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Summary

The steady discharge of a high-voltage corona is analyzed as a singular perturbation problem. The small parameter is the ratio of the length of the ionization region to the total gap length. By this method, current versus voltage characteristics can be calculated analytically.

Introduction

The equations governing the corona discharge which occurs when a thin electrode in a gas is raised to high voltage are nonlinear because the rate of ion production is a function of the electric field and the electric field is determined by the distribution of charge within the gas. However, a high electric field is required to ionize the gas, so ionization occurs only in a thin boundary layer around the cathode (Vatazhin and Grabovskii, 1983). It is natural to analyze corona discharge by a matched asymptotic analysis (Bender and Orszag, 1978). By doing so, we take advantage of the fact that the high electric field is generated by the small radius of the discharge electrode in order to find closed-form solutions for the corona.

Our method of analysis will be applied to the negative corona occurring in a coaxial geometry although the method also works for positive corona. Many analyses of negative corona have treated pulse formation in a negative point—positive plane configuration. These consisted primarily of numerical simulations (Morrow, 1985; Davies and Donne, 1980). Indeed, the Trichel pulses produced at negative points are of considerable interest. However, the present analysis deals only with the more-or-less continuous discharge produced by a corona wire (Gallo, 1977). It is unclear at present whether the asymptotic method can be applied to the highly unsteady Trichel pulse corona. Research in that direction is currently in progress.

Governing Equations

The dependent variables in this problem are the space charge density and the electric field. Let L be a characteristic dimension of the overall geometry, a a typical radius of the cathode, and φ_o the anode potential. The magnitude of electric field required to produce ionization will be E_i, and the total current in the corona will be I_o. Ultimately, I_o is to be found. On a fundamental level, E_i would be an ionization potential divided by the mean free path. Then the following nondimensional variables can be formed:

$$\left.\begin{array}{llll} x = \dfrac{x^*}{L}, & a = \dfrac{a^*}{L}, & n_\beta = \dfrac{n_\beta^* q b_e \varphi_o L}{I_o}, & \varphi = \dfrac{\varphi^*}{\varphi_o} \\[2ex] \alpha_a = L\alpha_a^*, & \alpha_i = \epsilon L \alpha_i^*, & b_\beta = \dfrac{b_\beta^*}{b_e^*} & \end{array}\right\} \quad (1)$$

where asterisks denote dimensional variables, q is the charge of an electron, n_β the number density of specie β (+ ions, − ions, or electrons), φ the electric potential, and α_i and α_a are ionization and attachment coefficients (α_i^* is Townsend's first coefficient, which we take to be $0(1/\epsilon)$). The b_β ($\beta = +, -,$ or e) are the mobility coefficients appearing in Ohm's law

$$v_\beta = \pm b_\beta E \quad (2)$$

where v_β is the drift velocity and the minus sign is used for electrons and negative ions. The nondimensional b_e equals 1. A small parameter equal to the scale of the ionization region divided by the scale of the overall geometry is given by

$$\epsilon = \frac{\varphi_o}{LE_i}$$

In order for discharge to occur, a must be $0(\epsilon)$.

We will consider a negative corona with electrons generated at the cathode surfaces by secondary emission; electron generation by photoionization is assumed to be negligible. Ions are produced within the gas by electron impact. In dimensionless form, the steady-state equations for conservation of species and electric flux are (Vatazhin and Grabovskii, 1983)

$$
\left.\begin{aligned}
\nabla^2\varphi &= J(n_e + n_- - n_+) \\
\nabla \bullet n_e v_e &= \frac{n_e|v_e|\alpha_i}{\epsilon} - n_e|v_e|\alpha_a \\
\nabla \bullet n_+ v_+ &= \frac{n_e|v_e|\alpha_i}{\epsilon} \\
\nabla \bullet n_- v_- - &= n_e|v_c|\alpha_a
\end{aligned}\right\} \tag{3a}
$$

with

$$
\mathrm{J} = \frac{4\pi I_o L}{\epsilon_g b_e^* \varphi_o^2} \tag{3b}
$$

and ϵ_g is the gas dielectric constant. The value of the nondimensional parameter J determines I_o when the other terms in equation (3b) are given. The electric field is

$$
E = -\nabla\varphi \tag{3c}
$$

In regions where $|E|$ is small compared to $1/\epsilon$, α_i will be zero. We will represent α_i by the form

$$
\alpha = \beta e^{-1/\epsilon|E|} \tag{4}
$$

(See Meek and Craggs, p. 60; note that β and $1/\epsilon$ are proportional to the gas pressure.) Thus, where $|E| = 0(1)$, α_i is exponentially small, and α_a will be taken to be independent of E.

Equations (3a) are to be solved with the boundary conditions

$$
\left.\begin{aligned}
\varphi &= 0 \quad \text{on the cathode} \\
\varphi &- 1 \quad \text{on the anode}
\end{aligned}\right\} \tag{5a}
$$

Because the reference current I_o is arbitrary, it can be defined so that the net positive ion current is -1 on the cathode:

$$
\int_{\text{cathode}} n_+ v_+ \bullet \hat{n}\, ds = -1 \tag{5b}
$$

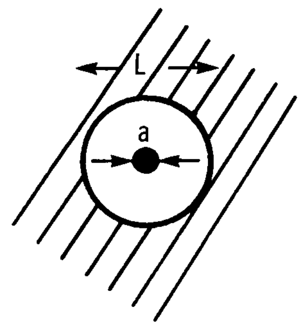
Cathode secondary emission occurs where positive ions knock electrons out of the surface. If γ is the coefficient of secondary emission, then

$$
\gamma n_+ v_+ \bullet \hat{n} = -n_e v_e \bullet \hat{n} \tag{5c}
$$

Also,

$$
\left.\begin{aligned}
n_- &= 0 \quad \text{on the cathode} \\
n_+ &= 0 \quad \text{on the anode}
\end{aligned}\right\} \tag{5d}
$$

Because of the exponential dependence (eq. (4)), α_i will be effectively zero, except within a distance of $0(\epsilon)$ around the cathode. Outside this region the production of ions by electron impact vanishes. Within the $0(\epsilon)$ boundary layer (or "sheath"), the spatial variation of φ is so rapid that the first of equations (3a) can be replaced by Laplace's equation; the contribution from space charge is negligible compared to the geometrical divergence of the electric field. To develop the asymptotic analysis, we will consider the coaxial geometry shown in the following sketch:



Asymptotic Analysis of Coaxial Discharge

Inner Region

Within the boundary layer we introduce the $0(1)$ scaled radial variable $\tilde{r} = r/\epsilon$. Then, with the notation $\pi_\beta = rn_\beta v_\beta$, to lowest order, equations (3a) become

$$
\frac{1}{\tilde{r}}\frac{d}{dr}\tilde{r}\frac{d\tilde{\varphi}}{d\tilde{r}} = 0 \tag{6a}
$$

$$
\left.\begin{aligned}
\frac{d}{d\tilde{r}}\tilde{\pi}_e &= \tilde{\pi}_e\alpha_i \\
\frac{d}{d\tilde{r}}\tilde{\pi}_+ &= \tilde{\pi}_e\alpha_i \\
\tilde{\pi}_- &= 0
\end{aligned}\right\} \tag{6b}
$$

in cylindrical coordinates. The solution to equation (6a) subject to $\varphi = 0$ on $\tilde{r} = \tilde{a}$ is

$$
\varphi = A\,\ln\left(\frac{\tilde{r}}{\tilde{a}}\right) \tag{7a}
$$

The constant A will be determined by matching with the outer solution.

The solution to equations (6b) with $\tilde{\pi}_+ = -1$ and $\tilde{\pi}_e = -\gamma\tilde{\pi}_+$ on $\tilde{r} = \tilde{a}$ is

$$\left.\begin{aligned}\tilde{\pi}_e &= \gamma e^{\int_{\tilde{a}}^{\tilde{r}} \alpha_i \, d\tilde{r}} \\ \tilde{\pi}_+ &= -1-\gamma+\gamma e^{\int_{\tilde{a}}^{\tilde{r}} \alpha_i \, d\tilde{r}}\end{aligned}\right\} \tag{7b}$$

In equation (7b) α_i is given by equation (4) with $E = -A/r$ from equations (3c) and (7a). Hence,

$$\int_{\tilde{a}}^{\tilde{r}} \alpha_i \, d\tilde{r}\,' = \beta \int_{\tilde{a}}^{\tilde{r}} e^{-\tilde{r}\,'/A} d\tilde{r}\,' = \beta A\left(e^{-\tilde{a}/A} - e^{-\tilde{r}/A}\right)$$

Outer Region

In the outer region, $r = 0(1)$. Equations (3a) then become

$$\frac{1}{r}\frac{d}{dr} r \frac{d\varphi}{dr} = J(n_e + n_-) \tag{8a}$$

$$\left.\begin{aligned}\frac{d}{dr}\pi_e &= -\alpha_a \pi_e \\ \frac{d}{dr}\pi_- &= \alpha_a \pi_e \\ \pi_+ &= 0\end{aligned}\right\} \tag{8b}$$

In order that π_+ in the inner region (eq. (7b)) match its value of 0 in the outer region, the relation

$$\int_{\tilde{a}}^{\infty} \alpha_i \, d\tilde{r} - \ln \frac{1+\gamma}{\gamma} \quad \text{or} \quad \beta A e^{-\tilde{a}/A} = \ln\left(\frac{1+\gamma}{\gamma}\right) \tag{9}$$

must be satisfied. This equation ultimately determines the value of J. In a sense, J is an "eigenvalue:" for a given value of radius a (or potential φ_o), a solution exists for only one J; this J determines the steady corona current I_o. In general, it is the condition (5b) which fixes J. When $J = 0$, equation (9) becomes Townsend's criterion for corona onset; it determines the value of a (or φ_o), at which the corona becomes self-sustained.

The solutions for π_- and π_e which match to equation (7b) are

$$\left.\begin{aligned}\pi_e &= (1+\gamma)e^{-\alpha_a r} \\ \pi_- &= (1+\gamma)(1 - e^{-\alpha_a r})\end{aligned}\right\} \tag{10}$$

Equation (9) was used to arrive at these forms. Equation (8a) can be rewritten using equation (2) as

$$\frac{d}{dr}\left[r\frac{d\varphi}{dr}\right]^2 = 2Jr\left(\pi_e + \frac{\pi_-}{b_-}\right) \tag{11}$$

On introducing equations (10), integrating equation (11), and matching to equation (7a), one finds

$$E = \frac{1}{r}\left\{2J(1+\gamma)\left[\frac{r^2}{2b_-} + \left(1 - \frac{1}{b_-}\right) \times \left(1 - e^{-\alpha_a r} - \alpha_a r e^{-\alpha_a r}\right)\Big/\alpha_a^2\right] + A^2\right\}^{1/2} \tag{12}$$

and

$$\varphi = 1 + \int_r^1 E \, dr$$

The condition that φ matches to the inner solution (7a) also requires that[1]

$$1 + \int_r^1 E \, dr = 0 \tag{13}$$

Equation (13) with equation (12) determine the value of A as a function of J. After substituting this $A(J)$ into equation (9), equation (9) determines J.

Equation (12) simplifies if electron attachment is either slow ($\alpha_a = 0$) or fast ($\alpha_a = \infty$). In the former case,

$$E = -\frac{1}{r}\left[J(1+\gamma)r^2 + A^2\right]^{1/2} \tag{14}$$

and the integral in equation (13) can be evaluated in closed form.

Figure 1 shows A as a function of J computed from equations (13) and (14). Figure 2 shows $\tilde{a}$ computed from equation (9) with the functions $A(J)$ shown in figure 1. One sees in each case that a maximum value of $\tilde{a}$ exists above which a corona discharge cannot be self-sustained. A maximum J exists above which the electrode radius becomes zero. One should not be mislead by this prediction of a maximum J: because $\tilde{a} = a^* E_i/\varphi_o$, when a^* is fixed the maximum J would occur as $\varphi_o \to \infty$. From equation (3b) the corresponding I_o tends to infinity like φ_o^2.

[1]Strictly a should be treated as a small parameter and A and J should be expanded in terms of ϵ; however, the present treatment is accurate to $0(\epsilon)$.

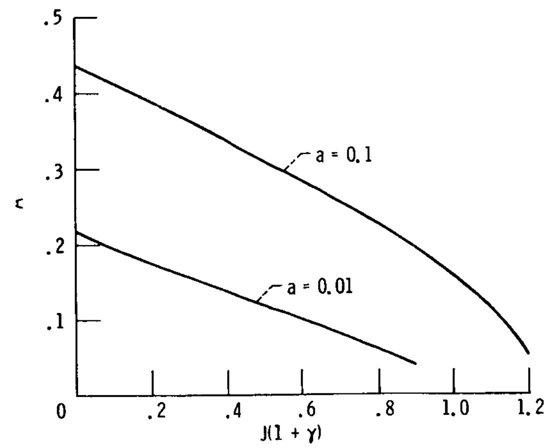


Figure 1.—Coefficient A of equation (7a) as function of current $J(1 + \gamma)$ for $\gamma = 0.01$.

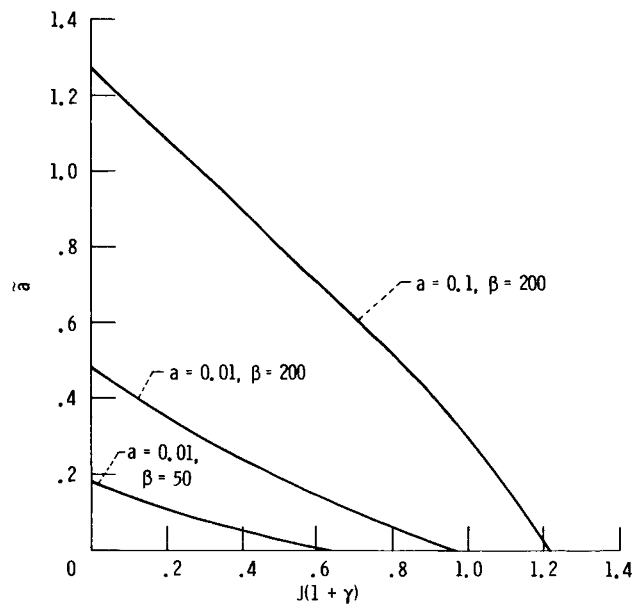


Figure 2.—Scaled electrode radius $\tilde{a}$ versus current J. For fixed dimensional radius a^*, $\tilde{a}$ represents one over the applied voltage.

Stability of the Corona

The stability to infinitessimal disturbances of the previous solution can be examined by introducing $\partial n_\beta/\partial t$ on the left side of equations (3a) and forming the linear perturbation equations. If the perturbation has the form $e^{st}F(r)$ where $s = 0(1)$, then the asymptotic stability equations can be solved in closed form for the case of equation (14). One finds that the eigenvalues s all lie in the left-half complex plane, indicating stability with respect to this type of exponentially growing, axisymmetric disturbances. At present, the stability to other types of perturbation is being studied in hopes that it will provide insight into corona pulsations.

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National Aeronautics and Space Administration
Cleveland, Ohio, July 1986

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15. Supplementary Notes

16. Abstract

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