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Stephen L. Keeling

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NASA Langley Research Center, Hampton, Virginia 23665

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Langley Research Center
Hampton, Virginia 23665

On Lipschitz Continuity of Nonlinear Differential Operators

Stephen L. Keeling*

Abstract

In connection with approximations for nonlinear evolution equations, it is standard to assume that nonlinear terms are at least locally Lipschitz continuous. However, it is shown here that $f = f(\mathbf{x}, \nabla u(\mathbf{x}))$ is Lipschitz continuous from the subspace $W^{1,\infty} \subset L_2$ into $W^{-1,2}$, and maps $W^{2,\infty}$ into $W^{1,\infty}$, if and only if f is affine with $W^{1,\infty}$ coefficients. In fact, a local version of this claim is proved.

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1 Introduction

This paper follows efforts to sharply estimate the convergence of some fully discrete approximations for semilinear parabolic partial differential equations [3]. At a certain point in the analysis, it is tempting to postulate that the semilinearity, viewed as a nonlinear operator, is Lipschitz continuous in a sense described below. However, it is proved here that this condition can hold if and only if the function in question is actually affine with respect to the argument for which Lipschitz continuity is assumed. Hence, while Lipschitz assumptions are standard in proving convergence of schemes for nonlinear evolution equations, generalizing them even very weakly to a function space setting may amount to linearizing the equation.

To establish some notation, suppose that Ω is a bounded domain in $\mathbf{R}^N$. For $1 \leq p \leq \infty$ and integers $m \geq 0$, let $W^{m,p}(\Omega)$ represent the well-known Sobolev spaces consisting of functions with distributional derivatives of order $\leq m$ in $L_p(\Omega)$. Also, $\|\cdot\|_{W^{m,p}(\Omega)}$ denotes the usual norm. Next, let $C_0^\infty(\Omega)$ consist of infinitely differentiable functions with support compactly contained in Ω. Completing the latter with respect to $\|\cdot\|_{W^{m,p}(\Omega)}$ produces the spaces $W_0^{m,p}(\Omega)$. Then, for $1 \leq p < \infty$, $p^{-1} + q^{-1} = 1$ and integers $m \geq 1$, define $W^{-m,q}(\Omega) \equiv W_0^{m,p}(\Omega)^*$ equipped with the norm:

$$\|v\|_{W^{-m,q}(\Omega)} \equiv \sup_{u \in W_0^{m,p}(\Omega)} |(u,v)|/\|u\|_{W^{m,p}(\Omega)} \qquad (u,v) \equiv \int_\Omega u(\mathbf{x})v(\mathbf{x})d\mathbf{x}.$$

Finally, let $L_p^N(\Omega)$ represent a Cartesian product of $L_p(\Omega)$ normed in the natural way, e. g. :

$$\|\mathbf{U}\|_{L_\infty^N(\Omega)} \equiv \max_{1 \leq i \leq N} \|U_i\|_{L_\infty(\Omega)}.$$

See Adams [1] for more details.

Now given a function $f : \mathbf{R}^{2\mathbf{N}} \to \mathbf{R}$, the generalized local Lipschitz postulate which would permit a stronger convergence theorem in [3] is that for some $u \in W^{2,\infty}(\Omega)$, and $\rho > 0$:

$$\text{(1.1.i)} \quad \begin{cases} \exists c_\rho > 0 \quad \text{such that} \quad \forall U_1, U_2 \in W^{1,\infty}(\Omega) \quad \text{satisfying} \quad \max_{m=1,2} \|\nabla U_m - \nabla u\|_{L_\infty^N(\Omega)} \leq \rho \\ \|f(\nabla U_2) - f(\nabla U_1)\|_{W^{-1,2}(\Omega)} \leq c_\rho \|U_2 - U_1\|_{L_2(\Omega)}. \end{cases}$$

However, with the additional assumption:

$$\text{(1.1.ii)} \quad \begin{cases} \forall v \in W^{2,\infty}(\Omega) \quad \text{satisfying} \quad \|\nabla v - \nabla u\|_{L_\infty^N(\Omega)} \leq \rho \\ f(\mathbf{x}, \nabla v(\mathbf{x})) \in W^{1,\infty}(\Omega) \end{cases}$$

it is shown in section 3 that (1.1.i) and (1.1.ii) are actually equivalent to the following:

$$\text{(1.2)} \quad \begin{cases} \exists \ \{f_m(\mathbf{x})\}_{m=0}^N \subset W^{1,\infty}(\Omega), \quad \mathbf{f} \equiv \langle f_1, f_2, \ldots, f_N \rangle^T \\ \text{such that} \quad \forall w \in W^{1,\infty}(\Omega) \quad \text{satisfying} \quad \|\nabla w - \nabla u\|_{L_\infty^N(\Omega)} \leq \rho \\ f(\mathbf{x}, \nabla w(\mathbf{x})) = f_0(\mathbf{x}) + \mathbf{f}(\mathbf{x}) \cdot \nabla w(\mathbf{x}). \end{cases}$$

This equivalence is established in Theorem 3.1 using techniques found in Dacorogna [2], where for example, Theorem 2.1 is proved. There are various aspects of the latter which impede its adaptation for the question at hand. However, most important is the fact that the set $\{\nabla U : U \in W^{1,\infty}(D)\}$ is not dense in $L_\infty^N(D)$ for $N \geq 2$, as demonstrated in Lemma 2.2. In spite of this, results in Chapter 4 of Morrey [4] can be distilled to obtain Theorem 2.2. For the significance of the arbitrariness of D, note that Morrey's proof requires sequential weak $\star$ continuity of $G(u, D)$ for vanishingly small hypercubes. On the other hand, (1.1.i) and (1.1.ii) are equivalent to (1.2) for a fixed, but arbitrarily bounded domain Ω. Finally, for [3], it is important not to append regularity assumptions to (1.1.i) and (1.1.ii), since for example, finite element approximation subspaces consisting of continuous piecewise linear functions are only in $W^{1,\infty}(\Omega)$. Nevertheless, Example 3 below shows that assuming additional regularity widens the class of functions for which the generalized local Lipschitz inequality holds.

2 Examples and Related Results

In this section, a few examples are offered to capture the spirit of claims made in the Introduction. The first two are intended to demonstrate the restrictive character of (1.1.i).

Example 1. Let $N = 1$, $\Omega \equiv (0,1)$, and $f(p) \equiv p^2$. Now, for arbitrary $\rho > 0$, a sequence $\{U_n\}_{n=1}^{\infty} \subset W^{1,\infty}(\Omega)$ is constructed in such a way that for a certain $u \in W^{2,\infty}(\Omega)$:

$$||D_x U_n - D_x u||_{L_\infty(\Omega)} = \rho \qquad \forall n$$

and:

$$||U_n - u||_{L_2(\Omega)} \quad \stackrel{n\to\infty}{\longrightarrow} \quad 0$$

while:

$$\begin{aligned} ||f(D_x U_n) - f(D_x u)||_{W^{-1,2}(\Omega)} &\geq |\textstyle\int_0^1 [f(D_x U_n(x)) - f(D_x u(x))]\phi(x)dx| \\ &= \rho^2 |\textstyle\int_0^1 \phi(x)dx| \qquad \forall \phi \in W_0^{1,2}(\Omega), \quad ||\phi||_{W^{1,2}(\Omega)} = 1. \end{aligned}$$

The plan is to construct a sequence of *saw-toothed* functions which converge to zero as f remains constant. First define the characteristic function for $[0, \frac{1}{2}]$:

$$\chi(x) \equiv \begin{cases} 1 & 0 \leq x \leq \frac{1}{2} \\ 0 & \frac{1}{2} < x \leq 1. \end{cases}$$

Now, let $U(x)$ be given by:

$$U(x) \equiv \rho x \chi(x) + \rho(1-x)[1-\chi(x)]$$

and extend this function by periodicity to $\mathbf{R}$ to obtain $\bar{U}(x)$. Similarly, let $\bar{\chi}(x)$ be the periodic extension of $\chi(x)$. Next, set:

$$U_n(x) \equiv n^{-1}\bar{U}(nx) \qquad \text{and} \qquad \chi_n(x) \equiv \bar{\chi}(nx) \qquad x \in [0,1]$$

so that:

$$D_x U_n(x) \stackrel{w}{=} \rho\chi_n(x) - \rho[1 - \chi_n(x)].$$

Finally, since:

$$f(D_x U_n(x)) = \rho^2 \qquad \text{a. e.}$$

the claim above follows with $u(x) \equiv 0$. ∎

In spite of the simplicity of Example 1, it may not be sufficiently satisfying because f is not monotone, or fails to meet some other favorite condition. So, Example 2 aspires for complete satisfaction but at a small cost. It requires the following Lemma which is also used in the next section. The proof of a special case is given here for completeness. (See Dacorogna [2].)

Lemma 2.1 *Let Q be a hypercube in $\mathbf{R}^N$ and suppose that $\chi \in L_\infty(Q)$. Extend χ by periodicity to $\mathbf{R}^N$ to obtain $\bar{\chi}$ and define $\chi_n(\mathbf{x}) \equiv \bar{\chi}(n\mathbf{x})$. Then the following holds:*

$$\chi_n \quad \underset{L_\infty(Q)}{\overset{\star}{\rightharpoonup}} \quad \frac{1}{\mu(Q)}\int_Q \chi(\mathbf{x})d\mathbf{x} \qquad \text{as } n \to \infty.$$

Proof: Only the case $N = 1$, and $Q \equiv [0,1]$ is considered here. Since the simple functions are dense in $L_1(Q)$, it suffices to show for example, that:

$$\int_0^\alpha \chi_n(x)dx \quad \overset{n\to\infty}{\longrightarrow} \quad \alpha\int_0^1 \chi(x)dx \qquad \forall\alpha \in [0,1].$$

This follows after taking the limit in:

$$\int_0^\alpha \chi_n(x)dx = n^{-1}\int_0^{n\alpha} \bar{\chi}(y)dy = [n\alpha]n^{-1}\int_0^1 \chi(y)dy + n^{-1}\int_{[n\alpha]}^{n\alpha} \chi(y)dy$$

where $[\cdot]$ represents the greatest integer function. ∎

Example 2. Except for the form of f, let every element of Example 1 be transported for use here. Now assume that $f(p)$ is any function which satisfies:

$$f(-\rho) + f(\rho) \neq 2f(0).$$

First, choose an arbitrary $\phi \in W_0^{1,2}(\Omega)$ with a nonzero average value and $\|\phi\|_{W^{1,2}(\Omega)} = 1$. Since $\phi \in L_1(\Omega)$ also, it follows from Lemma 2.1 that:

$$\begin{aligned}
\|f(D_x U_n) - f(D_x u)\|_{W^{-1,2}(\Omega)} \quad &\geq \quad |\textstyle\int_0^1 \{f(D_x U_n(x)) - f(0)\}\phi(x)dx| \\
&= \quad |\textstyle\int_0^1 \{f(\rho)\chi_n(x) + f(-\rho)[1 - \chi_n(x)] - f(0)\}\phi(x)dx| \\
&\overset{n\to\infty}{\longrightarrow} \quad |\{f(\rho)\tfrac{1}{2} + f(-\rho)[1 - \tfrac{1}{2}] - f(0)\}\textstyle\int_0^1 \phi(x)dx| \quad > \quad 0.
\end{aligned}$$

Hence, the left side cannot be made to vanish as $\|U_n\|_{L_2(\Omega)} \overset{n\to\infty}{\longrightarrow} 0$. ∎

These examples also suggest that the method of characteristic functions used to prove the following might be useful in proving Theorem 3.1. (See Dacorogna [2].)

Theorem 2.1 *Let $g : \mathbf{R}^N \to \mathbf{R}$ be continuous and define:*

$$G(\mathbf{U}, D) \equiv \int_D g(\mathbf{U}(\mathbf{x}))d\mathbf{x} \qquad \mathbf{U} \in L_\infty^N(D), \quad D \subset \mathbf{R}^N.$$

Then $G(\mathbf{U}, D)$ is sequentially weak $\star$ continuous for every $D \subset \mathbf{R}^N$ if and only if g is affine, i. e., for every $D \subset \mathbf{R}^N$:

$$G(\mathbf{U}_n, D) \quad \overset{n\to\infty}{\longrightarrow} \quad G(\mathbf{U}, D)$$

whenever:

$$\int_D \mathbf{U}_n(\mathbf{x}) \cdot \mathbf{\Phi}(\mathbf{x})d\mathbf{x} \quad \overset{n\to\infty}{\longrightarrow} \quad \int_D \mathbf{U}(\mathbf{x}) \cdot \mathbf{\Phi}(\mathbf{x})d\mathbf{x} \qquad \forall \mathbf{\Phi} \in L_1^N(D)$$

if and only if:

$$g(\lambda \mathbf{a} + (1-\lambda)\mathbf{b}) = \lambda g(\mathbf{a}) + (1-\lambda)g(\mathbf{b}) \qquad \forall \lambda \in [0,1], \quad \forall \mathbf{a}, \mathbf{b} \in \mathbf{R}^N.$$

■

Now, the next Lemma is presented to demonstrate the limits of Theorem 2.1 in connection with weak $\star$ convergence in $W^{1,\infty}(\Omega)$.

Lemma 2.2 *Let $N \geq 2$ and suppose D is any domain in $\mathbf{R}^N$. Then $\{\nabla U : U \in W^{1,\infty}(D)\}$ is not dense in $L_\infty^N(D)$.*

Proof. First, fix $\mathbf{x}_0 \in D$ and let $Q \subset D$ be a hypercube centered at $\mathbf{x}_0$. Then, note that since $W^{1,\infty}(Q) \hookrightarrow C^0(Q)$ [1], the set:

$$W_0 \equiv \{U \in W^{1,\infty}(Q) : \ U(\mathbf{x}_0) = 0\}$$

is a well-defined closed linear subspace of $W^{1,\infty}(Q)$. Also, when applied to gradients, $\|\cdot\|_{L_\infty^N(Q)}$ is actually a norm on W_0 equivalent to $\|\cdot\|_{W^{1,\infty}(Q)}$. If it were not so, there would exist a sequence $\{V_n\}_{n=1}^\infty \subset W_0$ such that:

$$\|V_n\|_{W^{1,\infty}(Q)} = 1 \qquad \forall n$$

while:

$$\|\nabla V_n\|_{L_\infty^N(Q)} \quad \overset{n\to\infty}{\longrightarrow} \quad 0.$$

Since the imbedding $W^{1,\infty}(Q) \hookrightarrow L_\infty(Q)$ is compact [1], there is a subsequence which converges in $L_\infty(Q)$ and hence in $W^{1,\infty}(Q)$. Further, the limit $V \in W_0$ must be constant and satisfy $\|V\|_{W^{1,\infty}(Q)} = 1$. However, this leads to a contradiction since $V \equiv V(\mathbf{x}_0) = 0$.

Now, choose a smooth $\mathbf{V} \in L_\infty^N(D)$ for which:

$$\partial_{x_2} V_1(\mathbf{x}) \neq \partial_{x_1} V_2(\mathbf{x}) \qquad \mathbf{x} \in Q$$

and suppose there exists a sequence $\{\nabla \tilde{U}_n\}_{n=1}^\infty \subset \{\nabla U : U \in W^{1,\infty}(D)\}$ such that:

$$\|\nabla \tilde{U}_n - \mathbf{V}\|_{L_\infty^N(D)} \quad \overset{n\to\infty}{\longrightarrow} \quad 0.$$

Then for $n \geq 1$, select $U_n \in W_0$ to satisfy:

$$\nabla U_n(\mathbf{x}) = \nabla \tilde{U}_n(\mathbf{x}) \qquad \mathbf{x} \in Q$$

so that:

$$\|\nabla U_n - \mathbf{V}\|_{L^N_\infty(Q)} \quad \overset{n\to\infty}{\longrightarrow} \quad 0.$$

Hence, $\mathbf{V}$ must be the gradient of some smooth $U \in W^0$. However, since $\partial^2_{x_2 x_1} U = \partial^2_{x_1 x_2} U$ cannot hold, the contradiction completes the proof. ∎

In spite of this Lemma, there is the following generalization of Theorem 2.1 [4].

Theorem 2.2 *Let $g : \mathbf{R}^{2N+1} \to \mathbf{R}$ be continuous and define:*

$$G(u, D) \equiv \int_D g(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x}))d\mathbf{x} \qquad u \in W^{1,\infty}(D), \quad D \subset \mathbf{R}^N.$$

Then $G(u, D)$ is sequentially weak $\star$ continuous for every $D \subset \mathbf{R}^N$ if and only if $g(\mathbf{x}, u, \mathbf{p})$ is affine with respect to $\mathbf{p}$, i. e., for every $D \subset \mathbf{R}^N$:

$$G(u_n, D) \quad \overset{n\to\infty}{\longrightarrow} \quad G(u, D)$$

whenever:

$$\int_D u_n(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} \quad \overset{n\to\infty}{\longrightarrow} \quad \int_D u(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} \qquad \forall\phi \in L_1(D)$$

and:

$$\int_D \nabla u_n(\mathbf{x}) \cdot \mathbf{\Phi}(\mathbf{x})d\mathbf{x} \quad \overset{n\to\infty}{\longrightarrow} \quad \int_D \nabla u(\mathbf{x}) \cdot \mathbf{\Phi}(\mathbf{x})d\mathbf{x} \qquad \forall\mathbf{\Phi} \in L_1^N(D)$$

if and only if:

$$g(\mathbf{x}, u, \lambda\mathbf{a} + (1-\lambda)\mathbf{b}) = \lambda g(\mathbf{x}, u, \mathbf{a}) + (1-\lambda)g(\mathbf{x}, u, \mathbf{b})$$

$$\forall\lambda \in [0,1], \quad \forall(\mathbf{x}, u) \in \mathbf{R}^{N+1}, \quad \forall\mathbf{a}, \mathbf{b} \in \mathbf{R}^N.$$

∎

Actually, with $W^{1,\infty}(D)$ viewed as a closed linear subspace $W \subset L_\infty^{N+1}(D)$, its predual is the quotient space $L_1^{N+1}(D)/W^\perp$. Nevertheless, the above is an equivalent formulation of sequential weak $\star$ continuity on $W^{1,\infty}(D)$.

Finally, the next example addresses the question of whether the class of functions satisfying (1.1.i) and (1.1.ii) can be widened by appending regularity assumptions.

Example 3. As in Example 1, let $N = 1$, $\Omega \equiv (0,1)$, and $f(p) \equiv p^2$. Now choose an arbitrary $u \in W^{2,\infty}(\Omega)$ and consider whether it is possible to show that for some $c_\rho > 0$:

$$\begin{cases} \forall U_1, U_2 \in W^{2,\infty}(\Omega) \quad \text{satisfying} \quad \max_{m=1,2} \|D_x U_m - D_x u\|_{W^{1,\infty}(\Omega)} \le \rho \\ \\ \qquad \|f(D_x U_2) - f(D_x U_1)\|_{W^{-1,2}(\Omega)} \le c_\rho \|U_2 - U_1\|_{L_2(\Omega)}. \end{cases}$$

That this holds with $c_\rho = 2\rho$ can be seen from the following calculation:

$$\begin{aligned} \|f(D_x U_2) - f(D_x U_1)\|_{W^{-1,2}(\Omega)} &= \sup_{\varphi \in W_0^{1,2}(\Omega)} \frac{|(D_x[U_2 - U_1], \varphi D_x[U_2 + U_1])|}{\|\varphi\|_{W^{1,2}(\Omega)}} \\ &= \sup_{\varphi \in W_0^{1,2}(\Omega)} \frac{|([U_2 - U_1], D_x\{\varphi D_x[U_2 + U_1]\})|}{\|\varphi\|_{W^{1,2}(\Omega)}} \\ &\le \|U_2 - U_1\|_{L_2(\Omega)} \|D_x[U_2 + U_1]\|_{W^{1,\infty}(\Omega)}. \end{aligned}$$

■

3 Demonstration of the Theorem

In this section, the equivalence advertised in the Introduction is established in the following.

Theorem 3.1 *Let Ω be a bounded domain in $\mathbf{R}^N$. Also, suppose $f : \mathbf{R}^{2\mathbf{N}} \to \mathbf{R}$, $u \in W^{2,\infty}(\Omega)$, and $\rho > 0$. Then (1.2) is necessary and sufficient for (1.1.i) and (1.1.ii).*

Proof: First, sufficiency is established. Condition (1.1.ii) follows immediately from (1.2). Now, let U_1 and U_2 satisfy:

$$\max_{m=1,2} ||\nabla U_m - \nabla u||_{L^N_\infty(\Omega)} \leq \rho.$$

Using (1.2), condition (1.1.i) is obtained as follows:

$$\begin{aligned} ||f(\nabla U_2) - f(\nabla U_1)||_{W^{-1,2}(\Omega)} &= \sup_{\varphi \in W_0^{1,2}(\Omega)} \frac{|(\mathbf{f} \cdot \nabla[U_2 - U_1], \varphi)|}{||\varphi||_{W^{1,2}(\Omega)}} \\ &= \sup_{\varphi \in W_0^{1,2}(\Omega)} \frac{|([U_2 - U_1], \nabla \cdot [\varphi \mathbf{f}])|}{||\varphi||_{W^{1,2}(\Omega)}} \\ &\leq c \max_{1 \leq i \leq N} ||f_i||_{W^{1,\infty}(\Omega)} ||U_2 - U_1||_{L_2(\Omega)}. \end{aligned}$$

For necessity, it is first shown that:

$$(3.1) \quad \begin{cases} \forall \lambda \in [0,1], \quad \forall \mathbf{x} \in \Omega, \quad \forall \mathbf{a}, \mathbf{b} \in \mathbf{R}^N \quad \text{such that} \quad \max\{||\mathbf{a}||_{\ell_\infty}, ||\mathbf{b}||_{\ell_\infty}\} \leq \rho \\ f(\mathbf{x}, \nabla u(\mathbf{x}) + [\lambda \mathbf{a} + (1-\lambda)\mathbf{b}]) = \lambda f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{a}) + (1-\lambda) f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{b}) \end{cases}$$

where $|| \cdot ||_{\ell_p}$ represents the usual norm on $\mathbf{R}^N$. Now fix $\lambda \in (0,1)$, $\mathbf{a}, \mathbf{b} \in \mathbf{R}^N$ satisfying:

$$(3.2) \qquad \max\{||\mathbf{a}||_{\ell_\infty}, ||\mathbf{b}||_{\ell_\infty}\} \leq \rho$$

and define:

$$\mathbf{c} \equiv (1-\lambda)(\mathbf{a} - \mathbf{b}) \qquad \mathbf{d} \equiv \lambda \mathbf{a} + (1-\lambda)\mathbf{b}.$$

Then, let Q be a hypercube containing Ω, with two of its $(N-1)$-dimensional faces F_1 and F_2 orthogonal to $\mathbf{c}$:

$$F_i \equiv \{\mathbf{x} \in Q : \ \mathbf{c} \cdot \mathbf{x} = \alpha_i\} \qquad i = 1,2.$$

Also, define:

$$(3.3) \qquad F_\lambda \equiv \{\mathbf{x} \in Q : \ \mathbf{c} \cdot \mathbf{x} = \alpha_\lambda\} \qquad \alpha_\lambda \equiv (1-\lambda)\alpha_1 + \lambda \alpha_2$$

and let the convex hull of $\{F_1, F_\lambda\}$ be represented by:

$$Q_\lambda \equiv \{\mathbf{x} \in Q : \ \mathbf{x} = t\mathbf{x}_1 + (1-t)\mathbf{x}_\lambda, \quad t \in [0,1], \quad \mathbf{x}_1 \in F_1, \ \mathbf{x}_\lambda \in F_\lambda\}.$$

Before proceeding, it is shown that:

$$\mu(Q_\lambda) = \lambda\mu(Q). \tag{3.4}$$

Let $\mathbf{q}_1 \in F_1$ and $\mathbf{q}_2 \in F_2$ be vertices forming an edge of Q, chosen so that $\|\mathbf{q}_2 - \mathbf{q}_1\|_{\ell_2}^N = \mu(Q)$. Then, $\mathbf{q}_\lambda \equiv \mathbf{q}_1 + \lambda(\mathbf{q}_2 - \mathbf{q}_1) \in F_\lambda$ since:

$$\mathbf{q}_\lambda \cdot \mathbf{c} = \alpha_1 + \lambda(\alpha_2 - \alpha_1) = \alpha_\lambda.$$

Thus:

$$\mu(Q_\lambda) = \|\mathbf{q}_\lambda - \mathbf{q}_1\|_{\ell_2}\|\mathbf{q}_2 - \mathbf{q}_1\|_{\ell_2}^{N-1} = \lambda\|\mathbf{q}_2 - \mathbf{q}_1\|_{\ell_2}^N = \lambda\mu(Q)$$

and (3.4) is obtained. Now on Q, define the characteristic function of Q_λ:

$$\chi(\mathbf{x}) \equiv \begin{cases} 1 & \mathbf{x} \in Q_\lambda \\ 0 & \mathbf{x} \in Q \backslash Q_\lambda. \end{cases}$$

Let $\bar{\chi}$ be the periodic extension of χ to $\mathbf{R}^N$, and define:

$$\chi_n(\mathbf{x}) \equiv \bar{\chi}(n\mathbf{x}) \qquad \mathbf{x} \in Q.$$

By (3.4) and Lemma 2.1:

$$\chi_n \underset{L_\infty(Q)}{\overset{*}{\rightharpoonup}} \frac{1}{\mu(Q)}\int_Q \chi(\mathbf{x})d\mathbf{x} \quad = \quad \lambda \qquad \text{as } n \to \infty. \tag{3.5}$$

Now on Q, define the ***hypertent*** function:

$$V_0(\mathbf{x}) \equiv (\mathbf{c}\cdot\mathbf{x} - \alpha_1)\chi(\mathbf{x}) - \lambda(1-\lambda)^{-1}(\mathbf{c}\cdot\mathbf{x} - \alpha_2)[1 - \chi(\mathbf{x})] \qquad \mathbf{x} \in Q.$$

By (3.3):

$$\lim_{Q_\lambda \ni \mathbf{x} \to F_\lambda} V_0(\mathbf{x}) = \alpha_\lambda - \alpha_1 = \lambda(\alpha_2 - \alpha_1) = -\lambda(1-\lambda)^{-1}(\alpha_\lambda - \alpha_2) = \lim_{Q\backslash Q_\lambda \ni \mathbf{x} \to F_\lambda} V_0(\mathbf{x}).$$

Hence, $V_0 \in C^0(Q)$ and:

$$\nabla V_0(\mathbf{x}) \overset{w}{=} \mathbf{c}\{\chi(\mathbf{x}) - \lambda(1-\lambda)^{-1}[1 - \chi(\mathbf{x})]\} \qquad \mathbf{x} \in Q.$$

Therefore, $V_0 \in W^{1,\infty}(Q)$. Further, since V_0 is constant on hyperplanes parallel to F_λ, and zero on F_1 and F_2, it can be extended periodically to $\mathbf{R}^N$ to obtain $\bar{V}_0 \in W^{1,\infty}(\mathbf{R}^N)$. Now on Q, define:

$$V_n(\mathbf{x}) = n^{-1}\bar{V}_0(n\mathbf{x}) \qquad \mathbf{x} \in Q$$

so that:

$$\nabla V_n(\mathbf{x}) \quad \overset{w}{=} \quad \mathbf{c}\{\chi_n(\mathbf{x}) - \lambda(1-\lambda)^{-1}[1 - \chi_n(\mathbf{x})]\} \qquad \mathbf{x} \in Q$$

and:

$$\|V_n\|_{L_2(\Omega)} \leq n^{-1}\mu(\Omega)\sup_{\mathbf{x}\in Q}|V_0(\mathbf{x})| \quad \overset{n\to\infty}{\longrightarrow} \quad 0.$$

Then on Ω, define:

$$U(\mathbf{x}) \equiv u(\mathbf{x}) + \mathbf{d}\cdot\mathbf{x} \qquad \text{and} \qquad U_n(\mathbf{x}) \equiv u(\mathbf{x}) + \mathbf{d}\cdot\mathbf{x} + V_n(\mathbf{x}) \qquad \mathbf{x}\in\Omega$$

so that:

$$\|U - U_n\|_{L_2(\Omega)} \overset{n\to\infty}{\longrightarrow} 0. \tag{3.6}$$

Also, note that by (3.2):

$$\|\nabla U - \nabla u\|_{L_\infty^N(\Omega)} = \|\mathbf{d}\|_{\ell_\infty} \leq \lambda\|\mathbf{a}\|_{\ell_\infty} + (1-\lambda)\|\mathbf{b}\|_{\ell_\infty} \leq \rho \tag{3.7}$$

and:

$$\begin{aligned} &\|\nabla U_n - \nabla u\|_{L_\infty^N(\Omega)} \\ &= \|\mathbf{d} + \nabla V_n\|_{L_\infty^N(\Omega)} = \|(\mathbf{d}+\mathbf{c})\chi_n + (\mathbf{d} - \lambda(1-\lambda)^{-1}\mathbf{c})[1-\chi_n]\|_{L_\infty^N(\Omega)} \\ &= \|\mathbf{a}\chi_n + \mathbf{b}[1-\chi_n]\|_{L_\infty^N(\Omega)} \leq \max\{\|\mathbf{a}\|_{\ell_\infty}, \|\mathbf{b}\|_{\ell_\infty}\} \leq \rho. \end{aligned} \tag{3.8}$$

Thus, according to (3.6) - (3.8), and (1.1.i):

$$\|f(\nabla U) - f(\nabla U_n)\|_{W^{-1,2}(\Omega)} \overset{n\to\infty}{\longrightarrow} 0$$

or:

$$\begin{aligned} &\int_\Omega [f(\mathbf{x}, \nabla u(\mathbf{x}) + \lambda\mathbf{a} + (1-\lambda)\mathbf{b}) - f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{d} + \nabla V_n(\mathbf{x})]\varphi(\mathbf{x})d\mathbf{x} \\ &\qquad \overset{n\to\infty}{\longrightarrow} 0 \qquad\qquad \forall\varphi\in W_0^{1,2}(\Omega). \end{aligned} \tag{3.9}$$

So, once it is established that:

$$\begin{aligned} &\int_\Omega [\lambda f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{a}) + (1-\lambda) f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{b}) - f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{d} + \nabla V_n(\mathbf{x})]\varphi(\mathbf{x})d\mathbf{x} \\ &\qquad \overset{n\to\infty}{\longrightarrow} 0 \qquad\qquad \forall\varphi\in W_0^{1,2}(\Omega) \end{aligned} \tag{3.10}$$

the claim (3.1) follows from (3.9) and (3.10). For (3.10), note that:

$$\begin{aligned} &f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{d} + \nabla V_n(\mathbf{x})) \\ &= f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{d} + \mathbf{c})\chi_n(\mathbf{x}) + f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{d} - \lambda(1-\lambda)^{-1}\mathbf{c})[1-\chi_n(\mathbf{x})] \\ &= f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{a})\chi_n(\mathbf{x}) + f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{b})[1-\chi_n(\mathbf{x})] \qquad \mathbf{x}\in\Omega. \end{aligned}$$

Now, for any $\varphi \in W_0^{1,2}(\Omega)$, by (1.1.ii) and (3.2):

$$\varphi_a(\mathbf{x}) \equiv \varphi(\mathbf{x}) f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{a}) \qquad \text{and} \qquad \varphi_b(\mathbf{x}) \equiv \varphi(\mathbf{x}) f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{b})$$

can be extended by zero to give:

$$\bar{\varphi}_a(\mathbf{x}),\ \ \bar{\varphi}_b(\mathbf{x}) \in L_1(Q).$$

Hence, with (3.5):

$$\int_\Omega f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{d} + \nabla V_n(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x}$$

$$= \int_Q \{\bar{\varphi}_a(\mathbf{x})\chi_n(\mathbf{x}) + \bar{\varphi}_b(\mathbf{x})[1 - \chi_n(\mathbf{x})]\}d\mathbf{x} \quad \stackrel{n\to\infty}{\longrightarrow} \quad \int_Q \{\bar{\varphi}_a(\mathbf{x})\lambda + \bar{\varphi}_b(\mathbf{x})[1-\lambda]\}d\mathbf{x}$$

$$= \int_\Omega \{\lambda f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{a}) + (1-\lambda) f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{b})\}\varphi(\mathbf{x})d\mathbf{x}$$

and (3.10) is obtained.

Condition (1.2) is now extracted from (1.1.ii) and (3.1). First, select any $\mathbf{v} \in \mathbf{R}^N$ for which:

$$\|\mathbf{v}\|_{\ell_\infty} \leq \rho.$$

Then with δ_{ij} denoting the Kronecker delta, let a basis $\{\mathbf{z}^i\}_{i=1}^N \subset \mathbf{R}^N$, and an $\varepsilon > 0$ be chosen arbitrarily but satisfying:

$$|z_j^i| = \varepsilon\delta_{ij} \qquad 1 \leq i,j \leq N$$

in addition to:

$$\|\mathbf{v} + s\mathbf{z}^i + t\mathbf{z}^j\|_{\ell_\infty} \leq \rho \qquad \forall s,t \in [0,1] \quad 1 \leq i,j \leq N.$$

Now, fix $\mathbf{x} \in \Omega$ and for convenience, take:

$$F(\mathbf{y}) \equiv f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{y}) \qquad \|\mathbf{y}\|_{\ell_\infty} \leq \rho.$$

With $h, s, t \in (0,1]$, the following is obtained from repeated applications of (3.1):

$$\begin{aligned}
\Delta^2_{ij}(h,s,t)F(\mathbf{v}) &\equiv h^{-1}\{s^{-1}[F(\mathbf{v} + s\mathbf{z}^i + h\mathbf{z}^j) - F(\mathbf{v} + h\mathbf{z}^j)] - t^{-1}[F(\mathbf{v} + t\mathbf{z}^i) - F(\mathbf{v})]\} \\
&= h^{-1}\{s^{-1}[(1-s)F(\mathbf{v} + h\mathbf{z}^j) + sF(\mathbf{z}^i + \mathbf{v} + h\mathbf{z}^j) - F(\mathbf{v} + h\mathbf{z}^j)] \\
&\qquad -t^{-1}[(1-t)F(\mathbf{v}) + tF(\mathbf{z}^i + \mathbf{v}) - F(\mathbf{v})]\} \\
&= h^{-1}\{(h-1)F(\mathbf{v}) - hF(\mathbf{z}^j + \mathbf{v}) + (1-h)F(\mathbf{z}^i + \mathbf{v}) + hF(\mathbf{z}^j + \mathbf{z}^i + \mathbf{v}) + F(\mathbf{v}) - F(\mathbf{z}^i + \mathbf{v})\} \\
&= 2\{F(\tfrac{1}{2}[\mathbf{z}^j + \mathbf{z}^i + \mathbf{v}] + \tfrac{1}{2}\mathbf{v}) - F(\tfrac{1}{2}[\mathbf{z}^j + \mathbf{v}] + \tfrac{1}{2}[\mathbf{z}^i + \mathbf{v}])\} \quad = \quad 0.
\end{aligned}$$

Hence:

$$\partial^2_{v_i v_j} f(\mathbf{x}, \nabla u(\mathbf{x}) + \mathbf{v}) \equiv 0 \qquad \forall \mathbf{x} \in \Omega, \quad \|\mathbf{v}\|_{\ell_\infty} \leq \rho, \quad 1 \leq i,j \leq N.$$

Now all that remains for (1.2) is establishing the regularity of the coefficients. For this, define:

$$v_k(\mathbf{x}) \equiv u(\mathbf{x}) + \rho x_k \qquad 1 \leq k \leq N$$

so that:

$$\|\nabla v_k - \nabla u\|_{L^N_\infty(\Omega)} \leq \rho.$$

Then according to (1.1.ii):

$$\rho f_k(\mathbf{x}) = f(\mathbf{x}, \nabla v_k(\mathbf{x})) - f(\mathbf{x}, \nabla u(\mathbf{x})) \quad \in \quad W^{1,\infty}(\Omega) \qquad 1 \leq k \leq N.$$

Also:

$$f_0(\mathbf{x}) = f(\mathbf{x}, \nabla u(\mathbf{x})) - \mathbf{f}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) \quad \in \quad W^{1,\infty}(\Omega).$$

Thus, (1.2) is obtained. ■

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16. Abstract

In connection with approximations for nonlinear evolution equations, it is standard to assume that nonlinear terms are at least locally Lipschitz continuous. However, it is shown here that $f = f(\mathbf{x}, \nabla u(\mathbf{x}))$ is Lipschitz continuous from the subspace $W^{1,\infty} \subset L_2$ into $W^{-1,2}$, and maps $W^{2,\infty}$ into $W^{1,\infty}$, if and only if f is affine with $W^{1,\infty}$ coefficients. In fact, a local version of this claim is proved.

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