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COMPARISON OF RELATIVISTIC EFFECTS IN BARYCENTRIC AND EARTH-CENTERED COORDINATES AND IMPLICATIONS FOR DETERMINATION OF GM FOR EARTH

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FINAL REPORT ON NASA CONTRACT NO. NASA NAG5-497

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I. Introduction and Summary

A. Discrepancies Between Experimental Results

This report summarizes the results of an investigation of relativistic effects which have an influence on the determination of GM_E (M_E is the mass of the Earth, G is the Newtonian gravitational constant.) The stimulus for this study has been an apparent discrepancy between values of GM_E reported by researchers who use different techniques and different data sets in the determination of this physical quantity.

B. Determinations of GM_E, 1983-85

Investigators who use Lunar Laser Ranging (LLR), together with relativistic dynamical models of the solar system including gravitational interactions between the sun, planets and other solar system bodies to determine GM_E in 1983 reported the value[1]:

$$(GM_E)_{TDB} = 398400.444 \pm 0.008 \text{ km}^3/\text{sec}^2 \; . \qquad (1)$$

From the point of view emphasized in this report, the distinguishing feature of this determination of GM_E is the use of a solar system barycentric system of coordinates and a time scale based on Barycentric Dynamical time (TDB).

On the other hand, GM_E can be determined by accurate ranging to LAGEOS (the Laser Geodynamics Satellite), a procedure which to a large extent is independent of other solar system bodies (except for tidal perturbations) and whose distinguishing feature is its use of a coordinate reference frame which is nearly locally inertial and a time scale based on the the SI second. The value of GM_E reported in 1985 by the Goddard/EG&G/RMS group of investigators[2] was:

$$(GM_E)_{LAGEOS} = 398400.434 \pm 0.002 \text{ km}^3/\text{sec}^2 \; . \qquad (2)$$

At the same time, the University of Texas group reported a value[3] of

$$(GM_E)_{LAGEOS} = 398400.440 \text{ km}^3/\text{sec}^2 . \tag{3}$$

As early as 1982, Misner[4] had considered the question of differences in units of length and mass between calculations with SI units in the nearly locally inertial frame, and calculations done with the barycentric system of coordinates and time scale. He did not specifically consider GM_E, but he concluded that for the sun's mass $(GM_S)_{TDB}$, the value determined in the barycentric system, should be given by:

$$(GM_S)_{TDB} = (GM_S)_{SI}/(1 + L) \tag{4}$$

where

$$L = 1.55047 \times 10^{-8}, \tag{5}$$

In Eq. (4), $(GM_S)_{SI}$ is the value of the sun's mass in SI units.

If a similar correction had been applied to the LLR value of $(GM_E)_{TDB}$, given in Eq. (1), it would have given an apparent SI value of 398400.450 km^3/sec^2. The fractional difference between this value and the value given by Eq. (2) would then have been 4×10^{-8}, which seemed surprisingly large. Since other relativistic effects possibly could be involved, a detailed examination of relativistic effects which might clarify the relationship between the different determinations of GM_E was undertaken.

C. Determination of GM_E; the current situation, 1987

Since the above determinations of GM_E were reported, the "best" values of GM_E obtained by the two methods have changed. We shall summarize the current situation by reporting here the currently accepted values.

1. The value of GM_E obtained from Lunar Laser Ranging changed significantly when the position of the equinox was regarded as a quantity to be determined and was solved for, rather than kept fixed in the sky. The current best value is[5]

$$(GM_E)_{TDB} = 398400.437 \pm .006 \text{ km}^3/\text{sec}^2 \;. \qquad (6)$$

2. The most recent LAGEOS solutions[6] carried out at Goddard and at the University of Texas give values in the range 398400.440 to 398400.441 km^3/sec^2. It is difficult to assess whether modeling or systematic effects might still influence the results appreciably, but an uncertainty larger than .002 or .003 km^3/sec^2 is regarded as unlikely. Thus, the present situation is that the experimental results from LAGEOS and LLR data agree well, if the correction suggested by Misner is used. Also, Hellings[7] has recently published theoretical results which agree with the correction suggested by Misner. Our more detailed study agrees with the results obtained by Misner and Hellings, and is described in the remainder of this report.

The theoretical results assume that all relevant relativistic effects have been correctly incorporated into the dynamical models of satellite and planetary motion and into time scale comparisons. If this is not so, then the results cannot be guaranteed. For example, it is well known that, due to the nonlinear (post-Newtonian relativistic) terms in the Schwarzschild field of the Earth, the perigee of an Earth satellite will precess by an angle of $3GM_E/[c^2a(1-e^2)]$ per revolution[8], where a is the semimajor axis of the satellite orbit and e is its eccentricity. For LAGEOS, at a semimajor axis of $a = 1.2\times10^9$ cm, this is equivalent to a motion of the perigee of 1.3 cm per revolution. Currently such effects are excluded from the dynamical model of LAGEOS's orbital motion[9] whereas the derivation of the correction factor $(1 + L)$ assumes that they are accounted for.

Thus while in principle one can derive the relativistic results on the basis of quite rigorous arguments, if in the actual data analysis programs some relativistic effects are omitted, the validity of the relativistic correction factor (1 + L)--insofar as it applies to the comparison of the experimental results--is obscured. The best remedy for this is to ensure that the dynamical models used to fit tracking data truly incorporate all known important relativistic effects.

C. Outline of Report

We describe in this section the contents of this report. Sections II through IV contain the detailed arguments and derivations leading to the result stated in Eq. (5), above. Since these arguments are lengthy a summary of the arguments is given in briefest possible form in Sect. V; the reader may thus turn to Sect. V for a paraphrase of the argument. Section VI contains some estimates of the consequences of neglecting or omitting certain important known relativistic effects.

In general relativity (GR) the interpretation of theoretical predictions requires great care. Many different coordinate systems may be used to describe physical phenomena. These coordinate systems may differ in the choices of length and/or time units, in their behavior at large distances from the sun, or in other ways. In this report we shall discuss Parametrized Post-Newtonian (PPN) coordinates; Eddington-Clark (EC) coordinates; a coordinate system based on barycentric dynamical time (TDB coordinates); and Local Inertial Coordinates. In spite of the many possibilities for choice of a coordinate reference frame, observations of physical quantities made using different coordinate systems must agree, when expressed in terms of appropriately defined and agreed-upon invariant quantities or observables such as proper lengths or proper times measured using standard clocks. The relationships between these different coordinate systems are discussed in Sect. III.

Knowledge of coordinate transformations between barycentric coordinates and local inertial coordinates is essential in the derivation of Eq. (5)[10]. The derivation of these coordinate transformations is contained in a paper entitled "Relativistic Effects in Local Inertial Frames," by N. Ashby and B. Bertotti[11], which was published in Phys. Rev. D34, 2246 (1986). This paper is incorporated as

Appendix A of the present report. We shall quote results from this paper as they are needed.

Notation. We use upper case letters such as X^μ to denote quantities measured in PPN, EC, or TDB coordinates as these are all systems with origin at the Solar system barycenter. Small letters such as x^μ denote quantities measured in local inertial frames. Greek indices run from 0 to 3 and latin indices run from 1 to 3. The signature of the metric tensor in vacuum is chosen to be (-1,1,1,1); other notation is as in the book by Weber[12].

II. Constancy of The Speed of Light

We now state an important assumption upon which subsequent discussion is to be based. Since much of the reasoning involves the relationships between different choices of time unit, the purpose of this assumption is to limit the different types of coordinate systems which may be chosen, thus simplifying the discussion.

Assumption: The speed of light is a defined quantity:

$$c = 299\ 792\ 458 \text{ meters/second} . \tag{7}$$

This means that we shall only consider coordinate systems in which the units of length and of time are related in such a way that Eq. (7) is satisfied. This defines the unit of length (the meter) in terms of the chosen time unit (the second.) The choice of unit of length is thus not independent of the unit of time. For example, one might consider a coordinate system in which a "new second" is twice as long as the SI second. Then the "new meter" would have to be, correspondingly, twice as long as the SI meter in order for the speed of light to have the value given by Eq. (7).

Some researchers[13] have investigated unit systems which would be inconsistent with Eq. (7) above. This seems likely to introduce additional confusion into an already complex situation and we shall not consider such possibilities here.

A consequence of the above assumption is that the usually quoted measure of mass, GM, (the gravitational constant times the mass), is not independent of the choice of unit of time. This can be seen by noting that the Schwarzschild mass parameter GM/c^2 corresponding to mass M, has units of length, while c has the same numerical value in all unit systems considered here.

III. Coordinate Time Scales

In this section we compare various coordinate systems which must be considered in arriving at Eq. (5).

A. The Metric Tensor

The invariant interval ds between two events in space-time whose coordinates differ by dX^μ ($\mu = 0,1,2,3$) is given by

$$-ds^2 = G_{\mu\nu} dX^\mu dX^\nu \, , \tag{8}$$

where $G_{\mu\nu}$ is the metric tensor. It is appropriate at this point to present several examples of the metric tensor, expressed in different coordinate systems.

B. Eddington-Clark Metric

Eddington and Clark[14] derived an approximate solution of Einstein's field equations using the "slow-motion, weak-field" approximation[10-11] (see also Appendix A) which is quite adequate for solar system dynamics. In this approximation scheme, a typical velocity of a solar system body is considered small compared to c, so V/c is a small parameter. Also, gravitational effects appear through a dimensionless measure of the gravitational potential, GM/c^2R, and in most cases in the solar system the order of magnitude of GM/c^2R is about the same as that of $(V/c)^2$. Thus we write:

$$O(GM/c^2R) \simeq O(V^2/c^2) \, .$$

The self-consistent solution of the field equations developed by Eddington and Clark then requires that G_{00} be calculated to $O(V^4/c^4)$, G_{0i} to $O(V^3/c^3)$, and G_{ij} to $O(V^2/c^2)$. The resulting metric is given below, although we shall not make very much use of it in the discussion.

$$G_{00} = -1 + 2U - 2U^2 + 4\sum_A \frac{GM_A V_A^2/c^4}{|\vec{X} - \vec{X}_A|} - \sum_A \frac{GM_A(\vec{X}-\vec{X}_A)\cdot\vec{A}_A/c^4}{|\vec{X} - \vec{X}_A|}$$

$$- \sum_A \frac{GM_A[(\vec{X} - \vec{X}_A)\cdot\vec{V}_A]^2/c^4}{|\vec{X} - \vec{X}_A|^3} - 2\sum_A \sum_B{}' \frac{GM_A GM_B/c^4}{|\vec{X} - \vec{X}_A||\vec{X}_A - \vec{X}_B|} ;$$

$$G_{0i} = -\sum_A \frac{4GM_A V_A^i/c^4}{|\vec{X} - \vec{X}_A|} ; \qquad (9)$$

$$G_{ij} = \delta_{ij}(1 + 2U) .$$

M_A is the mass having position $\vec{X}_A$, velocity $\vec{V}_A$ and acceleration $\vec{A}_A$ at coordinate time X^0. U is the Newtonian gravitational potential

$$U = \sum_A \frac{GM_A/c^2}{|\vec{X} - \vec{X}_A|} . \qquad (10)$$

The quantity $\vec{X}$ denotes the observation point. The prime in the double summation means the term B = A is omitted in the sum.

The above metric is of importance because it is the basis of Moyer's work[15] and because the equations of motion of solar system bodies based on the Eddington Clark metric have been incorporated into computer codes for computation of solar system ephemerides at the Jet Propulsion Laboratory. For reference we present here these equations of motion which include the leading relativistic (post-Newtonian) corrections to the Newtonian equations of motion. Using the abbreviations $\vec{X}_{EA} = \vec{X}_E - \vec{X}_A$, $R_{EA} = |\vec{X}_{EA}|$, where the subscript E represents the object of interest (particularly the earth), the equations of motion are:

$$A_E^k = -\sum_A{}' \frac{GM_A X_{EA}^k}{R_{EA}^3} \Big(1 - 4U_e + \frac{V_E^2}{c^2} - \sum_B{}' \frac{GM_B/c^2}{R_{BA}} + [2V_A^2 - 4\vec{V}_E\cdot\vec{V}_A]/c^2$$

$$- \frac{3}{2}\frac{(\vec{X}_{EA}\cdot\vec{V}_A/c)^2}{R_{EA}^2} - \frac{1}{2}\vec{X}_{EA}\cdot\vec{A}_A/c^2 \Big) + 4\sum_A{}' \frac{GM_A(\vec{X}_{EA}\cdot\vec{V}_E)(V_E^k - V_A^k)}{c^2 R_{EA}^3}$$

$$- 3\sum_A{}' \frac{GM_A(\vec{X}_{EA}\cdot\vec{V}_A)(V_E^k - V_A^k)}{c^2 R_{EA}^3} + \frac{7}{2}\sum_A{}' \frac{GM_A A_A^k}{c^2 R_{EA}}\ , \qquad (11)$$

where U_e is the negative of the gravitational potential due to external sources, evaluated at the position of M_E (see Appendix A, Eq. (A3).)

C. The PPN Metric

For comparison with the EC metric, we present here the PPN metric[16] including the parameters γ and β, which is sufficiently general to provide the comparisons we need.

$$G_{00} = -1 + 2U - 2\beta U^2 + (2\gamma+1)\sum_A \frac{GM_A V_A^2/c^4}{|\vec{X} - \vec{X}_A|} \qquad (12)$$

$$- \sum_A \frac{GM_A[(\vec{X} - \vec{X}_A)\cdot\vec{V}_A]^2/c^4}{|\vec{X} - \vec{X}_A|^3} + 2(1-2\beta)\sum_A \sum_B{}' \frac{GM_A GM_B/c^4}{|\vec{X} - \vec{X}_A||\vec{X}_A - \vec{X}_B|}\ ;$$

$$G_{0i} = -\frac{1}{2}(4\gamma+3)\sum_A \frac{GM_A V_A^i/c^3}{|\vec{X} - \vec{X}_A|} - \frac{1}{2}\sum_A \frac{GM_A[(\vec{X}-\vec{X}_A)\cdot\vec{V}_A](X^i - X_A^i)}{c^3|\vec{X} - \vec{X}_A|^3}\ ;$$

$$G_{ij} = \delta_{ij}(1 + 2\gamma U) \ .$$

Note that this metric does not contain the acceleration term which is present in the EC metric. This difference between the forms of the EC and the PPN solutions can be resolved by means of a gauge transformation which transforms the PPN metric into a metric which contains the acceleration term and which has the same form as the metric of EC. The two metrics are physically equivalent in that they describe the same distribution of masses. However they appear different because they involve different sets of clocks. In particular, the EC metric can be derived from the PPN metric by a coordinate transformation:

$$X^0 \rightarrow X^0 + \frac{1}{2} \sum_A \frac{GM_A[(\vec{X} - \vec{X}_A)\cdot\vec{V}_A]/c^3}{|\vec{X} - \vec{X}_A|} \qquad (13)$$

with no change in spatial coordinates. This resetting of the clocks transforms the PPN metric into:

$$G_{00} = -1 + 2U - 2\beta U^2 + (2\gamma+2)\sum_A \frac{GM_A V_A^2/c^4}{|\vec{X} - \vec{X}_A|} - \sum_A \frac{GM_A(\vec{X} - \vec{X}_A)\cdot\vec{A}_A/c^4}{|\vec{X} - \vec{X}_A|}$$

$$- \sum_A \frac{GM_A[(\vec{X} - \vec{X}_A)\cdot\vec{V}_A]^2/c^4}{|\vec{X} - \vec{X}_A|^3} + 2(1-2\beta)\sum_A \sum_B{}' \frac{GM_A GM_B/c^4}{|\vec{X} - \vec{X}_A||\vec{X}_A - \vec{X}_B|} \ ;$$

$$G_{0i} = -\,2(\gamma+1)\sum_A \frac{GM_A V_A^i/c^2}{|\vec{X} - \vec{X}_A|} \ , \qquad (14)$$

with no change in G_{ij}. As can readily be seen, this metric reduces to the EC metric exactly when $\gamma = \beta = 1$, and in general has the same analytical form as that of EC including the acceleration term.

The EC metric is a solution to Einstein's field equations which satisfies causal boundary conditions corresponding to retarded potentials such as the well-known Lienard-Wiechert potentials of electromagnetic theory. Resetting the clocks by means of the gauge transformation, Eq. (13), produces an acceleration term in the PPN metric, as can be verified by straightforward application of the usual tensor transformation laws. Since general relativity is generally covariant with respect to arbitrary tensor transformation laws, the PPN metric is a solution to the field equations corresponding to the same distribution of source masses which, however, arises from boundary conditions corresponding to half retarded, half-advanced potentials.

Since the PPN and EC metrics are physically equivalent, we shall base our subsequent discussion on the EC metric because Moyer's development of the equations of motion, and the implementation of these equations of motion in computer code, is based on the EC metric. It is important to point out however, that the PPN metric, although it appears different from the EC metric, gives rise to equations of motion which are of precisely the same form as Eqs. (11).

D. TDB Coordinates

We shall now discuss the introduction of TDB coordinates, which differ by a scale factor from the EC coordinates--or barycentric coordinates--given in Eqs. (9). The argument presented below is similar to that presented by Hellings[7], who uses PPN coordinates rather than EC coordinates. To $O(V^2/c^2)$ the two metrics are identical, however, so the arguments are almost equivalent.

We begin with the Eddington-Clark metric keeping only terms of $O(V^2/c^2)$; to this order the EC metric is:

$$G_{00} = -(1 - 2U) \ ; \quad G_{0i} = 0 \ ; \quad G_{ij} = \delta_{ij}(1 + 2U) \tag{15}$$

Then the invariant interval, Eq. (8), is to this order

$$-ds^2 = -(1 - 2U)(dX^0)^2 + (1 + 2U)(dX^2 + dY^2 + dZ^2) \tag{16}$$

and U is given by Eq. (10).

Now standard clocks at rest on the surface of the Earth, which define the SI second, beat more slowly than the coordinate clocks represented by the variable X^0 in the above equation, due to relativistic effects. These are principally gravitational redshifts due to gravitational potentials of the sun and the Earth, and time dilation (second-order Doppler shifts) of Earth-Borne clocks due to the Earth's orbital motion. If we compute the long-term time average of ds/dX^0 for such an Earth-borne clock, then to $O(V^2/c^2)$ we have:

$$\langle \frac{ds}{dX^0} \rangle = \langle 1 - U - V^2/2c^2 \rangle \equiv 1 - L \; , \qquad (17)$$

where V is the barycentric coordinate velocity of the Earth-borne clock.

The factor L has been computed in detail by Misner and by Hellings. Misner[4] has summarized the contributions and his summary is repeated here for convenience. The main contributions to L are as follows: from the barycentric velocity and solar potential at the Earth-Moon barycenter, 1.480594×10^{-8}; from the Earth's potential at the equator, 0.069535×10^{-8}; from the time-averaged potential due to Jupiter, 0.000181×10^{-8}; from the Earth's rotational velocity at the equator, 0.000120×10^{-8}, plus still smaller effects due to the potentials of Saturn and the moon. The value of L is thus:

$$L = 1.55047 \times 10^{-8} \; . \qquad (18)$$

Hellings[7] gives a value $L = 1.55052\times10^{-8}$ which is slightly larger. This may partly be due to choice of a different averaging interval but also due to Hellings' inclusion of potentials from other solar system bodies such as the minor planets and asteroids. For purposes of this report the difference between these two values is insignificant.

The TDB time coordinate is introduced by means of the following scale transformation in time:

$$X^0_{TDB} = (1 - L)X^0 \qquad (19)$$

Thus TDB clocks beat at the same average rate as Earth-borne clocks, from the point of view of an observer in the barycentric system.

The new metric is therefore:

$$- ds^2 = - (1-2U)(dX^0_{TDB})^2/(1-L)^2 + (1+2U)(dX^2+dY^2+dZ^2) \qquad (20)$$

which can be rewritten:

$$- (1-L)^2 ds^2 = -(1-2U)(dX^0_{TDB})^2 + (1+2U)[(1-L)^2(dX^2+dY^2+dZ^2)] \; . \qquad (21)$$

The left side can be interpreted as the invariant interval measured in TDB units. If we introduce

$$s_{TDB} = (1 - L)s \; , \qquad (22)$$

then the metric becomes:

$$- (ds_{TDB})^2 = -(1-2U)(dX^0_{TDB})^2 + (1+2U)[(1-L)^2(dX^2+dY^2+dZ^2)] . \qquad (23)$$

In order for this invariant interval to be cast into the same form as the interval in Eq. (16), we must do two things: (a) we must introduce new scaled spatial coordinates,

$$X^k_{TDB} = (1-L)X^k \; ; \qquad (k = 1,2,3) \qquad (24)$$

and (b) we must ensure that the dimensionless quantity U retains its form and magnitude unchanged in TDB coordinates. Since c does not change, this means that

$$U = \sum_A \frac{GM_A/c^2}{|\vec{X} - \vec{X}_A|} = \sum_A \frac{GM_A(1-L)/c^2}{|\vec{X}_{TDB} - \vec{X}_{A_{TDB}}|} = U_{TDB} \qquad (25)$$

and so in TDB coordinates we must identify the mass parameter by:

$$(GM)_{TDB} = (1-L)GM \; . \qquad (26)$$

with all these changes, the theoretical description of the physics of planetary and satellite motion and of the motion of electromagnetic signals (photons) is of the same mathematical form in TDB coordinates, to $O(V^2/c^2)$, as it is in EC coordinates:

$$c_{TDB} = c; \quad U_{TDB} = U \;;$$

$$-(ds_{TDB})^2 = -(1-2U_{TDB})(dX^0_{TDB})^2$$
$$+ (1+2U_{TDB})(dX^2_{TDB}+dY^2_{TDB}+ dZ^2_{TDB}) \;. \qquad (27)$$

At this point it may be objected that the metric in Eqs. (13) was written only to lowest order whereas the difference between U and U_{TDB} in Eq. (27) involves terms of $O(V^4/c^4)$ only, therefore the argument leading to Eq. (26) cannot be considered as rigorous because of the possibility that higher order terms might modify the result. We shall therefore give two additional arguments showing the conclusion (26) is valid.

First if we look at the full metric tensor, Eq. (9), including terms of $O(V^4/c^4)$, then it may be verified by inspection that each term in G_{00} is unchanged in form by the combination of transformations

$$c_{TDB}= c \;; \quad X^{\mu}_{TDB} = (1-L)X^{\mu} \;; \quad (GM)_{TDB} = (1-L)GM \;. \qquad (28)$$

These imply:

$$V^k_{TDB}/c = dX^k_{TDB}/dX^0{}_{TDB} = dX^k/dX^0 = V^k/c \;, \text{ and} \qquad (29)$$

$$(GM_A A^k_A)_{TDB} = [(1-L)GM_A(1-L)A^k_A/(1-L)^2] = GM_A A^k_A \;. \qquad (30)$$

Therefore $(G_{00})_{TDB} = G_{00}$. A similar argument holds for G_{ij}. For the contributions to the metric arising from G_{0i}, we have:

$$[G_{0i}dX^0dX^i]_{TDB} = -4\sum_A \{ \frac{GM_A(V^i_A/c)}{|\vec{X} - \vec{X}_A|} dX^0dX^i\}_{TDB}$$

$$= -4 \sum_A \{\frac{(1-L)\, GM_A(V_A^i/c)}{(1-L)|\vec{X} - \vec{X}_A|}\,(1-L)^2 dX^0 dX^i\}$$

$$= (1-L)^2 G_{0i} dX^0 dX^i \quad . \qquad (31)$$

The factor $(1-L)^2$ in the above expression is just the factor needed to make the following equation valid:

$$-(ds_{TDB})^2 = (G_{\mu\nu})_{TDB} dX^{\mu}_{TDB} dX^{\nu}_{TDB} \quad . \qquad (32)$$

A second argument which is more powerful and convincing may be obtained by examining the equations of motion, Eqs. (11), term by term. Using the scale changes of Eq. (28), we imagine multiplying every term in Eq. (11) by $(1-L)^{-1}$. The left side becomes $(A^k_E)_{TDB}$ and each term on the right including all post-Newtonian terms then may be transformed into its TDB counterpart.

The value of this approach is that it may be seen that the equations of motion take the same mathematical form in Eddington-Clark coordinates, as they take in TDB coordinates.

A simple explanation of the result (26) is as follows: A TDB clock beats more slowly, by the factor (1-L), than an EC clock. Therefore to maintain a universally defined numerical value for the speed of light c, the unit of length in TDB coordinates must be physically longer than the length unit in EC coordinates. Thus since $(GM/c^2)_{TDB}$ represents a physical length as measured using a TDB meter stick, the numerical value of $(GM/c^2)_{TDB}$ will be less than it is in EC coordinates. The speed of light c is the same in the two unit systems, however, hence:

$$(GM)_{TDB} = (1-L)GM \quad . \qquad (26)$$

E. The SI Second; Standard Clocks

The relation between TDB coordinates and EC coordinates may be further clarified by considering the respective units of time in terms of atomic clocks based on Cesium.

A "Standard Clock" is an atomic clock using transitions between certain levels of Cesium. Specifically, the length of the SI second is defined as:

$$1 \text{ SI second} \equiv 9\ 192\ 631\ 770 \text{ cycles of Cesium.} \qquad (33)$$

A standard SI second may be realized by any observer who is at rest with respect to such a clock and near it, by measuring the above number of cycles of Cesium in any one of a number of identically constructed Cesium clocks.

F. Standard Clocks in TDB Coordinates

Choice of the number of cycles defining the time unit as in Sect. E. above is a matter of convention. For purposes of later discussion we shall define a different standard clock in which the time unit is slightly longer. We shall refer to these clocks as "TDB standard clocks" because the time unit will agree with the unit of TDB time. This alternate system of standard clocks is defined by requiring the length of the TDB second to be:

$$1 \text{ TDB second} \equiv (1+L)\times(9\ 192\ 631\ 770) \text{ cycles of Cesium.} \qquad (34)$$

where L is given by Eq. (5). The reason for this choice of L has been discussed previously. Thus the TDB second is:

$$1 \text{ TDB second} \equiv 9\ 192\ 631\ 912.5 \text{ cycles of Cesium.} \qquad (35)$$

The standard TDB clock thus beats more slowly than the standard SI clock discussed in Sect. E., above.

G. Reinterpretation of EC Metric In Terms of TDB Coordinates

It was seen in Sect. III.D. above, that the metric tensor and equations of motion, expressed in TDB coordinates, have the same form as the metric tensor and equations of motion, respectively, in EC coordinates. The lunar laser ranging data analysis and solar system ephemerides work at JPL in fact uses the interpretation of Moyer's equations of motion in terms of TDB coordinates[7].

If we adopt this interpretation, we can obtain a very compelling derivation of Eq. (5). Let us suppose that in the equations of motion, Eqs. (11), all quantities are to be considered as expressed in TDB coordinates, with the standard TDB clocks beating at the rate given by Eqs. (34) and (35). Then we may also reinterpret the coordinate transformations derived in Appendix A as providing the metric in local inertial coordinates in terms of TDB units. Under this coordinate transformation to local inertial coordinates the mass parameter $(GM_E)_{TDB}$ does not change. The proof is given in Appendix A. This value $(GM_E)_{TDB}$ would then be obtained by fitting the LAGEOS ranging experiments, if TDB standard clocks were used on Earth. But such clocks are not used, clocks based on the SI second are used. These clocks run more rapidly than do the TDB clocks, by the factor (1+L). Therefore

$$(GM_E)_{SI} = (1+L)\times(GM_E)_{TDB}$$

which again yields Eq. (26) since $L \ll 1$.

It should be stressed again that the results proved in Appendix A provide a crucial link in this argument, namely that if in the barycentric EC metric, Eq. (9), the Earth has mass parameter GM_E/c^2, then after transforming to Local Inertial Coordinates the mass parameter is numerically unchanged. In both coordinate systems the same standard clocks are used. This will be discussed in more detail in the next section.

IV. Local Inertial Coordinates; Coordinate Transformations

The purpose of this section is to summarize the main results obtained in Appendix A for purposes of this report. In Appendix A, one begins with the EC metric in barycentric coordinates, in which the mass parameter of the Earth is represented by GM_E, measured using conventionally selected standard rods and standard clocks. The Earth is in free fall along a geodesic in space-time. The position of the Earth becomes the origin of a freely falling, local inertial frame in which the time coordinate can be directly related to time elapsed on a freely falling standard clock and the space coordinates are proper distances, measured using standard rods. Coordinate transformations are constructed between barycentric coordinates and local inertial coordinates, and applied to the EC metric. It is found that the term in the metric tensor, in the local inertial frame arising from the Earth's potential is described by the same mass parameter. There is no mass parameter change upon transforming from barycentric coordinates to local inertial coordinates.

This is because the same type of standard clocks are used to measure time both in barycentric coordinates and in local inertial coordinates. In the case of barycentric EC coordinates, a standard clock at rest at infinity (where $G_{00} = -1$) will determine the rate of advance of coordinate time. In local inertial coordinates, an identically constructed clock at rest on the surface of Earth determines the rate of advance of coordinate time. Since identical clocks are used, the mass parameter GM_E of the Earth has the same numerical value in the two coordinate systems.

This result--proved in detail in Appendix A--shows that if the same unit of time (in terms of cycles of Cesium) is used in barycentric and local inertial coordinates, then the mass parameters of the Earth in the two coordinate systems must agree.

The result, Eq. (5), may then seem paradoxical since in 1976 the IAU passed a resolution[17] requiring that, on the average, the TDB second has to be the same length as the SI second. However as shown in Sections III. E. and III. F., standard TDB clocks and standard SI clocks when compared side by side, would beat at different rates. This explains Eq. (5).

V. Summary

Lunar Laser Ranging data analysis at JPL utilizes a system of units based on the TDB second, and solar system dynamics is described in a barycentric system of coordinates. LAGEOS ranging data analysis utilizes a system of units based on the SI second, and a locally inertial system of coordinates. From the point of view of an observer at rest in the barycentric system, the TDB second is on the average the same length as the SI second, but due to relativistic effects these SI clocks beat more slowly than standard SI clocks would beat if they were at rest in the barycentric system. The SI unit of time is therefore shorter than the TDB unit of time. The SI unit of length is also shorter than the TDB unit of length, in order to maintain the same numerical value of c. Thus a mass parameter GM/c^2 having dimensions of length, will have a larger numerical value in SI units than in TDB units. Hence $GM_{TDB} = (1-L)GM_{SI}$.

VI. Implications of Neglecting Relativistic Effects

A. The Nonlinear Schwarzschild Field of the Earth

Let us assume that in ranging to an earth satellite such as LAGEOS, the mass of the Earth is determined from observations of two quantities: the period of revolution of the satellite around the Earth, and the semimajor axis of the satellite's orbit. For simplicity we shall assume the orbit is circular, and assum that ranging measurements give directly the radius of the orbit. Eq. (77) in Appendix A is Kepler's third law and includes relativistic correction terms. If the higher order relativistic corrections are retained, which arise from the nonlinear contributions due to the gravitational field of the Earth itself, then:

$$GM_E = 4\pi^2R^3/T^2(1 + 3GM_E/c^2R) \; . \tag{36}$$

Since for LAGEOS' orbit $3GM_E/c^2R \approx 1.1\times10^{-9}$, the relativistic correction in parentheses in Eq. (36) is very small. Currently this correction is neglected[9]. If included it could affect the determination of the Earth's mass by about 0.0004 km^3/sec^2. This would be a systematic effect of magnitude about 20% of the error quoted in Eq. (2) and is probably sufficiently large that it should be accounted for. The correction is in such a direction that the agreement between the two sides of Eq. (5) would be improved.

B. Precession of Perigee of LAGEOS

As mentioned in the introduction, the rate of precession of perigee of LAGEOS due to the nonlinear Schwarzschild field of the Earth is $3GM_E/c^2a(1-e^2)$ per revolution. The period of revolution is $2\pi\sqrt{[a^3/GM_E]}$, so the rate of perigee precession per second due to relativity is:

$$\frac{3(GM_E)^{3/2}}{2\pi c^2 a^{5/2}(1-e^2)} \approx 8.5 \times 10^{-14} \text{ rad/sec} \; , \tag{37}$$

where we have used $a \approx 1.2 \times 10^7$ meters, $e \approx 0$.

On the other hand there is a much larger contribution to the perigee precession rate due to the Earth's oblateness. Considering only the quadrupole moment coefficient $J_2 = -\sqrt{5}\times C_{20}$ of the Earth[18], the secular part of the precession rate[19] is:

$$\frac{3J_2R_e^2}{2a^2(1-e^2)^2}\sqrt{[GM_E/a^3]} \simeq 2.2 \times 10^{-7} \text{ rad/sec} \quad (38)$$

where $R_e \simeq 6.38\times10^6$ meters is the equatorial radius of the Earth, and $C_{20} = -484.16499\times10^{-6}$ (see reference 18).

We observe from these results that the relativistic perigee precession effect is only 3.9×10^{-7} of that due to the Earth's quadrupole moment. However the uncertainty in J_2 is 0.6 parts per million[18]. Since relativistic effects are not currently accounted for, the expected relativistic precession could contribute to a systematic error in the rate of perigee precession which is about 60% of that which could arise from an error in J_2 equal to the quoted uncertainty. In this case the neglect of relativistic effects has direct implications for the determination of J_2 (or C_{20}) rather than GM_E.

C. Precession of the Nodal Line; Geodetic Precession

The phenomenon of geodetic precession[11] implies that the orbital plane of LAGEOS will precess by an amount:

$$3\pi GM_{SUN}/c^2R \simeq 9.2\times10^{-8} \text{ rad/sec} = 19 \text{ marcsec/yr} , \quad (38)$$

where R is the radius of the Earth's orbit about the sun and $GM_{SUN}/c^2 \simeq 1.46\times10^3$ meters is the Sun's Schwarzschild radius. This precession would give rise to a motion of the nodal line by about 9.2 cm per month at a distance $a = 1.2\times10^7$ meters from Earth.

The quadrupole moment of the Earth also causes secular precession of the nodal line, which[19] to lowest order in eccentricity e is:

$$\frac{3J_2R_e^2}{2a^2}\cos I\sqrt{[GM_E/a^3]} \simeq 7.2\times10^{-8} \text{ rad/sec} \simeq 4.7\times10^8 \text{marcsec/yr} \quad (39)$$

where we have used $I \simeq 109°$ for the orbital inclination of LAGEOS. The fractional uncertainty in J_2 of 0.6 parts per million would thus correspond to a nodal precession rate of about 280 milliarcseconds per year, which is significantly larger than the geodetic precession rate. This relativistic effect is therefore so small that it probably does not need to be considered yet in the modelling of LAGEOS observations, because of the large uncertainty in J_2.

VII. Conclusions

In sum, the principal implications of relativity for the determination of GM_E stem from the different choices of units of time, as expressed through Eqs. (4), (5), and (26). Additional relativistic effects arise from the nonlinear Schwarzschild field of the Earth could contribute systematically to the determination of the Earth's mass, and quadrupole moment, by amounts which vary from 20% to 60% of the currently quoted errors in these quantities, respectively, and should be taken into account in the orbit modelling.

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