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ANALYTICAL FORMULATION OF ORBITER-PAYLOAD COUPLED BY TRUNNION JOINTS WITH COULOMB FRICTION

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ABSTRACT

An orbiter and its payload substructure are linked together by five trunnion joints which have thirty degrees-of-freedom. Geometric compatibility conditions require fourteen of the interface physical coordinates of the orbiter and payload be equal to each other and the remaining sixteen are free to have relative motions under Coulomb friction. This report presents the component modes synthesis method using fourteen inertia relief attachment modes for the formulation of the coupled system. The exact nonlinear friction function is derived based on the characteristics of the joints. Formulation is applicable to an orbiter that carries any number of payload substructures.

ACKNOWLEDGMENTS

It has been a great pleasure for me to work with a former colleague, Mr. Robert Ryan, after twenty-one years and a very talented young engineer Dr. John Townsend who has accomplished so much in the field of structural dynamics in such a short time. With their help, this summer fellowship gave me a very enjoyable summer.

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NOMENCLATURE

A_a, A_{--}, a_a	inertia relief attachment modes, Eqs. (2.14), (2.21)
B_k, B_{--}, b_k	free-free normal kept modes, Eqs. (2.12), (2.21)
D_o, D_p, D	defined by Eq. (2.33)
f_f	friction force
f_s	applied force on substructure
I	unit matrix
k, $\bar{k}$, K	stiffness, generalized stiffness, system stiffness matrices
k_a, k_b, k_r, k_s	spring constants of joint shafts, Eq. (2.6)
k_o, k_p	number of kept normal modes of orbiter and payload, respectively
m, $\bar{m}$, M	mass, generalized, system mass matrices, respectively
N_n	normal force on nth free coordinates
0	zero matrix
p_a	generalized coordinate of attachment modes
p_k	generalized coordinate of normal modes
q	set of independent generalized coordinates, Eq. (2.20)
S	transformation matrix of p to q
u_b	coordinates of built-in end of joint shaft opposite to u_f
u_f	free coordiantes of joint nodes
u_j	constraint coordinates of joint nodes
u_s	coordinates of interior nodes of substructure

Greek Alphabets:

Λ	diagonal matrix of ω^2
μ, $\bar{\mu}$	kinematic and static coefficients

θ	angular displacement
ω	natural frequency of substructure

Superscripts:

o, p	orbiter and payload
$(\ )^T$	matrix transpose
$(\ )^{-1}$	matrix inverse

I. INTRODUCTION

1.1 Objective of This Research

An orbiter and its payload substructure are linked together by five trunnion joints: two primary joints support load in the axial (x) and normal (z) directions, two stabilizing joints transfer forces in the normal direction and the keel joint carries load in the transverse (y) direction. The joint shafts of the payload and the support bearings of the orbiter are free to have motions relative to each other under Coulomb friction. The objective of this research is to make analytical formulation of this nonlinearly coupled system for dynamic response.

1.2 Method of Approach

Component modes synthesis has been accepted as the most efficient method for coupling substructures of large degrees-of-freedom for dynamic analysis in the last decade. The five nodes of the joints have thirty DOF's (3 translations and 3 rotations) of which the fourteen restrained coordinates are the interface coordinates and the remaining sixteen coordinates free to move under Coulomb friction are a subset of the interior coordinates. The orbiter and its payload are treated as two free-free substructures for which two sets of free-free normal modes are generated by their finite element models. These modes constitute the generalized coordinates. Fourteen inertia relief attachment modes for the interface coordinates form the dependent generalized coordinates.

The friction forces acting on the free coordinates depend on whether the joints are in stuck state (no relative motion) or in a motion state. Any number of coordiantes may be stuck and the others in motion. Nonlinear friction functions will be derived for the joints. A set of motion equations of the coupled system in terms of the generalized coordinates are obtained by using the geometric compatibility conditions and equilibrium of the joint interaction forces.

1.3 Component Modes Synthesis Method

Since Hurty[1] first proposed the method of coupling large substructures by component modes synthesis for dynamic analysis in 1965, numerous papers have been published, such as references 2, 3, and 4, for extension and modification of the original concept. Among the various versions of this method, the inertia relief attachment modes for unconstraint substructures and the residual attachment modes for constraint substructures have proven to be the most accurate methods up-to-date.

An outline and examples of the various approaches of the component modes synthesis can be found in chapter 19 of Craig's book[5]. Because of the limited pages allowed for this report, the detail of this method will not be presented, but rather, the readers are referred to reference 5. The formulation of the coordinates transformation matrix and the generalized mass and stiffness matrices in chapter 2 of this report are very similar to that given in Craig's book. Some of the Greek alphabets used in the equations in Craig's book are replaced by English alphabets for typing convenience.

1.4 Description of Substructures

Figure 1 shows the payload structure and its finite element model is shown in Figure 2. This structure is to be linked to an orbiter by five joints which can be seen from these figures. The joint shafts and the support bearings can be seen in Figure 3. Coordinates x, y, and z are axes along the body axis, transverse, and normal directions of the orbiter as shown in Figure 1.

It is important for engineering designers to determine the loads on the payload structure and its interfacing structure accurately during the orbiter launching phase. The existing computing program has not been analytically formulated for dealing with frictional joints and is based on some simple assumptions on the joint friction forces. It is doubtful that such a simple model can produce accurate data for the designers. This is the motivation for this research.

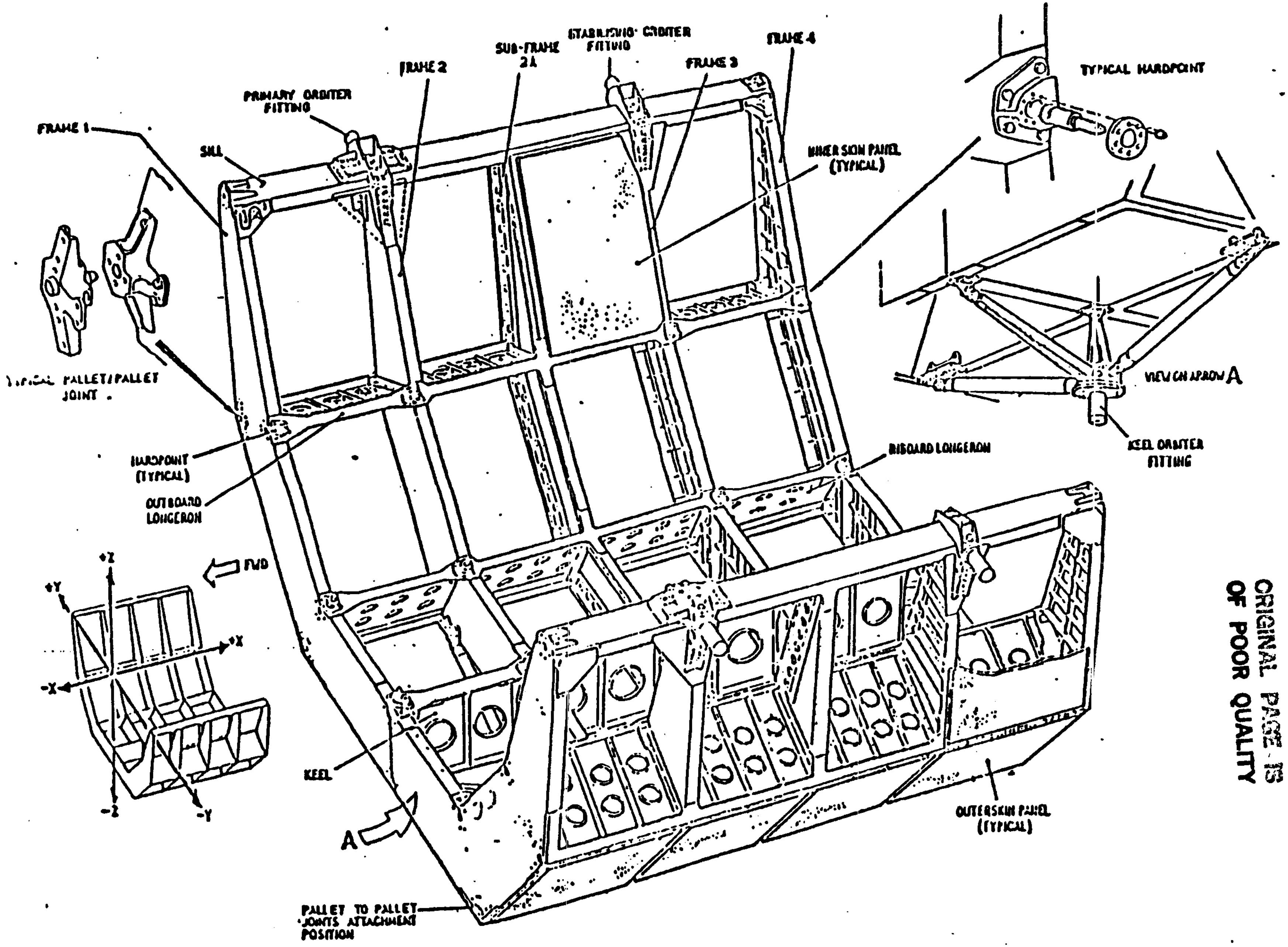


Figure 1. Payload Structure.

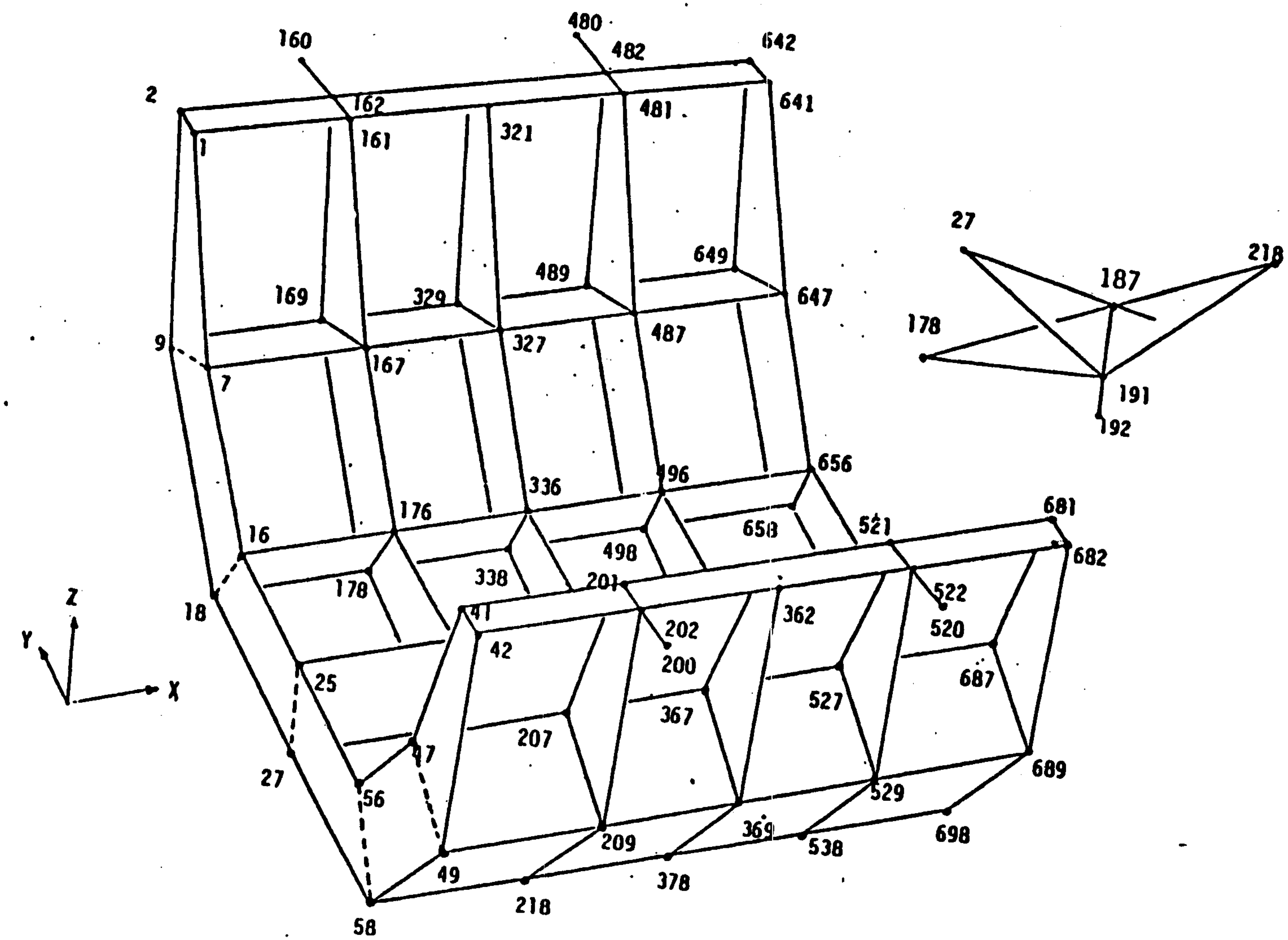
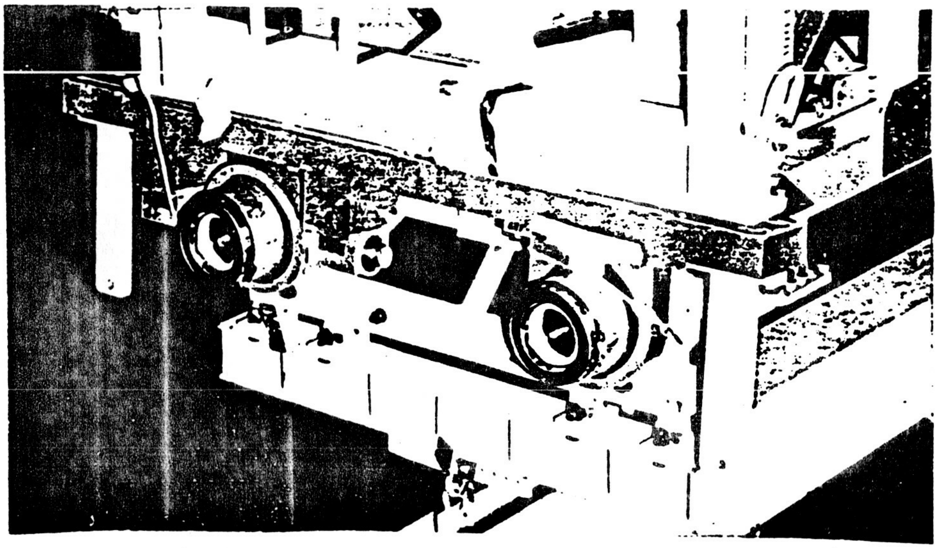
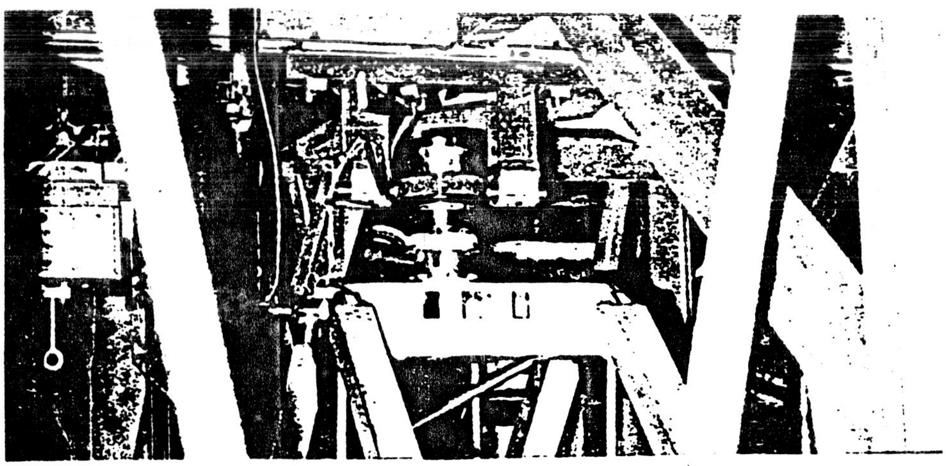


Figure 2. Finite element model of payload structure.



(a) Primary (left) and stabilizing (right) joints.



(b) Keel joint.

Figure 3. Trunnion joints of payload.

II. ANALYSIS

2.1 Characteristics of Trunnion Joints

An orbiter may carry one or more payload substructures and each of them is attached to the orbiter by five trunnion joints. As shown in Figure 3(a), the two primary joints support loads in x and z directions while the joint shafts are free to slide and rotate relative to the support bearings mounted on the orbiter. The two stabilizing joints shown in Figure 3(a) are designed to transfer load along z direction only and the joint shafts may have sliding and rotation motions in x and y directions relative to the orbiter. The keel joint which is located at the bottom of the payload structure as shown in Figure 3(b), has the same function as the stabilizing joint except the direction of y and z are interchanged.

Consider that each interface node of the orbiter and payload has six degrees-of-freedom (3 translations and 3 rotations) which are denoted by u_x, u_y, u_z, θ_x, θ_y, and θ_z. The geometric compatibility, or constraint, conditions for the five joints are summarized in Table 1. An inertia relief attachment mode denoted by p_a is assigned for each constraint physical coordinate, u_j, as shown in Figure 4 and listed in the third column of Table 1. The coordinates which have free motion relative to the support, u_f, are given in the last column. Discussion of inertia relief attachment modes will be given later.

2.2 Coordinate Numbering for Joint Nodes

Let u_i and u_j denote the physical coordinates of the interior and interface nodes, respectively. As shown in Table 1, among the 30 coordinates of the joint nodes there are 16 coordinates that have free motions (u_f) which must be treated as a subset of u_i and the remaining 14 coordinates form the set u_j. These physical coordinates and the generalized coordinates for the attachment modes are arranged in sequence as shown in Table 2.

Table 1. Characteristics of Trunnion Joints

Joint	u_j Constraint	p_a Attachment Mode	u_f Free Motion
Primary	$u_x^o = u_x^p$	x - force	u_y
	$u_z^o = u_z^p$	z - force	θ_y
	$\theta_x^o = \theta_x^p$	x - moment	
	$\theta_z^o = \theta_z^p$	z - moment	
Stabilizing	$u_z^o = u_z^p$	z - force	u_x
	$\theta_z^o = \theta_z^p$	z - moment	u_y
			θ_x
			θ_y
Keel	$u_y^o = u_y^p$	y - force	u_x
	$\theta_y^o = \theta_y^p$	y - moment	u_z
			θ_x
			θ_z
Total Number	14	14	16

Superscripts "o" and "p" denote orbiter and payload respectively.

Special notation is given to another set of physical coordinates which is a subset of u_i, for the convenience of formulation of joint friction forces. These are the coordinates of the nodes of the joint shafts at the built-in end of the payload structure. The coordinates at the built-in end opposite to u_{nf} at the free end is denoted by u_{nb} as shown in Figure 4.

In Figure 4 parts (a) and (b) depict the pairs of bearing and shaft of the two primary joints, parts (c) and (d) are for the stabilizing joints, and part (c) is for the keel joint. Nodes B_1 through B_5 represent nodes of the joint shaft at the built-in end. Other symbols are:

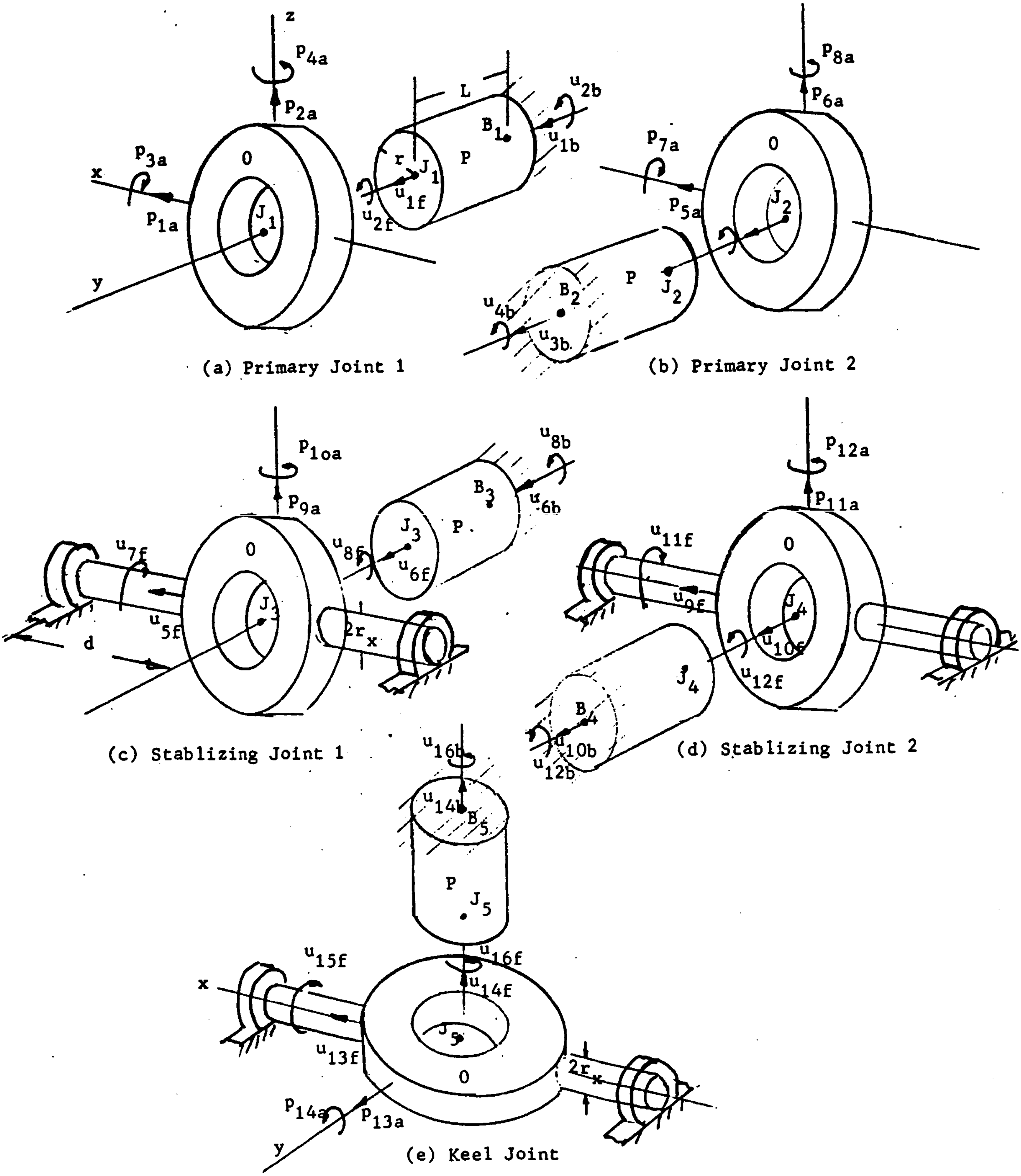


Figure 4. Coordinate system of trunnion joints.

Table 2. Sequence of Joints Coordinates

Sequence	u_f (joint)	u_j (joint)	p_a (joint)
1	u_y (1)	u_x (1)	f_x (1)
2	θ_y (1)	u_z (1)	f_z (1)
3	u_y (2)	θ_x (1)	M_x (1)
4	θ_y (2)	θ_z (1)	M_z (1)
5	u_x (3)	u_x (2)	f_x (2)
6	u_y (3)	u_z (2)	f_z (2)
7	θ_x (3)	θ_x (2)	M_x (2)
8	θ_y (3)	θ_z (2)	M_z (2)
9	u_x (4)	u_z (3)	f_z (3)
10	u_y (4)	θ_z (3)	M_z (3)
11	θ_x (4)	u_z (4)	f_z (4)
12	θ_y (4)	θ_z (4)	M_z (4)
13	u_x (5)	u_y (5)	f_y (5)
14	u_z (5)	θ_y (5)	M_y (5)
15	θ_x (5)	---	---
16	θ_z (5)	---	---

2d = distance between supports of the shaft in x direction

L = length of joint shaft

r = radius of joint shaft

r_x = radius of bearing support in x direction

The above dimensions may have different values for different joints. The numbering system for all coordinates and forces are given according to Table 2.

2.3 Friction Force Acting On Free Coordinate

2.3.1 Primary Trunnion Joint 1

If the axial elastic force in the joint shaft is smaller than the axial friction resistance, there will be no sliding motion between the shaft and the bearing. With the aid of Figure 4(a), one may state the following. If

$$|u_{1f}^{p} - u_{1b}|\ k_a < \bar{\mu}N \tag{2.1a}$$

then,

$$u_{1f}^{p} = u_{1f}^{o} \quad \text{(stuck)} \tag{2.1b}$$

$$f_{1f}^{p} = -f_{1f}^{o} = k_a(u_{1f}^{p} - u_{1b}) \tag{2.1c}$$

where

$\bar{\mu}$ = static coefficient of friction

k_a = axial stiffness of shaft = $\pi r^2 E/L$

E = Young's modulus of elasticity

N = pressure between shaft and bearing = $[(p_{1a})^2 + (p_{2a})^2]^{1/2}$

Once the sliding starts, one may state that if

$$u_{1f}^{p} \neq u_{1f}^{o}\ , \quad \text{(sliding)} \tag{2.2a}$$

then

$$f_{1f}^{p} = -f_{1f}^{o} = k_a(u_{1f}^{p} - u_{1b}) = \mu N \tag{2.2b}$$

When the elastic force in the joint shaft becomes smaller than the friction force,

$$k_a|u_{1f}^{p} - u_{1b}| < \mu N\ , \tag{2.2c}$$

then sliding stops,

$$u_{1f}^{p} = u_{1f}^{o} \quad \text{(stuck)} \tag{2.2d}$$

where μ is the kinematic coefficient of friction.

In dealing with the coordinate u_{2f}, two sets of equations similar to Equations (2.1) and (2.2) may be written by replacing subscript "1" by "2", N by rN, and k_a by $k_s = \pi r^4 G/2L$. The spring constant represents the shear stiffness of the shaft and G is the modulus of shear. The equations for the above two cases and the cases to be discussed for other joints can be represented by a simple figure as shown in Figure 5. Let

$$x_n = u_{nf}^{p} - u_{nb} \qquad n - 1,2,\ldots,16$$

$$f_{nf}^{p} = - f_{nf}^{o} = k_n x_n \tag{2.3}$$

The function f_{nf} and the "stuck" and "sliding" conditions are shown in Figure 5.

As x_n increasing reaches point A where

$$f_{nf} = k_n x_n = \bar{\mu} r_n N_n \tag{2.4}$$

relative motion of the coordinate u_{nf} starts; and then, the friction force drops immediately to point B where

$$f_{nf} = k_n x_n = \bar{\mu} r_n N_n \tag{2.5}$$

If elastic force in the joint shaft decreasing passes point B, relative motion stops (stuck state). To the negative side points A' and B' are the equivalent points of A and B, respectively. It is important to note that the magnitude of N_n is a function of the generalized coordinates p_a which determine the magnitudes of A and B. Next, with the aid of Figure 4, the notations k, N, and r are derived for n equal to 1 through 16.

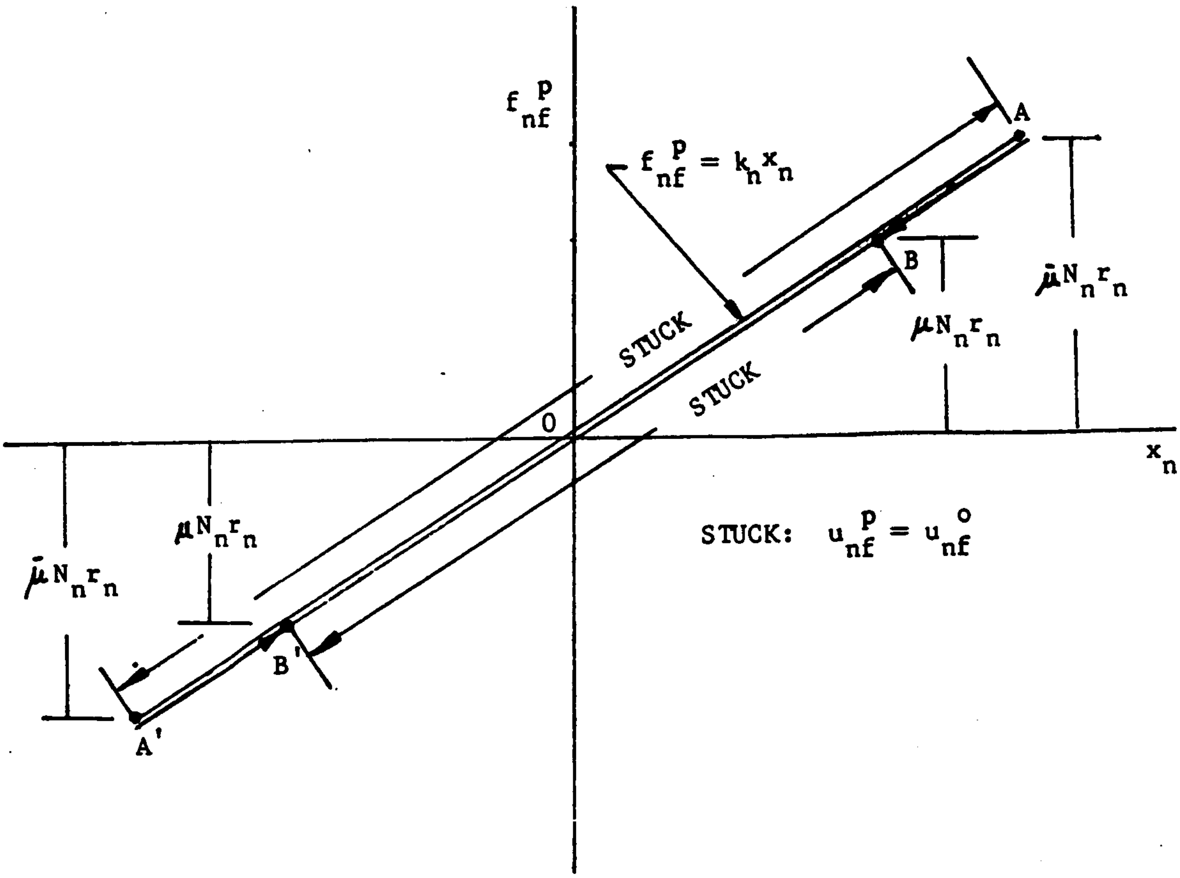


Figure 5. General function of friction in joints

2.3.2 General Function of Friction in Joints

The following spring constants of the joint shaft are defined:

$$k_a = \pi r^2/EL \text{ (axial)}$$

$$k_s = \pi r^4 G/2L \text{ (shear)} \qquad k_{sx} = \pi r_x^4 G/2L \tag{2.6}$$

$$k_r = 4EI/L \text{ (end moment with zero deflection)}$$

$$k_b = 12EI/L^3 \text{ (end deflection with zero slope)}$$

One may define the following for Equations (2.3), (2.4), and (2.5):

$k_n = k_s$	$n = 1,3,6,10,14$
$k_n = k_r$	$n = 2,4,8,12,16$
$k_n = k_{sx}$	$n = 5,9,13$
$k_n = k_{rx}$	$n = 7,11,15$

$$r_n = 1 \qquad n = 1,3,6,10,14$$

$$r_n = r \qquad n = 2,4,8,12,16$$

$$r_n = r_x \qquad n = 5,7,9,11,13,15$$

$$N_n = [(p_{1a})^2 + (p_{2a})^2]^{\frac{1}{2}} \qquad n = 1,2$$

$$N_n = [(p_{5a})^2 + (p_{6a})^2]^{\frac{1}{2}} \qquad n = 3,4$$

$$N_n = [(p_{9a})^2 + (p_{10a}/d)^2]^{\frac{1}{2}} \qquad n = 5,7$$

$$N_n = [(p_{11a})^2 + (p_{12a}/d)^2]^{\frac{1}{2}} \qquad n = 9,11$$

$$N_n = [(p_{13a})^2 + (p_{14a}/d)^2]^{\frac{1}{2}} \qquad n = 13,15$$

$$N_n = \begin{cases} [(p_{9a})^2 + (\mu N_5)^2]^{\frac{1}{2}} & \text{if } u_{5f}^p = u_{5f}^o \\ [(p_{9a})^2 + (k_b x_5)^2]^{\frac{1}{2}} & \text{if } u_{5f}^p = u_{5f}^o \end{cases} \qquad n = 6,8$$

$$N_n = \begin{cases} [(p_{11a})^2 + (\mu N_9)^2]^{\frac{1}{2}} & \text{if } u_{9f}^p \neq u_{9f}^o \\ [(p_{11a})^2 + (k_b x_9)^2]^{\frac{1}{2}} & \text{if } u_{9f}^p = u_{9f}^o \end{cases} \qquad n = 10,12$$

$$N_n = \begin{cases} [(p_{13a})^2 + (\mu N_{13})^2]^{\frac{1}{2}} & \text{if } u_{13f}^p \neq u_{13f}^o \\ [(p_{13a})^2 + (k_b x_{13})^2]^{\frac{1}{2}} & \text{if } u_{13f}^p = u_{13f}^o \end{cases} \qquad n = 14,16$$

2.4 Component Modes for Unrestraint Substructures

2.4.1 Physical Coordinates

Consider that the orbiter and its payload are two unrestraint substructures to be coupled together by the 14 interface coordinates described in Section 2.1. Now, let's begin the formulation from the equation of motion of a substructure,

$$m\ddot{u} + ku = f \tag{2.7}$$

The mass and stiffness matrices m and k are generated by finite element models, and f denotes external forces applied to the substructure. Note that the superscripts "o" for orbiter and "p" for payload will be used when the situation requires to identify the substructure, otherwise, expressions without these superscripts are applicable to both. The physical coordinates u are separated into two subsets: u_i for interior nodes and u_j for interface coordinates. As described in Section 2.2, the free coordinates of the joint nodes, u_f, belongs to a subset of u_i. Thus, u_i consists of u_s, the physical coordinates of interior nodes, and u_f. The m and k matrices must be rearranged and consistent with u in the form,

$$u = [u_s \; u_f \; \vdots \; u_j]^T = [u_i \; \vdots \; u_j]^T \tag{2.8}$$

and so is the matrix f,

$$f = [f_s \; f_f \; \vdots \; f_j]^T = [f_i \; \vdots \; f_j]^T \tag{2.9}$$

where

f_s = external load acting on interior nodes

f_f = friction force acting on u_f

f_j = external load acting on $u_j = 0$

The dimension of these elements are as follows:

u_f : 16x1 u_j : 14x1 (for orbiter and payload)

and dimension for u_s is arbitrary and different for orbiter and payload.

2.4.2 Free-Free Normal Modes

Solving eigenvalues and eigenvectors of Equation (2.7) for free vibrations, one obtains six rigid-body modes (zero frequency) and free-free elastic normal modes as follows:

$$(k - \omega_i^2 m) b_i = 0 \tag{2.10}$$

with

$$b_r^T m b_r = 1 \qquad r = 1, 2, \ldots, 6 \tag{2.11}$$

$$b_e^T m b_e = 1$$

$$b_e^T k b_e = \omega_e^2 \qquad e = 1, 2, \ldots, k \tag{2.12}$$

where b_r and b_e are normalized nx1 eigenvectors of rigid-body and elastic modes respectively, n is the dimension of u and k is the number of elastic modes to be kept in the formulation ($k \leq n - 6$). The modes higher than k are dropped to reduce the size of the coupled structural system. Now, denote the matrix of the eigenvectors of the kept elastic modes by

$$B_k = (b_1 \; b_2 \; \ldots \; b_k) \tag{2.13}$$

and let p_k be the generalized coordinate associated with the normal mode b_k. Note that the orbiter and payload may have different values of k, i.e., $k_o \neq k_p$.

2.4.3 Inertia Relief Attachment Modes

An attachment mode a_a is a set of physical coordiantes u due to a unit force applied to one of the interface coordinates, u_j, while forces on all the rest of the interface coordinates are zero. Let p_a be a set of generalized coordinates associated with the attachment mode a_a ($a = 1, 2, \ldots, 14$), as shown in Figure 4. Let matrix A_a denote the set of attachment modes,

$$A_a = (a_1 \; a_2 \; \ldots \; a_{14}) \tag{2.14}$$

Since the orbiter and its payload are unrestraint structures, the attachment modes are due to a self-equilibrated force system,

$$f_e = f - m\ddot{u}_r$$

which is the result of applied and rigid-body inertia forces. Hence, A_a is called inertia relief attachment modes. For more detail the reader is referred to Reference 5, pages 479-487.

By definition, the inertia relief attachment modes may be obtained from the equation,

$$A_a = G_e F_a \tag{2.15}$$

where the elastic flexibility matrix may be formulated from

$$G_e = B_e \Lambda_{ee}^{-1} B_e^T \qquad \Lambda_{ee}^{-1} = \mathrm{diag}(1/\omega_e^2) \tag{2.16}$$

and matrix F_a represents unit force at each coordinate u_j,

$$F_a = \begin{bmatrix} 0_{ij} \\ --- \\ I_{jj} \end{bmatrix} \qquad \begin{array}{l} 0_{ij} = \text{ixj zero matrix} \\ \\ I_{jj} = \text{jxj unit matrix} \end{array} \tag{2.17}$$

Thus, Equations (2.15), (2.16), and (2.17) result in

$$A_a = B_k (\Lambda_{kk}^{-1} B_{ak}^T) \tag{2.18}$$

Note that if only the first k elastic modes are kept for formulation of G_e, and $B_e = B_k$ as defined by Equation (2.13), and

$$\Lambda_{kk}^{-1} = \begin{bmatrix} 1/\omega_1^2 & & \\ & \ddots & \\ & & 1/\omega_k^2 \end{bmatrix} \tag{2.19}$$

it can be seen that the contribution due to the neglected high frequency modes is small, if k is sufficiently large.

It is easy to show that Equation (2.16) yields the flexibility. From Equation (2.12)

$$\Lambda_{ee}^{-1} = [B_e^T k_e B_e]^{-1} = B_e^{-1} k_e^{-1} (B_e^T)^{-1} \tag{2.20}$$

Premultiplying B_e and postmultiplying B_e^T to both sides of the above equation leads to

$$B_e \Lambda_{ee}^{-1} B_e^T = k_e^{-1} = G_e$$

2.4.4 Coordinate Transformation

It can be seen that the physical coordinate u is the result of the generalized coordinates p_k and the action of the generalized interface forces p_a. Thus,

$$u = (B_k \; A_a) \begin{bmatrix} p_k \\ p_a \end{bmatrix} \qquad (2.21a)$$

or in partitioned forms,

$$\begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{bmatrix} B_{ik} & A_{ia} \\ B_{jk} & A_{ja} \end{bmatrix} \begin{bmatrix} p_k \\ p_a \end{bmatrix} \qquad (2.21b)$$

and

$$\begin{bmatrix} u_s \\ u_f \\ u_j \end{bmatrix} = \begin{bmatrix} B_{sk} & A_{sa} \\ B_{fk} & A_{fa} \\ B_{jk} & A_{ja} \end{bmatrix} \begin{bmatrix} p_k \\ p_a \end{bmatrix} \qquad (2.21c)$$

2.5 Equation of Motion of Structural System

2.5.1 Equation of Motion in Terms of Generalized Coordinates

Substitution of the coordinate transformation and premultiplication of the transpose of the transformation matrix given by Equation (2.21) to Equation (2.7) results in the equation of motion in terms of the generalized coordinates,

$$\begin{aligned} \bar{m}_{kk}\ddot{p}_k + \bar{k}_{kk}p_k &= B_{sk}^T f_s + B_{fk}^T f_f + B_{jk}^T p_a \\ \bar{m}_{aa}\ddot{p}_a + \bar{k}_{aa}p_a &= A_{sa}^T f_s + A_{fa}^T f_f + A_{ja}^T p_a \end{aligned} \qquad (2.22)$$

Making use of the orthogonality conditions given by Equations (2.11) and (2.12) and taking into account that A_a is a linear combination of the normal eigenvectors B_k, it can be shown that the generalized mass and stiffness matrices in the above equation can be written in the following forms,

$$\bar{m}_{kk} = I_{kk} \quad , \qquad \bar{m}_{ka} = \bar{m}_{ak}^{T} = 0$$

$$\bar{m}_{aa} = A_a^T m A_a = B_{jk} \Lambda_{kk}^{-1} \Lambda_{kk}^{-1} B_{jk}^{T} \tag{2.23}$$

$$\bar{k}_{kk} = \Lambda_{kk} \quad , \qquad \bar{k}_{ka} = \bar{k}_{ak}^{T} = 0$$

$$\bar{k}_{aa} = A_a^T k A_a = B_{jk} \Lambda_{kk}^{-1} B_{jk}^{T} = A_{ja} \tag{2.24}$$

2.5.2 Equations of Constraints

Two sets of constraints on the interface coordiantes of the two substructures (orbiter and payload) that must be satisfied are

Geometric Compatibility: $u_j^o - u_j^p = 0 \qquad j = 1,2,\ldots,14$ (2.25)

Equilibrium of Interface: $p_a^o + p_a^p = 0 \qquad a = 1,2,\ldots,14$ (2.26)

Rewriting Equation (2.25) in terms of the generalized coordinates given by Equation (2.21) and putting Equations (2.25) and (2.26) into a single matrix equation, one obtains

$$Cp = 0 \tag{2.27}$$

where

$$p = [p_a^o \; p_a^p \quad p_k^o \; p_k^p]^T \tag{2.28}$$

$$C = \left[\begin{array}{cc|cc} A_{ja}^{o} & -A_{ja}^{p} & B_{jk}^{o} & -B_{jk}^{p} \\ I & I & 0 & 0 \end{array}\right] = [c_{aa} \;\vdots\; c_{ak}] \tag{2.29}$$

Let q denote the set of independent generalized coordinates p_k^o and p_k^p and express the set of dependent generalized coordinates p_a^o and p_a^p in terms of q. It gives

$$\begin{bmatrix} p_a^o \\ p_a^p \end{bmatrix} = - c_{aa}^{-1} c_{ak} \, q \qquad q = \begin{bmatrix} p_k^o \\ p_k^p \end{bmatrix} \tag{2.30}$$

where

$$c_{aa}^{-1} = \begin{bmatrix} k_1 & k_1 A_{ja}^{p} \\ -k_1 & I - k_1 A_{ja}^{p} \end{bmatrix} \qquad k_1 = (A_{ja}^{o} + A_{ja}^{p})^{-1} \tag{2.31}$$

2.5.3 Transformation of p to q

The complete set of the generalized coordinates p given by Equation (2.28) can be expressed in terms of the set of independent generalized coordinates q given by Equation (2.30) by the transformation,

$$p = Sq \tag{2.32}$$

where matrix S can be readily obtained by using Equations (2.29) and (2.31) in the form,

$$S = \begin{bmatrix} D_o & D_p \\ -D_o & -D_p \\ I & 0 \\ 0 & I \end{bmatrix} \qquad \begin{aligned} D_o &= -\,k_1 B_{jk}^{o} \\ D_p &= k_1 B_{jk}^{p} \end{aligned} \qquad D = [D_o \; D_p] \tag{2.33}$$

2.6 Equation of Motion of Coupled Structural System

2.6.1 Equation of System in Terms of p

Writing equations for orbiter and payload and putting together according to Equation (2.28), one obtains an equation of motion of the coupled structural system in the form,

$$\bar{m}\ddot{p} + \bar{k}p = P_s + P_f + P_a \tag{2.34}$$

where the elements are given in the following. The mass and stiffness matrices are

$$\bar{m} = \begin{bmatrix} \bar{m}_{aa}^{o} & & & \\ & \bar{m}_{aa}^{p} & & \\ & & \bar{m}_{kk}^{o} & \\ & & & \bar{m}_{kk}^{p} \end{bmatrix} \qquad \bar{k} = \begin{bmatrix} \bar{k}_{aa}^{o} & & & \\ & \bar{k}_{aa}^{p} & & \\ & & \bar{k}_{kk}^{o} & \\ & & & \bar{k}_{kk}^{p} \end{bmatrix} \tag{2.35}$$

The matrix P_s which denotes the external applied load, is

$$P_s = [A_{sa}^{o} \quad 0 \quad B_{sk}^{o} \quad 0]^T f_s^{o} \tag{2.36}$$

and note that the external applied load on the payload is zero. The matrix P_f which represents friction load acting on the joint free coordinates u_f, is

$$P_f = [A_{fa}^{o} \quad -A_{fa}^{p} \quad B_{fk}^{o} \quad -B_{fk}^{p}]^T f_f^{o} \tag{2.37}$$

and note that $f_f^p = -f_f^o$. The term P_a is due to the constraint forces acting on the interface coordiantes u_j. With $p_a^p = -p_a^o$, one has

$$P_a = [A_{ja}^{o} \quad -A_{ja}^{p} \quad B_{jk}^{o} \quad -B_{jk}^{p}]^T p_a^{o} \tag{2.38}$$

2.6.2 Elimination of the Dependent Generalized Coordinates

Substitution of Equation (2.32) into Equation (2.34) and premultiplication of the resulting equation by S^T leads to equation of motion of the coupled structural system in terms of the independent generalized coordinates q in the form,

$$M\ddot{q} + Kq = Q_s f_s^o + Q_f f_f^o \tag{2.39}$$

The elements in the above equation are given in the following. The system mass matrix may be written in the form,

$$M = S^T \bar{m} S = \begin{bmatrix} M_{oo} & M_{op} \\ M_{po} & M_{pp} \end{bmatrix} \tag{2.40}$$

where

$$M_{oo} = I_{kk}^{o} + (B_{jk}^{o})^T m_1 B_{jk}^{o} \quad , \quad M_{op} = M_{po}^{T} = -(B_{jk}^{o})^T m_1 B_{jk}^{p}$$

$$M_{pp} = I_{kk}^{p} + (B_{jk}^{p})^T m_1 B_{jk}^{p} \quad , \quad m_1 = k_1^T(\bar{m}_{aa}^{o} + \bar{m}_{aa}^{p})k_1$$

where

$$k_1 = (A_{ja}^{o} + A_{ja}^{p})^{-1}$$

The system stiffness matrix has the form,

$$K = S^T \bar{K} S = \begin{bmatrix} K_{oo} & K_{op} \\ K_{po} & K_{pp} \end{bmatrix} \tag{2.41}$$

where

$$K_{oo} = \Lambda_{kk}^{o} + (B_{jk}^{o})^T \bar{k}_1 B_{jk}^{o} , \qquad K_{op} = K_{po}^{T} = - (B_{jk}^{o})^T \bar{k}_1 B_{jk}^{p}$$

$$K_{pp} = \Lambda_{kk}^{p} + (B_{jk}^{p})^T \bar{k}_1 B_{jk}^{p} , \qquad \bar{k}_1 = k_1^T(\bar{k}_{aa}^{o} + \bar{k}_{aa}^{p}) k_1$$

The matrix Q_s,

$$Q_s = S^T P_s = \begin{bmatrix} D_o^T (A_{sa}^{o})^T + (B_{sk}^{o})^T \\ D_p^T (A_{sa}^{o})^T \end{bmatrix} \tag{2.42}$$

The matrix Q_f,

$$Q_f = S^T P_f = \begin{bmatrix} D_o^T (A_{fa}^{o} + A_{fa}^{p})^T + (B_{fk}^{o})^T \\ D_p^T (A_{fa}^{o} + A_{fa}^{p})^T - (B_{fk}^{p})^T \end{bmatrix} \tag{2.43}$$

The matrix product $S^T P_a = 0$. With the aid of Equations (2.29) through (2.33), Equation (2.38) can be rewritten in the form,

$$P_a = C^T \begin{bmatrix} Dq \\ 0 \end{bmatrix}$$

Then

$$S^T P_a = (CS)^T \begin{bmatrix} Dq \\ 0 \end{bmatrix}$$

It can be seen from Equations (2.27) and (2.32) that

$$CS = 0 \quad \text{and} \quad S^T P_a = 0 \quad .$$

2.6.3 Equation of Motion of Coupled Structural System

Now, the formulation of equation of motion of the coupled orbiter and payload system, Equation (2.39), is completed. This equation is nonlinear as a result of the friction force being a nonlinear function of the independent variables. Therefore, the usual modal analysis of free vibrations can not be applied. However, one may examine three special cases from Equation (2.39).

Case 1. Free vibration with frictionless joints ($\bar{\mu} = \mu = 0$)

$$M\ddot{q} + Kq = 0 \tag{2.44}$$

Case 2. Forced response with frictionless joints

$$M\ddot{q} + Kq = Q_o f_s^o \tag{2.45}$$

Case 3. Free vibration with frictional joints

$$M\ddot{q} + Kq = Q_f f_f^o \tag{2.46}$$

Both cases 1 and 2 are linear and can be treated by the common approach. The third case is nonlinear and its natural frequencies depend on initial conditions and amplitudes of motion.

2.7 Orbiter Coupled with Two Payload Substructures

Any number of payloads may be coupled with the orbiter at the same time. The modification of the formulation is simple and straightforward. To be specific, an orbiter with two payloads is illustrated in the following.

2.7.1 Physical Coordinates Transformation

The transformation given by Equation (2.21) is expanded as follows.

(1) Coordinates of the orbiter

$$\begin{bmatrix} u_s \\ u_{f1} \\ u_{f2} \\ u_{j1} \\ u_{j2} \end{bmatrix} = \begin{bmatrix} B_{sk} & A_{sa_1} & A_{sa_2} \\ B_{f_1k} & A_{f_1a_1} & A_{f_1a_2} \\ B_{f_2k} & A_{f_2a_1} & A_{f_2a_2} \\ B_{j_1k} & A_{j_1a_1} & A_{j_1a_2} \\ B_{j_2k} & A_{j_2a_1} & A_{j_2a_2} \end{bmatrix} \begin{bmatrix} p_k \\ p_{a_1} \\ p_{a_2} \end{bmatrix} = \begin{bmatrix} B_{sk} & A_{sa} \\ B_{fk} & A_{fa} \\ B_{jk} & A_{ja} \end{bmatrix} \begin{bmatrix} p_k \\ p_a \end{bmatrix} \tag{2.47}$$

where

u_{f_1} = free coordinates of joints of payload 1

u_{f_2} = free coordinates of joints of payload 2

u_{j_1} = interface coordinates of joints of payload 1

u_{j_2} = interface coordinates of joints of payload 2

(2) Coordinates of payloads

$$\begin{bmatrix} u_s^{p_1} \\ u_s^{p_2} \\ u_f^{p_1} \\ u_f^{p_2} \\ u_j^{p_1} \\ u_j^{p_2} \end{bmatrix} = \begin{bmatrix} B_{sk}^{p_1} & 0 & A_{sa}^{p_1} & 0 \\ 0 & B_{sk}^{p_1} & 0 & A_{sa}^{p_2} \\ B_{fk}^{p_1} & 0 & A_{fa}^{p_1} & 0 \\ 0 & B_{fk}^{p_2} & 0 & A_{fa}^{p_2} \\ B_{jk}^{p_1} & 0 & A_{ja}^{p_1} & 0 \\ 0 & B_{jk}^{p_2} & 0 & A_{ja}^{p_2} \end{bmatrix} \begin{bmatrix} p_k^{p_1} \\ p_k^{p_2} \\ p_a^{p_1} \\ p_a^{p_2} \end{bmatrix} \tag{2.48}$$

where superscripts "p_1" and "p_2" denote payloads 1 and 2, respectively.

2.7.2 The Generalized Corodinates

The generalized coordinates defined by Equation (2.28) are increased and arranged in the form,

$$p = [p_{a_1}^{o} \quad p_{a_2}^{o} \quad p_a^{p_1} \quad p_a^{p_2} \mid p_k^{o} \quad p_k^{p_1} \quad p_k^{p_2}]^T \tag{2.49}$$

$$q = [p_k^{o} \quad p_k^{p_1} \quad p_k^{p_2}]^T \tag{2.50}$$

2.7.3 Other Symbols Used in the Formulation

According to Equations (2.47) and (2.48), the following are defined:

$$A_{jk}^{o} = \begin{bmatrix} A_{j_1k}^{o} \\ \\ A_{j_2k}^{o} \end{bmatrix} \qquad A_{jk}^{p} = \begin{bmatrix} A_{jk}^{p_1} & 0 \\ \\ 0 & A_{jk}^{p_2} \end{bmatrix} \tag{2.51}$$

These expressions also apply to A_{ja}, A_{fa}, B_{jk}, B_{ja}, and B_{fa}.

III. OUTLINE FOR NUMERICAL COMPUTATION

3.1 Elements of Formulation

Formulation of equation of motion of the orbiter and payload coupled structural system has been presented in Chapter II. Here, the essential elements for numerical computation will be discussed. Suppose that the finite element models for the orbiter and payload are available from NASTRAN, the first thing one must do is to identify the physical coordinates according to Table 2 and Equation (2.21). Then, do the following:

(1) Make modal analysis of substructures to generate B (2.21)

(2) Form inertia relief attachment modes A (2.18)

(3) Compute the generalized mass and stiffness matrices:
$\bar{m}$ (2.23), $\bar{k}$ (2.24), M (2.40), K (2.41)

(4) Formulate the loading functions: Q_s (2.42), Q_f (2.43)

(5) Program the friction force functions, Section 2.3.

3.2 Friction Force in Terms of the Independent Coordinates

The friction force function derived in Section 2.3 must be expressed in terms of the independent coordinate q for the solution of the equation of motion. The elastic deformation of the joint shaft x_n given by Equation (2.3) may be written by making use of Equations (2.21) and (2.33) in the form,

$$x = u_f^p - u_b^p = (B_{fk}^p p_k^p + A_{fa}^p p_a^p) - (B_{bk}^p p_k^p + A_{ba}^p p_a^p)$$

$$= \begin{bmatrix} 0 & B_{fk}^p - B_{bk}^p \\ & (A_{ba}^p - A_{fa}^p)D \end{bmatrix} q \qquad (3.1)$$

The normal force acting on the joints which causes friction is a function of p_a which is given by Equation (2.33),

$$p_a^o = Dq \qquad (3.2)$$

As shown by Sections 2.2 and 2.3 and the above equations, the friction in the joints is a nonlinear function of q.

3.3 Dimensions of Matrices

In the following the dimensions of each symbol will be given in the parenthesis following the symbol.

3.3.1 Physical Coordinates and Generalized Coordinates

Let s_o and s_p denote the degrees-of-freedom of the interior nodes, and k_o and k_p denote the number of kept normal nodes for the orbiter and payload, respectively. Then,

u_s^o $(s_o x1)$	u_f^o (16x1)	u_j^o (14x1)
u_s^p $(s_p x1)$	u_f^p (16x1)	u_j^p (14x1)
p_k^o $(k_o x1)$	p_a^o (14x1)	
p_k^p $(k_p x1)$	p_a^p (14x1)	
q $[(k_o + k_p)x1]$	p $[(k_o + k_p + 28)x1]$	

3.3.2 Transformation Matrix A and B

Orbiter: $i = s_o + 16$ $j = 16$, $k = k_o$, $f = 16$, $a = 14$

Payload: $i = s_p + 16$ $j = 16$ $k = k_p$, $f = 16$, $a = 14$

3.3.3 Other Matrices

f_s^o $(s_o x1)$	Q_o $[(k_o + k_p)xs_o]$
f_f^o (16x1)	Q_f $[(k_o + k_p)x16]$

VI. CONCLUSIONS

1. The formulation is exact, i.e., there is no assumption or approximation in dealing with the friction force in the joints.

2. Formulation applies to structural systems with any number of payloads coupled with an orbiter.

3. The equation of motion of the coupled orbiter and payload system can be used for the following cases:
   (a) Free vibration of a system with frictionless joints (linear problem)
   (b) Load response of a system with frictionless joints (linear)
   (c) Load response of a system with frictional joints (nonlinear)

4. For free vibration of the system with frictional joints, the natural frequencies depend on initial conditions and amplitudes of vibration. The usual eigenvalue numerical procedure can not apply, since the system is nonlinear.

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