbf{u} = 0 \quad \text{on } \Gamma \tag{1.3}$$

where $\mathbf{u}$ is the velocity field, p the pressure, $\mathbf{f}$ the given body force, and ν the given constant kinematic viscosity. In (1.1) the constant density has been absorbed into the pressure. Whenever u and p represent non-dimensionalized variables, then ν is the inverse of the Reynolds number Re.

Following our detailed discussion of the approximation of solutions of (1.1)-(1.3) by finite element methods, we will consider other incompressible flow formulations, especially as they concern boundary conditions other than (1.3).

I.1 - *Function spaces, norms and forms*

In order to introduce a Galerkin type weak formulation through which a finite element approximation is determined, we first need to define some function spaces, associated norms and forms involving functions belonging to those spaces. Lucid and more detailed accounts concerning these spaces may be

found in, e.g., [BaC,GR2,OR].

First we denote by $L^2(\Omega)$ the space of functions which are square integrable over Ω and which is equipped with the inner product and norm

$$(p,q) = \int_\Omega pq \quad \text{and} \quad \|q\|_0 = (q,q)^{1/2} ,$$

respectively. We then define the constrained space

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \mid \int_\Omega q = 0 \right\} .$$

Thus $L_0^2(\Omega)$ consists of square integrable functions with zero mean over Ω. This space is used in connection with the pressure; such a constraint is needed since it is clear from (1.1)-(1.3) that the pressure can be determined only up to an arbitrary constant. Other constraints, e.g., fixing the pressure at a given point, may be used instead without effecting any appreciable change in the results discussed below. Next we define the Sobolev spaces

$$H^k(\Omega) = \left\{ q \in L^2(\Omega) \mid D^s q \in L^2(\Omega) \text{ for } s=1,\ldots,k \right\}$$

where D^s denotes any and all derivatives of order s. Thus $H^k(\Omega)$ consists of square integrable functions all of whose derivatives of order up to k are also square integrable. $H^k(\Omega)$ comes equiped with the norm

$$\|q\|_k = \left\{ \|q\|_0^2 + \sum \|D^s q\|_0^2 \right\}^{1/2}$$

where the summation extends over all possible derivatives of order k or less. Clearly $H^0(\Omega)=L^2(\Omega)$. Of particular interest is the space $H^1(\Omega)$ consisting of

functions with one square integrable derivative and the subspace

$$H_0^1(\Omega) = \{ q \in H^1(\Omega) \mid q = 0 \text{ on } \Gamma \}$$

whose elements have one square integrable derivative over Ω and which vanish on the boundary Γ. These spaces have the associated norm

$$\|q\|_1 = \left(\|q\|_0^2 + \sum_{i=1}^{d} \|\frac{\partial q}{\partial x_i}\|_0^2 \right)^{1/2} . \tag{1.4}$$

We note that for functions belonging to $H_0^1(\Omega)$ the semi-norm

$$|q|_1 = \left(\sum_{i=1}^{d} \|\frac{\partial q}{\partial x_i}\|_0^2 \right)^{1/2} \tag{1.5}$$

is actually a norm equivalent to (1.4) and thus, for such functions, (1.5) may be used instead of (1.4).

For vector valued functions we use the spaces

$$H^k(\Omega) = [H^k(\Omega)]^d = \{ \mathbf{v} \mid v_i \in H^k(\Omega) \text{ for } i=1,\ldots,d \}$$

and

$$H_0^1(\Omega) = [H_0^1(\Omega)]^d = \{ \mathbf{v} \mid v_i \in H_0^1(\Omega) \text{ for } i=1,\ldots,d \} .$$

For example, $H^k(\Omega)$ consists of vector valued functions each of whose components belongs to $H^k(\Omega)$. $H^k(\Omega)$ is equiped with the norm

$$\|\mathbf{v}\|_k = \left(\sum_{i=1}^{d} \|v_i\|_k^2 \right)^{1/2} ;$$

alternately, $H_0^1(\Omega)$ has the norm

$$|v|_1 = \Big(\sum_{i=1}^{d} |v_i|_1^2 \Big)^{1/2} .$$

Also, the inner product for functions belonging to $L^2(\Omega)=H^0(\Omega)=[L^2(\Omega)]^d$ is given by

$$(u,v) = \int_\Omega u\cdot v$$

where there is no ambiguity possible resulting from using the same notation for both the inner product of scalar and vector valued functions.

We now define the bilinear forms

(1.6) $$a(u,v) = \nu \int_\Omega \text{gradu:gradv} \quad \text{for all } u,\ v\in H_0^1(\Omega)$$

and

(1.7) $$b(v,q) = -\int_\Omega q\,\text{div}v \quad \text{for all } v\in H_0^1(\Omega) \text{ and } q\in L_0^2(\Omega),$$

and the trilinear form

(1.8) $$c(w,u,v) = \int_\Omega w\cdot\text{gradu}\cdot v \quad \text{for all } u,v,w\in H_0^1(\Omega).$$

In (1.6) and (1.8) we have that $(\text{gradu})_{ij}=\partial u_j/\partial x_i$ and

$$\text{gradu:gradv} = \sum_{i,j=1}^{d} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} \quad \text{and} \quad w\cdot\text{gradu}\cdot v = \sum_{i,j=1}^{d} w_j \frac{\partial u_i}{\partial x_j} v_i .$$

Using the bilinear form $b(\cdot,\cdot)$, we can define the subspace

$$Z = \{ \mathbf{v} \in H_0^1(\Omega) \mid b(\mathbf{v},q) = 0 \quad \text{for all } q \in L_0^2(\Omega) \}$$

which consists of (weakly) *divergence free functions*, i.e., functions whose divergence is orthogonal to all $L_0^2(\Omega)$ functions. Certainly any divergence free function, in the strong sense, belongs to Z.

I.2 - *A Galerkin type weak formulation*

The most commonly used weak formulation of (1.1)-(1.3) is the following. Given $f \in L^2(\Omega)$, we seek $\mathbf{u} \in H_0^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that

$$a(\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{f},\mathbf{v}) \quad \text{for all } \mathbf{v} \in H_0^1(\Omega) \tag{1.9}$$

$$b(\mathbf{u},q) = 0 \quad \text{for all } q \in L_0^2(\Omega). \tag{1.10}$$

By virtue of (1.10) we see that the solution $\mathbf{u}$ belongs to Z.

We note that $L^2(\Omega)$ is not the largest function space for the data $\mathbf{f}$ such that the problem (1.9)-(1.10) makes sense; indeed, all that is required of the data is that the right hand side of (1.9) be bounded and this is possible for some functions which are not square integrable. However, for our purposes, $f \in L^2(\Omega)$ is sufficiently general.

It can be easily verified that whenever a pair $\mathbf{u},p$ satisfies (1.9)-(1.10) and is sufficiently smooth to allow for the appropriate integrations by parts, then $\mathbf{u},p$ is also a solution of (1.1)-(1.3). Of course, (1.9)-(1.10) admit solutions which are not sufficiently smooth to be solutions of (1.1)-(1.3); hence the terminology *weak formulation* and *generalized solution* are applied to

(1.9)-(1.10) and their solution, respectively. On the other hand, it is also clear that any solution of (1.1)-(1.3), i.e., a *strong solution*, satisfies (1.9)-(1.10).

For the weak formulation (1.9)-(1.10), the boundary condition (1.3) is an *essential* one, i.e., it must be imposed on the candidate solution functions. Below, in section IV.3, we will discuss the *natural* boundary conditions associated with the weak formulation (1.9)-(1.10).

We will not enter into details concerning the existence, uniqueness, continuous dependence on data and regularity of solutions of (1.9)-(1.10). Such results may be found in, e.g., the definitive treatise of Teman [Te]. Furthermore, many of these results are similar to those discussed below for the approximate problem.

II - The Finite Element Problem and the Div-stability Condition

II.1 - *The discrete finite element problem*

Once the Galerkin formulation (1.9)-(1.10) is established, the approximate problem which determines the finite element solution is defined in the usual manner. First one chooses the approximating finite element spaces, or more precisely, a family of finite element spaces, $\mathbf{V}^h$ and S^h for the velocity and pressure, respectively. Here h is a parameter which is usually related to the size of the grid associated with the finite element partitioning of Ω. Then one requires that (1.9)- (1.10) hold for functions belonging to these finite dimensional spaces, i.e., one seeks $\mathbf{u}^h \in \mathbf{V}^h$ and $p^h \in S^h$ such that

$$a(\mathbf{u}^h,\mathbf{v}^h) + c(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h) + b(\mathbf{v}^h,p^h) = (\mathbf{f},\mathbf{v}^h) \quad \text{for all } \mathbf{v}^h \in \mathbf{V}^h \tag{2.1}$$

and

$$b(u^h, q^h) = 0 \qquad \text{for all } q^h \in S^h. \tag{2.2}$$

If V^h and S^h are subspaces of the underlying infinite dimensional spaces of (1.9)- (1.10), i.e., if $V^h \subset H_0^1(\Omega)$ and $S^h \subset L_0^2(\Omega)$, then the finite element solution defined by (2.1)-(2.2) is said to be *conforming*. Otherwise, i.e., if $V^h \not\subset H_0^1(\Omega)$ and/or $S^h \not\subset L_0^2(\Omega)$, then the method is said to be *non-conforming*. We will restrict our attention to examples of the former.

Once one chooses specific bases for V^h and S^h, (2.1)-(2.2) are equivalent to a *nonlinear system of algebraic equations*. Indeed, if $\{q_j(\mathbf{x})\}$, $j=1,\ldots,J$ and $\{\mathbf{v}_k(\mathbf{x})\}$, $k=1,\ldots,K$, denote bases sets for S^h and V^h, respectively, we may then write

$$p^h = \sum_{j=1}^{J} \alpha_j q_j(\mathbf{x}) \quad \text{and} \quad \mathbf{u}^h = \sum_{k=1}^{K} \beta_k \mathbf{v}_k(\mathbf{x})$$

for some constants α_j, $j=1,\ldots,J$, and β_k, $k=1,\ldots,K$. Substituting into (2.1)-(2.2) then yields

$$\sum_{k=1}^{K} a(\mathbf{v}_k, \mathbf{v}_\ell)\, \beta_k + \sum_{k,m=1}^{K} c(\mathbf{v}_m, \mathbf{v}_k, \mathbf{v}_\ell)\, \beta_k\, \beta_m$$

$$+ \sum_{j=1}^{J} b(\mathbf{v}_\ell, q_j)\, \alpha_j = (\mathbf{f}, \mathbf{v}_\ell) \quad \text{for } \ell=1,\ldots,K \tag{2.3}$$

and

$$(2.4) \qquad \sum_{k=1}^{K} b(\mathbf{v}_k, q_i)\, \beta_k = 0 \quad \text{for } i=1,\ldots,J.$$

which constitute a nonlinear algebraic, in fact, quadratic, system of $J+K$ equations for the $J+K$ unknowns α_j, $j=1,\ldots,J$, and β_k, $k=1,\ldots,K$. Note that the the discrete continuity equation (2.2) yields the $J\times K$ rectangular *linear* system (2.4).

II.2 - *The div-stability condition*

In the positive definite case, e.g., for the equations of linear elasticity, the mere inclusion of the finite element spaces within the underlying function spaces is essentially sufficient to assure that the approximations are well defined and are as accurate as possible for the type of finite elements functions being used. Here the inclusions $\mathbf{V}^h \subset H_0^1(\Omega)$ and $S^h \subset L_0^2(\Omega)$ are not by themselves sufficient to produce stable, meaningful approximations. We find ourselves in the realm of what are known as *mixed finite element methods*.

There are number of conditions which the elements belonging to the finite element spaces should satisfy. Most of them, e.g., the boundedness of the various bilinear and trilinear forms, are easily satisfied by conforming finite element spaces. The one condition which presents a problem has the following mathematical realization:

given any $q^h \in S^h$,

$$(2.5) \qquad \sup_{0 \neq \mathbf{v}^h \in \mathbf{V}^h} \left(\frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_1} \right) \geq \gamma \|q^h\|_0$$

where the constant $\gamma>0$ may be chosen independent of h and of the particular choice of $q^h \in S^h$.

This condition may be equivalently expressed in the form:

given any $q^h \in S^h$ there exists a non-zero $\mathbf{v}^h \in \mathbf{V}^h$ such that

$$b(\mathbf{v}^h, q^h) \geq \gamma \|q^h\|_0 \|\mathbf{v}^h\|_1 \tag{2.6}$$

where the constant $\gamma>0$ may be chosen independent of h and of the particular choice of $q^h \in S^h$.

Of course, for each q^h a different $\mathbf{v}^h$ may be used in order to satisfy (2.6).

The condition (2.5), or equivalently (2.6), is variously known as the *Ladyzhenskaya-Babuska-Brezzi* or the *LBB* or the *inf-sup* condition, the latter designation following from the third equivalent form:

there exists a $\gamma>0$, independent of h, such that

$$\inf_{0 \neq q^h \in S^h} \sup_{0 \neq \mathbf{v}^h \in \mathbf{V}^h} \left(\frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_1 \|q^h\|_0} \right) \geq \gamma \, . \tag{2.7}$$

We will refer to any of the equivalent statements (2.5)-(2.7) as the condition for *div-stability*. Note that these have nothing to do with the non-linearity of the Navier-Stokes equations and, in fact, the possible problems its satisfaction poses is shared by the linear equations of Stokes flow.

Associated with the finite element spaces $\mathbf{V}^h$ and S^h and the bilinear form

$b(\cdot,\cdot)$ we have the subspace

$$Z^h = \left(v \in V^h \mid b(v^h, q^h) = 0 \quad \text{for all } q^h \in S^h \right) .$$

of *discretely divergence free functions*. In general $Z^h \not\subset Z$, even when $V^h \subset H_0^1(\Omega)$ and $S^h \subset L_0^2(\Omega)$, i.e., discretely solenoidal functions are not necessarly solenoidal. This is, of course, entirely analogous to the finite difference case, e.g., a function satifying a difference approximation to the incompressibility constraint is not in general solenoidal. A measure of the "angle" between the spaces Z^h and Z is given by

$$\Theta = \sup_{\substack{z^h \in Z^h \\ \|z^h\|_1 = 1}} \inf_{z \in Z} \|z - z^h\|_1 . \tag{2.8}$$

In general, $0 \le \Theta \le 1$, which is easily seen by observing that for $Z^h \in Z$, $\Theta = 0$, and that by choosing $z = 0$, $\Theta = 1$.

Note that because of (2.2), the approximate velocity $u^h \in Z^h$, i.e., u^h is discretely solenoidal. However, since in general $Z^h \not\subset Z$, $\mathrm{div} u^h \neq 0$. Loosly speaking, the div-stability condition (2.5) ensures, as $h \to 0$ at least, that discretely solenoidal functions tend to solenoidal functions.

II.3 - *Error estimates and other results concerning the approximate solution*

We now present some of the available mathematical results concerning the solution u^h, p^h of the finite element problem (2.1)-(2.2). Here we *assume* that the chosen finite element spaces V^h and S^h satisfy the div-stability condition

(2.5). Subsequently, we will look into the issue of verifying that condition. The summary presented is based on the detailed analysis found in [CR, JR, GR1, GR2, and GP].

First off, for any $f \in L_0^2(\Omega)$, (2.1)-(2.2) has a solution u^h, p^h, provided that the div-stability condition (2.5) holds. However, one can prove that the solution is unique only for "sufficiently small" data f or "sufficiently large" viscosity ν. More precisely, let

$$\kappa = \sup_{u^h, v^h, w^h \in V^h} \left(\frac{a(w^h, u^h, v^h)}{|u^h|_1 |v^h|_1 |w^h|_1} \right) .$$

For standard choices of finite element spaces κ can be shown to be independent of h and, in fact, depends only on $\Omega \subset R^d$ and d. Then, one can show that (2.1)-(2.2) has a unique solution whenever

$$\frac{\kappa}{\nu^2} \sup_{v^h \in V^h} \left(\frac{\int_\Omega f \cdot v^h}{|v^h|_1} \right) \leq 1 .$$

This condition is very similar to the one which is needed to show the uniqueness of the solution of (1.9)-(1.10) and in fact the latter implies the former, i.e., whenever (1.9)-(1.10) can be shown to have a unique solution, then, provided the div- stability condition is satisfied, (2.1)-(2.2) also have a unique solution.

When one can show that (1.9)-(1.10) has a unique solution, it can also be shown that the finite element solution of (2.1)-(2.2) converges to that

solution. In addition, something can be said about the convergence of the finite element solution even when (1.9)-(1.10) does not posses a unique solution. For details, see [GR1 and GR2].

Error estimates can also be derived. Provided that the div-stability condition is satisfied, we have that

$$\|\mathbf{u} - \mathbf{u}^h\|_1 \le C_1 \inf_{\mathbf{v}^h \in \mathbf{V}^h} \|\mathbf{u} - \mathbf{v}^h\|_1 + C_2 \Theta \inf_{q^h \in S^h} \|p - q^h\|_0 \tag{2.9}$$

and

$$\|p - p^h\|_0 \le C_3 \inf_{\mathbf{v}^h \in \mathbf{V}^h} \|\mathbf{u} - \mathbf{v}^h\|_1 + C_4 \inf_{q^h \in S^h} \|p - q^h\|_0 \tag{2.10}$$

where Θ is defined in (2.8) and C_i, $i=1,..,4$, are constants independent of h. These estimates are optimal for the "graph norm" $\|\mathbf{u}\|_1 + \|p\|_0$ of functions belonging to $H_0^1(\Omega) \times L_0^2(\Omega)$ in the sense that the rate of convergence of the finite element solution, measured in this norm, is the same as that of the best approximation to $\mathbf{u}$ and p out of $\mathbf{V}^h$ and S^h, respectively.

If the solution of (1.9)-(1.10), or more precisely, of the linearized adjoint problem corresponding to (1.9)-(1.10), is sufficiently regular, then one can obtain an improved velocity error estimate in the $L^2(\Omega)$-norm, namely

$$\|\mathbf{u} - \mathbf{u}^h\|_0 \le C_5 h \|\mathbf{u} - \mathbf{u}^h\|_1 \tag{2.11}$$

where again C_5 is independent of h.

We see that once the div-stability condition is satisfied, the error in the finite element approximation depends only on the ability to approximate in the

chosen finite element subspaces. In general, (2.9)-(2.10) indicate that the velocity and pressure errors are coupled. Furthermore, one finds that it is efficient to equilibrate the rates of convergence of the two terms on the right hand side of (2.9)-(2.10). For this reason, one would like to use, for example, polynomials of one degree higher for the velocity components than those used for the pressure. As a final comment, we note that the constants appearing in (2.9)-(2.10) are in general proportional to $1/\gamma$ where γ is the stability constant appearing in (2.5).

II.4 - *Verifying the div-stability condition*

For particular choices of $\mathbf{V}^h$ and S^h, it is usually *not* an easy matter to verify that the div-stability condition holds. To accomplish this task for families of such spaces is even more difficult. Here, we sketch three techniques for verifying the div-stability condition.

a) Fortin's method - One seemingly attractive method of showing that the div- stability condition holds is due to Fortin. He has shown [F] that the div-stability condition (2.5) is equivalent to the existence of a linear operator Π^h from $\mathbf{H}_0^1(\Omega) \to \mathbf{V}^h$ such that given any $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$

$$b(\Pi^h \mathbf{v}, q^h) = b(\mathbf{v}, q^h) \quad \text{for all } q^h \in S^h$$

and

$$\|\Pi^h \mathbf{v}\|_1 \leq C \|\mathbf{v}\|_1$$

where the constant $C>0$ may be chosen independent of h and of the particular choice of $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$. Thus the task of verifying the div-stability condition (2.5) is reduced to the task of showing the existence of the operator Π^h; unfortunately, although the latter task has been accomplished in a few specific

settings, in general, it is also a difficult thing to do.

b) Verfürth's method - Verfürth [V1] has developed a method for verifying the div-stability condition (2.5) which applies to the case of *continuous* discrete pressure spaces.

Specifically, if $S^h \subset H^1(\Omega) \cap L^2_0(\Omega)$, he starts out by combining the inverse inequality, see, e.g., [Ci],

$$|v^h|_1 \leq C_1 h^{-1} \|v^h\|_0 \quad \text{for all } v^h \in V^h \,, \tag{2.12}$$

and the result

$$\sup_{0 \neq v^h \in V^h} \frac{b(v^h, q^h)}{\|v^h\|_0} \geq C_2 |q^h|_1 \quad \text{for all } q^h \in S^h \tag{2.13}$$

to yield

$$\sup_{0 \neq v^h \in V^h} \frac{b(v^h, q^h)}{|v^h|_1} \geq \frac{C_2}{C_1} h |q^h|_1 \quad \text{for all } q^h \in S^h \,. \tag{2.14}$$

The inequality (2.13) can be shown to hold for many element pairs involving continuous discrete pressure fields; see, e.g., [BP]. Note that (2.13) has a similar appearance to the div-stability condition (2.5), but that it involves the "wrong" norms.

Next, one combines the result, which can be found in, e.g., [GR1,GR2, L]: given any $q^h \in S^h \subset L^2_0(\Omega)$, there exists a $w \in H^1_0(\Omega)$ such that $\mathrm{div} w = q^h$ and $|w|_1 \leq C_3 \|q^h\|_0$, with the approximation theoretic assumption concerning the space V^h: for any $w \in H^1(\Omega)$ there exists a $w^h \in V^h$ such that

(2.15) $$|w-w^h|_k \le C_4 h^{1-k}|w|_1 \quad \text{for } k=0,1 \ ,$$

to yield

(2.16) $$\sup_{0\neq v^h\in V^h} \frac{b(v^h,q^h)}{|v^h|_1} \ge C_5 - C_6 h|q^h|_1 \quad \text{for all } q^h\in S^h \text{ with } \|q^h\|_0=1.$$

Verfürth then shows that the div-stability condition (2.5) follows from (2.14) and (2.16) provided the constants $C_1,\ldots,C_6$ are independent of h.

Thus the main task of applying his method, once the inverse inequality (2.12) and the approximation theoretic result (2.15) have been shown to hold for the discrete velocity space V^h, is to show that (2.13) is valid.

c) The Boland-Nicolaides method - A more useful method, in the sense of having wide applicability and relative ease of use, has been developed by Boland and Nicolaides [BN1]. One difficulty with verifying the div-stability condition (2.5) is its *global* nature; Boland and Nicolaides have shown how to *localize* the difficult part of the verification process.

Specifically, consider a subdivision of Ω into disjoint *macro-elements* Ω_r, $r=1,\ldots,R$, each of which consists of one or a few elements in the the finite element triangulation associated with V^h and S^h. The number of elements within a macro- element is independent of h, i.e., as we refine the mesh the macro-elements are also refined so that they always contain the same number of elements. Let Γ_r denote the boundary of the macro-element Ω_r.

Now, first suppose that the div-stability condition holds for the pair V^h and S^h *locally* over a macro-element, i.e., there exists a constant $\hat{\gamma}>0$, independent of h and of the particular choice of macro-element, such that

$$\sup_{0\neq \mathbf{v}^h\in \mathbf{V}_r^h} \left(\frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_1} \right) \geq \hat{\gamma}\|q^h\|_0 \quad \text{for all } q^h\in S_r^h \tag{2.17}$$

where

$$\mathbf{V}_r^h = \left(\mathbf{v}\in \mathbf{V}^h|_{\Omega_r} \mid \mathbf{v}=\mathbf{0} \text{ on } \Gamma_r \right) \quad \text{and} \quad S_r^h = \left(q\in S^h|_{\Omega_r} \right).$$

Since $\mathbf{V}_r^h$ and S_r^h have fixed small dimension, independent of h, (2.17) may often be verified by a direct computation.

Second, suppose that the div-stability condition holds *globally* for the spaces $\tilde{\mathbf{V}}^h$ and $\tilde{S}^h$ where

$$\left\{ \begin{array}{l} \tilde{\mathbf{V}}^h \subset \mathbf{V}^h \\ \tilde{S}^h = \left(\begin{array}{l} L_0^2(\Omega) \text{ piecewise constant functions with respect} \\ \qquad\qquad \text{to the macro-elements } \Omega_r,\ r=1,\ldots,R \end{array} \right) \end{array} \right. , \tag{2.18}$$

i.e., suppose that there exists a constant $\tilde{\gamma}>0$, independent of h, such that

$$\sup_{0\neq \mathbf{v}^h\in \tilde{\mathbf{V}}^h} \left(\frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_1} \right) \geq \tilde{\gamma}\|q^h\|_0 \quad \text{for all } q^h\in \tilde{S}^h . \tag{2.19}$$

Sumarizing the Boland-Nicolaides method, suppose we know that the pair $\mathbf{V}^h, S^h$ is *locally div-stable* with constant $\hat{\gamma}$ independent of h, i.e., in the sense of (2.17). Further, suppose that the *comparison* spaces $\tilde{\mathbf{V}}^h, \tilde{S}^h$, which satisfy (2.18), are *globally div-stable* with constant $\tilde{\gamma}$ independent of h, i.e., in the sense of (2.19). Then the spaces $\mathbf{V}^h, S^h$ are *globally div-stable* with a constant γ independent of h. Thus, through the use of the comparison spaces the div-stability of the pair $\mathbf{V}^h, S^h$ need only be checked locally, i.e., over a macro-element.

This method has been succesfully used, e.g., in [BC,BN1,BN3,GR2], to show the div-stability of a variety of well known elements and some novel ones as well, both in two and three dimensions.

II.5 - *Examples of unstable spaces including the bilinear-constant pair.*

There are different ways in which arbitrarily chosen finite element spaces may fail to satisfy the div-stability condition. Here we discuss some of these and then give specific examples, focusing on the much studied and much misunderstood bilinear velocity-constant pressure pair.

The most catastrophic type of failure is for (2.2), or equivalently (2.4), to imply that $\mathbf{u}^h=\mathbf{0}$, i.e., the only discretely solenoidal field belonging to $\mathbf{V}^h$ is the zero vector. The approximate solution is useless since, of course, $u^h=0$ cannot be a good approximate solution of the Navier-Stokes equations. This type of situation can usually be detected by a counting argument, i.e., the discrete divergence matrix $b(\mathbf{v}_k,q_j)$, $j=1,\ldots,J$ and $k=1,\ldots,K$, appearing in (2.4) has more independent rows than columns.

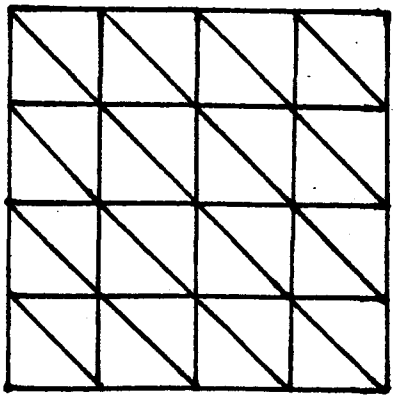
Less catastrophic is the situation wherein for one or a few, but not all, $q^h\in S^h$ we have that $b(\mathbf{v}^h,q^h)=0$ for all $\mathbf{v}^h\in\mathbf{V}^h$ so that $\gamma=0$ in (2.5). This kind of failure of the div-stability condition is usually easy to detect since it results, in practice, in the discrete divergence matrix being rank deficient. Furthermore, if these type of pressure modes q^h are the sole reason for the invalidity of (2.5), one may often still obtain, through a filtering process, useful approximations.

A more subtle failure of the div-stability condition is the case where for at least some $q^h\in S^h$

$$(2.20) \qquad C_1 h \|q^h\|_0 \leq \sup_{0 \neq v^h \in V^h} \left(\frac{b(v^h, q^h)}{\|v^h\|_1} \right) \leq C_2 h \|q^h\|_0$$

for some constants C_1 and C_2 independent of h. In this case $\gamma = O(h)$ where γ is the constant appearing in (2.5). In practice this may result in a loss of accuracy, especially for the pressure approximations. Such instabilities are harder to detect because, of course, computations are usually carried out using a finite value of h. In particular no problems such as those caused by rank deficient approximations to the continuity equation are encountered. This type of situation points out the dangers of calculating on only one grid and of not at least performing serious mesh refinement studies. It also points out the usefulness of rigorous results concerning the stability, or lack thereof, of finite element spaces.

a) - <u>An unstable linear-constant pair</u> - An example of the first and most catastrophic instability is the following seemingly natural choice for the velocity and pressure finite element spaces. Let Ω be a square which is triangulated as in the figure below. For the velocity approximations we choose piecewise linear functions with respect to the given triangulation which are continuous over $\bar{\Omega}$ and which vanish on Γ. For the discrete pressures we choose piecewise constant functions with respect to the same triangulation and having zero mean over Ω. Clearly $V^h \subset H_0^1(\Omega)$ and $S^h \subset L_0^2(\Omega)$. For this choice the only discrete velocity field $u^h \in V^h$ satisfying the discrete incompressibility constraint (2.2) is $u^h = 0$, i.e., the only discretely solenoidal velocity field is the zero

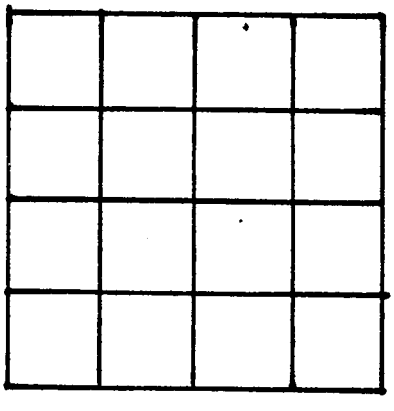


vector! One easily sees that if there are N cells to side, that the number of equations in (2.4) is $J=\dim(S^h)=2N^2-1$ which is greater than the number of columns $K=\dim(V^h)=2(N-1)^2$.

In the above example we see that the discrete incompressibilty condition (2.2) imposes too many constraints relative to the available velocity degrees of freedom. In fact, $\dim(S^h)>\dim(V^h)$. In order to remedy the situation one must, at least, increase the dimension of V^h relative to that of S^h.

b) The bilinear-constant element pair - We next consider the bilinear velocity- constant pressure pair which is often refered to as the Q_1-P_0 element pair. Again consider the case of Ω being a square and consider the "triangulation" of the figure below. We now choose V^h to consist of piecewise bilinear functions with respect to this triangulation which are continuous over $\bar{\Omega}$ and which vanish on Γ. For S^h we choose piecewise constant functions over the same triangulation and which have zero mean over Ω. Once again the inclusions $V^h \subset H_0^1(\Omega)$ and $S^h \subset L_0^2(\Omega)$ hold. The simple counting argument used for the first example does not yield any definitive information since $\dim(V^h)=2(N-1)^2$, the same as before, while now $\dim(S^h)=N^2-1$.

It is well known, e.g., see [F, BH, SGLGE, JP, GNP], that this bilinear-constant element pair exhibits the disastrous "checkerboard" mode, i.e., for the particular discrete pressure field $q^h \in S^h$ which is +1 in the "red squares" and -1 in the "black squares" we have that $b(v^h,q^h)=0$ for all $v^h \in V^h$. This is an example of the second type of instability discussed above. The single "bad" pressure mode can be easily filtered out, and therefore some have suggested that once this mode



is taken care of, the bilinear-constant element pair can be safely used.

However, this is not the whole story for the bilinear-constant element pair. Boland and Nicolaides [BN2] have shown that there exist other pressure modes for which (2.20) is satisfied. The left hand inequality of (2.4) was previously known [JP], at least in the different context of penalty methods. Of course, the left inequality does not imply the right, and certainly doesn't imply that for those modes the stability constant $\gamma=O(h)$. However, Boland and Nicolaides have shown that this is indeed the case. Moreover, they have shown [BN3] that there exist data f for which the pressure approximations do not converge and that it is also possible to set up problems for which the velocity approximations do not converge as well. At the least, since the constants in the error estimates (2.9)-(2.11) are proportional to γ^{-1}, there will likely be a loss of accuracy due to these pressure modes. Their conclusions are worth noting, especially in view of the fact that the bilinear- constant element pair, with the checkerboard mode filtered out, has been used on numerous ocassions in "practical" computations.

III - Finite Element Spaces for the Primitive Variable Formulation

In this section we discuss pressure and velocity finite element spaces which have been rigorously shown to satisfy the div-stability condition. There are many such pairs known, especially for two dimensional problems; therefore we will restrict our attention to pairs which have proven to be of the most practical utility.

Throughout, $P_k(\mathcal{D})$ denotes the space of polynomials of degree less than equal to k with respect to the set $\mathcal{D}\subset R^d$ and $[P_k(\mathcal{D})]^d$ denotes the space of d-

vector valued functions each of whose components belong to $P_k(\mathcal{D})$. Analogous definitions hold for $Q_k(\mathcal{D})$ and $[Q_k(\mathcal{D})]^d$ in the case of functions which are polynomials of degree less than or equal to k in each of the coordinate directions, e.g., $Q_1(\mathcal{D})$ denotes piecewise bilinear functions with respect to the set $\mathcal{D}$. Likewise we define the spaces $C^k(\mathcal{D})$ and $[C^k(\mathcal{D})]^d$ of k-times continuously differentiable functions with respect to the set $\mathcal{D}$.

For the most part, the results below hold for polygonal domains in R^2 and polyhedral domains in R^3. Through the use of, e.g., isoparametric elements, they will also hold for domains with curved bounddaries provided the latter satisfy the usual smoothness criteria. Furthermore, we assume that all subdivisions of Ω into finite elements which are employed below satisfy the standard conditions. For details concerning these issues, one may consult, e.g., [Ci].

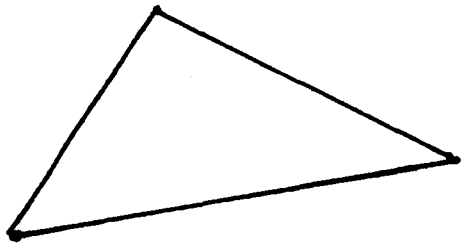
III.1 - *Piecewise linear and bilinear velocity fields*

We begin with some examples involving piecewise linear or bilinear velocity fields with respect to a subdivision of Ω into triangles or rectangles, respectively. In all cases the discrete velocity fields are continuous over $\bar{\Omega}$. In combination with these type of velocity finite element spaces we will consider both discontinuous piecewise constant and continuous, over $\bar{\Omega}$, piecewise linear pressure fields. Every element pair listed satisfies the div-stability condition (2.5). Moreover, provided the solution $\mathbf{u},p$ of (1.9)-(1.10) satisfies $\mathbf{u}\in H^2(\Omega)\cap H_0^1(\Omega)$ and $p\in H^1(\Omega)\cap L_0^2(\Omega)$, the following error estimates for the discrete solution $\mathbf{u}^h,p^h$ of (2.1)-(2.2) hold uniformily in h:

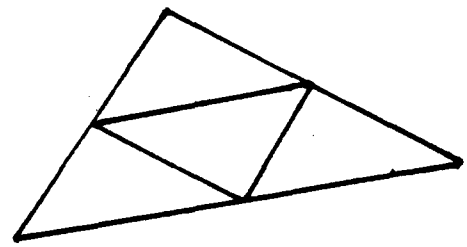
$$(3.1)\qquad \begin{cases} \|\mathbf{u}-\mathbf{u}^h\|_1 = O(h) \\ \|\mathbf{u}-\mathbf{u}^h\|_0 = O(h^2) \\ \|p-p^h\|_0 = O(h) \end{cases}$$

Thus, these elements yield first order accurate pressure approximations and second order accurate velocity approximations.

a) Piecewise constant pressures I - For the linear-constant element pair mentioned in section II.5 the discrete continuity equation overconstrained the approximate velocity field. However, by employing different grids for the pressure and velocity fields, the linear-constant element pair may be made stable. For example, consider a given triangulation $\mathcal{T}_h$ of a polygonal domain Ω into triangles. Then divide each triangle in $\mathcal{T}_h$ into four triangles by joining the midsides, thus defining a refined triangulation $\mathcal{T}_{h/2}$. An example is provided in the figure below.



A pressure triangle in $\mathcal{T}_h$



The four associated velocity triangles in $\mathcal{T}_{h/2}$

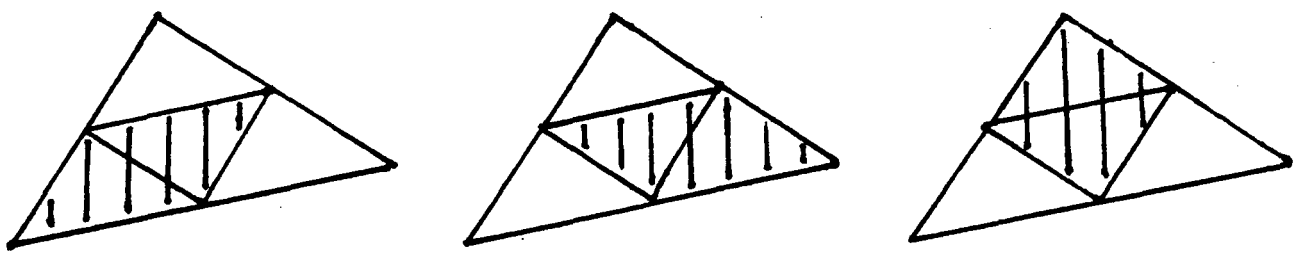
Now define

$$\text{(3.2)} \qquad \begin{cases} S^h = \left\{ q \mid q \in P_0(\Delta) \ , \ \Delta \in \mathcal{T}_h \ ; \ \int_\Omega q = 0 \right\} \\ \mathbf{V}^h = \left\{ \mathbf{v} \mid \mathbf{v} \in [P_1(\Delta)]^2 \ , \ \Delta \in \mathcal{T}_{h/2} \ ; \ \mathbf{v} \in [C^0(\bar{\Omega})]^2 \ ; \ \mathbf{v} = \mathbf{0} \text{ on } \Gamma \right\} \end{cases}$$

so that the pressure is sought among piecewise constants with respect to the

triangulation $\mathcal{T}_h$ while the velocity is sought among continuous piecewise linear fields with respect to the finer triangulation $\mathcal{T}_{h/2}$. The pair of finite element spaces defined by (3.2) are known to satisfy the div-stability condition (2.5) and thus yield optimally accurate approximations satisfying (3.1)

b) Piecewise constant pressures II - For the unstable linear-constant element pair of section II.5 there was one velocity element for each pressure element; for the stable linear-constant element pair (3.2) there are four velocity triangles for each pressure triangle. Stable linear-constant element pairs may be defined wherein the ratio of discrete pressures to velocities is not so high. For example, let the velocity space $\mathbf{V}^h$ be as in (3.2); now define the pressure space S^h through the following choice of basis. For each triangle of $\mathcal{T}_h$ we define three basis functions, namely piecewise constants which are unity in the shaded areas in figure below and zero in the unshaded areas. Of course, outside the particular triangle of $\mathcal{T}_h$, the basis functions vanish as well. This pressure space consists of three out of the



four possible piecewise constants associated with the four triangles in $\mathcal{T}_{h/2}$ contained within a single triangle in $\mathcal{T}_h$. Moreover, there are essentially three times as many pressure degrees of freedom for this choice of S^h as there are

for the choice made in (3.2). However, this element pair is also stable, i.e., satisfies the div-stability condition (2.5) and the error estimates (3.1).

c) Piecewise linear pressures - One may also couple a piecewise linear velocity element with a piecewise linear pressure element and still satisfy the div-stability condition (2.5) and the estimates (3.1). Such a pair was introduced in [BP], analyzed there and in [V1], and is given by

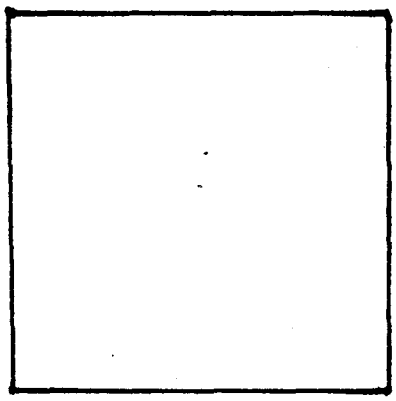
$$(3.3)\qquad \begin{cases} S^h = \left\{ q \mid q\in P_1(\Delta)\ ,\ \Delta\in\mathcal{T}_h \ ;\ q\in C^0(\Omega) \ ;\ \int_\Omega q = 0 \right\} \\ V^h = \text{as in (3.2).} \end{cases}$$

Due to the coupling between the pressure and velocity errors one cannot take advantage of the better approximating ability of the linear pressure space. Thus, insofar as the rates of convergence, this linear-linear element pair is no better than the stable linear-constant element pairs. However, in practical calculations we have found this to be the best element combination involving linear velocity fields, better in the sense of giving more accuracy for useful values of h. Furthermore, this linear-linear element pair usually results in fewer unknowns, for the same grid, than do the linear-constant pairs. For example, suppose the pressure triangulation $\mathcal{T}_h$ is given by the first figure of section II.4 with N intervals on each side. Thus there are $2N^2$ triangles in $\mathcal{T}_h$ and the element pair (3.2) has $2N^2-1$ pressure unknowns; on the hand, the number of nodes in this triangulation is only $(N+1)^2$ and thus the piecewise linear pressure space of (3.3) has only $(N+1)^2-1$ degrees of freedom. Both element pairs have $2(2N-1)^2$ velocity unknowns so that the linear-linear element pair (3.3) has roughly N^2 less degrees of freedom, for the same grid, as does the linear-constant element pair (3.2).

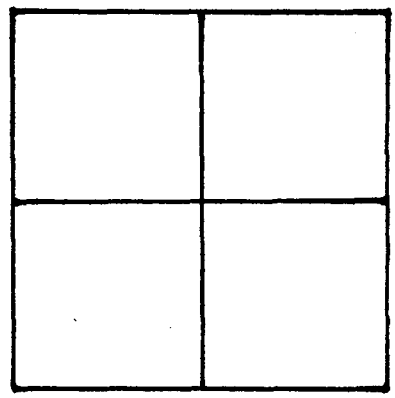
d) Piecewise bilinear velocity fields - Entirely analagous to the triangular elements described above, we have the following elements involving

bilinear velocity fields with respect to rectangular elements. More general quadrilateral elements may be found from these through, e.g., isoparametric mappings.

We start with a subdividivision $\mathcal{Z}_h$ of Ω into rectangles, or more generally quadrilaterals. Subsequently we divide each rectangle into four smaller rectangles by joining the midsides, thus creating another subdivision $\mathcal{Z}_{h/2}$ of Ω into rectangles. See the figure below. In all three velocity-pressure element pairs



A pressure rectangle in $\mathcal{Z}_h$



The four associated velocity rectangles in $\mathcal{Z}_{h/2}$

about to be described we choose the approximating velocity space to consist of piecwise bilinear functions with respect to the subdivision $\mathcal{Z}_{h/2}$ which are continuous over $\bar{\Omega}$ and which vanish on Γ, i.e.,

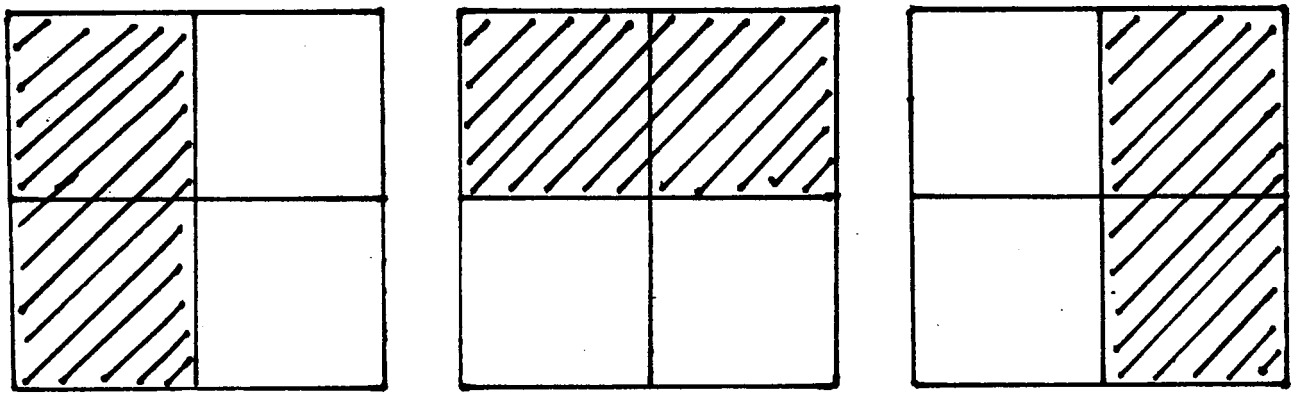
$$(3.4) \qquad \mathbf{V}^h = \left(\mathbf{v} \mid \mathbf{v} \in [Q_1(\square)]^2 \ , \ \square \in \mathcal{Z}_{h/2} \ ; \ \mathbf{v} \in [C^0(\bar{\Omega})]^2 \ ; \ \mathbf{v} = \mathbf{0} \text{ on } \Gamma \right).$$

For the first pressure space we choose piecewise constants with respect to the larger quadrilaterals of the subdivision $\mathcal{Z}_h$ and which have zero mean over

Ω, i.e.,

$$S^h = \{\, q \mid q \in Q_0(\square),\ \square \in \mathcal{Z}_h \ ;\ \int_\Omega q = 0 \,\}\ .$$

As indidicated in the figure below, for the second pressure space we choose three



out of the four possible piecewise constants associated with the rectangles belonging to $\mathcal{Z}_{h/2}$ and which have zero mean over Ω. Finally, the third pressure space consists of piecewise bilinear functions with respect to the subdivision $\mathcal{Z}_h$ which are continuous over $\bar{\Omega}$ and have zero mean over Ω, i.e.,

$$S^h = \{\, q \mid q \in Q_1(\square),\ \square \in \mathcal{Z}_h \ ;\ q \in C^0(\bar{\Omega}) \ ;\ \int_\Omega q = 0 \,\} \tag{3.5}$$

The three velocity-pressure elements just described satisfy the div-stability condition (2.5) and the error estimates (3.1). Similar to the case for triangles and for the same reasons, the prefered element pair involving bilinear velocities is (3.4) coupled with (3.5), i.e., the bilinear velocity-bilinear pressure pair.

III.2 - *The Taylor-Hood element pair*

We next turn to quadratic and biquadratic approximate velocity fields. Suppose we have a triangulation $\mathcal{T}_h$ of Ω. Then, the Taylor-Hood element pair [TH] is defined by

$$(3.6)\qquad \begin{cases} V^h = \left(\mathbf{v} \mid \mathbf{v}\in[P_2(\Delta)]^2 \ , \ \Delta\in\mathcal{T}_h \ ; \ \mathbf{v}\in[C(\bar{\Omega})]^2 \ ; \ \mathbf{v}=\mathbf{0} \text{ on } \Gamma \right) \\ S^h = \left(q \mid q\in P_1(\Delta) \ , \ \Delta\in\mathcal{T}_h \ ; \ q\in C(\bar{\Omega}) \ ; \ \int_\Omega q=0 \right) . \end{cases}$$

Note that we are now basing V^h and S^h on the same grid but on different degree polynomials, in contrast to (3.3), which uses the same degree polynomials but different grids. The element pair (3.6) satisfies the div- stability condition (2.5). Furthermore, if the solution $(\mathbf{u},p)$ of (1.9)-(1.10) has the indicated smoothness, then the following error estimates hold uniformily in h:

$$(3.7)\qquad \left\{ \begin{array}{l} \|\mathbf{u}-\mathbf{u}^h\|_1 = O(h^{m-1}) \\ \|\mathbf{u}-\mathbf{u}^h\|_0 = O(h^m) \\ \|p-p^h\|_0 = O(h^m) \end{array} \right\} \text{ whenever } \left\{ \begin{array}{c} \mathbf{u}\in H^m(\Omega)\cap H_0^1(\Omega) \\ \text{and} \\ p\in H^{m-1}(\Omega)\cap L_0^2(\Omega) \end{array} \right\}, \ m=2 \text{ or } 3.$$

These results have been established by many authors, including [BP,V1,BN1]. We see from (3.7) that if $\mathbf{u}\in H^3(\Omega)\cap H_0^1(\Omega)$ and $p\in H^2(\Omega)\cap L_0^2(\Omega)$ then, in L^2-norms, we have third order accurate velocity approximations and second order accurate pressure approximations. This is an improvement over any of the elements involving linear velocities.

One should note that the number of degrees of freedom, both of velocity and pressure type, associated with the use of (3.6) is identical to that associated with the use of (3.3), the most efficient linear velocity element. In fact, the structure of the discrete system resulting from a Taylor-Hood discretization is

in every way identical to that resulting from the use of (3.3). Therefore, the solution times for the Taylor-Hood and the linear-linear discrete systems are roughly the same if one uses the same pressure triangulation in both cases. Of course, the Taylor-Hood element pair will yield better accuracy than the linear-linear pair, provided the exact solution is sufficiently smooth.

On the other hand, on the same grid, the assembly costs of Taylor-Hood will in general be higher since one needs to use higher order quadrature rules to integrate the higher degree polynomial integrands resulting from the Taylor-Hood element pair. For many solvers, the assembly time is overwhelmed by the solution time; therefore the increased assembly cost associated with (3.6) is not a serious drawback. Of course, this is further mitigated by the fact that for the same accuracy, one may use a coarser grid for (3.6) than for (3.3).

Summarizing, provided the exact solution is sufficiently smooth, the Taylor-Hood element pair, when compared to any of the linear velocity elements, yields better accuracy for essentially the same work, or alternately, will yield a desired level of accuracy for less cost.

For rectangles or quadrilaterals we have the analogous pair

$$(3.8)\qquad \begin{cases} \mathcal{V}^h = \left(\mathbf{v} \mid \mathbf{v}\in[Q_2(\square)]^2 \ , \ \square\in\mathcal{Z}_h \ ; \ \mathbf{v}\in[C(\bar{\Omega})]^2 \ ; \ \mathbf{v}=\mathbf{0} \text{ on } \Gamma \right) \\ S^h = \left(q \mid q\in Q_1(\square) \ , \ \square\in\mathcal{Z}_h \ ; \ q\in C(\bar{\Omega}) \ ; \ \int_\Omega q=0 \right) \end{cases}$$

where $\mathcal{Z}_h$ denotes a subdivision of Ω into rectangles. This element pair satisfies the div-stability condition (2.5) and the error estimates (3.7).

One may well ask if further efficiencies may be gained by using higher order elements, e.g., cubic velocities coupled with quadratic pressures. Here one needs to consider the trade-off between the increased accuracy of higher order elements and the increased complexity of those elements. As in other settings, e.g., structural mechanics, one generally finds that the optimum

seems to be achieved by quadratic elements. Furthermore, it is questionable that in general settings the exact solution of the Navier-Stokes equations is sufficiently smooth to enable the potential better accuracy of higher order elements. In our overall experience, we have found the best choice of velocity-pressure elements to be the Taylor-Hood element pair (3.6), or its quadrilateral counterpart (3.8).

III.3 - *Divergence free elements*

Ideally, one would like to choose the finite element spaces V^h and S^h so that the functions belonging to $\mathbf{V}^h$ are at least discretely divergence free. Certainly if the functions belonging to $\mathbf{V}^h$ are divergence free then they are discretely divergence free as well, i.e., $\mathrm{div}\mathbf{v}^h=0$ for all $\mathbf{v}^h\in\mathbf{V}^h$ implies that $\mathbf{V}^h=\mathbf{Z}^h$. Such a case effects a great simplification since the velocity and pressure uncouple. Indeed, we need only solve

$$a(\mathbf{u}^h,\mathbf{v}^h) + c(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h) = (\mathbf{f},\mathbf{v}^h) \quad \text{for all} \quad \mathbf{v}^h\in\mathbf{V}^h$$

for the discrete velocity field $\mathbf{u}^h$ since in this case the term $b(\mathbf{v}^h,q^h)$ in (2.1) vanishes for any $\mathbf{v}^h\in\mathbf{V}^h=\mathbf{Z}^h$. Also, since $\mathbf{Z}^h\in\mathbf{Z}$, note that in the velocity estimate (2.9), $\Theta=0$ so that the velocity error depends only on the ability to approximate in $\mathbf{V}^h$.

Unfortunately, although there are known some finite element pairs such that the functions in $\mathbf{V}^h$ are at least locally divergence free, these have proven to be impractical, and we will not consider them here. We do mention that one obvious method of generating divergence free discrete vector fields is to take the curl of a piecewise polynomial field, i.e., of a piecewise polynomial streamfunction. One problem with this approach is that if one wants a

conforming velocity field, i.e., $V^h \subset H_0^1(\Omega)$, then the discrete streamfunction field must be chosen to be continuously differentiable over Ω. In R^2 this, of course, necessitates the use of at least quintic streamfunctions over triangles, or cubic polynomials over macro-elements, e.g., the Clough-Toucher element. Non-conforming velocity fields can also be generated in this manner. See [Ca,CN,GR1, and GR2] for details.

III.4 - *Three dimensional elements*

Compared to the two dimensional setting, there are known much fewer stable element pairs for three dimensional problems. However, there is great interest in this subject and therefore there has been substantial recent progress. Here we mention a few of the known stable three dimensional elements.

In the first place, the three dimensional analogue of the Taylor-Hood element is known to be stable in 3-D; this may be shown by the methods of Verfürth or Boland-Nicolaides. Specifically, we subdivide Ω into tetrahedrons and use continuous piecewise quadratic polynomials for the velocity and continuous piecewise linear polynomials for the pressure. The accuracy of this combination is the same as in the two dimensional case.

Next we consider linear-constant elements. Again, subdivide Ω into tetrahedrons. For the pressure space we choose piecewise constants with respect to this initial subdivision. Now we subdivide each tetrahedron into 12 smaller tetrahedrons by first joining the centroid of the faces to the vertices, and then joining the centroid of the large tetrahedron to the vertices and the centroids of the faces. For the velocity space we choose continuous piecewise linear polynomials with respect to the smaller tetrahedrons.

Another stable linear-constant element pair is defined by first subdividing

Ω into rectangular prisms, or more generally, into distortions of such prisms. For the pressure space we choose piecewise constants over the rectangular subregions. We subdivide each rectangular prism into 24 tetrahedrons by first drawing the two diagonals of each face, then joining the centroid of the prism to the vertices and to the six intersection points of the face diagonals.

Both these linear-constant element pairs are known to be stable and yield the same accuracy results as those for the two dimensional linear-constant pairs. See [BoC] for details.

IV - Alternate Weak Forms and Boundary Conditions

In this section we examine some variants of the weak formulation (1.9)-(1.10), mostly from the viewpoint of how different boundary conditions may be incorporated into a finite element method using primitive variables. We again emphasize that there are many radically different weak formulations involving u and p which we will not be able to consider; we are restricting ourselves to variants of the most commonly used weak formulation.

Before considering boundary conditions, we briefly consider an alternate formulation of the convection term in (1.9).

IV.1 - *An alternate formulation of the convection term*

For the purpose of simplifying the analysis of the approximate solution it can be useful to introduce a slightly different weak formulation wherein the trilinear form $c(\cdot,\cdot,\cdot)$ appearing in (1.9) is replaced by the skew-symmetrized form introduced by Teman

(4.1) $$\tilde{c}(w,u,v) = \frac{1}{2}\left(c(w,u,v) - c(w,v,u) \right).$$

One may easily verify that $\tilde{c}(u,u,v)=c(u,u,v)$ whenever $\mathrm{div}u=0$ in Ω and $u\cdot n=0$ on Γ, where n denotes the outward normal to Γ. Therefore, due to (1.2)-(1.3), it seems irrelevant whether one uses (1.8) or (4.1) in a weak formulation of the Navier- Stokes equations. From an analysis point of view, the advantage of (4.1) is that $\tilde{c}(w,u,v)=-\tilde{c}(w,v,u)$ for any $u,v,w\in H^1(\Omega)$ while the analogous result for (1.8) holds only when $\mathrm{div}w=0$ in Ω and one of $u=0$, $v=0$ or $w\cdot n=0$ on Γ.

We emphazise that, insofar as the accuracy of the approximations is concerned, it makes no difference whether one uses (1.8) or (4.1); we merely point out that many of the results concerning finite element approximations of solutions of (1.1)- (1.3) were first obtained through the use of (4.1). On the other hand, any implementation of (4.1) will result in more computational work than the analogous implementation of (1.8).

IV.2 - *Inhomogeneous velocity boundary conditions*

There are many different ways to treat inhomogeneous velocity boundary conditions. In practice, the overwhelming choice is to use the boundary interpolant. We describe this method for polygonal domains $\Omega\subset R^2$; entirely analogous ideas may be used in three dimensions and for domains with curved sides, the latter through the aid of, e.g., isoparametric elements.

Consider the boundary condition

(4.2) $$u=g \text{ on } \Gamma$$

and the set

(4.3) $$V_g = \left(u\in H^1(\Omega) \mid u \text{ satisfies (4.2)} \right).$$

Note that $V_0=H_0^1(\Omega)$. The weak formulation which we will discretize is as follows: seek $u\in V_g$ and $p\in L_0^2(\Omega)$ such that (1.9) and (1.10) hold. Note that the test function v still belongs to $H_0^1(\Omega)$, i.e., $v=0$ on Γ.

In order to pose our discrete problem we choose finite element spaces $V^h\subset H^1(\Omega)$ and $S^h\subset L_0^2(\Omega)$. We denote by $V^h|_\Gamma$ the restriction of V^h to the boundary Γ, i.e., $V^h|_\Gamma$ consist of functions defined on Γ and which can agree with the boundary values of at least one function belonging to V^h. The finite element functions belonging to V^h, being, for example, piecewise polynomials, cannot in general satisfy the boundary condition (4.2); certainly, in general $g\notin V^h|_\Gamma$. Therefore we choose an approximation to g, which we denote by g^h, belonging to $V^h|_\Gamma$. The most common choice for g^h, and the one we consider here, is the interpolant of g in $V^h|_\Gamma$.

This choice is trivial to implement, which at least partially accounts for its popularity. For example, suppose V^h is a Lagrange finite element space, i.e., one whose degrees of freedom are exclusively function values at points. Let $\{v_k\}$, $k=1,\ldots,\bar{K}$ denote the usual finite element basis for V^h. Let the first K of these basis functions be associated with interior nodes x_k so that for $k=1,\ldots,K$, $v_k=0$ for $x\in\Gamma$. The remaining basis function $\{v_k\}$, $k=K+1,\ldots,\bar{K}$, are associated with nodes x_k lying on Γ. In practical implementations there are more efficient node numbering schemes than the one we are using; however, the latter simplifies the explanations being attempted here.

Choosing g^h to be the boundary interpolant of g is then equivalent to writing

$$u^h(x) = \sum_{k=1}^{K} \beta_k v_k(x) + \sum_{k=K+1}^{\bar{K}} g(x_k) v_k(x) \,. \tag{4.4}$$

In (4.4) β_k, $k=1,\ldots,K$, are the unknown coefficients to be determined; the

coefficients of the basis functions associated with boundary nodes are simply set equal to g evaluated at the corresponding node. Note that (4.4) implies that

$$g^h(x) = \sum_{k=K+1}^{\bar{K}} g(x_k)v_k(x) \quad \text{for} \quad x\in\Gamma \ .$$

The contribution to u^h emanating from the second summation of (4.4) becomes part of the data of the discrete system of equations.

Once an approximation g^h is chosen, one may define the set

$$V_g^h = \{ \ v\in V^h \ | \ v=g^h \text{ on } \Gamma \ \} \ .$$

Note that V_0^h is the finite element subspace of $H_0^1(\Omega)$ used in conjunction with the homogeneous boundary condition (1.3); also, clearly $V_g^h \subset H^1(\Omega)$ is not a subset of V_g. Now, the approximate problem may be defined as follows: seek $u^h\in V_g^h$ and $p^h\in S^h\subset L_0^2(\Omega)$ such that (2.1)-(2.2) hold for all $v^h\in V_0^h$ and $q^h\in S^h$, respectively. Again, the test functions v^h vanish on the boundary Γ.

The whole discussion of the div-stability condition (2.5) carries over intact to the case of the inhomogeneous boundary (4.2); in (2.5) we still use the subspace V_0^h of finite element velocity fields which vanish on the boundary. Results analagous to those of section II.3 can be derived in a fairly straightforward manner with the exception of some technicalities encountered for the $L^2(\Omega)$-error estimate for the velocity approximation. See [GP, FGP, GR2] for details.

In particular, if g^h is chosen to be the boundary interpolant of g in $V^h|_\Gamma$, then all the results, e.g., error estimates, concerning the finite element spaces discussed in section III are essentially still valid for the

inhomogeneous velocity boundary condition (4.2). Again, see [GP, FGP and GR2] for details.

IV.3 - *Alternate boundary conditions and formulations of the viscous term*

In this section we examine how different choices for the viscous term in (1.1) affect the natural boundary conditions of corresponding weak formlations. Some of this material can be found in [GLN].

Due to (1.2), when ν is constant, the viscous term in (1.1) may be written in the various equivalent forms

(4.5.1) $\qquad \nu\Delta \mathbf{u} =$

(4.5.2) $\qquad \operatorname{div}\Big[\, \nu\big(\, (\operatorname{grad} \mathbf{u}) + (\operatorname{grad} \mathbf{u})^T \,\big)\Big] =$

(4.5.3) $\qquad -\nu \operatorname{curl}(\operatorname{curl}\mathbf{u}) =$

(4.5.4) $\qquad \nu\big(\, \operatorname{grad}(\operatorname{div}\mathbf{u}) - \operatorname{curl}(\operatorname{curl}\mathbf{u}) \,\big)$.

Although these different realizations are equivalent insofar as the partial differential equations are concerned, we shall see that each generates a different numerical method.

If for some reason ν is not constant or $\operatorname{div}\mathbf{u}\neq 0$, then only (4.5.2) may be used. Indeed, (4.5.2) is the form of the viscous term which arises naturally in the derivation of the Navier-Stokes equations from the principle of conservation of linear momentum and the Cauchy-Poisson constitutive equation. The other three forms (4.5.1), (4.5.3) and (4.5.4) are derived from (4.5.2) with the aid of (1.2) and the assumption that ν=constant. In (1.1) we have used

(4.5.1) only because this is the most popular choice in the literature; all of the results obtained so far hold equally well if one chooses (4.5.2) instead. As will be seen from the discussion below, (4.5.2) is, in general, to be prefered to (4.5.1).

Denote two segments of the boundary Γ by Γ_n and Γ_τ. These segments may be empty, are not necessarily disjoint and, in fact, may be equal. Now, for fixed given functions g_n and g_τ, define the set

$$V_g = \left(\mathbf{v}\in H^1 \mid \mathbf{v}\cdot\mathbf{n}=g_n \text{ on } \Gamma_n \;;\; \mathbf{n}\times\mathbf{v}\times\mathbf{n}=g_\tau \text{ on } \Gamma_\tau \right)$$

and the spaces

$$V_0 = \left(\mathbf{v}\in H^1 \mid \mathbf{v}\cdot\mathbf{n}=0 \text{ on } \Gamma_n \;;\; \mathbf{v}\times\mathbf{n}=0 \text{ on } \Gamma_\tau \right)$$

and

$$S = L_0^2(\Omega) \text{ if } \Gamma_n=\Gamma \;,\; S=L^2(\Omega) \text{ otherwise.}$$

where $\mathbf{v}\cdot\mathbf{n}$ denotes the component of $\mathbf{v}$ normal to the boundary Γ and $\mathbf{n}\times\mathbf{v}\times\mathbf{n}=\mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\mathbf{n}$ is the projection of $\mathbf{v}$ onto the plane tangent to Γ. In the definition of V_0 we may use $\mathbf{v}\times\mathbf{n}=0$ due to the relation $\mathbf{v}\times\mathbf{n}=\mathbf{n}\times(\mathbf{n}\times\mathbf{v}\times\mathbf{n})$, i.e., $\mathbf{n}\times\mathbf{v}\times\mathbf{n}=0$ implies that $\mathbf{n}\times\mathbf{v}=0$. In R^2, $\mathbf{n}\times\mathbf{v}\times\mathbf{n}=\mathbf{v}\cdot\tau$ where τ is the unit tangent vector to Γ.

Suppose that we wish to specify the boundary conditions

(4.6.1) $\qquad \mathbf{u}\cdot\mathbf{n}=g_n \quad \text{on } \Gamma_n$

and

(4.6.2) $\qquad \mathbf{n}\times\mathbf{u}\times\mathbf{n}=g_\tau \quad \text{on } \Gamma_\tau$.

i.e., the normal velocity on Γ_n and the tangential velocity on Γ_τ, respectively. For all the weak formulations which we will consider involving any of the choices in (4.5), (4.6) will be *essential boundary conditions*. Thus the trial solution functions $\mathbf{u}$ will satisfy (4.6), i.e., $\mathbf{u}\in\mathbf{V}_g$, and the test functions satisfy $\mathbf{v}\in\mathbf{V}_0$.

Consider the following weak formulation: for i=1,2,3 or 4, seek $\mathbf{u}\in\mathbf{V}_g$ and $p\in S$ such that

$$a_i(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) + c(\mathbf{u},\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}) + d(\mathbf{v}) \quad \text{for all } \mathbf{v}\in\mathbf{V}_0 \tag{4.7}$$

and

$$b(\mathbf{u},q) = 0 \quad \text{for all } q\in S. \tag{4.8}$$

Here, $b(\cdot,\cdot)$ and $c(\cdot,\cdot,\cdot)$ remain as in (1.7) and (1.8), respectively, and $\mathbf{f}$ continues to denote the body force appearing in the momentum equation. The linear functional $d(\cdot)$ is given by

$$d(\mathbf{v}) = \int_{\Gamma/\Gamma_n} r\mathbf{v}\cdot\mathbf{n} + \int_{\Gamma/\Gamma_\tau} \mathbf{s}\cdot\mathbf{v}\times\mathbf{n} \tag{4.9}$$

where the functions r and $\mathbf{s}$ are additional data for the problem. In (4.9), for example, Γ/Γ_n denotes the complement of Γ_n in Γ, i.e., $x\in\Gamma/\Gamma_n$ implies that $x\in\Gamma$ but $x\notin\Gamma_n$. Also, since $\mathbf{v}$ is an arbitrary test function, in direction $\mathbf{v}\times\mathbf{n}$ can be taken to be vectors spanning the tangent plane to Γ.

The bilinear forms $a_i(\cdot,\cdot)$, i=1,...,4, depend on the choice made in (4.5) and, corresponding to the four choices possible in (4.5), are given by

$$a_1(\mathbf{u},\mathbf{v}) = \nu\int_\Omega \text{grad}\mathbf{u}:\text{grad}\mathbf{v} \tag{4.10.1}$$

$$(4.10.2) \qquad a_2(\mathbf{u},\mathbf{v}) = \frac{1}{2}\int_\Omega \nu\left(\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^T\right):\left(\operatorname{grad}\mathbf{v} + (\operatorname{grad}\mathbf{v})^T\right)$$

$$(4.10.3) \qquad a_3(\mathbf{u},\mathbf{v}) = \nu\int_\Omega (\operatorname{curl}\mathbf{u})\cdot(\operatorname{curl}\mathbf{v})$$

and

$$(4.10.4) \qquad a_4(\mathbf{u},\mathbf{v}) = \nu\int_\Omega (\operatorname{curl}\mathbf{u})\cdot(\operatorname{curl}\mathbf{v}) + (\operatorname{div}\mathbf{u})(\operatorname{div}\mathbf{v}) \ .$$

In the customary manner, should $\mathbf{u}$ and p be sufficiently smooth, one can, through formal integration by parts procedures, ascertain what differential equation problem the weak formulation (4.7)-(4.8) corresponds to. To begin with, we know that the boundary conditions (4.6) are satisfied since these are being required of the candidate trial functions $\mathbf{u}$. We also find that the differential equations (1.1) and (1.2) are satisfied, where in (1.1) the viscous term is replaced according to (4.5), depending on which choice is made in (4.10). Finally, one finds the *natural boundary conditions* corresponding to the particular weak formulation. We will now discuss these in some detail for each possible choice in (4.10).

Corresponding to the paired choices (4.5.1) and (4.10.1) we have the natural boundary conditions

$$(4.11.1) \qquad p - \nu\mathbf{n}\cdot\operatorname{grad}\mathbf{u}\cdot\mathbf{n} = r \quad \text{on } \Gamma/\Gamma_n \qquad \text{and} \qquad \nu\mathbf{n}\cdot\operatorname{grad}\mathbf{u}\times\mathbf{n} = \mathbf{s} \quad \text{on } \Gamma/\Gamma_\tau .$$

Unfortunately, these boundary conditions have *no physical meaning*. Thus the choice (4.5.1), or equivalently (4.10.1), can only be used in conjunction with the boundary condition (4.6) specified on all of Γ, i.e., $\mathbf{u}$ given on $\Gamma_n=\Gamma_\tau=\Gamma$.

Next, consider the choices (4.5.1) and (4.10.1). The natural boundary conditions are then

$$(4.11.2)\qquad \begin{cases} -p + \nu \mathbf{n}\cdot\left(\text{grad}\mathbf{u} + (\text{grad}\mathbf{u})^T\right)\cdot\mathbf{n} = -r & \text{on } \Gamma/\Gamma_n \quad \text{and} \\ \nu \mathbf{n}\cdot\left(\text{grad}\mathbf{u} + (\text{grad}\mathbf{u})^T\right)\times\mathbf{n} = -\mathbf{s} & \text{on } \Gamma/\Gamma_\tau \ . \end{cases}$$

Thus $-r$ and $-\mathbf{s}$ are the normal and tangential stresses, respectively, on the boundary. Then, for the choice (4.10.2), the possible combinations of boundary conditions at a point on the boundary Γ are as follows: we may specify the velocity, or we may specify the normal velocity and the tangential stress, or we may specify the tangential velocity and the normal stress. The latter combinations are useful, e.g., for free surface problems or at artificial outflow boundaries. Details may be found in [GLN].

The third choice (4.5.3), or (4.10.3), yields the natural boundary conditions

$$(4.11.3)\qquad p = r \ \text{ on } \Gamma/\Gamma_n \qquad \text{and} \qquad \boldsymbol{\omega} = \mathbf{s}/\nu \ \text{ on } \Gamma/\Gamma_\tau$$

so that r and $\mathbf{s}$ are the pressure p and ν times the vorticity $\boldsymbol{\omega}=\text{curl}\mathbf{u}$, respectively, on the boundary. The possible combinations of boundary conditions are now: we may specify the velocity, or we may specify the normal velocity and the vorticity, or we may specify the tangential velocity and the pressure. The pressure is often used as an outflow condition; the vorticity is useful in exterior problems when matching to an inviscid irrotational flow since it is well known that the vorticity decays to its value at infinity faster than does the velocity. Again, details may be found in [GLN].

Unfortunately, although the boundary conditions associated with the use of (4.10.3) can be useful, in practice we cannot employ this particular

formulation of the viscous term. The reason for this is that the choice (4.10.3) requires the use of divergence free finite element velocity fields in order for the form $a_3(\cdot,\cdot)$ to be coercive on Z^h. This condition is also needed to guarantee the stability of the approximations and, for the other three cases (4.10.1), (4.10.2) and (4.10.4), is trivially satisfied for any choice of conforming discrete velocity space.

Fortunately, the boundary conditions (4.11.3) are *approximately* the natural boundary conditions associated with the choice (4.11.4). In fact, for (4.10.4), we have the natural boundary conditions

$$(4.11.4) \qquad p - \nu \mathrm{div}\mathbf{u} = r \quad \text{on } \Gamma/\Gamma_n \qquad \text{and} \qquad \omega = s/\nu \quad \text{on } \Gamma/\Gamma_\tau \ .$$

The second of these is identical to the second of (4.11.3). If ν is "small", and/or if we assume the incompressibility constraint holds up to portions of the boundary where the normal velocity is *not* specified, then $(p-\nu\mathrm{div}\mathbf{u})$ is essentially equal to p. Thus we recover, at least approximately, the first boundary condition of (4.11.3).

In summary, when one has velocity and/or stress boundary conditions, one should use (4.11.2) in (4.7) and when one has velocity and/or pressure and/or vorticity boundary conditions the choice (4.11.4) is preferable.

The discretization of (4.7)-(4.8) follows the usual procedures once one chooses the finite element spaces for the velocity and the pressure approximations. The natural boundary conditions are automatically acounted for by the inclusion of the linear functional $d(\cdot)$ in (4.7). The essential boundary conditions on the components of the velocity can be enforced in a manner analogous to that described in section IV.2 for the case where the complete velocity is specified on the whole boundary. All material relating to the div-

stability condition (2.5) is essentially still valid, and thus, insofar as that condition is concerned, the particular choices of finite elements discussed in section III may still be used.

In actuality, there are very few rigorous error estimates available for boundary conditions other than the velocity. For polygonal or polyhedral domains Ω, the error estimates of section II.3 are still valid. However, for domains with *curved boundaries*, using the type of weak formulations discussed here may result in a loss of accuracy. For example, for (4.10.2) with normal velocity and tangential stress boundary conditions, it was shown by Verfürth [V2] that there is a loss of accuracy due to a Babuska type paradox, i.e., the limit of solutions of problems posed on polygonal approximations to $\Omega \subset R^2$ is not the solution of the problem posed on Ω. Verfürth [V3] has also shown how through the use of additional Lagrange multipliers on the boundary, a different weak formulation yields optimal accuracy.

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16. Abstract

We survey some mathematical aspects of finite element methods for incompressible viscous flows, concentrating on the steady primitive variable formulation. We address the discretization of a weak formulation of the Navier-Stokes equations; we then consider the div-stability condition, whose satisfication insures the stability of the approximation. Specific choices of finite element spaces for the velocity and pressure are then discussed. Finally, the connection between different weak formulations and a variety of boundary conditions is explored.

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