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Philip Hall

William D. Lakin

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INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

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Langley Research Center
Hampton, Virginia 23665

# The fully nonlinear development of Görtler vortices in growing boundary layers

by Philip Hall, Mathematics Department, University of Exeter
and William D. Lakin, Mathematics Department, Old Dominion University.

## Abstract

The fully nonlinear development of small wavelength Gõrtler vortices in a growing boundary layer is investigated using a combination of asymptotic and numerical methods. The starting point for the analysis is the weakly nonlinear theory of Hall (1982b) who discussed the initial development of small amplitude vortices in a neighbourhood of the location where they first become linearly unstable. That development is unusual in the context of nonlinear stability theory in that it is not described by the Stuart-Watson approach. In fact the development is governed by a pair of coupled nonlinear partial differential evolution equations for the vortex flow and the mean flow correction. Here the further development of this interaction is considered for vortices so large that the mean flow correction driven by them is as large as the basic state. Surprisingly it is found that such a nonlinear interaction can still be described by asymptotic means. It is shown that the vortices spread out across the boundary layer and effectively drive the boundary layer. In fact the system obtained by writing down the equations for the fundamental component of the vortex generate a differential equation for the basic state. Thus the mean flow adjusts so as to make these large amplitude vortices locally neutral. Moreover in the region where the vortices exist the mean flow has a 'square-root' profile and the vortex velocity field can be written down in closed form. The upper and lower boundaries of the region of vortex activity are determined by a free-boundary problem involving the boundary layer equations. In general it is found that this region untimately includes almost all of the original boundary layer and much of the free-stream. In this situation the mean flow has essentially no relationship to the flow which exists in the absence of the vortices.

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## Introduction

Our concern is with large amplitude Görtler vortices in viscous incompressible flows over walls of variable curvature. Much of the recent interest in the Görtler instability has been motivated by practical problems such as Laminar Flow Control or the flow over turbine blades. A particular cause for concern in Laminar Flow Control is the question of whether Görtler vortices are likely to induce premature transition because of their effect on the receptivity of the original boundary layer to Tollmien-Schlichting waves or crossflow vortices. As yet little progress has been made with the latter problem but the corresponding internal fully developed flow problem has been discussed by Hall and Bennett (1986), Bennett and Hall, (1987) and Hall and Smith (1987). Thus, for example, Bennett and Hall (1987) showed that fully nonlinear Taylor vortices in a curved channel can have a massive influence on the growth of lower branch Tollmien-Schlichting waves. The absence of a fully nonlinear description of Görtler vortices in external flows means that a similar interaction problem cannot yet be attempted for these flows. Here we shall show how the required large amplitude vortex flows can be described asymptotically. However it is perhaps useful if we first describe earlier work on the Görtler problem.

The original work of Görtler (1940) and the later work of, for example, Hammerlin (1955, 1956), Smith (1955) and Floryan and Saric (1979) did not take care of boundary layer growth in a selfconsistent manner. Unlike the corresponding Tollmien-Schlichting instability problem, where the fact that instability occurs at relatively high Reynolds numbers renders parallel flow calculations reasonably accurate, the neglect of non-parallel effects in the Görtler problem leads to inconsistent and, in some cases, physically absurd results. Thus as extreme examples some of the results predicted by these calculations showed instability at zero Görtler number or zero wavenumber.

In fact at high wavenumbers the above theories gave consistent results and Hall (1982a) showed that in this regime an asymptotic solution of the non-parallel problem is possible. The vortices are found to become linearly unstable at a particular downstream location and to concentrate themselves at some depth in the boundary layer. This depth corresponds to the position where Rayleigh's inviscid instability criterion for the boundary layer is most violated. At $O(1)$ wavenumbers no asymptotic or self-consistent parallel flow calculation is possible and the linear instability partial differential equations must be solved numerically as was done by Hall (1983). These equations are parabolic in the downstream variable and so can be solved by a marching procedure and the neutral location for a disturbance inserted at a given position can be calculated. Hall (1983) showed that the neutral curve produced by such a calculation depends on the nature of the initial disturbance. However each such neutral curve has none of the obvious anomalies of the parallel flow neutral curves and at high wavenumbers the different neutral curves merge into the asymptotic result.

The initial nonlinear development of Görtler vortices of small wavelength was discussed by Hall (1982b). It was found that the nonlinear interaction is dominated by a 'mean-field' type theory in which the fundamental mode and mean flow correction reinforce each other. The higher harmonics play no role in this interaction so the Stuart-Watson approach is not applicable. In order to motivate the asymptotics to be used for much larger vortices it is instructive to summarize the essential details of the calculation of Hall (1982b)

Suppose then that a vortex of nondimensional wavenumber $\epsilon^{-1}$ is locally neutrally stable at the downstream location $x = \bar{x}$ of a two-dimensional boundary layer flow $(\bar{u}, \bar{v})$. If the most unstable part of the boundary layer is at a depth $y = \bar{y}$ it is necessary to define variables $X$ and $\varsigma$ by

$$X = \frac{(x - \bar{x})}{\epsilon}, \qquad \varsigma = \frac{y - \bar{y}}{\epsilon^{\frac{1}{2}}} \tag{1.1a, b}$$

and the total downstream velocity component expands in the neighbourhood of $(\bar{x}, \bar{y})$ as

$$\begin{aligned} u = \bar{u}(\bar{x}, \bar{y}) &+ \epsilon X \bar{u}_x(\bar{x}, \bar{y}) + \epsilon^{\frac{1}{2}} \varsigma \bar{u}_y(\bar{x}, \bar{y}) + \cdots + \epsilon^{\frac{3}{2}} u_M(X, \varsigma) \\ &+ \epsilon^{\frac{3}{2}} [U_1(X, \varsigma) \exp(\frac{iz}{\epsilon}) + C.C.] + \cdots . \end{aligned} \tag{1.2}$$

For a boundary layer which has a local Görtler number increasing faster than the fourth power of the local wavenumber the evolution equation for $u_M$, the mean flow correction, and the fundamental $U_1$ take the form

$$\left( \frac{\partial^2}{\partial \varsigma^2} - \frac{\partial}{\partial X} \right) u_M = \frac{\partial}{\partial \varsigma} \mid U_1 \mid^2,$$

$$\left\{ \frac{\partial^2}{\partial \varsigma^2} - \frac{2}{3} \frac{\partial}{\partial X} - \frac{1}{4} \varsigma^2 + X \right\} U_1 = 2 U_1 \frac{\partial u_M}{\partial \varsigma}. \tag{1.3a, b}$$

The linearized form of (1.3) shows that $U_1$ can be expressed in terms of parabolic cylinder functions. Otherwise (1.3) must be solved numerically by a marching procedure. However at large values of $X$ it was shown by Hall (1982b) that $u_M$ and $U_1$ develop a surprisingly simple asymptotic structure. In fact $u_M$ and $U_1$ can be written down explicitly in terms of $X$ and a similarly variable $\xi$ in a finite interval $\xi_1 < \xi < \xi_2$. Near $\xi_1, \xi_2$ the fundamental satisfies a nonlinear Airy equation and is reduced to zero exponentially whilst $u_M$ persists above $\xi_2$ and below $\xi_1$. Of crucial importance is the fact that when this structure develops $u_M \sim X^{3/2}, U_1 \sim X^{1/2}$ so that the total mean flow correction $\epsilon^{3/2} u_M$ is comparable with the basic state for $X \sim 1/\epsilon$. Thus if $x - \bar{x}$ is $O(1)$ the vortices will drive a mean flow correction as big as the original mean state.

The remarkable feature of the latter regime is that the nonlinear Görtler equations can still be solved asymptotically in this region. Thus we are able to describe asymptotically a large amplitude disturbance capable of altering at zeroth order the basic state. The structure of the large $X$ solution of Hall (1982b) continues to describe the flow but with the major change that the depth of the fluid where vortex activity persists is now $O(1)$. In this layer the mean flow is driven by the vortices and indeed is determined as a solvability condition on the equations for the fundamental there. The downstream mean velocity component in this layer then has a simple square root profile and the mean equations drive a finite amplitude vortex. Thus there is an exact reversal of the usual roles of the equations for the mean and fundamental obtained by Fourier- analysing the spanwise dependence of the disturbance. The situation is not unlike the scenario postulated some years ago for turbulent flows by Malkus (1956). We recall that Malkus argued that the 'mean' part of a turbulent flow would organize itself so that any 'modes' were marginally stable. We shall refer to the layer where the vortex activity is concentrated as $I$. This layer is bounded by 'transition' layers of depth $\epsilon^{2/3}$ where the vortices are again reduced to zero as solutions of a nonlinear Airy equation. We refer to these layers as $IIa, b$ respectively and the remainder of the flow is denoted by regions $IIIa, b$ as shown in Figure 1.

In regions $IIIa, b$ there is no spanwise dependence for the flow and the velocity field satisfies the boundary layer equations. The solutions of these equations must be matched with those emanating from $IIa, b$ whose positions are unknown functions of the downstream variables. Thus a complete description of the flow requires the numerical solution of a free-boundary partial differential system.

The region $I$ of vortex activity in general grows as the flow moves downstream until it eventually occupies almost all of the extent of the boundary layer which would exist in the absence of the vortices. In fact the only part free of vortices is a thin layer at the wall and the vortices even take over part of the free stream. At this stage the flow has effectively no relationship with the flow which would exist in the absence of the vortices. The procedure adopted in the rest of this paper is as follows: in §2 the nonlinear Görtler vortex equations are derived. In §3 an asymptotic solution of these equations for large amplitude small wavelength vortices is given. In $4 an asymptotic solution of the initial and ultimate downstream stages of the free-boundary problem obtained in §3 is discussed. In §5 a numerical scheme which we have used to solve this free boundary problem is described. Finally in §6 we discuss our results and draw some conclusions.

## 2. Formulation of the nonlinear Görtler equations.

We consider the flow of a viscous fluid of kinematic viscosity $\nu$, density $\rho$, over a wall of variable concave curvature $a^{-1}\chi(X/L)$. Here $X$ denotes distance along the wall, $a$ is a typical radius of curvature and $L$ is a typical length scale along the wall. If $U_0$ is a typical flow velocity we define a Reynolds number $R_E$ by

$$R_E = \frac{U_0 L}{\nu}, \tag{2.1}$$

and a curvature parameter $\delta$ by

$$\delta = \frac{L}{a}. \tag{2.2}$$

We confine our attention to the limit $R_E \to \infty$ with the Görtler number $G$ defined by

$$G = 2R_E^{\frac{1}{2}}\delta \tag{2.3}$$

held fixed. We take $(X, Y, Z)$ to be co-ordinates along the wall, normal to the wall and in the spanwise direction respectively. If $(U, V, W)$ denotes the corresponding velocity vector we define dimensionless co-ordinates $(x, y, z)$ and velocity $(u, v, w)$ by

$$(x, y, z) = L^{-1}(X, YR_E^{\frac{1}{2}}, ZR_E^{\frac{1}{2}})$$

and

$$(U, V, W,) = U_0(u, vR_E^{-1/2}, wR_E^{-1/2}).$$

We restrict our analysis to flows with $u \to 1, y \to \infty$ so that the pressure $P$ can be written in the form

$$P = \rho \frac{U_0^2}{R_E} p \tag{2.4}$$

The Navier-Stokes equations for the flow can be written in the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$\nabla u = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},$$

$$\nabla v - \frac{1}{2}G\chi u^2 - \frac{\partial p}{\partial y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z},$$

$$\nabla w - \frac{\partial p}{\partial z} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}, \tag{2.5a, b, c, d}$$

where $\nabla \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and terms of relative order $R_E^{-1/2}$ have been neglected. We further note that we have assumed that the flow is steady.

In the absence of any Görtler vortex we can write $(u, v, w) = (\bar{u}, \bar{v}, 0)$ in which case $\bar{u}, \bar{v}$ satisfy

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \frac{\partial^2 \bar{u}}{\partial y^2},$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0, \tag{2.6}$$

$$\bar{u} = \bar{v} = 0, y = 0, \qquad \bar{u} \to 1, \qquad v \to \infty, \qquad \bar{u} = 1, x = 0, y > 0.$$

Thus $(\bar{u}\ \bar{v}\ 0)$ is just a Blasius boundary layer until a Görtler vortex begins to grow at some downstream location

Finally in this section we note that our analysis can easily be modified to take care of pressure gradient driven flows. In that case $P$ in (2.4) must be altered so as to contain an $x$ dependent component of size $\rho U_0^2$ and the boundary conditions in (2.6) must be modified.

We shall obtain a solution of (2.5) which satisfies

$$u = v = w = 0, \qquad y = 0.$$

$$u \to 1, y \to \infty. \tag{2.7a, b}$$

In fact we shall see that when $y \to \infty, v \to v(x), w \to 0$; in general $v(x)$ is not the $y$ velocity component at infinity of a Blasius boundary layer so the higher order outer 'inviscid' problem associated with (2.5) is modified by the instability.

## 3. The evolution equations for large amplitude Görtler vortices

We now develop an asymptotic solution of (2.5) valid in the limit of small vortex wavelength. We suppose that the boundary layer $(\bar{u}(x,y), \bar{v}(x,y))$ becomes linearly unstable to Görtler vortices at $x = x^*$. Furthermore we assume that the curvature function $\chi(x)$ is such that the flow becomes more unstable with increasing $x$. Thus if the boundary layer is a Blasius boundary layer we require that $\chi$ increases more rapidly than $x^{\frac{1}{2}}$. This restriction is discussed by Hall (1983) and is a direct consequence of the scaling of the right hand branch of the neutral curve for Görtler vortices.

The discussion of §2 suggests that beyond $x = x^*$ the flow will support fully nonlinear Görtler vortices. The flowfield is therefore split up as shown in Figure 1. The vortex activity is therefore confined between $y_1(x)$ and $y_2(x)$. The layers denoted by regions $IIa, b$ are required in order to smooth out the algebraically decaying vortices in region $I$. Later we will see that these layers must be of thickness $\epsilon^{2/3}$ in the limit $\epsilon \to 0$. We expand the Görtler number $G$ in the form

$$G = G_0\epsilon^{-4} + G_1\epsilon^{-3} + \cdots, \tag{3.1}$$

whilst in region $I$ the appropriate expansions of $u, v, w$ and $p$ are

$$\begin{aligned}
u &= \bar{u}_0 + \epsilon\bar{u}_1 + \cdots + \{\epsilon E(U_0^1 + \epsilon U_1^1 + \cdots) + \epsilon^2 E^2(U_0^2 + \epsilon U_1^2 + \cdots)\cdots + C.C.\} \\
v &= \bar{v}_0 + \epsilon\bar{v}_1 + \cdots + \{\bar{\epsilon}^1 E(V_0^1 + \epsilon V_1^1 + \cdots) + \epsilon^0 E^2(V_0^2 + \epsilon V_1^2 + \cdots)\cdots + C.C.\} \\
w &= E(W_0^1 + \epsilon W_1^1 + \cdots) + \epsilon E^2(W_0^2 + \epsilon W_1^2 + \cdots)\cdots + C.C. \\
p &= \bar{p}_0 + \epsilon\bar{p}_1 + \cdots + \{\bar{\epsilon}^1 E(P_0^1 + \epsilon P_1^1 + \cdots) + \epsilon^0 E^2(V_0^2 + \epsilon V_1^2 + \cdots)\cdots + C.C.\}
\end{aligned} \tag{3.2a, b, c, d}$$

Here 'C.C.' denotes 'complex conjugate' whilst $E = \exp(iz/\epsilon)$. In the absence of a vortex the flow reduces to the basic state $w = p = 0, u = \bar{u}, v = \bar{v}$. We note that the only $z$ dependence in (3.2) is through $E$ so that $\bar{u}_0$,$\bar{v}_0$, etc. are functions of $x$ and $y$ only.

The expansions (3.1), (3.2) are then substituted into (2.5) and like powers of $\epsilon$ for each Fourier component are equated. The mean flow $(\bar{u}_0, \bar{v}_0)$ is determined from the zeroth order continuity and $x$ momentum equations. The pressure field $\bar{p}_0$ is then found from the normal momentum equation. The equations to determine $\bar{u}_0$ and $\bar{v}_0$ are

$$\frac{\partial \bar{u}_0}{\partial x} + \frac{\partial \bar{v}_0}{\partial y} = 0,$$

$$\bar{u}_0\frac{\partial \bar{u}_0}{\partial x} + \bar{v}_0\frac{\partial \bar{u}_0}{\partial y} - \frac{\partial^2 \bar{u}_0}{\partial y^2} = V_0^1\frac{\partial \tilde{U}_0^1}{\partial y} + \tilde{V}_0^1\frac{\partial U_0^1}{\partial y} - W_0 i\tilde{U}_0^1 + \tilde{W}_0^1 iU_0^1. \tag{3.3a, b}$$

Thus the boundary layer equations are now driven by the nonlinear interaction of the Görtler vortex with itself. The zeroth order equations for the fundamental terms in the core are

$$\frac{\partial V_0^1}{\partial y} + iW_0^1 = 0,$$
$$U_0^1 + V_0^1 \frac{\partial \bar{u}_0}{\partial y} = 0, \qquad (3.4a,b,c,d)$$
$$V_0^1 + G_0 \chi U_0^1 \bar{u}_0 = 0,$$
$$-iP_0^1 = W_0^1.$$

The consistency of equations (3.4b,c) requires that throughout the core

$$G_0 \chi \bar{u}_0 \frac{\partial \bar{u}_0}{\partial y} = 1, \qquad (3.5)$$

and (3.3) then becomes

$$\frac{\partial \bar{u}_0}{\partial x} + \frac{\partial \bar{v}_0}{\partial y} = 0,$$
$$\bar{u}_0 \frac{\partial \bar{u}_0}{\partial x} + \bar{V}_0 \frac{\partial \bar{u}_0}{\partial y} - \frac{\partial^2 u_0}{\partial y^2} = -2\frac{\partial}{\partial y}\{\frac{\partial \bar{u}_0}{\partial y} \mid V_0^1 \mid^2\}. \qquad (3.6a,b)$$

Thus (3.5) determines the mean flow streamwise velocity component in the core whilst (3.6b) then determines $\mid V_0^1 \mid^2$. It follows that in the core the boundary layer flow is now being forced by the vortex which from (3.6b) is itself driven by the boundary layer. This is, of course, the exact reverse of the roles played by these equations in the linear or weakly nonlinear descriptions of this problem. In fact the interaction described by (3.5) and (3.6) is not unlike that postulated some years ago by Malkus (1956) for turbulent flows. In the latter theory it was argued that the 'mean flow' for a turbulent flow organizes itself so as to make any possible instability marginally stable. In the large amplitude Görtler problem this mean flow adjustment is achieved by centrifugal effects and is described by (3.5). We can integrate (3.5), (3.6) to give

$$\bar{u}_0 = \frac{\sqrt{a(x)+2y}}{\sqrt{G_0 \chi}}, \qquad (3.7a,b)$$

$$\bar{v}_0 = \frac{-a'(x)\sqrt{a(x)+2y}}{2\sqrt{G_0 \chi}} + \frac{\{a(x)+2y\}^{\frac{3}{2}}\chi'}{6\sqrt{G_0 \chi}\chi} - b(x)$$

where $a(x)$ and $b(x)$ are arbitrary functions of $x$ and a dash denotes a derivative with respect to $x$. The function $\mid V_0^1 \mid^2$ is found by integrating (3.6) to give

$$B(x) + 2\frac{\partial \bar{u}_0}{\partial y} \mid V_0^1 \mid^2 = \frac{b\sqrt{a+2y}}{\sqrt{G_0 \chi}} + \frac{1}{12}\frac{(a+2y)^2 \chi'}{G_0 \chi^2} + \frac{1}{\sqrt{a+2y}\sqrt{G_0 \chi}}, \qquad (3.8)$$

where $B(x)$ is another function of $x$ to be determined. The function $| V_0^1 |^2$ cannot be negative so $y_1$ and $y_2$ which determine the edges of $I$ satisfy (3.8) with $V_0^1 = 0$. If $B(x)$ is then eliminated we obtain

$$\frac{b\sqrt{a+2y_1}}{\sqrt{G_0\chi}} + \frac{1}{12}\frac{(a+2y_1)^2\chi'}{G_0\chi^2} + \frac{1}{\sqrt{a+2y_1}\sqrt{G_0\chi}} = \frac{b\sqrt{a+2y_2}}{\sqrt{G_0\chi}} + \frac{1}{12}\frac{(a+2y_2)^2\chi'}{G_0\chi^2} + \frac{1}{\sqrt{a+2y_2}\sqrt{G_0\chi}}. \tag{3.9}$$

The above equation is not of course sufficient to determine $a, b, y_1$ and $y_2$ so we are not yet able to determine the location of the layers $IIa, b$. The thickness of these layers is determined by a balance between diffusion across the layers and convection in the streamwise direction. This balance shows that the layers are of thickness $\epsilon^{2/3}$ so that in $IIa$ we write

$$\xi = \frac{\{y - y_2\}}{\epsilon^{2/3}}.$$

Hence in $IIa$ we replace $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ by $\frac{\partial}{\partial x} - \frac{y_2'}{\epsilon^{2/3}}\frac{\partial}{\partial \xi}$ and $\frac{1}{\epsilon^{2/3}}\frac{\partial}{\partial \xi}$ respectively. We can see from (3.8) that $| V_0' |^2 \sim y_2 - y$ when $y \to y_{2-}$ in which case the fundamental velocity components and pressure in (3.2) decrease by $O(\epsilon^{1/3})$ where $y - y_2 = O(\epsilon^{2/3})$. The appropriate expansions in $IIa$ therefore take the form

$$u = \bar{\bar{u}}_0 + \epsilon^{2/3}\bar{\bar{u}}_1 + \cdots + \{\epsilon^{4/3}E(U_{01} + \epsilon^{2/3}U_{11} + \cdots) + \epsilon^{8/3}E^2(U_{02} + \epsilon^{2/3}U_{12} + \cdots) + \cdots + C.C.\}$$

$$v = \bar{\bar{v}}_0 + \epsilon^{2/3}\bar{\bar{v}}_1 + \cdots + \{\epsilon^{-2/3}E(V_{01} + \epsilon^{2/3}V_{11} + \cdots) + \epsilon^{2/3}E^2(V_{02} + \epsilon^{2/3}V_{12} + \cdots) + \cdots + C.C.\}$$

$$\begin{aligned} w &= \epsilon^{-1/3}E(W_{01} + \epsilon^{2/3}W_{11} + \cdots) + \epsilon^{1/3}E^2(W_{02} + \epsilon^{2/3}W_{12} + \cdots) + \cdots + C.C.\} \\ p &= \bar{\bar{p}}_0 + \epsilon^{2/3}\bar{\bar{p}}_1 + \cdots + \{\epsilon^{-4/3}E(P_{01} + \epsilon^{2/3}P_{11} + \cdots) \\ &\quad + \epsilon^{2/3}E^2(P_{02} + \epsilon^{2/3}P_{12} + \cdots) + \cdots + C.C.\} \end{aligned} \tag{3.10}$$

The coefficient in the above expansions are functions of $x$ and $\xi$.

It has been anticipated above that the first harmonic functions also decrease in size in $IIa$; this decrease is forced by the form of the equations for $U_0^2$, $V_0^2$ in the core. The appropriate equations are

$$4U_0^2 + V_0^2\frac{\partial \bar{u}_0}{\partial y} = F_1, \tag{3.11a,b}$$

$$4V_0^2 + G_0\chi U_0^2\bar{u}_0 = F_2,$$

where $F_1$ and $F_2$ are quadratic in $U_0^1$, $V_0^1$. The forcing terms are therefore $O(\epsilon^{2/3})$ in $IIa$ and so $U_0^2$, $V_0^2$, $W_0^2$, $P_0^2$ must be rescaled in that layer as indicated in (3.10). We note that (3.11) can always be solved in the core since $\bar{u}_0$ satisfies (3.5). A similar analysis applies to the higher harmonics so we conclude that the fundamental effectively drives the mean flow and all the higher harmonics. The situation is quite different from the usual type of weakly nonlinear based on the Stuart-Watson method where the mean flow remains essentially identical to that present before any instability occurs.

We now return to the solution of (2.3) in the transition layer $IIa$. The first two terms in the expansion of the mean flow in this layer satisfy

$$\frac{\partial^2 \overline{\overline{u}}_0}{\partial \xi^2} = 0, \qquad \frac{\partial^2 \overline{\overline{u}}_1}{\partial \xi^2} = 0$$

and the solutions of these equations which match with the solution in the core are

$$\overline{\overline{u}}_0 = \frac{\sqrt{a + 2y_2}}{\sqrt{G_0}\chi}, \qquad \overline{\overline{u}}_1 = \frac{\xi}{\sqrt{G_0}\chi\sqrt{a + 2y_2}}. \tag{3.12a, b}$$

The function $\overline{\overline{u}}_2$ is forced by the fundamental terms in (3.10) so that we cannot proceed further without solving for the latter terms. The equations to determine $(U_{01}, V_{01})$ and $(U_{11}, V_{11})$ are found to be

$$\begin{aligned} U_{01}\frac{+V_{01}}{\sqrt{G_0}\chi\sqrt{a+2y_2}} &= 0, \\ V_{01} + U_{01}\sqrt{G_0}\chi\sqrt{a+2y_2} &= 0, \\ U_{11} + \frac{V_{11}}{\sqrt{G_0}\chi\sqrt{a+2y_2}} &= -V_{01}\frac{\partial \overline{\overline{u}}_2}{\partial \xi} + \frac{\partial^2 U_{01}}{\partial \xi^2}, \\ V_{11} + U_{11}\sqrt{G_0}\chi\sqrt{a+2y_2} &= \frac{-\sqrt{G_0}\chi\xi}{\sqrt{a+2y_2}}U_{01} + 2\frac{\partial^2 V_{01}}{\partial \xi^2}. \end{aligned} \tag{3.13a, b, c, d}$$

The equations (3.13a,b) are of course always consistent but (3.13c,d) are only consistent if an orthogonality condition holds. To obtain this condition we must proceed to higher order in the mean flow expansion.

The first order solution for $\overline{v}_0$ is found to be

$$\overline{v}_0 = \frac{-a'\sqrt{a+2y_2}}{2\sqrt{G_0}\chi} + \frac{\{a+2y_2\}^{3/2}\chi'}{6\sqrt{G_0}\chi\chi} - b,$$

whilst $\overline{\overline{u}}_2$ satisfies

$$\frac{\partial^2 \overline{\overline{u}}_2}{\partial \xi^2} = \frac{-\chi'[a^2 + 2y_2]}{3\chi^2 G_0} - \frac{b}{\sqrt{G_0}\chi\sqrt{a+2y_2)}} - \frac{2}{\sqrt{G_0}\chi\sqrt{a+2y_2}}\frac{\partial}{\partial \xi} \mid V_{01} \mid^2$$

which can be integrated once to give

$$\frac{\partial \overline{\overline{u}}_2}{\partial \xi} = -\xi\left[\frac{\chi'[a+2y_2]}{3\chi^2 G_0} + \frac{b}{\sqrt{G_0}\chi\sqrt{a+2y_2}}\right] - \frac{2 \mid V_{01} \mid^2}{\sqrt{G_0}\chi\sqrt{a+2y_2}} + f. \tag{3.14}$$

Here $f(x)$ is another function which can only be determined at higher order; for our purposes here it is not required here. Having determined $\frac{\partial \bar{\bar{u}}_2}{\partial \xi}$ we can now write down the appropriate solvability condition for (3.13b):

$$\frac{\partial^2 V_{01}}{\partial \xi^2} + S(x)\xi V_{01} = \frac{2}{3} \mid V_{01} \mid^2 V_{01}\sqrt{a+2y_2}\sqrt{G_0\chi} + G_0\chi(a+2y_2)V_{01}f. \quad (3.15)$$

where

$$S(x) = \frac{1}{a+2y_2} - [\frac{\chi'(a+2y_2)^2}{3\chi} + b\sqrt{G_0\chi}\sqrt{a+2y_2}].$$

This equation is a particular form of the second Painleve transcendent and has been shown by Hastings and Mcleod (1978) to have a solution such that

$$S\xi \sim \frac{2}{3} \mid V_{01} \mid^2 \sqrt{a+2y_2}\sqrt{G_0\chi}, \qquad \xi \to -\infty, \quad \mid V_{01} \mid \to 0, \qquad \to \infty$$

It follows that in $IIa$ the fundamental terms decay to zero so that the finite amplitude Görtler vortex is trapped below region $IIIa$. We note that a similar analysis for the higher harmonics shows that these functions also decay exponentially to zero in $IIa$. However the mean flow is virtually unaltered by the presence of $IIa$, thus the first two terms in the expansion of the mean flow in $IIa$ are simply obtained by expanding the mean flow in $I$ in terms of $\xi$. This means that the mean flow in $IIIa$ must to zeroth order have $\bar{u}, \bar{u}_y$ and $\bar{v}$ defined by the coreflow solution evaluated with $y = y_2$.

An identical analysis to that above shows that the $z$-dependent part of the flow is reduced to zero exponentially in $IIb$. Hence in $IIIb$ there is only a mean velocity field. Thus in $IIIa, b$ we write

$$u = \bar{u} + O(\epsilon^{2/3}), \qquad v = \bar{v} + O(\epsilon^{2/3}), \qquad w = p = 0, \quad (3.16)$$

so that $(\bar{u}, \bar{v})$ defined in $(0, y_1)$ and $(y_2, \infty)$ satisfies

$$\begin{aligned} \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} &= \frac{\partial^2 \bar{u}}{\partial y^2}, \\ \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} &= 0. \end{aligned} \quad (3.17)$$

These equations must be solved subject to

$$\bar{u} = \bar{v} = 0, \qquad y = 0, \quad (3.18)$$

$$\bar{u} \to 1, \qquad y \to \infty, \quad (3.19)$$

and

$$\bar{u} = \frac{\sqrt{a+2y_j}}{\sqrt{G_0\chi}}, \qquad \bar{u}_y = \frac{1}{\sqrt{G_0\chi}\sqrt{a+2y_j}}, \quad (3.20)$$

$$\bar{v} = -\frac{a'\sqrt{a+2y_j}}{2\sqrt{G_0\chi}} + \frac{\{a+2y_j\}^{3/2}\chi'}{6\sqrt{G_0\chi}\chi} - b, \qquad y = y_j, \qquad j = 1,2. \tag{3.21}$$

The equations (3.17) – (3.21) together with (3.9) specify a free boundary problem for $y_1$,$y_2$ and the functions $a(x)$,$b(x)$. Clearly no analytical solution to this problem is available. In fact it is also necessary to specify something about the 'upstream' nature of the instability. This amounts to finding an asymptotic form for the solution of this system close to the value of $x$ where the original boundary layer becomes linearly unstable. This will also be discussed in §5 where a large $x$ solution of the system will also be developed. In §5 a scheme which we have used to solve the system numerically will be described. However there is a special case where a similarity solution of the system can be generated. This corresponds to the case when the undisturbed boundary layer is Blasius flow and $\chi$ is proportional to $x^{1/2}$. Though this particular case is perhaps not of much practical importance its solution is instructive in that it suggests how to solve the full free-boundary partial differential system numerically. For that reason we now indicate how the similarity solution can be calculated.

Consider then the curvature distribution $\chi(x)$ defined by

$$\chi(x) = \sqrt{2x}. \tag{3.22}$$

If the undisturbed flow is a Blasius boundary layer the local Görtler varies with $x^2$ as does the fourth power of the local spanwise wavenumber. This is consistent with the scaling of the right hand branch of the neutral curve discussed by Hall (1982a). Hence on the basis of linear theory the flow is either stable or unstable for all values of $x$. We define the similarity variable $\eta$ by

$$\eta = \frac{y}{\sqrt{2x}}, \tag{3.23}$$

and seek a solution of (3.17) by writing

$$\bar{u} = f'(\eta), \qquad \bar{v} = \frac{1}{\sqrt{2x}}\{\eta f' - f\}. \tag{3.24}$$

The functions $a$ and $b$ then take the form

$$a = \tilde{a}\sqrt{2x}, b = \frac{\tilde{b}}{\sqrt{2x}}. \tag{3.25}$$

We can then show that the free boundary problem specified by (3.9), (3.17) and (3.21) is equivalent to

$$f''' + ff'' = 0, \tag{3.26}$$

on

$$0 < \eta < \eta_1, \eta_2 < \eta < \infty$$

subject to

$$f = f' = 0, \qquad \eta = 0, \tag{3.27}$$

and

$$f'(\infty) = 1, \tag{3.28}$$

together with the matching conditions

$$f'(\eta_j) = \frac{\sqrt{\tilde{a} + 2\eta_j}}{\sqrt{G_0}},$$

$$f(\eta_j) = \eta_j f'(\eta_j) - \left\{ \frac{\tilde{a}\sqrt{\tilde{a} + 2\eta_j}}{2\sqrt{G - 0}} + \frac{(\tilde{a} + 2\eta_j)^{3/2}}{6\sqrt{G_0}} - \tilde{b} \right\}, \qquad j = 1, 2, \quad (3.29a, b)$$

and the scaled form of (3.9):

$$\frac{\tilde{b}\sqrt{\tilde{a} + 2\eta_1}}{\sqrt{G_0}} + 1/12 \frac{\{\tilde{a} + 2\eta_1\}^2}{G_0} + \frac{1}{\sqrt{\tilde{a} + 2\eta_1}\sqrt{G_0}} =$$
$$\frac{\tilde{b}\sqrt{\tilde{a} + 2\eta_2}}{\sqrt{G_0}} + \frac{1}{12} \frac{\{\tilde{a} + 2\eta_2\}^2}{G_0} + \frac{1}{\sqrt{\tilde{a} + 2\eta_2}\sqrt{G_0}} \tag{3.30}$$

Here we have replaced $\frac{y_j}{\sqrt{2x}}$ by $\eta_j$ for $j = 1,2$. Thus in the special case $\chi = \sqrt{2x}$ we have a free boundary ordinary differential system to solve. The unknowns are $\tilde{a}, \tilde{b}$ and $\eta_1, \eta_2$, the boundaries of the region where a finite amplitude vortex exists. These constants and the function $f$ were obtained by following the procedure shown below.

(i) Integrate (3.26) forward from $\eta = 0$ with $f(0) = f'(0) = 0$ and an initial guess for $f''(0)$. This integration is stopped at $\eta = \eta_1$ where $f'(\eta_1)f''(\eta_1) = \sqrt{\frac{2}{G_0}}$.

(ii) Using $f(\eta_1), f'(\eta_1)$ calculate $\tilde{a}, \tilde{b}$ from (3.29a,b) with $j = 1$.

(iii) Now calculate $\eta_2$ from (3.30).

(iv) Finally integrate (3.26) from $\eta = \eta_2$ to some suitably large value of $\eta$ with $f(\eta_2), f'(\eta_2)$ defined by (3.29a,b) with $j = 2$ and using

$$f'(\eta_2)f''(\eta_2) = \sqrt{\frac{2}{G_0}}$$

.

(v) If $f'(\infty) \neq 1$ we return to step (i) and alter the initial value of $f''$ at the wall in some suitable manner.

The above procedure was found to converge when used with Newton's method to update the value of $f''$ at the wall. We found that, as expected, a solution exists only for $G_0 > 4.2$ the neutral value of $G_0$. Beyond $G_0 = 4.2$ the values of $\eta_1$ and $\eta_2$ respectively decrease and increase from their limiting value $\eta_1 = \eta_2 = 1.56$ when $G_0 = 4.2$. At larger values of $G_0, \eta_2$ varies linearly with $G_0$ whilst $\eta_1$ goes to zero like $G_0^{-3}$. These scalings can be recovered by an asymptotic investigation of (3.26) – (3.30) in the limit $G_0 \to \infty$.

Such a calculation shows that for $G_0 >> 1$, $\eta_2 \simeq \frac{1}{2}G_0$, $\eta_1 \simeq .12G_0^{-3}$ and the value of $\eta_2$ predicted in this limit is shown in Figure 2 where we have shown the result of a full numerical solution of (3.26) – (3.30).

Thus for the particular case $\chi = \sqrt{(2x)}$ the partial differential system governing the development of the mean flow can be reduced to an ordinary differential one. Moreover the results shown in Figure 2 suggest the type of behaviour for $y_1, y_2$ which we should expect for the general problem when $x$ increases beyond its neutral value. At large values of $x$ we shall see that a structure essentially identical to that shown in Figure 2 for $G_0 >> 1$ is set up. Finally we note that for other similarity boundary layers we can always choose a particular curvature distribution $\chi(x)$ which leads to an ordinary differential system for the mean flow.

## 4. The initial and ultimate downstream developments of the instability.

We first describe the development of the instability close to the downstream location where the boundary layer $(\bar{u}, \bar{v}, 0)$ becomes linearly unstable. The description of the flow in this region is similar to that given by Hall (1982b) in the weakly nonlinear regions so we shall not give all the details here. Suppose then that vortex with wavenumber $\epsilon^{-1}$ is linearly unstable for $x > x^*$ and the instability originates in a layer of thickness $\epsilon^{1/2}$ centred on $y = y^*$. The weakly nonlinear theory of Hall (1982b) suggests that the finite amplitude vortex occupies a region of depth $O(x - x^*)^{1/2}$ for $| x - x^* | << 1$. We therefore expand $y_1$ and $y_2$ in the form

$$y_1 = y^* - (x - x^*)^{\frac{1}{2}}\tilde{y} + O(x - x^*),$$

$$y_2 = y^* + (x - x^*)^{\frac{1}{2}}\tilde{y} + O(x - x^*), \tag{4.1a,b}$$

It is convenient for us to now define a similarly variable $\xi$ by

$$\xi = \frac{y - y^*}{(x - x^*)^{1/2}} = \frac{Y}{X^{1/2}} \tag{4.2}$$

so that

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial X} - \frac{\xi}{2X}\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial y} \to \frac{1}{X^{1/2}}\frac{\partial}{\partial \xi}.$$

In region $I$ we express $(\bar{u}_0, \bar{v}_0)$ which satisfies (3.5) in the form

$$\bar{u}_0 = \bar{u} + X^{3/2}u_M(\xi) + \cdots$$

$$\bar{v}_0 = \bar{v} + Xv_M(\xi) + \cdots . \tag{4.3a,b}$$

Here $(\bar{u}, \bar{v}, 0)$ is again the boundary layer flow which exists in the absence of the instability. In the neighbourhood of $(x^*, y^*)$ $\bar{u}, \bar{v}$, and the curvature $\chi$ expand as

$$\bar{u} = u_{00} + X^{\frac{1}{2}}\xi u_{10} + X\{\xi^2 u_{20} + u_{01}\} + X^{\frac{3}{2}}\xi u_{11} + \cdots,$$

$$\bar{v} = v_{00} + X^{\frac{1}{2}}\xi v_{10} + X\{\xi^2 v_{20} + v_{01}\} + X^{\frac{3}{2}}\xi v_{11} + \cdots,$$

$$\chi = \chi_0 + X\chi_1 + X^2\chi_2 + \cdots. \qquad (4.4a,b,c)$$

If we substitute $(\bar{u}_0, \bar{v}_0)$ from (4.3) into (3.5) and use (4.4) then, equating like powers of $X^0$, $X^{\frac{1}{2}}$, $X$ we obtain

$$G_0\chi_0 u_{00}u_{10} = 1,$$

$$u_{10}^2 + 2u_{00}u_{20} = 0,$$

$$\frac{-\chi_1}{G_0\chi_0^2 u_{00}} = \frac{\partial u_M}{\partial \xi} + u_{11} + \frac{3\xi^2 u_{10}u_{20}}{u_{00}} + \frac{u_{01}u_{10}}{u_{00}}. \qquad (4.5a,b,c)$$

Equation (4.5a) is satisfied if $x^*$ is the neutral location whilst (4.5b) holds because $y^*$ in the linear theory is chosen so that

$$\frac{\partial}{\partial y}(\bar{u}\frac{\partial \bar{u}}{\partial y}) = 0, \qquad y = y^*.$$

Equation (4.5c) can be integrated once to give

$$u_M = \lambda_0\xi - \lambda_1\xi^3 \qquad (4.6)$$

where

$$\lambda_0 = -u_{11} - \frac{\chi_1 u_{10}}{\chi_0} - \frac{u_{01}u_{10}}{u_{00}}$$

and

$$\lambda_1 = \frac{u_{10}u_{20}}{u_{00}} \qquad (4.7a,b).$$

We have anticipated in (4.7) that $u_M$ is an odd function of $\xi$; this would otherwise have been determined at a later stage.

The Görtler vortex function $V_0^1$ in region I expands as

$$| V_0^1 |^2 = XF(\xi) + \cdots$$

and $F(\xi)$ can then be found from the zeroth order approximation to (3.6b). This equation can be integrated once and $F(\xi)$ is found to vanish at $\xi = \xi_\pm$ where

$$| \xi_\pm |^2 = \frac{C + \lambda_0}{3\lambda_1 + \frac{1}{2}u_{00}\lambda_0}. \qquad (4.8)$$

Here $C$ is an unknown constant of integration to be determined later. It follows from (4.1) and (4.8) that

$$\xi_- = -\tilde{y}, \qquad \xi_+ = \tilde{y}.$$

In region IIa the forcing of the mean flow ceases and $| V_0^1 |$ is reduced to zero exponentially in the manner described in §3. It remains for us to determine the constant $C$ appearing in (4.8). We can write down the mean flow in regions $IIIa, b$ in the form

$$\bar{u} = \overline{\overline{u}} + X^{\frac{3}{2}}U_M(\xi) + \cdots$$

$$\bar{v} = \bar{\bar{v}} + XV_M(\xi) + \cdots \tag{4.9}$$

It is easy to show that $U_M(\xi)$ and $V_M(\xi)$ are odd and even functions of $\xi$ respectively so it is enough for us to consider $IIIa$ where $\xi \geq \xi_+$. The functions $U_M$ and $V_M$ satisfy the equations

$$\frac{d^2U_M}{d\xi^2} + \frac{1}{2}u_{00}\left\{\xi\frac{d}{d\xi} - 3\right\}U_M = 0$$

$$\frac{\partial V_M}{\partial \xi} + \frac{3}{2}U_M - \xi\frac{dU_M}{d\xi} = 0 \tag{4.9a, b}$$

and in order to match with the core solution (4.6) $U_M$ must satisfy

$$U_M = \lambda_0\xi_+ - \lambda_1\xi_+^3, \qquad U_M' = \lambda_0 - 3\lambda_1\xi_+^2, \qquad \xi = \xi_+. \tag{4.10}$$

Since $U_M$ must also tend to zero when $\xi \to \infty$ (4.9a), (4.10) constitute a nonlinear eigenvalue problem for $\xi_+$. After some manipulation we find that

$$U_M = \frac{(\lambda_0\xi_+ - \lambda_1\xi_+^3)\exp\{-\frac{u_{00}}{8}[\xi^2 - \xi_+^2]\}U(\frac{7}{2}, \sqrt{\frac{u_{00}}{2}}\xi)}{U(\frac{7}{2}, \sqrt{\frac{u_{00}}{2}}\xi_+)},$$

$$V_M = \xi U_M - \frac{5}{2}\int_\infty^\xi U_M d\xi, \tag{4.11a, b}$$

where $U(a, x)$ is a parabolic cylinder function. The eigenrelation for $\xi_+$ is then found to be

$$\sqrt{\frac{u_{00}}{2}}\left\{\frac{4U(\frac{9}{2}, \frac{\sqrt{u_{00}}}{2}\xi_+)}{U(\frac{7}{2}, \sqrt{\frac{u_{00}}{2}}\xi_+)} + \sqrt{\frac{u_{00}}{2}}\xi_+\right\} = \frac{\lambda_0 - 3\lambda_1\xi_+^2}{\lambda_0\xi_+ - \lambda_1\xi_+^3}, \tag{4.12}$$

and $C$ is then given by (4.8). Moreover $y_1$ and $y_2$ are then known correct to order $(x - x*)^{1/2}$

The functions $a(x)$ and $b(x)$ are found by comparing (3.7) and (4.3) with $\bar{u}, \bar{v}, \chi$ given by (4.4). Thus, for example, we find that $a(x)$ is given by

$$a(x) = G_0\chi\left\{u_{00}^2 - \frac{2y*}{G_0\chi_0} + 2Xu_{00}[u_{01} - \lambda_0 y * - u_{11}] + \cdots\right\} \tag{4.13}$$

for $X << 1$. Thus we can construct an asymptotic solution of the free boundary partial differential system of §3. This asymptotic solution can of course be used to start off a numerical solution of this system by a marching procedure.

Next we suppose that the curvature distribution function becomes large when $x \to \infty$. More precisely we suppose $\chi(x)/x^{\frac{1}{2}} \to \infty, x \to \infty$ so that the flow is unstable on the basis of linear theory when $x \to \infty$. In this limit it is convenient to perform an asymptotic analysis directly on the solutions obtained in §3. For definiteness we assume that $\chi = x^M, M > \frac{1}{2}$ when $x \to \infty$.

The scaling for $\chi \to \infty$ is suggested by the large $G_0$ solution discussed in §3 and can be inferred from (3.7). It is easy to show that a solution of the 'upper problem' in *IIIa* requires that $y_2 >> x^{\frac{1}{2}}$ and that $\bar{u}_0 = O(1)$ when $y = y_2$. Thus *I* must be of depth $O(\chi)$; the core solution for $y = O(\chi)$ can then be written

$$\bar{u}_0 = \sqrt{\frac{2y}{G_0 x^M}} + \cdots \tag{4.14}$$

and if we take

$$y_2 = \frac{G_0 x^M}{2} + \cdots$$

$\bar{u}_0 \to 1 + \cdots$ when $y \to y_2-$. We then find that

$$\bar{v}_0 \to \frac{G_0 M x^{M-1}}{6} + \cdots,$$

when $y \to y_2-$.Thus a solution of (3.17) for $(\bar{u}, \bar{v})$ in the upper layer *IIIa* takes the form

$$\bar{u} = 1 + \cdots$$

$$\bar{v} = \frac{G_0 M x^{M-1}}{6} + \cdots. \tag{4.15a, b}$$

*If*

$$y_2 \neq \frac{G_0 x^M}{2} + \cdots$$

the resulting upper layer problem for $(\bar{\bar{u}}, \bar{\bar{v}})$ does not have a solution. The depth of the region *IIIb* can now be inferred from equation (3.9). The dominant term on the right hand side of this equation is $O(x^{M-1})$ so a balance with a comparable term on the left hand side can only be achieved if $a + y_1 \sim X^{2-3M}$. Thus $y_1$ and $a$ must be $O(x^{2-3M})$ and we therefore write

$$y_1 = y_{10} x^{2-3M} + \cdots$$

$$a = a_0 x^{2-3M} + \cdots \tag{4.16a, b}$$

The solution for $(\bar{u}, \bar{v})$ in *IIIb* can then be developed in the form

$$\bar{u} = N x^{1-2M} (y x^{3M-2}) + \cdots,$$

$$\bar{v} = -(M-1) x^{M-2} \frac{y^2}{2} + \cdots , \tag{4.17a, b}$$

Here $N$ is an unknown constant to be determined. Finally the continuity of the normal velocity component at $y_1$ requires that we expand $b$ in the form

$$b = b_0 x^{2-5M} + \cdots. \tag{4.17c}$$

The conditions (3.20), (3.21) for $j = 1$ then require that

$$\frac{1}{G_0} = N^2 y_{10},$$

$$\frac{\sqrt{a_0 + 2y_{10}}}{\sqrt{G_0}} = N y_{10},$$

and

$$-\frac{(M-1)}{2} y_{10}^2 = -b_0 + \frac{(3M-2)\sqrt{a_0 + 2y_{10}}}{2\sqrt{G_0}} + \frac{M}{6}\frac{(a_0 + 2y_{10})^{\frac{3}{2}}}{\sqrt{G_0}}. \qquad (4.18a, b, c)$$

It follows from (4.18a,b) that $a_0 = -y_{10}$ so it remains for us to determine $y_{10}$. The required equation follows from the zeroth order approximation to (3.9) which yields

$$y_{10} = \frac{9G_0}{M^2}$$

and then $N$ and $b_0$ follow from (4.17a,c) respectively Thus in the limit $x \to \infty$ with $\chi \sim x^M$, $M > 1/2$ the locations $y_1$ and $y_2$ have the asymptotic forms

$$y_1 = \frac{9G_0}{M^2} x^{2-3M} + \cdots$$

$$y_2 = \frac{G_0 M x^M}{2} + \cdots \qquad (4.19a, b)$$

and between these positions the mean velocity component in the $x$ direction is given by the square root form (4.14). Thus for a curvature distribution $\chi = x^M$ the initial and far downstream regions where a finite amplitude vortex exists are as indicated in Figure 3 for $M > 1/2$. The intermediate region can only be completed by a numerical procedure of the type we discuss in the next section.

## 5. A numerical scheme for the determination of the free boundaries $y_1$,$y_2$

We shall now outline a scheme which we have used to solve the problem specified by (3.17) – (3.20). For convenience we drop the '-' notation for $\bar{u}, \bar{v}$ and assume that a solution of the problem is known for $x \leq \tilde{x}$. The scheme which we have used can be used to advance this solution downstream to $\tilde{x} + \tilde{\epsilon}$ for sufficiently small values of $\tilde{\epsilon}$. In general the state for $x \leq \tilde{x}$ must be calculated using the approach of §4 in a neighbourhood of where the vortices first become unstable.

The first step in our calculation is to define a variable $\varsigma$ by

$$\varsigma = \frac{y}{y_j}, j = 1, y \leq y_1, \qquad j = 2, y \geq y_2,$$

so that (3.17) is now to be solved on $(0,\infty)$ in terms of $\varsigma$ with boundary conditions at 0, and $\infty$ together with 'jump' conditions at $\varsigma = 1$. Thus (3.17) is now written in the form

$$u\frac{\partial u}{\partial x} = \frac{1}{y_j^2}\frac{\partial^2 u}{\partial \varsigma^2} - \frac{v}{y_j}\frac{\partial u}{\partial \varsigma} + \frac{y_j'}{y_j}\varsigma u\frac{\partial u}{\partial \varsigma}$$

$$\frac{\partial v}{\partial \varsigma} = y_j\left\{-\frac{\partial u}{\partial x} + \varsigma\frac{y_j'}{y_j}\frac{\partial u}{\partial \varsigma}\right\}, \qquad (5.1a,b)$$

where $j = 1$ for $\varsigma < 1$ and $j = 2$ for $\varsigma > 1$. The required boundary conditions are

$$u = v = 0, \qquad \varsigma = 0, \qquad u \to 1, \qquad \varsigma \to \infty \qquad (5.2)$$

and the 'discontinuous' conditions at $\varsigma = 1$ may be written

$$u_\pm = \frac{\sqrt{a + 2y_j}}{\sqrt{G_0\chi}}, \qquad u_{\varsigma\pm} = \frac{y_j}{\sqrt{G_0\chi}\sqrt{a+2y_j}},$$

$$v_\pm = -\frac{a'\sqrt{a+2y_j}}{2\sqrt{G_0\chi}} + \frac{\{a+2y_j\}^{\frac{3}{2}}\chi'}{6\sqrt{G_0\chi}\,\chi} - b. \qquad (5.3a,b,c)$$

Here the $\pm$ signs correspond to the limits $\varsigma \to 1_+$, $\varsigma \to 1_-$ and the index $j = 2$ is associated with the $+$ sign and $j = 1$ with the negative sign. Finally the system is completed by the jump condition (3.9).

We first advance the solution of (5.1a) for $\varsigma$ in the range $[0,1]$. This is done using the scheme

$$u_n\tilde{u}_n - \frac{\tilde{\epsilon}}{h^2\tilde{y}_j^2}\{\tilde{u}_{n+1} - 2\tilde{u}_n + \tilde{u}_{n-1}\} =$$
$$u_n^2 - \frac{\tilde{\epsilon}v_n}{2y_jh}\{u_{n+1} - u_{n-1}\} + \frac{\tilde{\epsilon}\varsigma_n y_j' u_n}{2y_jh}\{u_{n+1} - u_{n-1}\}. \qquad (5.4)$$

where $h$ is the vertical grid spacing and a tilde denotes a quantity evaluated at the position $\tilde{x} + \tilde{\epsilon}$. The index $n$ refers to a quantity at the grid point $\varsigma = nh$. In order to solve the tridiagonal system (5.4) we must make a guess for $\tilde{y}_j$ and set $y_j' = (\tilde{y}_j - y_j)/\tilde{\epsilon}$. When solving (5.4) we satisfy the required condition on $u$ at $\varsigma = 0$ and (5.3b) with $j = 1$. The continuity equation (5.1b) is then discretized as

$$\frac{\tilde{v}_{n+1} - \tilde{v}_{n-1}}{2h} = y_j\left\{\frac{u_n - \tilde{u}_n}{\tilde{\epsilon}} - \frac{\varsigma_n y_j'}{\tilde{y}_j}\frac{[\tilde{u}_{n+1} - \tilde{u}_{n-1}]}{2h}\right\} \qquad (5.5)$$

so that $\tilde{v}$ can be determined at $\tilde{x} + \tilde{\epsilon}$ for $0 \le \varsigma \le 1$. However the equation for $\tilde{v}$ is only of first order in $\varsigma$ so only the boundary condition at $\varsigma = 0$ can be satisfied during this procedure. Thus the solution of (5.1) for $0 \le \varsigma \le 1$ has been calculated at $\tilde{x} + \tilde{\epsilon}$ but, as yet, (5.3a,c) with $j = 1$ have not been satisfied. However these conditions are now used to obtain an improved value of $\tilde{y}_1$ and a value for $\tilde{a}$ by writing these conditions in the form

$$2\tilde{y}_1 = G_0\chi\tilde{u}_-^2 - \tilde{a}$$

$$\tilde{v}_- + \tilde{b} = \frac{(\tilde{a}-a)}{\tilde{\epsilon}}\frac{\sqrt{\tilde{a}+2\tilde{y}_1}}{2\sqrt{G_0\chi}} + \frac{\{\tilde{a}+2\tilde{y}_1\}^{\frac{3}{2}}\chi'}{6\sqrt{G_0\chi}\chi}$$

and iterating until $\tilde{a}$ and $\tilde{y}_1$ converge. Here $\tilde{b}$ is the current guess for $b$ at $x+\tilde{\epsilon}$. The scheme used to find $u, v$ at $x+\tilde{\epsilon}$ for $0 \le \varsigma \le 1$ can be applied in a similar manner to the region $\varsigma \ge 1$. The $u$ equation is solved subject to $u \to 1, \varsigma \to \infty$ and (5.3b) with $j = 2$, whilst the $v$ equation is solved subject to (5.2c) with $j = 2$. Finally (3.9), (5.3c) are written in the form

$$\tilde{b}\Big\{\frac{\sqrt{\tilde{a}+2\tilde{y}_1}}{\sqrt{G_0\chi}} - \frac{\sqrt{\tilde{a}+2\tilde{y}_2}}{\sqrt{G_0\chi}}\Big\} = \frac{\chi'}{12G_0\chi^2}\{(\tilde{a}+2\tilde{y}_2)^2 - (\tilde{a}+2\tilde{y}_1)^2\}$$
$$+ \frac{1}{\sqrt{G_0\chi}}\Big\{\frac{1}{\sqrt{\tilde{a}+2\tilde{y}_2}} - \frac{1}{\sqrt{\tilde{a}+2\tilde{y}_1}}\Big\},$$
$$2\tilde{y}_2 = G_0\chi\tilde{u}_+^2 - \tilde{a}.$$

The second of these equations determines a new value for $\tilde{y}_2$ and the first one then determines a value for $\tilde{b}$. Thus we now have values for $u, v, a, b, y_1, y_2$ at $x+\tilde{\epsilon}$. We then repeat the whole procedure using the new values of $\tilde{a}, \tilde{b}, \tilde{y}_1, \tilde{y}_2$ obtained in the first iteration until converged values of these quantities are obtained.

The above numerical scheme was found to converge for sufficiently small values of $\tilde{\epsilon}$ the step length in the $x$ direction. It was found that $h = .1, \tilde{\epsilon} = .005$ gave a stable scheme for the cases investigated and produced values for $y_1, y_2$ and the other flow quantities correct to two decimal places. The first flow considered had $G_0 = 5., \chi = \sqrt{2x}$ and the free boundary value problem of §3 was integrated for $x > .5$. At $x = .5$ the initial values of $y_1, y_2$ etc. were calculated from the similarity solution of §3. In Figure 4 we have shown the values of $y_1, y_2$ calculated using the scheme outlined in this section. In the same Figure we have indicated the values of $y_1, y_2$ predicted by the similarity solution of §3. We see that there is excellent agreement between the results from the different solution methods.

As an example of a non 'self-similar' Görtler vortex we considered the case $\chi = 2x, G_0 = 4.176$. This curvature distribution has a basic state which is linearly unstable beyong $x^* = .5$. The small $(x - x^*)$ asymptotic solution of §4 was used to generate initial values of $y_1, y_2, a, b$ and the velocity profiles at $x = .51$. The numerical scheme described in this section was then used to advance the solution beyond $x = .51$. The results obtained for $y_1$ and $y_2$ are shown in Figure 5 for $.51 < x < 1.8$. In this Figure we have also shown the corresponding results predicted by the small $x - x^*$ and large $x$ asymptotic solutions of §4. We see that, apart from an apparently constant horizontal translation, the numerical scheme predicts values of $y_1$ and $y_2$ which rapidly approach their large $x$ amplitude values. This translation is not unexpected since the large $x$ solution of §4, to the order given, has an arbitrary origin for $x$.

In Figure 6 we have shown the streamwise velocity component produced by our scheme for the numerical solution of the free boundary problem in §3. The Blasius flow appropriate to this position and the large $x$ asymptotic solution for $u$ are also shown. It can be seen that at $x = 1.8$ the asymptotic and numerical solutions are virtually identical. Thus at larger values of $x$ we can approximate $u$ by the asymptotic solution of §4. In Figure 6d we have shown the result of making this assumption to calculate $u$ at $x = 7.5, 15$. We see that by this step the boundary layer has been substantially thickened by the effect of the vortices. This is because the asymptotic solution has $y_2 \sim \chi \sim x$ whilst the undisturbed boundary layer grows like $x^{\frac{1}{2}}$.

In Figures 7 and 8 we have shown the eigenfunctions $V_0^1$ and $U_0^1$ at the downstream locations $x = .6, 1.0, 1.4, 1.8$. The corresponding large $x$ asymptotic solutions for $V_0^1$ and $U_0^1$ can be derived using the analysis of §4. For $x = 1.8$ such a calculation produces results virtually identical to the ones shown. In Figure 8 we see that when $x$ increases the streamwise velocity component of theGörtler vortex develops a shear layer near the lower boundary. The development of the shear layer is caused by the fact that the transition layer $IIb$ approaches the wall when $x \to \infty$. Thus $U_0^1 \sim \dfrac{V_0^1}{\sqrt{a + 2y_1}}$ increases rapidly from zero to an O(1) value in the neighbourhood of $y_1$.

The implication of the above calculations is that we can reasonably expect that the large $x$ solution of§4 will give accurate predictions for the vortex induced flow quantities at relatively small values of $x$. Thus in practical situations we might expect to obtain sensible results by using that approach rather than the numerical scheme for the free boundary problem. Moreover such an approach produces velocity profiles whose lower stability to Tollmien-Schlichting waves can be readily investigated.

## 6. Conclusions

It is perhaps useful at this stage to remind the reader that the small wavelength approximation wehave used does not make our calculation physically unrealistic. This is because it is known experimentally that when Görtler vortices develop their wavelength is conserved as they move downstream. Thus for a growing boundary layer the effective wavenumber increases in the downstream direction so that a small wavelength approximation eventually becomes justifiable. We have no reason to suppose that in the non linear case the downstream position where the small wavelength results approach the O(1) wavelength results will differ significantly from that for the linear case.

The discussion in §3-5 has been concerned with flows for which $\chi$ increases at least as quickly as $x^{\frac{1}{2}}$ when $x$ increases. Otherwise the basic state will become linearly stable at a finite value of $x$ and the structure we have found will terminate at some value of $x$. The termination of the finite interval of vortex activity is simply the 'mirror-image' of the small $(x - x*)$ solution of §4. This result can be confirmed from the weakly nonlinear theory of Hall (1982b) so that at some value $x$, say $x^+$, $y_1$ and $y_2$ will have the asymptotic form $y^+ \pm y^{++}(x^+ - x)^{\frac{1}{2}}$ where $y^+, y^{++}$ are constants. In Figure 9 we have shown the development of $y_1, y_2$ for $G_0 = 4.176$ with $\chi = 2x, \quad x \leq 1$, $\chi = 4x^2 - 3x^3, \quad x \geq 1$. We see that the region of vortex activity which begins at linear neutral position $x = .5$ stops at $x \simeq 1.5$. In the absence of a finite amplitude vortex beyond $x = .5$ this flow is linearly stable for $x > 1.854$. Thus the presence of the vortices in the range $.5 < x < 1.5$ produces a boundary layer which is stable in a regime where it would have been unstable if the vortices had not developed.

However the effect of the vortices on the boundary layer does not end when the layer of vortex activity terminates. This is because the initial velocity distribution at $x = x^+$ for the subsequent boundary layer will in general be quite different from that appropriate to the undisturbed flow. Furthermore there is no reason to suppose that $y^+$ should equal $y^*$ the location of the vortices according to linear theory applied to the undisturbed state. Thus the decay of the vortices at $x = x^+$ does not in any sense allow the boundary layer to return to its undisturbed state. It follows that Görtler vortices might have a significant effect on separation subsequent to a region of concave curvature. Indeed it is known from the work of Hall and Bennett (1986) that triple-deck flows can support Taylor-Görtler vortices so the properties of these flows might also be significantly altered in the presence of vortices.

In some flows it is possible that there will be several intervals in $x$ where vortices can develop. We might expect that the steady boundary layer over a wavy wall might support vortices at regular intervals along the wall. In the intervals where vortices do not develop the basic state will in general be altered from its undisturbed state by the vortex activity in the previous undisturbed interval.

The most surprizing feature of our calculation is that the fully nonlinear state driven by large amplitude Görtler vortices can be described in a relatively simple manner. The major effect of the vortices is to gradually expand into the boundary layer to give the mean flow there a simple square root profile. If the location increases faster than $x^{\frac{1}{2}}$ this layer thickens until it occupies the whole of the boundary layer apart from a thin layer at the wall. In addition the layer of vortex activity expands into the free stream and thus thickens the undisturbed boundary layer.

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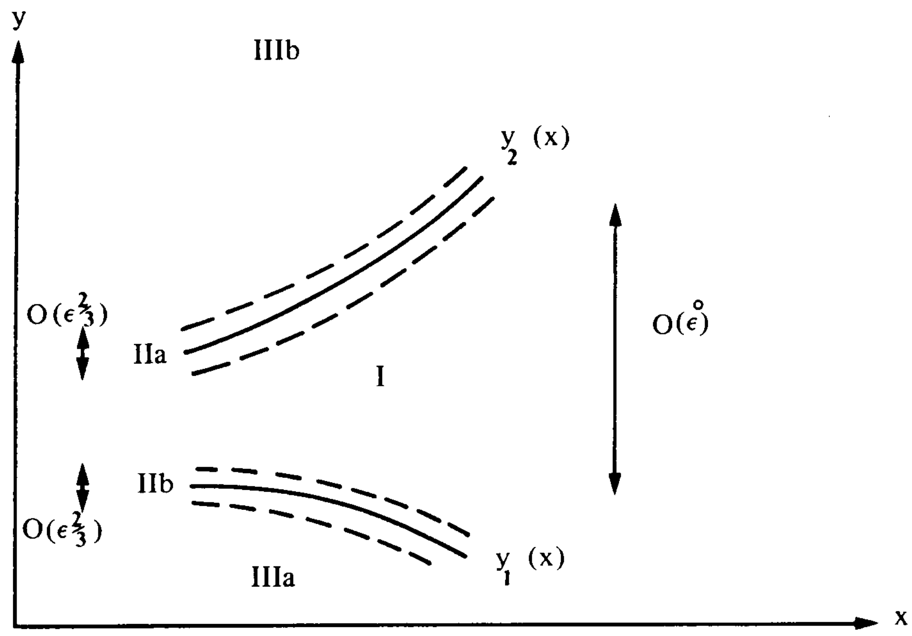


Figure 1. The different regions beyond the downstream position of neutral stability.

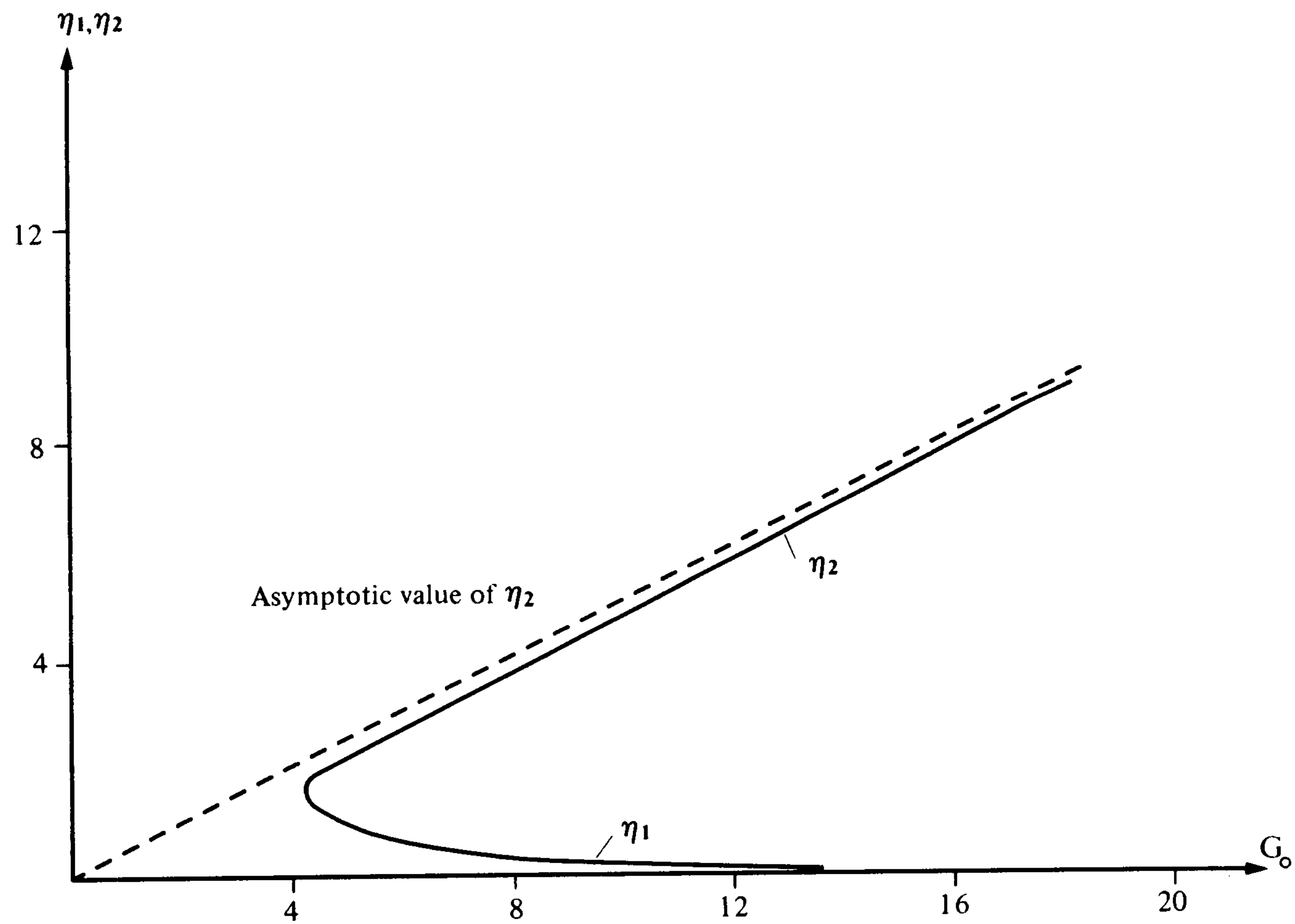


Figure 2. The values of $\eta_1, \eta_2$ for the special case $\chi = \sqrt{2}x$ .

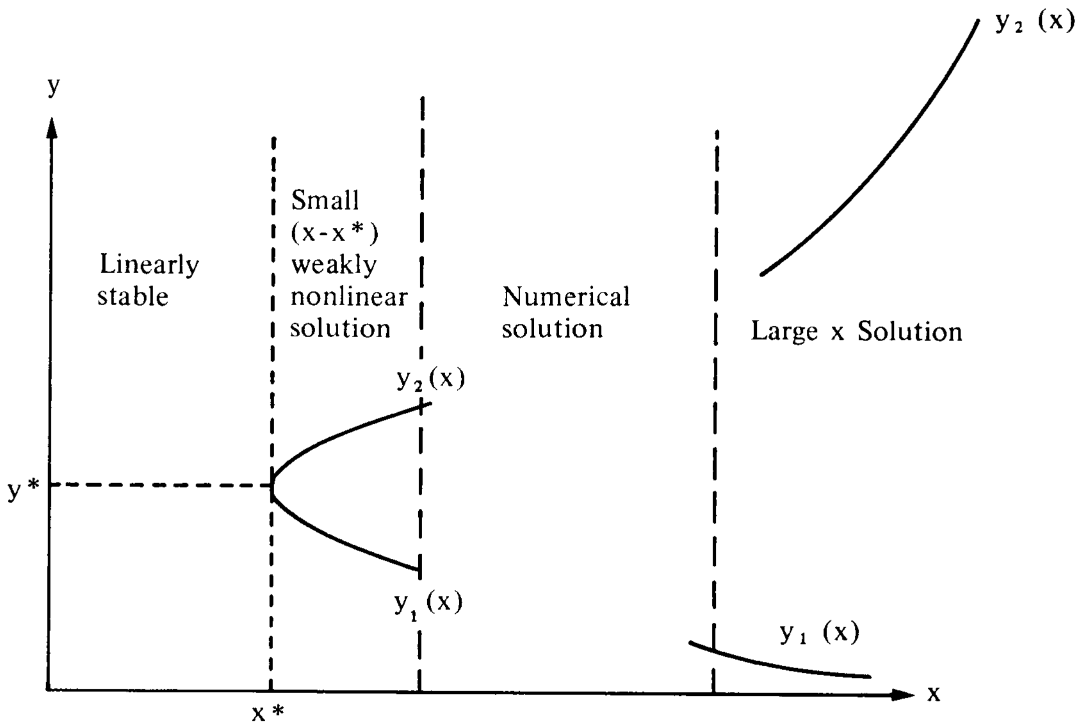


Figure 3. The different regions of vortex activity as the vortices develop in the downstream direction for $\chi \sim x^{M}$, $M > \frac{1}{2}$

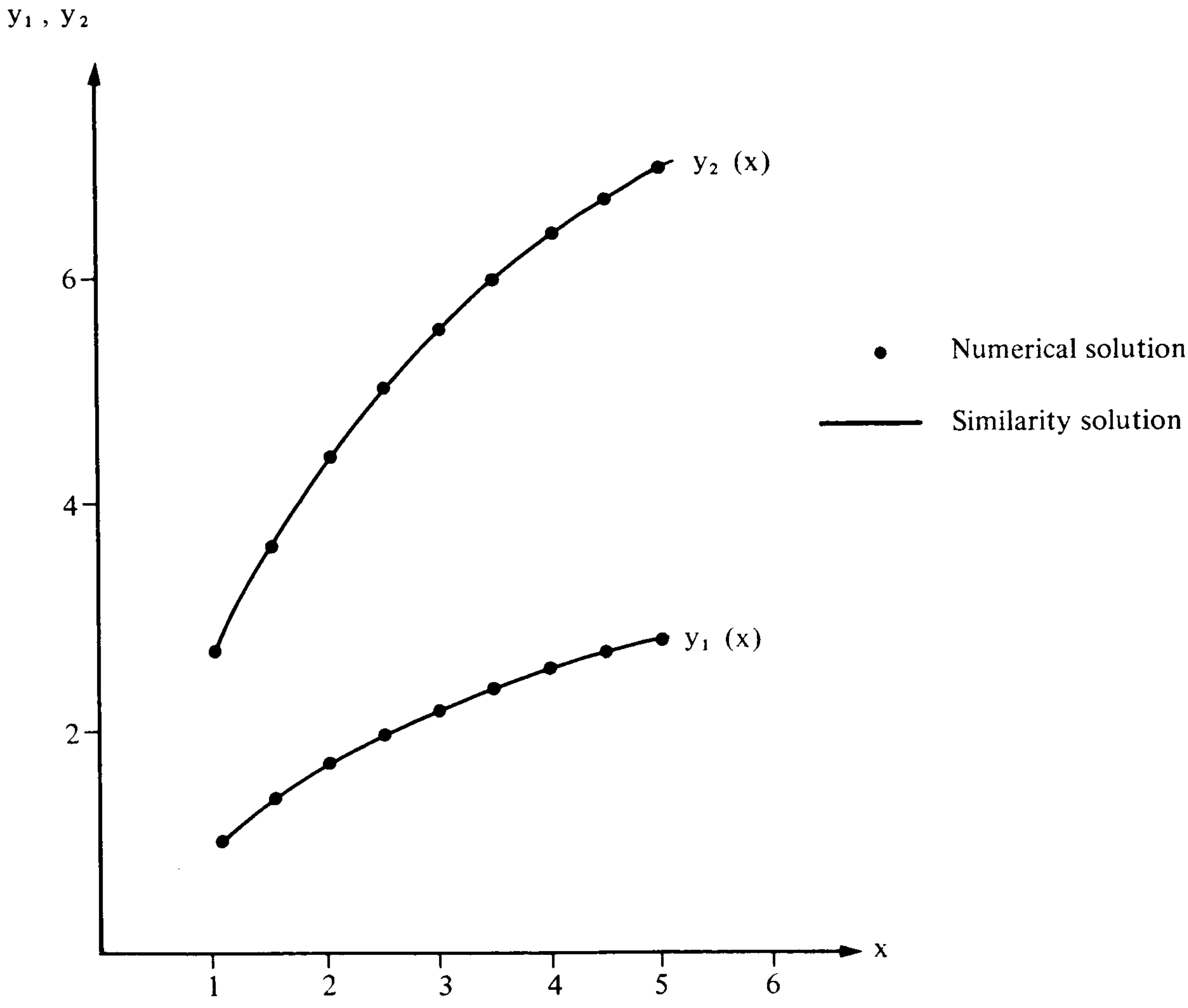


Figure 4. Comparison between the similarity solution of §3 with the numerical scheme of §4 applied to the case $\chi = \sqrt{2}x$, $g_0 = .5$

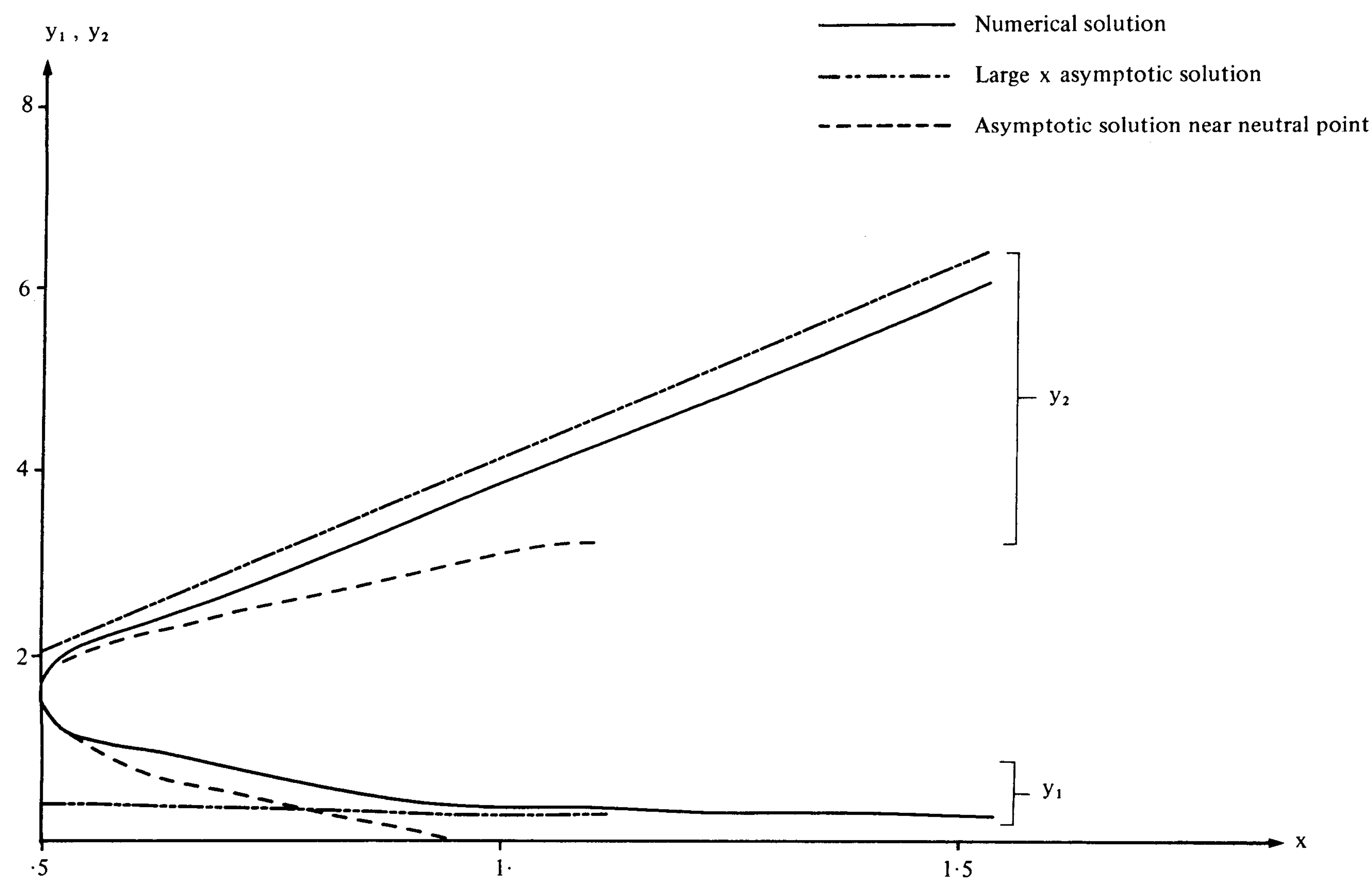


Figure 5. The development of $y_1$ and $y_2$ with $x$ for the case $\chi = 2x$, $g_0 = 4.176$.

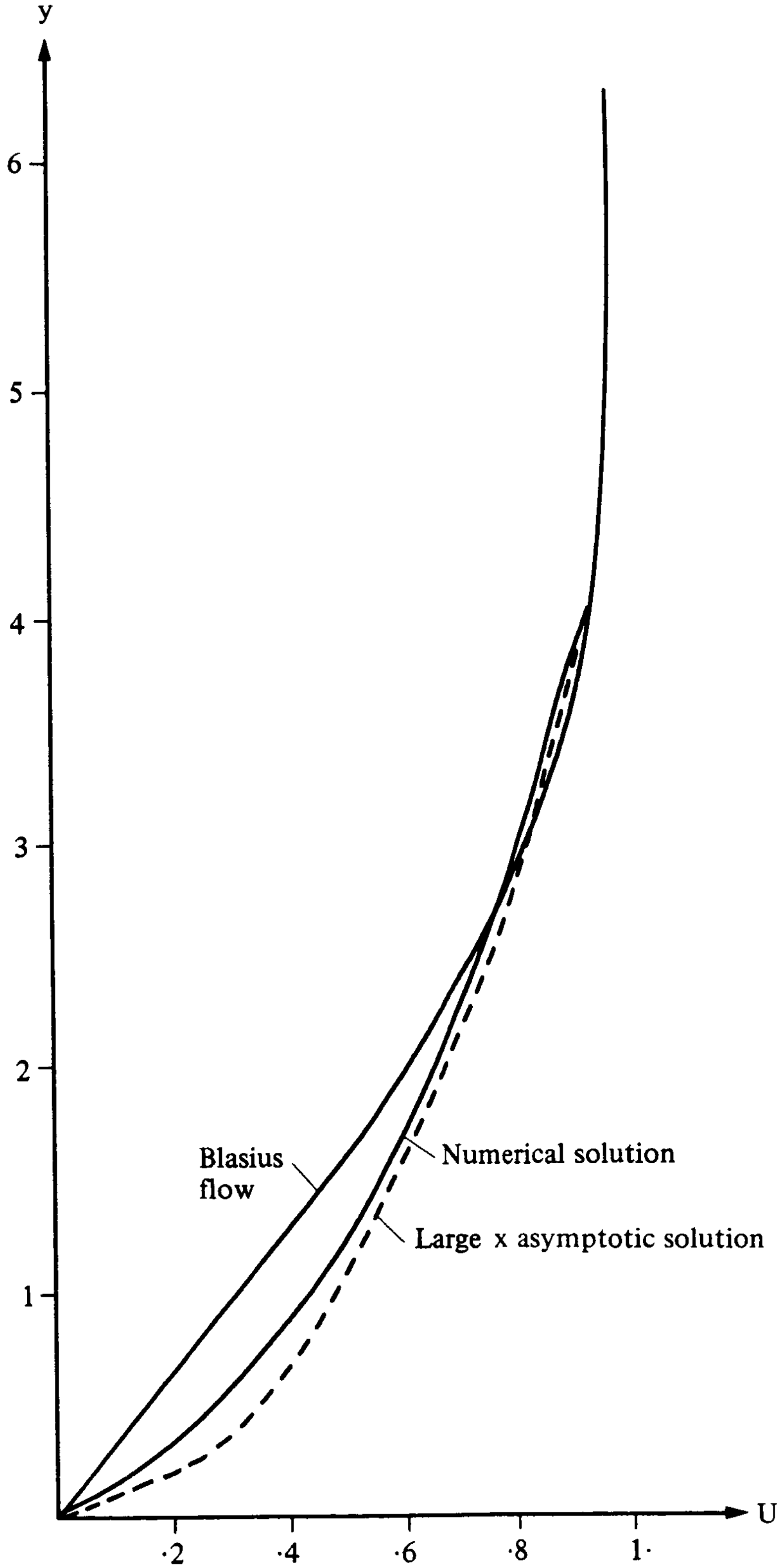


Figure 6a. The streamwise velocity component corresponding to $\chi = 2x$, $g_0 = 4.176$, $x = 1$.

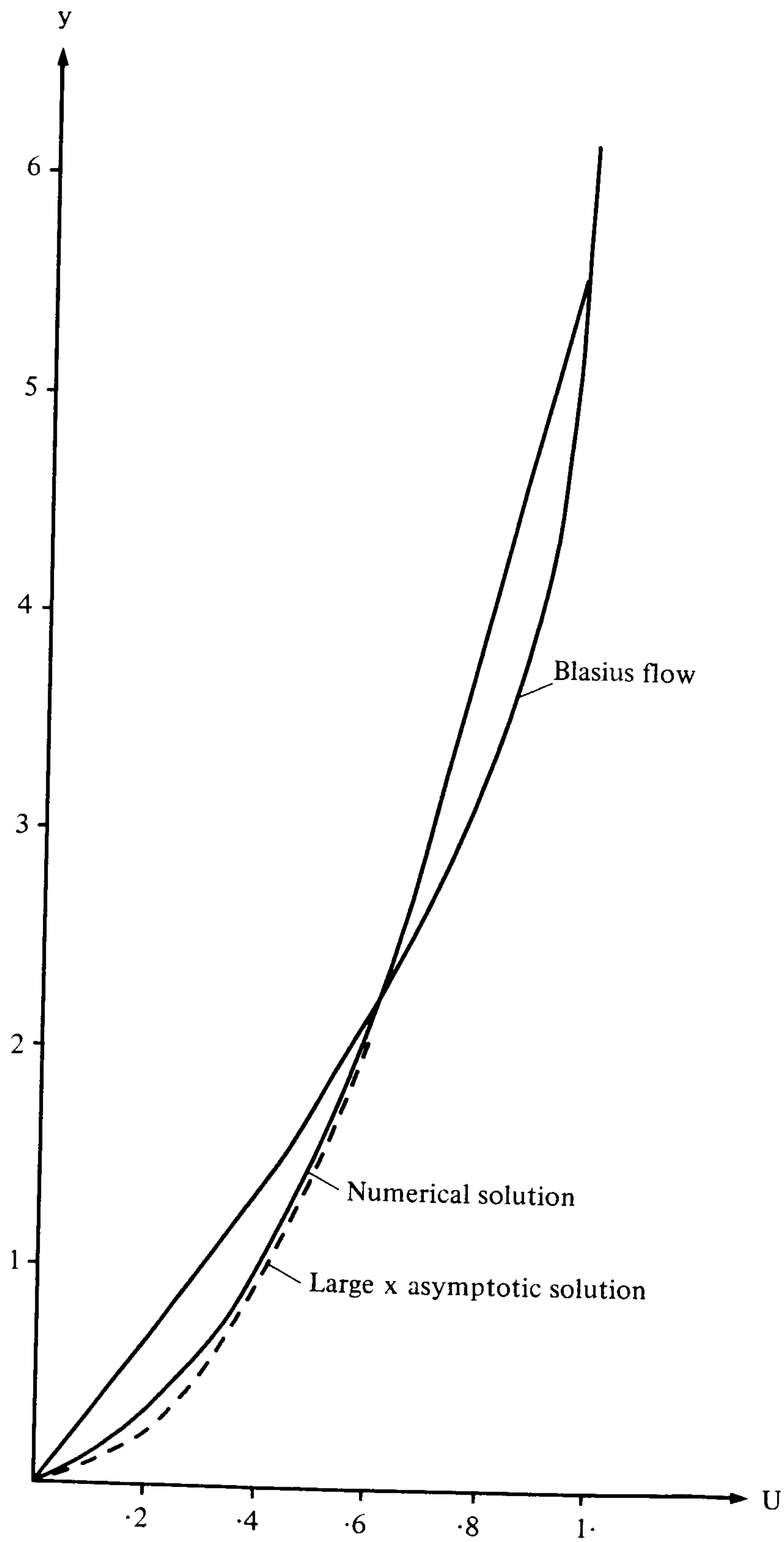


Figure 6b. The streamwise velocity component corresponding to $\chi = 2x$, $g_0 = 4.176$, $x = 1.4$

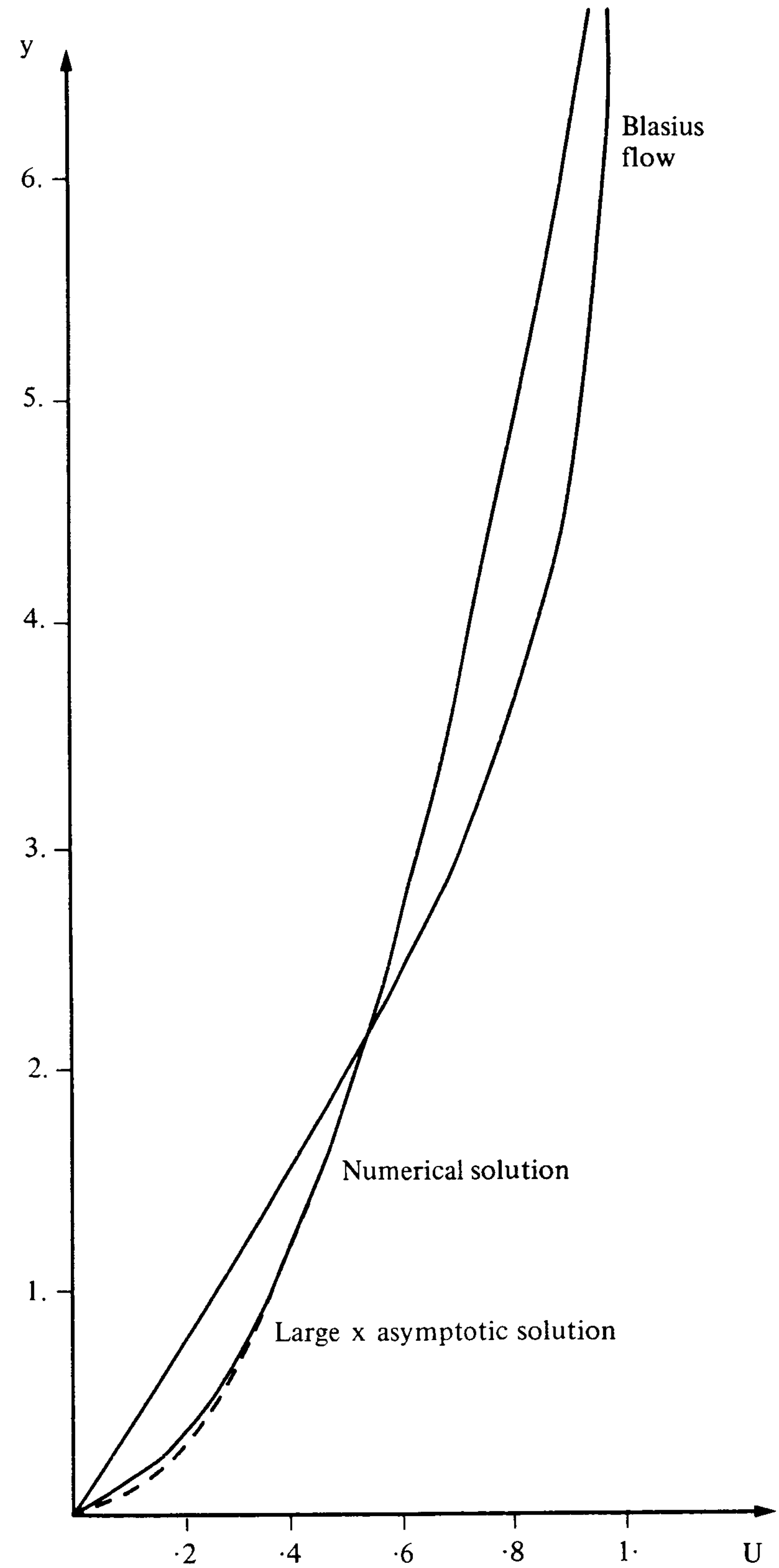


Figure 6c. The streamwise velocity component corresponding to $\chi = 2x$, $g_0 = 4.176$, $x = 1.8$

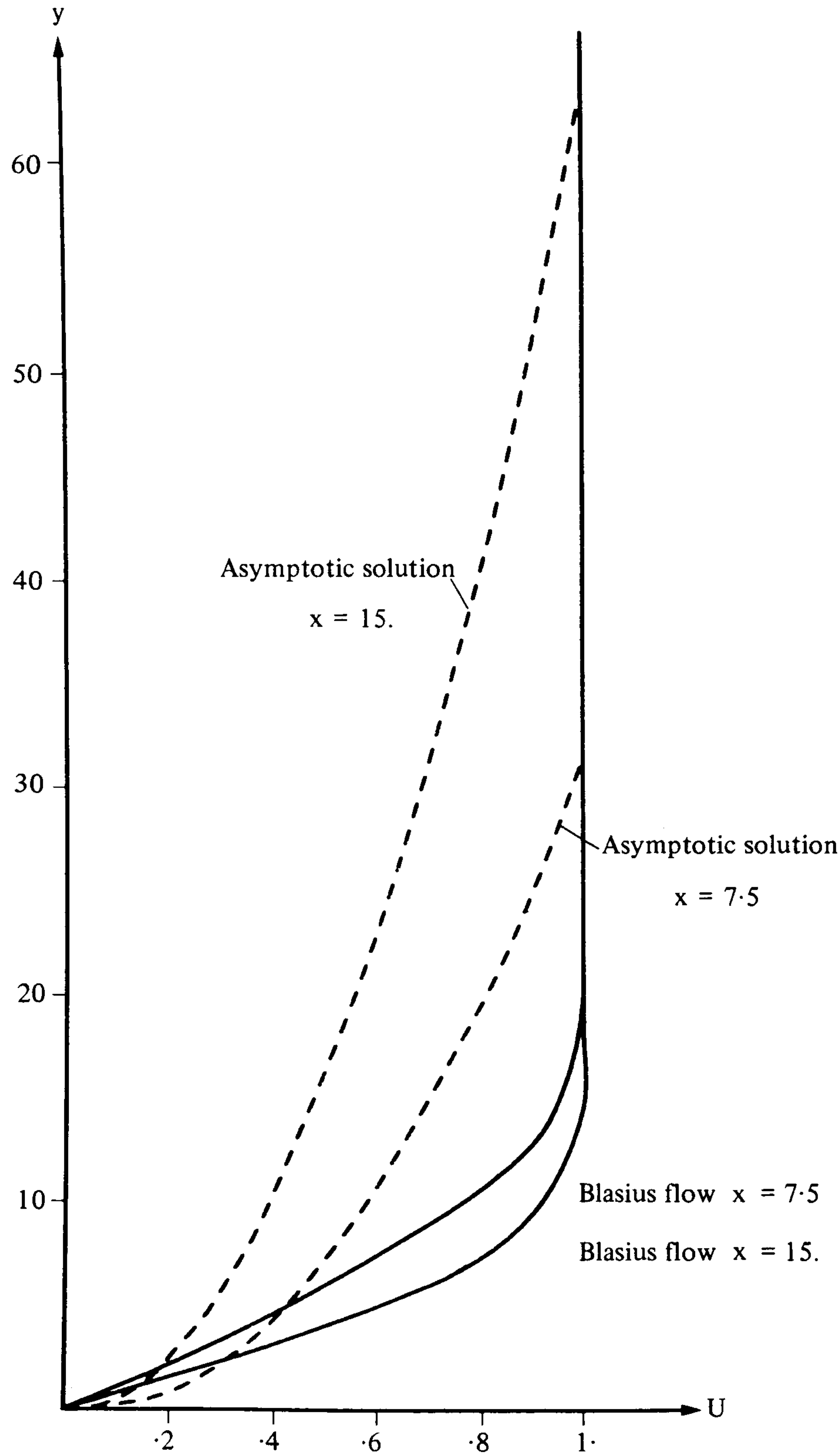


Figure 6d. The streamwise velocity component corresponding to $\chi = 2x$, $g_0 = 4.176$, $x = 7.5,\ 15.$

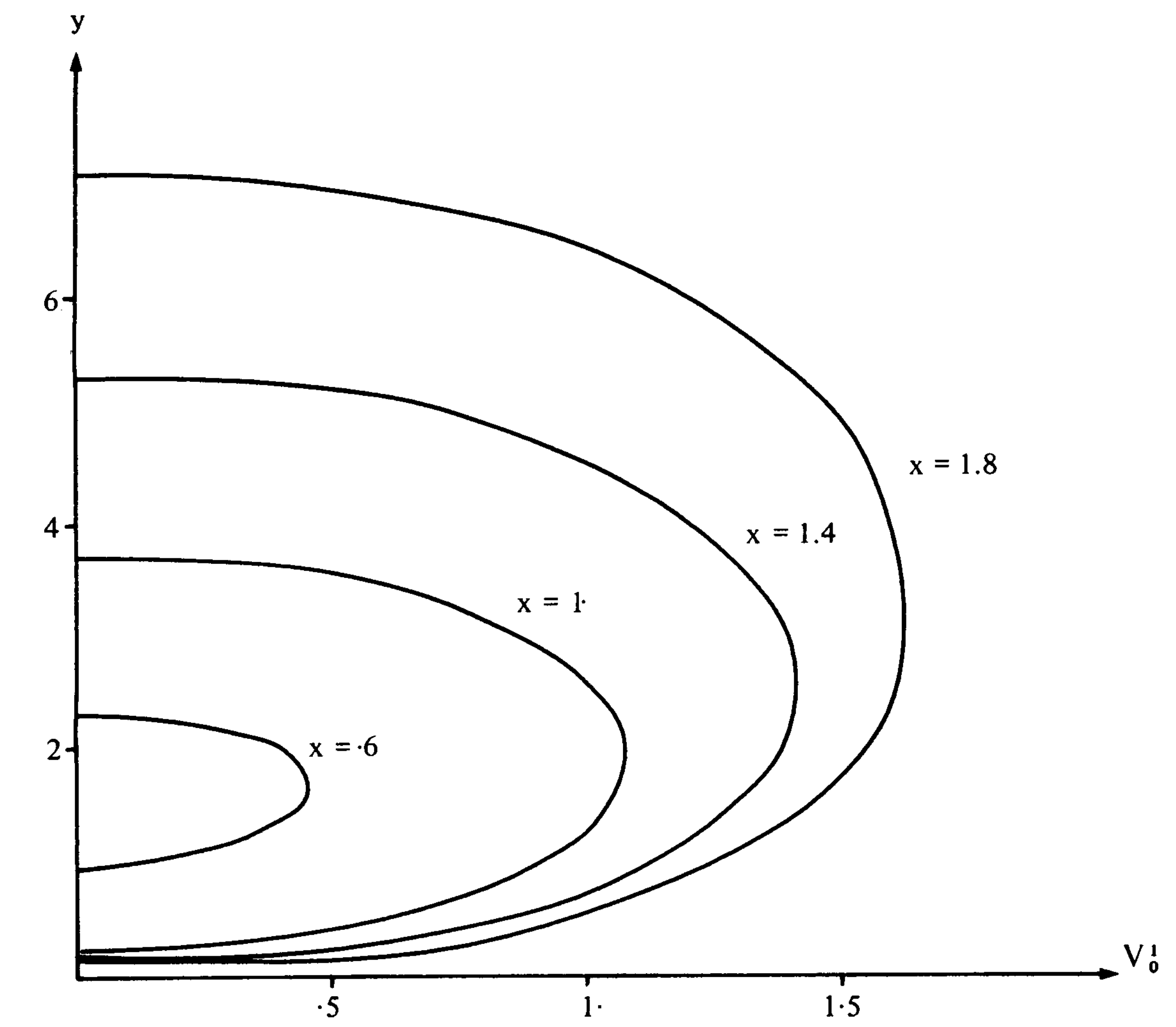


Figure 7. The eigenfunctions $V_0{}^1$ at the downstream location $x = .6, 1., 1.4, 1.8$ for the case $\chi = 2x$, $g_0 = 4.176$.

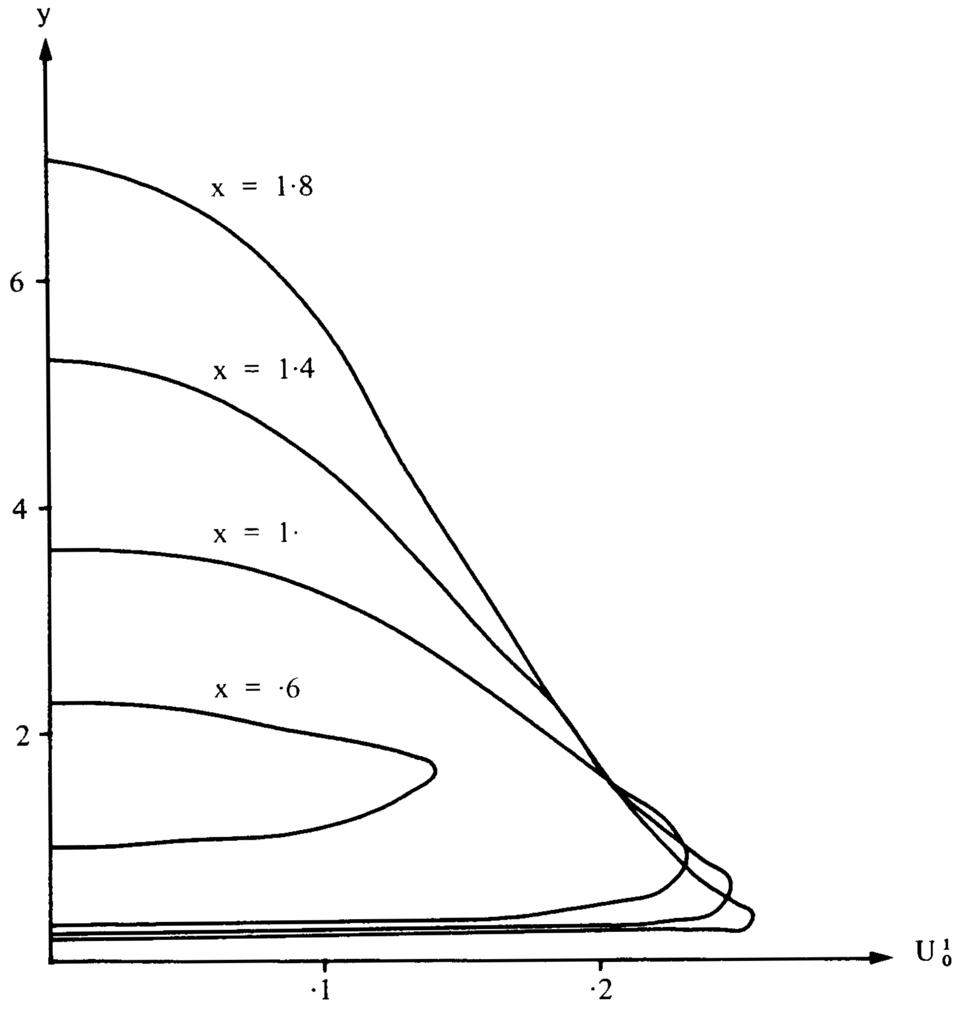


Figure 8. The eigenfunctions $U_0^{\ 1}$ at the downstream ocation
x = .6, 1., 1.4, 1.8 for the case $\chi$ = 2x, $g_0$ = 4.176.

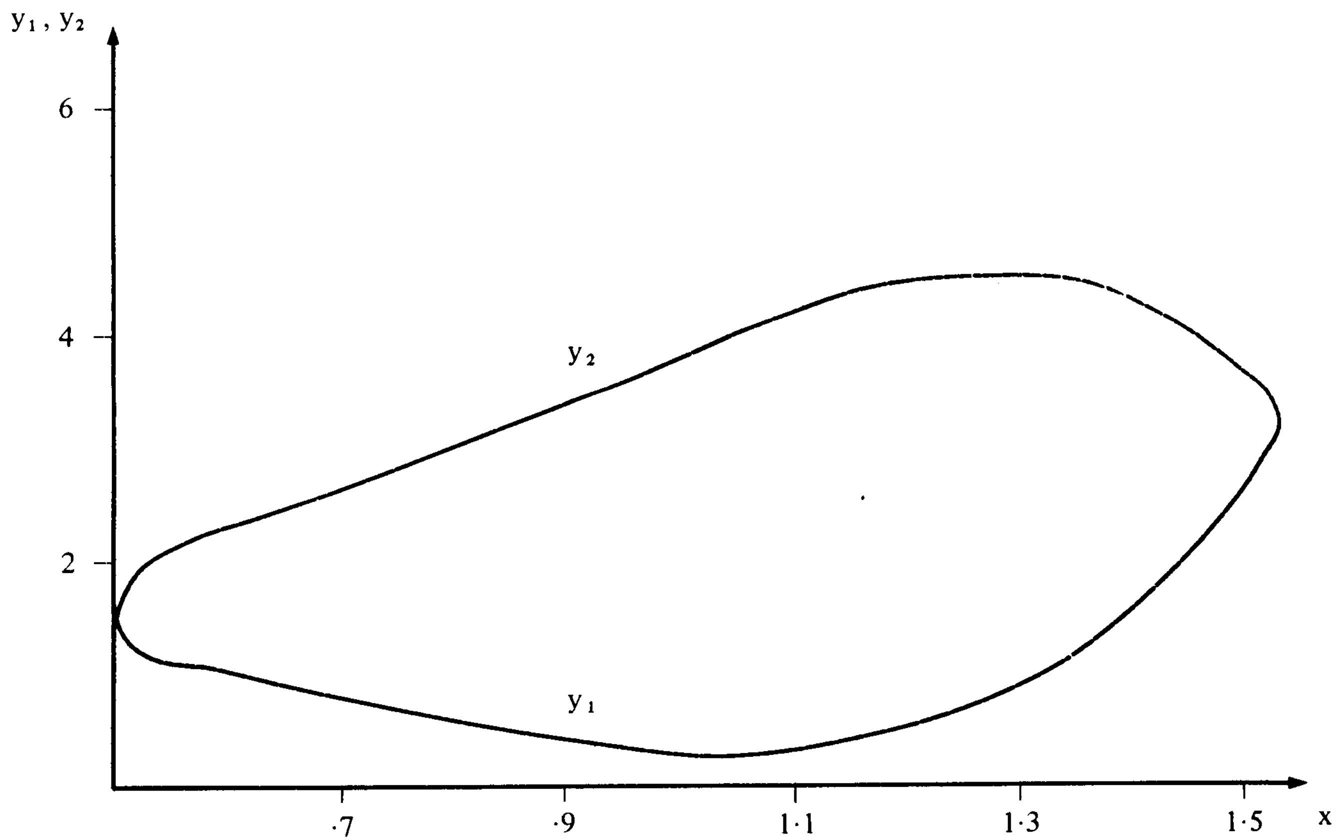


Figure 9. The development of $y_1$ and $y_2$ for the case $g_0 = 4.176$, $\chi = 2x$, $x \leqslant 1$, $\chi = 4x^2 - 2x^3$, $x \geqslant 1$.

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16. Abstract

The fully nonlinear development of small wavelength Görtler vortices in a growing boundary layer is investigated using a combination of asymptotic and numerical methods. The starting point for the analysis is the weakly nonlinear theory of Hall (1982b) who discussed the initial development of small amplitude vortices in a neighbourhood of the location where they first become linearly unstable. That development is unusual in the context of nonlinear stability theory in that it is not described by the Stuart-Watson approach. In fact the development is governed by a pair of coupled nonlinear partial differential evolution equations for the vortex flow and the mean flow correction. Here the further development of this interaction is considered for vortices so large that the mean flow correction driven by them is as large as the basic state. Surprisingly it is found that such a nonlinear interaction can still be described by asymptotic means. It is shown that the vortices spread out across the boundary layer and effectively drive the boundary layer. In fact the system obtained by writing down the equations for the fundamental component of the vortex generate a differential equation for the basic state. Thus the mean flow adjusts so as to make these large amplitude vortices locally neutral. Moreover in the region where the vortices exist the mean flow has a 'square-root' profile and the vortex velocity field can be written down in closed form. The upper and lower boundaries of the region of vortex activity are determined by a free-boundary problem involving the boundary layer equations. In general it is found that this region untimately includes almost all of the original boundary layer and much of the free-stream. In this situation the mean flow has essentially no relationship to the flow which exists in the absence of the vortices.

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