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ALTERNATIVE METHODS FOR CALCULATING SENSITIVITY OF OPTIMIZED DESIGNS TO PROBLEM PARAMETERS

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OUTLINE

Optimum sensitivity is defined as the derivative of the optimum design with respect to some problem parameter, P. The problem parameter is usually fixed during optimization, but may be changed later. Thus, we can use optimum sensitivity to estimate the effect of changes in loads, materials or constraint bounds on the design without expensive re-optimization.

Here, we will discuss the general topic of optimum sensitivity, identify available methods, give examples, and identify the difficulties encountered in calculating this information in nonlinear constrained optimization.

1. NEEDS
2. DEFINITIONS
3. AVAILABLE METHODS
4. EXAMPLES
5. CONCLUSIONS

THE NEED FOR OPTIMUM SENSITIVITY

In many situations, we not only want to find the optimum, but we also want to know how sensitive the optimum is relative to a certain parameter (i.e. how stable the optimum is).

When parameter P changes, optimum sensitivity can be used to estimate the changes in the optimum design variables and objective function without expensive re-optimization.

In multi-level optimization, we need the derivative of the lower level optimum with respect to the upper level design variables.

1. FIND THE CHANGE IN THE OPTIMUM DESIGN DUE TO CHANGES IN LOADS, MATERIALS, OR OTHER DESIGN SPECIFICATIONS
2. AVOID RE-OPTIMIZATION
3. PROVIDE NEEDED INFORMATION FOR MULTI-LEVEL OPTIMIZATION

THE DEFINITION OF OPTIMUM SENSITIVITY

The mathematical definition of optimum sensitivity is given here. What makes this unique from what we usually define as sensitivity analysis is that there is an implied inequality constrained sub-problem. Because of this, it is possible that the optimum sensitivity may not be continuous at $P = P^0$.

OPTIMUM SENSITIVITY

$$DF^*/DP = \lim_{\Delta P \to 0} [F(\underline{X}^* + \underline{\Delta X}^*, P + \Delta P) - F(\underline{X}^*, P)]/\Delta P$$

$$\underline{DX}^*/DP = \lim_{\Delta P \to 0} [\underline{\Delta X}^*/\Delta P]$$

WHERE $F(\underline{X}^* + \underline{\Delta X}, P + \Delta P)$ IS FOUND FROM;

MINIMIZE $F(\underline{X}, P + \Delta P)$

SUBJECT TO;

$$G_J(\underline{X}, P + \Delta P) \leq 0 \qquad J=1,M$$

$$X_I^L \leq X_I \leq X_I^U \qquad I=1,N$$

AVAILABLE METHODS

Several methods have been proposed to estimate the optimum sensitivity of a design with respect to parameter P. Each of these methods contains certain assumptions, and these assumptions can be incorrect in some cases. The methods to be discussed here are listed below.

1. BASED ON THE KUHN-TUCKER NECESSARY CONDITIONS FOR AN OPTIMUM
2. BASED ON THE CONCEPT OF A FEASIBLE DIRECTION
3. BASED ON A LINEAR PROGRAMMING METHOD
4. BASED ON A FULL SECOND-ORDER APPROXIMATION

METHOD 1

The assumption contained in this method is that all of the constraints that are critical at the optimum will remain critical when P changes infinitesimally.

Differentiation of the Kuhn-Tucker conditions gives n equations.

The assumption gives another K equation, where K is the number of critical constraints at the optimum.

This method requires second-order information.

Because of the assumption that all critical constraints remain critical, this method does not recognize the discontinuity which may exist in the optimum sensitivity.

This method gives no assurance that the answer obtained is correct.

METHOD 1: BASED ON THE KUHN-TUCKER CONDITIONS

AT $\underline{X}^*$ $\quad G_J(\underline{X}^*) = 0 \quad J \in K$

$$\underline{\nabla}F(\underline{X}^*) + \sum_{J \in K} \lambda_J \underline{\nabla} G_J(\underline{X}^*) = \underline{0}$$

THIS LEADS TO THE SOLUTION OF THE FOLLOWING SET OF EQUATIONS;

$$\begin{bmatrix} A_{NxN} & B_{NxK} \\ B^T_{KxN} & 0_{KxK} \end{bmatrix} \begin{Bmatrix} S \\ \Delta\lambda \end{Bmatrix} + \begin{Bmatrix} C_{Nx1} \\ D_{Kx1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

WHERE

$$A_{IK} = \frac{\partial^2 F(\underline{X}^*)}{\partial X_I \partial X_K} + \sum_{J \in K} \lambda_J \frac{\partial^2 G_J(\underline{X}^*)}{\partial X_I \partial X_K}$$

WITH SIMILAR EXPRESSIONS FOR B_{IK}, C_I AND D_I

METHOD 2

This method treats the parameter as a new design variable. This enlarges the design space to n+1. The assumption contained in this method is that, in the expanded design space, the maximum improvement or minimum degradation in the design is sought.

This method seeks the constrained steepest descent direction in n+1 space to give DX^*/DP, and from this DF^*/DP is calculated directly.

This method requires only first-order sensitivity information.

This method accounts for possible discontinuity of the total derivative.

As with the first method, there is no assurance that the result obtained is correct.

LINEAR METHOD BASED ON FEASIBLE DIRECTIONS

LINEAR APPROXIMATION: LET $X_{N+1} = P$

MINIMIZE $F(\underline{X}) = F(\underline{X}^*) + \underline{\nabla}F(\underline{X}^*)\cdot\underline{S}$

SUBJECT TO;

$$G_J(\underline{X}^*) + \underline{\nabla}G_J(\underline{X}^*)\cdot\underline{S} \leq 0 \qquad J \varepsilon K$$

$$\underline{S} \text{ BOUNDED}$$

WHERE $S_I = X_I - X_I^*$ IS EQUIVALENT TO $S_I = \partial X_I/\partial P$

EQUIVALENT PROBLEM:

MINIMIZE $\underline{\nabla}F(\underline{X}^*)\cdot\underline{S}$

SUBJECT TO;

$$\underline{\nabla}G_J(\underline{X}^*)\cdot\underline{S} \leq 0 \qquad J \varepsilon K$$

$$\underline{S}\cdot\underline{S} \leq 1$$

METHOD 2B

Two extended forms are available to deal with the possible discontinuity of the optimum sensitivity. This is necessary because the direction in which P is changed will determine the value of the sensitivity. If the value is different, depending on the sign of delta-P, then this indicates that the design will follow one subset of constraints if P is increased but a different set if P is decreased.

This method for dealing with the potential discontinuities of the optimum sensitivity is somewhat dependent on the choice of the parameter C. Numerical difficulties can be encountered in deciding the correct value of C.

DEALING WITH DISCONTINUITY DEPENDENT ON THE SIGN OF P

$\Delta P \geq 0$

MINIMIZE $\quad \underline{\nabla F}(\underline{X}^*) \cdot \underline{S} - C \cdot S_{N+1}$

SUBJECT TO;

$$\underline{\nabla G}_J(\underline{X}^*) \cdot \underline{S} \leq 0 \qquad J \in K$$

$$\underline{S} \cdot \underline{S} \leq 1$$

$\Delta P \leq 0$

MINIMIZE $\quad \underline{\nabla F}(\underline{X}^*) \cdot \underline{S} + C \cdot S_{N+1}$

SUBJECT TO;

$$\underline{\nabla G}_J(\underline{X}^*) \cdot \underline{S} \leq 0 \qquad J \in K$$

$$\underline{S} \cdot \underline{S} \leq 1$$

METHOD 3

Here, we create a Taylor series expansion for the objective function and the critical constraints. Taking the limit as delta-P goes to zero and keeping the lowest order terms only produces the optimum sensitivity according to the original definition. This process requires that we pay close attention to whether delta-P approaches zero from the positive or negative side.

This method requires solving two resultant LP problems.

This method requires first-order sensitivities only.

If the number of the constraints is less than the number of design variables, the LP problems do not have a unique solution.

If a unique solution exists, it is always the correct solution.

LINEAR PROGRAMMING APPROACH

USING THE DEFINITION OF OPTIMUM SENSITIVITY;

MINIMIZE $\underline{\nabla}F(\underline{X}^*)\cdot\underline{\Delta X} + \partial F(\underline{X}^*)/\partial P\cdot\Delta P + O(\underline{\Delta X},\Delta P)$

SUBJECT TO;

$$G_J(\underline{X}^*,P) + \underline{\nabla}G_J(\underline{X}^*)\cdot\underline{\Delta X} + \partial G_J/\partial P\cdot\Delta P + O(\underline{\Delta X},\Delta P) \leq 0 \qquad J=1,M$$

KEEP THE LOWEST ORDER TERMS WHEN $\Delta P \rightarrow 0$. THIS LEADS TO;

IF $\Delta P \rightarrow +0$ $(\Delta P > 0)$:

MINIMIZE $\underline{\nabla}F(\underline{X}^*)\cdot\underline{\Delta X}/DP + \partial F(\underline{X}^*)/\partial P$

SUBJECT TO;

$$\underline{\nabla}G_J(\underline{X}^*)\cdot\underline{\Delta X}/DP + \partial G_J/\partial P \leq 0 \qquad J \in K$$

IF $\Delta P \rightarrow -0$ $(\Delta P < 0)$:

MINIMIZE $\underline{\nabla}F(\underline{X}^*)\cdot\underline{\Delta X}/DP - \partial F(\underline{X}^*)/\partial P$

SUBJECT TO;

$$\underline{\nabla}G_J(\underline{X}^*)\cdot\underline{\Delta X}/DP - \partial G_J/\partial P \leq 0 \qquad J \in K$$

METHOD 4

Just as with method 1, this method requires second derivatives. However, here the second-order information is used directly as an approximate optimization task.

The parameter P may be treated as an independent design variable, or the change in P may be specified.

If a small change in P is specified, the method becomes a finite difference method. When delta-P goes to zero, this method gives the exact answer to a second order approximation.

The set, K, of retained constraints can include all critical and near critical constraints, or even the entire set of constraints. Therefore, as P is changed, a totally new set of constraints can become critical.

Within the limits of numerical precision, this method will always give the correct solution. The disadvantage is that this problem has a quadratic objective and constraints and so must be solved by nonlinear programming. It is, however, quite efficient since it is an explicit problem.

If an attempt is made to simplify this method by linearizing it, the result is the set of two LP problems given in method 3.

FULL SECOND-ORDER APPROXIMATION

SOLVE THE FOLLOWING EXPLICIT APPROXIMATE PROBLEM:

FIND THE CHANGE $\underline{S}$ THAT WILL

MINIMIZE $F(\underline{X}^*,P) + \underline{\nabla}F(\underline{X}^*,P)\cdot\underline{S} + 0.5\underline{S}^T H_F \underline{S}$

SUBJECT TO;

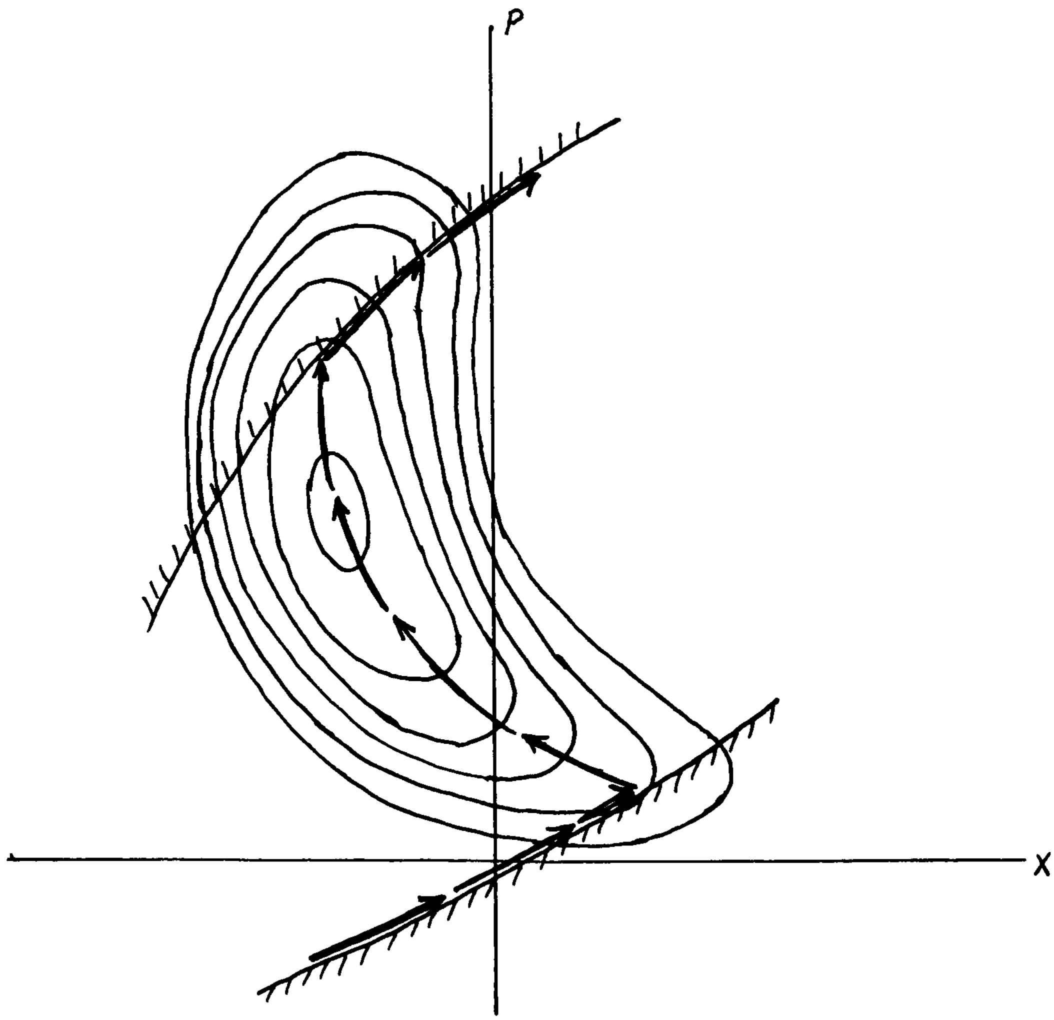
$$G_J(\underline{X}^*,P) + \underline{\nabla}G_J(\underline{X}^*,P)\cdot\underline{S} + 0.5\underline{S}^T H_J \underline{S} \leq 0 \quad J \in K$$

$\underline{S}$ BOUNDED

AN OPTIMUM CURVE

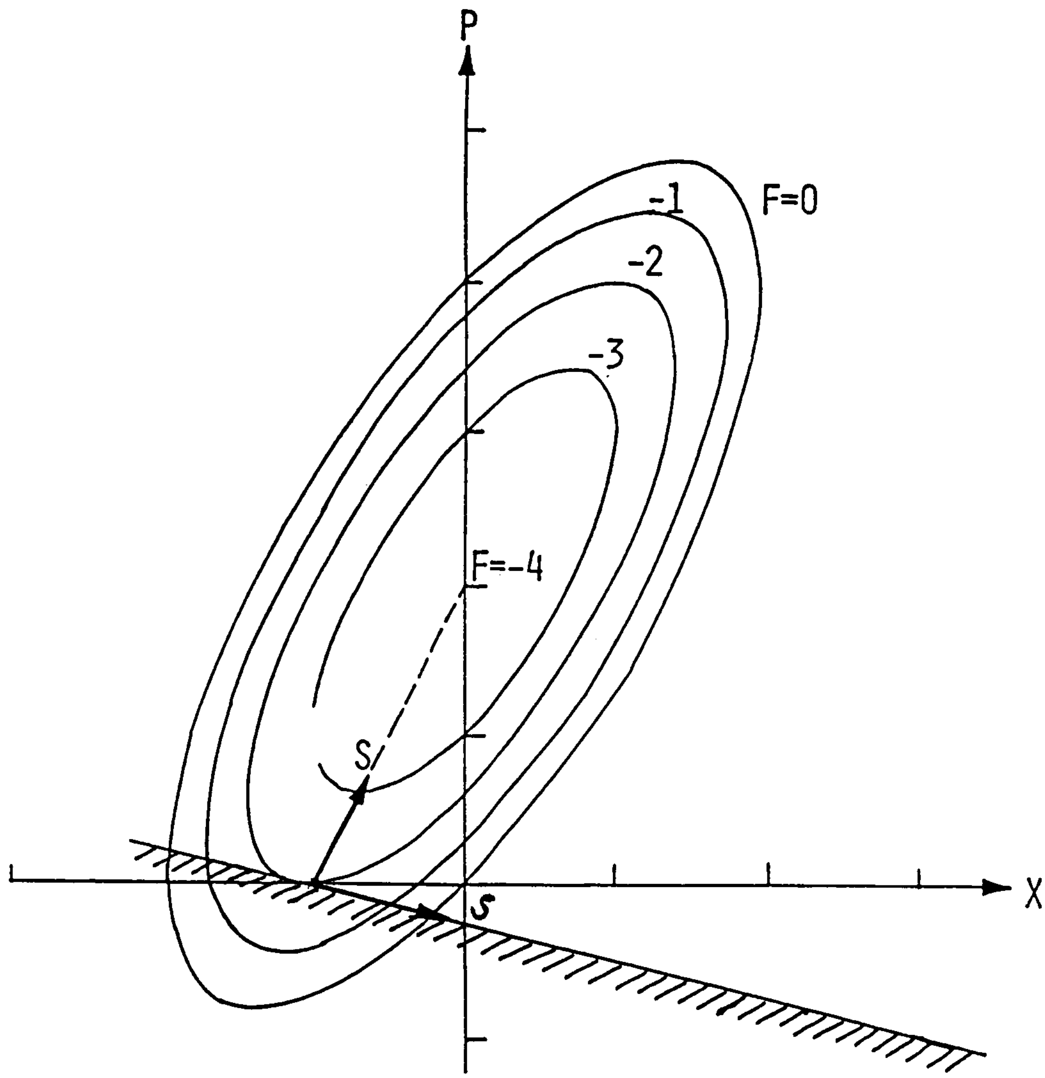
When P changes, the optimum points X^* form a curve in n+1 space. The optimum sensitivity DX^*/DP is represented by the tangent of this curve. The curve can be nonsmooth, so DX^*/DP can be discontinuous.

An infinitesimal change in P may cause the curve to leave a currently critical constraint. This demonstrates the potentially discontinuous nature of optimum sensitivity.



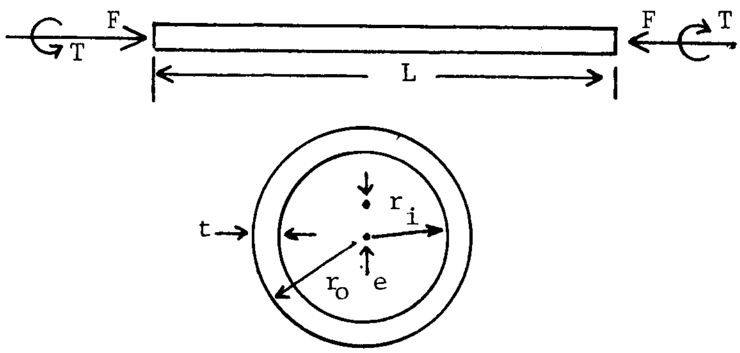
DISCONTINUOUS DERIVATIVES

This is another graphical example of the discontinuous derivative problem. In this case, the constrained optimum is found for P=0 to lie on the constraint boundary. Now if P is increased, the optimum sensitivity will point to the unconstrained minimum. On the other hand, if P is decreased, the optimum sensitivity follows the constraint. Since the total derivative is the scalar product of the gradient of the objective function with the vector $\underline{S}$, it is clear that the optimum sensitivity is not continuous at $\underline{X}^*$.



ROTATING SHAFT OPTIMIZATION

This example demonstrates the usefulness of optimum sensitivity as an engineering approach to frequency domain avoidance. Assuming it is required that the rotating shaft not vibrate in the domain between 2.8 and 3.5 Hz, the shaft is first optimized with respect to all other constraints. Then the sensitivity with respect to the fundamental frequency is calculated and a new optimum design is projected with a frequency below 2.8 Hz and with a frequency above 3.5 Hz. From this it appears that it is far more economical to drive the frequency up than to drive it down. However, this was not known in advance and so it was not known whether the frequency should be bounded from above or below. Thus, optimum sensitivity provides one means of dealing with a problem in which the design space is disjoint.



OBJECTIVE: MINIMUM WEIGHT. CONSTRAINTS: STRESS, DISPLACEMENT, EULER BUCKLING, SHELL BUCKLING. PARAMETER P: THE FIRST NATURAL FREQUENCY.

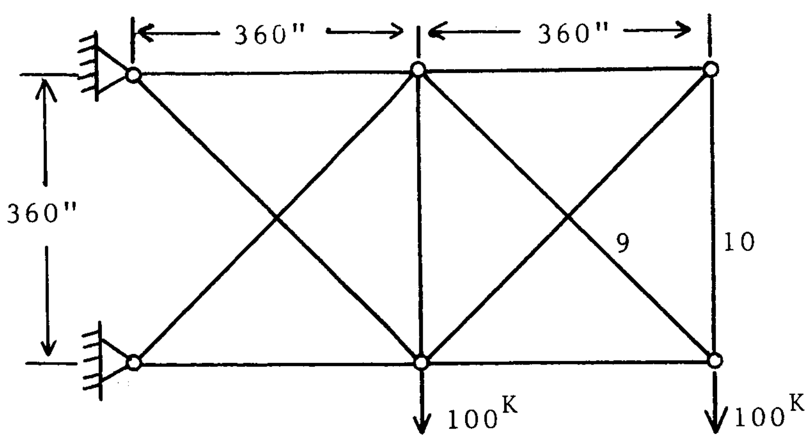
THE OPTIMUM WITHOUT ANY FREQUENCY CONSTRAINTS: $\omega_1 = 3.1$, $W^* = 27,242$

OPTIMUM SENSITIVITY

		METHOD 2	METHOD 3	RE-OPTIMIZE
$\omega_1 \leq 2.8$	W^*	+5,278	+5,278	+6,429
	X_1	-1.65	-1.65	-1.72
	X_2	+0.33	+0.33	+0.44
$\omega_1 \geq 3.5$	W^*	+417	+417	+169
	X_1	+0.17	+0.17	+0.17
	X_2	-0.003	-0.003	-0.011

10-BAR TRUSS

Here the common 10-bar truss was optimized and the sensitivity was calculated with respect to the allowable stress in member 9. It is known that the weight of this structure can be reduced by increasing this allowable stress to a value of 37.5 ksi, but beyond that, no weight reduction is possible. At the initial optimum, member 10 was at its lower bound. Method 1 assumed, incorrectly, that it would stay there, while method 2 recognized that this member dimension should be increased. Using method 3, the allowable stress in member 9 was allowed to change as an independent variable and this method projected that the optimum allowable stress is 38.2 ksi, quite close to the actual value of 37.5 ksi. The case at the bottom of the figure is for optimization at the 37.5 ksi value and shows the discontinuity of the optimum sensitivity.



OBJECTIVE: MINIMUM WEIGHT. CONSTRAINTS: STRESS, MINIMUM GAGE.
PARAMETER P: STRESS LIMIT IN MEMBER 9. INITIAL OPTIMUM σ_9 = 30 KSI, W^* = 1545

SENSITIVITY

CASE 1:

PARAMETER	METHOD 1	METHOD 2	METHOD 3
$DF^*/D\sigma_9$	-240.5	-238.4	-178.6
S_{10}	0.00	0.17	0.16
σ_9	---	---	38.2

CASE 2: σ_9 =37.5 KSI, W^* = 1498, METHOD 2.

$\Delta\sigma_9$ 0 $DF^*/D\sigma_9$ = 215.7 $W^*(35)$ = 1512 CALCULATED $W^*(35)$ = 1511

$\Delta\sigma_9$ 0 $DF^*/D\sigma_9$ = 0.00

CONCLUSIONS

Optimum sensitivity in linear programming is a common and widely used tool. Research in optimum sensitivity for nonlinear problems has not been this successful and it has been shown here that none of the methods is completely satisfactory. Methods 1-3 often do not provide the correct answer, while method 4 requires second-order information that may be costly to obtain, as well as the nonlinear optimization of the approximating functions.

The reasons for these difficulties are now beginning to be understood. If the optimum design is fully constrained and unique, the optimum sensitivity can be reliably calculated, just as in linear programming. However, if the design is not fully constrained (fewer active constraints than design variables), the optimum sensitivity using first-order information will not be unique and second-order information is essential. Unfortunately, this is the usual case in engineering design. The reason that first-order information is inadequate is that the higher order terms cannot be ignored as delta-P goes to zero in the limit.

The need to calculate the optimum sensitivity is a clear one and often justifies considerable effort. It is this information that is needed to make many fundamental design decisions. Therefore, improved understanding of these concepts is useful in the search to extract the maximum information from the optimization process.

1. IN GENERAL, THERE ARE SITUATIONS WHERE NONE OF THE AVAILABLE METHODS EXCEPT THE FULL SECOND-ORDER APPROXIMATION WILL GIVE THE CORRECT ANSWER

2. ITERATIVE METHODS USING FIRST-AND SECOND-ORDER INFORMATION SHOULD BE INVESTIGATED

3. IF SECOND-ORDER INFORMATION IS AVAILABLE, METHOD 4 WILL PROVIDE USEFUL ENGINEERING WHICH ACCOUNTS FOR NEARBY CONSTRAINTS THAT MAY BECOME CRITICAL WHEN PARAMETER P IS CHANGED

4. FURTHER RESEARCH IS NEEDED; THE USEFULNESS OF OPTIMUM SENSITIVITY HAS BEEN CLEARLY SHOWN IN PAST WORK