N87-20935

TOPOLOGICAL CONSTRAINTS AND THE EXISTENCE OF FORCE-FREE FIELDS

S. K. Antiochos
E. O. Hulburt Center for Space Research, Naval Research Lab

INTRODUCTION

A fundamental problem in plasma theory is the question of the existence of MHD equilibria. Given some initial configuration of the field and plasma, and given that the system can evolve only via ideal MHD, one asks whether the system can reach a static equilibrium state. This question obviously has great relevance to fusion devices; hence, it has received considerable attention for closed magnetic configurations (e.g. Moffatt 1985). There is reason to believe that the question of equilibria may also be very important to the study of the solar corona. Parker (1983a,b), in particular, has argued that the underlying cause for coronal heating is the lack of well-behaved magnetic equilibria. As a result of photospheric motions the coronal magnetic field must be distorted into a complex three-dimensional pattern. Parker argues that such a complex field topology can have no well-behaved equilibria in general. He further argues that the effect of this lack of well-behaved equilibria is to lead to the formation of current sheets. Since the corona is not perfectly ideal, the current sheets will dissipate rapidly, thereby heating the corona.

These arguments have some support from the recent work on equilibria in closed field geometries. Moffatt (1985) has shown that for topologically complex geometries, the magnetic field will evolve towards equilibrium configurations that, in general, have discontinuities, specifically current sheets. This occurs even for an evolution that is completely ideal, in which case the field for all finite times must be well-behaved. Moffatt's point is that the equilibrium state will be achieved only at infinite time so that discontinuities can, and usually will be created. Since this evolution is basically the one hypothesised by Parker in his coronal heating model, Moffatt's results are clearly strong support for the central points of this model. However, there is a critical difference between the types of topology considered by Moffatt and the corona. The coronal field lines are not closed; from the viewpoint of coronal equilibria the lines can be considered to terminate at the photospheric boundary. Therefore, one has to include the boundary conditions imposed by the photospheric motions on the possible evolution of the field. So far this has not been done, consequently Moffatt's results by themselves do not definitively settle the question of whether the solar coronal field has well-behaved equilibria in general.

Recently, Parker (1986) has presented a new proof for non-equilibrium based on topological arguments. This proof is limited to force-free

PRECEDING PAGE BLANK NOT FILMED

fields; however, this is not a significant restriction since force-free fields are believed to be a good approximation to the coronal field. They have been widely used in the past (e.g.Sturrock and Woodbury 1967; Barnes and Sturrock 1972; Sakurai 1979; Aly 1984; Yang, Sturrock and Antiochos 1986). Parker first argues that the coronal force-free field can generally be expressed in terms of two scalar functions. Although this is not true for arbitrarily complex field, it should be valid for the case where the field begins in some simple state and then evolves by ideal, finite motions. We make the same restriction in this work and consider only fields that can be described by two well-behaved Euler potentials (e.g. Stern 1966). Next, Parker points out that if the positions of the field-line footpoints at the photospheric boundary is fixed, then the pattern of wrappings and windings of the field lines in the corona imposes constraints on the possible evolution, and hence on the possible equilibria of the field. These constraints are due to the topology of the coronal field, which in turn is due to the history of the photospheric footpoint motions. In order for the field to be in equilibrium the two potentials must satisfy the two independent force-free equations as well as all the topological constraints. However, Parker argues that since the pattern of coronal wrappings and windings is arbitrary, one of the potentials is essentially fixed throughout by the topological constraints. This leaves only one free function to satisfy the two force equations; hence, the problem is overdetermined and no well-behaved solutions exist in the general case.

It is evident from these arguments and also from Moffatt's work that the issue of topological constraints is of crucial importance for the problem of the existence of equilibria. In this paper we will show that, countrary to Parker's claim, the topological constraints do not overdetermine the force-free problem. The source of the discrepancy between his results and ours is that we find that the topological constraints do not fix the value of one of the potentials in the corona. We show below that all the constraints are incorporated in the positions of the footpoints at the photospheric boundary, which fixes only the boundary values of the potentials. Since the force-free problem is naturally a Dirichlet-type boundary value problem, there is no reason to expect it to be overdetermined on the basis of these constraints.

Our result that the footpoint positions include all the topological constraints may appear counter-intuitive. If one considers the field as a collection of strings, then for a particular set of positions of the strings' footpoints, the strings should be able to wrap around each other in an infinite number of distinct patterns. Hence, the topology of the strings is clearly not determined solely by their footpoint positions. However, there is a key difference between the magnetic lines and a collection of strings. The field lines that we are considering must have a smooth and continuous distribution throughout the corona and on the boundary. Furthermore, all field lines must be connected to the photosphere. It is these requirements, that the field be smooth and well-connected throughout, that forces a one-to-one relationship between the footpoint positions and the coronal wrapping pattern.

CORONAL WRAPPING PATTERN

In this section we will show how the wrapping of any two field lines about each other can be determined from the footpoint positions, which is commonly referred to as the "connectivity". The discussion in this short contribution will necessarily be somewhat heuristic; a more rigorous and thorough demonstration of this result is given in Antiochos (1986).

First let us reemphasize that we are considering only those fields that can be written in terms of two well-behaved Euler potentials,

$$\vec{B} = \nabla\alpha \times \nabla\beta$$

This form is very convenient for expressing the connectivity. Note that α and β are constant along each field line and, hence, label each line. The footpoint positions is given simply by the value of α and β at the photospheric boundary.

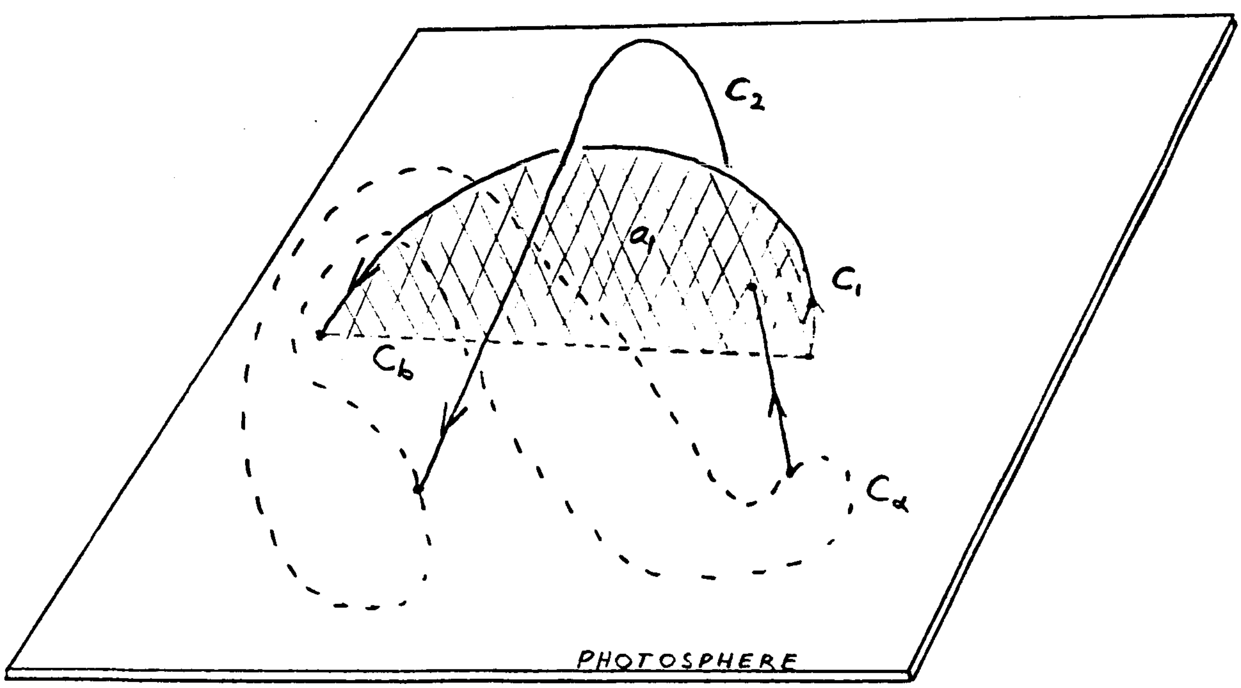

Figure 1. Illustration of curves used to determine the wrapping of field line C_2 about C_1 . Coronal field lines are indicated by solid lines, curves lying on the photospheric boundary (here shown as a plane) are indicated by broken lines. The area $\vec{a}_1$ enclosed by field line C_1 and the boundary curve C_b is shaded.

Now consider any two field lines such as those illustrated in Figure 1. The coronal field lines are labelled as C_1 and C_2 . It seems intuitively obvious from the Figure that field line C_2 wraps once around C_1 . We can make this wrapping concept rigorous by connecting the two footpoints of C_1 on the photospheric boundary by a straight line, labelled C_b , and then considering the area $\vec{a}_1$ enclosed by the closed curve consisting of C_1 and C_b . We define the value of the wrapping of C_2 about C_1 to be given by sum of the number of times that C_2 intersects area $\vec{a}_1$, with each intersection assigned a value of +1 or -1 depending on the direction of the intersection. The wrapping of C_2 about C_1 has a value of -1 for the lines in the Figure 1.

Given the values of the potentials α and β throughout the corona, we can calculate the exact positions of the field lines and consequently the wrapping. However, the field line positions obviously contain a great deal more information than is required to determine the wrapping. We will now show that the values of α and β solely on the photospheric boundary are sufficient to fix the wrapping. The two different field lines C_2 and C_1 have a different value for at least one of the Euler potentials; assume that it is α so that $\alpha_1 \neq \alpha_2$. Consider the surface $\alpha = \alpha_2$. If the connectivity is well-behaved everywhere in the corona and on the photosphere, we expect the Euler potentials to define a set of simple, well-behaved surfaces in this domain. The intersections of these surfaces with the photospheric boundary define the contours of constant α or constant β on the photosphere. In order to have a well-defined connectivity each contour of constant α must intersect each β contour at exactly two points on the photosphere; consequently, we expect that each contour consists of a single closed curve. For example, the boundary contour for $\alpha = \alpha_2$ is shown in Figure 1 as the closed curve C_α . Note that it can be considered to consist of two parts, each of which connects the two foopoints of field line C_2 . Now the key point is that field line C_2 lies completely on the surface $\alpha = \alpha_2$, while field line C_1 nowhere intersects this surface. This implies that with no change in the wrapping, we are free to deform C_2 down along the constant α surface until it just coincides with one of the parts of curve C_α . Since both C_b and C_α lie on the photospheric boundary the number of times that this deformed curve crosses area $\vec{a}_1$ must equal the number of times that either part of C_α crosses C_b . But this number depends only on the straight line C_b that connects the footpoints of C_1 and the contour of constant α on the boundary; therefore the wrapping can be determined solely by the values of α and β on the boundary.

We conclude that for a given set of footpoint positions the wrapping pattern in the corona is completely fixed. Contrary to the assumption implicit in Parker's arguments, one is not free to arbitrarily prescribe a wrapping pattern. Note that the wrapping pattern does not include all the possible topological features of the field (e.g. Berger 1986); however, by extending the arguments above it can be shown (Antiochos 1986) that all the topological features created by the footpoint motions of well-behaved fields can be determined from the connectivity. Hence, the topological constraints are included in the boundary conditions on the Euler potentials and impose no additional restrictions on possible equilibria. Although this does not prove that equilibria always exist, it does show that the force-free problem is not overdetermined and that the existence of equilibria is still an open question.

ACKNOWLEDGEMENTS

This work was supported in part by a grant from the Office of Solar and Heliosperic Physics at NASA.

REFERENCES

Aly, J. J. 1984, Ap. J., 283, 349.
Antiochos, S. K. 1986, Ap. J., (submitted).
Barnes, C. W. and Sturrock, P. A. 1972, Ap. J., 174, 659.
Berger, M. A. 1986, (these proceedings).
Moffatt, H. K. 1985, J. Fluid Mech., 159, 359.
Parker, E. N. 1986, Geophys. Ap. Fluid Dyn., 34, 243.
Parker, E. N. 1983a, Ap. J., 264, 635.
Parker, E. N. 1983b, Ap. J., 264, 642.
Sakurai, T. 1979, Pub. Ast. Soc. Japan, 31, 209.
Stern, D. P. 1966, Space Sci. Rev., 6, 147.
Sturrock, P. A. and Woodbury, E. T. 1967, "Plasma Astrophysics", (ed. P. A. Sturrock, New york: Academic Press), p. 155.
Yang, W. H., Sturrock, P. A. and Antiochos, S. K. 1986, Ap. J., (in press)