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The Binary Weight Distribution of the Extended $(2^m, 2^m-4)$ Code of Reed-Solomon Code over $GF(2^m)$ with Generator Polynomial $(x-\alpha)(x-\alpha^2)(x-\alpha^3)$*

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ABSTRACT: Consider an (n,k) linear code with symbols from $GF(2^m)$. If each code symbol is represented by a binary m-tuple using a certain basis for $GF(2^m)$, we obtain a binary (nm,km) linear code, called a binary image of the original code. In this paper, we present a lower bound on the minimum weight of a binary image of a cyclic code over $GF(2^m)$ and the weight enumerator for a binary image of the extended $(2^m,2^m-4)$ code of Reed-Solomon code over $GF(2^m)$ with generator polynomial $(x-\alpha)(x-\alpha^2)(x-\alpha^3)$ and its dual code, where α is a primitive element in $GF(2^m)$.

1. Introduction

Let $\{\beta_1, \beta_2, \cdots, \beta_m\}$ be a basis of the Galois field $GF(2^m)$. Then each element z in $GF(2^m)$ can be expressed as a linear sum of $\beta_1, \beta_2, \cdots, \beta_m$ as follows:

$$z = c_1\beta_1 + c_2\beta_2 + \cdots + c_m\beta_m,$$

where $c_i \in GF(2)$ for $1 \le i \le m$. Thus z can be represented by the m-tuple $(c_1, c_2, \cdots, c_m)$ over GF(2). Let C be an (n,k) linear block code with symbols from the Galois field $GF(2^m)$. If each code symbol of C is represented by the corresponding m-tuple over the binary field GF(2) using the basis $\{\beta_1, \beta_2, \cdots, \beta_m\}$ for $GF(2^m)$, we obtain a binary (mn, mk) linear block code, called a binary image of C. The weight enumerator of a binary image of C is called a binary weight enumerator of C. In general, a binary weight enumerator depends on the choice of basis. A basis $\{\beta_1, \beta_2, \cdots, \beta_m\}$ is called a polynomial basis, if there is an element $\beta \in GF(2^m)$

such that $\beta_j = \beta^{j-1}$ for $1 \le j \le m$. A polynomial basis will be said to be primitive, if β is primitive.

Let α be a primitive element of $GF(2^m)$, and let $n = 2^m-1$. For $1 \le k < n$, let RS_k denote the (n, k) Reed-Solomon code over $GF(2^m)$ with generator polynomial $(x-\alpha)(x-\alpha^2)\cdots(x-\alpha^{n-k})$ [1], let $RS_{k,e}$ denote the (n, k) Reed-Solomon code over $GF(2^m)$ with generator polynomial $(x-1)(x-\alpha)(x-\alpha^2)\cdots(x-\alpha^{n-k-1})$, and let ERS_k be the extended (n+1, k) code of RS_k. The dual code of RS_k is $RS_{n-k,e}$, and the dual code of ERS_k is ERS_{n+1-k}.

Binary weight enumerators for RS_{n-i} with $1 \le i \le 2$, $RS_{n-i,e}$ with $2 \le i \le 3$ and ERS_{n-i} with $1 \le i \le 2$ were presented in [2], and those for $RS_{2,e}$, the dual code of RS_{n-2}, and RS_3, the dual code of $RS_{n-3,e}$, were derived in [3,4]. These binary weight enumerators are independent of the choice of basis.

In section 2, the binary image of the dual code of a linear code C over $GF(2^m)$ by using the complementary basis of a basis $\{\beta_1, \beta_2, \cdots, \beta_m\}$ is shown to be the dual code of the binary image of C by using basis $\{\beta_1, \beta_2, \cdots, \beta_m\}$. In section 3, a lower bound on the minimum weight of a binary image of a cyclic code over $GF(2^m)$. In section 4, the binary weight enumerator of ERS_4 is derived for a class of bases including the complementary bases of primitive polynomial bases. By Theorem 1 the binary weight enumerator for ERS_{n-3} is obtained. This approach can be readily extended to derive the binary weight enumerator for ERS_5.

2. Binary Images of Linear Block Codes over $GR(2^m)$

Let C be an (n,k) linear code with symbols from $GF(2^m)$. Let $C^{(b)}$ denote the binary (nm,km) linear code obtained from C by representing each code symbol by the corresponding m-tuple over GF(2) using the basis $\{\beta_1, \beta_2, \cdots, \beta_m\}$ for $GF(2^m)$. Let $\{\delta_1, \delta_2, \cdots, \delta_m\}$ be the complementary (or dual) basis of $\{\beta_1,\beta_2,\cdots,\beta_m\}$, i.e.,

$$Tr(\beta_i\delta_j) = 0 , \quad \text{for } i \ne j,$$

$$Tr(\beta_i\delta_i) = 1 ,$$

where Tr(x) denotes the trace of the field element x [5,p.117]. Let C^D be the dual code of C. Let $C^{D(b)}$ denote the binary $(nm,(n-k)m)$ linear code obtained from C^D by representing each code symbol by a binary m-tuple over GF(2) using the complementary basis $\{\delta_1, \delta_2, \cdots, \delta_m\}$ of $\{\beta_1, \beta_2, \cdots, \beta_m\}$. Then we have Theorem 1.

<u>Theorem 1</u>: $C^{D(b)}$ is the dual code of $C^{(b)}$.

<u>Proof</u>: Let $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ be codewords of C and C^D respectively. Then

$$\sum_{i=1}^{n} a_i b_i = 0 . \tag{1}$$

Let

$$a_i = \sum_{j=1}^{m} a_{ij}\beta_j , \tag{2}$$

$$b_i = \sum_{j=1}^{m} b_{ij}\delta_j . \tag{3}$$

It follows from (1) to (3) that

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij}\beta_j \right)\left(\sum_{h=1}^{m} b_{ih}\delta_h \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{h=1}^{m} a_{ij}b_{ih}\beta_j\delta_h = 0 . \tag{4}$$

Taking the trace of both sides of (4), we have

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{h=1}^{m} a_{ij}b_{ih}Tr(\beta_j\delta_h) = 0 . \tag{5}$$

Since $Tr(\beta_j\delta_h) = 0$ for $j \neq h$ and $Tr(\beta_j\delta_j) = 1$, it follows from (5) that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}b_{ij} = 0 . \tag{6}$$

Equation (6) implies that $C^{D(b)}$ is the dual code of $C^{(b)}$. ΔΔ

For a basis $\{\beta_1, \beta_2, \cdots, \beta_m\}$ for $GF(2^m)$ and an n-tuple $\bar{v} = (v_1, v_2, \cdots, v_n)$ over $GF(2^m)$, let $\bar{v}_j$ be defined as

$$\bar{v}_j = (v_{1j}, v_{2j}, \cdots, v_{nj}), \quad \text{for } 1 \le j \le m , \tag{7}$$

where $v_i = \sum_{j=1}^{m} v_{ij}\beta_j$ with $v_{ij} \in GF(2)$ for $1 \le i \le n$. If $\{\delta_1, \delta_2, \cdots, \delta_m\}$

is the complementary basis of $\{\beta_1, \beta_2, \cdots, \beta_m\}$, then $\bar{v}_j$ is represented as

$$\bar{v}_j = (Tr(\delta_j v_1), Tr(\delta_j v_2), \cdots, Tr(\delta_j v_n)) , \qquad (8)$$

and $\bar{v}_j$ is called the δ_j component vector of $\bar{v}$. The binary weight of $\bar{v}$, denoted $|\bar{v}|_2$, is given by

$$|\bar{v}|_2 = \sum_{j=1}^{m} |\bar{v}_j|_2 . \qquad (9)$$

3. Binary Images of Cyclic Codes over $GF(2^m)$

Let n be a positive integer which divides 2^m-1. If s is the smallest number in a cyclotomic coset mod n over $GF(2^m)$, s is called the representative of the coset and the coset is denoted by Cy(s). Let m(s) denote the number of integers in Cy(s). For a subset I of $\{0,1,2, \cdots ,n-1\}$, $\bar{I}$ denotes the set union of those cosets which have a nonempty intersection with I, and Rc(I) denotes the set of the representatives of cyclotomic cosets in $\bar{I}$.

Let γ be an element of order n in $GF(2^m)$. For a subset I of $\{0,1,2, \cdots ,n-1\}$, let C(I) be the cyclic code of length n over $GF(2^m)$ with check polynomial

$$\prod_{i \in I} (x - \gamma^i).$$

and let $C_b(I)$ be the binary cyclic code of length n with check polynomial

$$\prod_{i \in \bar{I}} (x - \gamma^i).$$

For a polynomial $f(X) = \sum_{i=0}^{n-1} a_i X^i$ with $a_i \in GF(2^m)$, let $v[f(X)]$ and $ev[f(X)]$ be defined by

$$v[f(X)] = (f(1), f(\gamma), f(\gamma^2), \cdots , f(\gamma^{n-1})) , \qquad (10)$$

and

$$ev[f(X)] = (f(0), f(1), f(\gamma), \cdots , f(\gamma^{n-1})) . \qquad (11)$$

It follows from (8) and (9) that

$$| v[f(X)] |_2 = \sum_{j=1}^{m} | v[Tr(\delta_j f(X))] |_2 , \quad (12)$$

$$| ev[f(X)] |_2 = \sum_{j=1}^{m} | ev[Tr(\delta_j f(X))] |_2 . \quad (13)$$

For a subset I of {0,1,2, ··· ,n-1}, let P(I) be defined by

$$P(I) = \{ \sum_{i\varepsilon I} a_i X^i \mid a_i \varepsilon GF(2^m) \text{ for } i\varepsilon I \} .$$

As is well-known[5],

$$C(I) = \{v[f(X)] | f\varepsilon P(I)\} .$$

It follows from (8),(10) and the definitions of C(I) and $C_b(I)$ that for $\bar{v}$ = v[f(x)] ε C(I), the δ_j component vector of $\bar{v}$, denoted $\bar{v}_j$, is given by

$$\bar{v}_j = v[Tr(\delta_j f(X))] , \qquad 1 \le j \le m , \quad (14)$$

and

$$\bar{v}_j \ \varepsilon \ C_b(I) . \quad (15)$$

As is also known [5],

$$C_b(I) = \{v[\sum_{i\varepsilon Rc(I)} Tr_{m(i)}(a_i X^i)] | a_i \ \varepsilon \ GF(2^{m(i)}) \text{ for } i \ \varepsilon \ Rc(I) \} , \quad (16)$$

where

$$Tr_j(X) = X + X^2 + \cdots + X^{2^{j-1}} .$$

Polynomial f(X) ε P(I) can be expressed as

$$f(X) = \sum_{i\varepsilon Rc(I)} \sum_{q\varepsilon Q(i,I)} a_{i2^q} X^{i2^q} , \quad (17)$$

where $i2^q$ is taken modulo n and

$$Q(i,I) = \{q \mid p{:}i2^q \equiv p \pmod n),\ p \in I \text{ and } 0 \le q < m(i)\} .$$

It follows from (17) that for $1 \le j \le m$

$$Tr(\delta_j f(X)) = \sum_{i \in Rc(I)} Tr_{m(i)}(b_{ji} X^i) , \tag{18}$$

where

$$b_{ji} = Tr^{(m(i))} \Big(\sum_{q \in Q(i,I)} \delta_j^{2^{m(i)-q}} a_{i2^q}^{2^{m(i)-q}} \Big) , \ i \in Rc(I) , \tag{19}$$

where for a divisor h of m

$$Tr^{(h)}(X) = X + X^{2^h} + X^{2^{2h}} + \cdots + X^{2^{m-h}} . \tag{20}$$

Note that

$$b_{ji} \in GF(2^{m(i)}) . \tag{21}$$

It follows from (14) and (18) that for $1 \le j \le m$

$$\bar{v}_j = v\Big[\sum_{i \in Rc(I)} Tr_{m(i)} (b_{ji} X^i) \Big] . \tag{22}$$

For $i \in Rc(I)$, let $\tilde{C}_i$ be defined by

$$\tilde{C}_i \overset{\Delta}{=} \Big\{ (b_1, b_2, \cdots, b_m) \mid b_j = Tr^{(m(i))}\Big(\sum_{q \in Q(i,I)} \delta_j^{2^{m(i)-q}} a_q' \Big) ,$$

$$1 \le j \le m,\ a_q' \in GF(2^m) \Big\} . \tag{23}$$

Note that the following matrix D over $GF(2^m)$ is invertible [5,p.117] :

$$D \overset{\Delta}{=} \begin{bmatrix} \delta_1 & \delta_1^2 & \delta_1^{2^2} & \cdots & \delta_1^{2^{m-1}} \\ \delta_2 & \delta_2^2 & \delta_2^{2^2} & \cdots & \delta_2^{2^{m-1}} \\ & & & & \\ \delta_m & \delta_m^2 & \delta_m^{2^2} & \cdots & \delta_m^{2^{m-1}} \end{bmatrix} . \tag{24}$$

If $Tr^{(m(i))}\left(\sum_{q\epsilon Q(i,I)} \delta_j^{2^{m(i)-q}} a'_q \right) = 0$ for $1 \le j \le m$, then

$$a'_q = 0, \text{ for } q \in Q(i,I) . \tag{25}$$

Hence $\tilde{C}_i$ is a linear (m, #Q(i,I)m/m(i)) code over $GF(2^{m(i)})$, where #M denotes the number of elements in set M.

For a code C, let mw[C] denote the minimum weight of C. Then the following theorem holds.

Theorem 2 : For $i \in I$,

$$mw[C(I)^{(b)}] \ge \min \{ mw[\tilde{C}_i]mw[C_b(I)], mw[C(I-\overline{\{i\}})^{(b)}] \}, \tag{26}$$

where $mw[C(I-\overline{\{i\}})^b] = \infty$, if $I \subseteq \overline{\{i\}}$.

Proof: If follows from (19) and (25) that $b_{ji} = 0$ for $1 \le j \le m$ if and only if $a_h = 0$ for $h \in I \cap \overline{\{i\}}$. Suppose that there is an integer $h \in I \cap \overline{\{i\}}$ such that $a_h \ne 0$. Then the weight of $(b_{1i}, b_{2i}, \cdots, b_{mi})$ is at least $mw[\tilde{C}_i]$. Hence there are at least $mw[\tilde{C}_i]$ nonzero codewords of $C_b(I)$ in $\{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_m\}$ where $\bar{v}_j$ is given by (22). Then this theorem follows from (12). ΔΔ

The following lemma holds for $\tilde{C}_i$.

Lemma 1: Suppose that $m(i) = m$ and there are integers h and s such that $0 \le h < m$, $0 < s \le m$ and

$$Q(i,I) = \{q \mid m-q \equiv h+j (\bmod m),\ 0 \le q < m \text{ and } 0 \le j < s\}.$$

Then $\tilde{C}_i$ is a maximum distance separable (m,s) code over $GF(2^m)$.

Proof: Consider a polynomial F(X) over $GF(2^m)$ of the following form:

$$F(X) = \sum_{q \in Q(i,I)} c_q X^{2^{m-q}}.$$

Then,

$$F(X)^{2^{m-h}} = \sum_{j=0}^{s-1} c_{m-h-j}^{2^{m-h}} X^{2^j},$$

where the suffix of a coefficient is taken modulo m. Since $F(X)^{2^{m-h}}$ is a linearized polynomial of degree 2^{s-1} or less [5], the zeros of F(X) in $GR(2^m)$ form a subspace of $GF(2^m)$ whose dimension is at most s-1. Hence at most s-1 elements of $\{\delta_1, \delta_2, \cdots, \delta_m\}$ can be roots of F(X). It follows from the definition of $\tilde{C}_j$ that $mw[\tilde{C}_i] = m-s+1$.

Since $\#Q(i,I) = s$, $\bar{C}_i$ is a maximum distance separable (m,s) code.

ΔΔ

Example 1: For an integer m greater than 2, let $n = 2^m - 1$, and let $I = \{1,2,3,4\}$. Then C(I) is $RS_{4,e}$, $Q(3,I) = \{0\}$, and $Q(1,I') = \{0,1,2\}$ where $I' = I - \overline{\{3\}}$. It is known [6,7] that

$$\begin{aligned} mw[C_b(I')] &= 2^{m-1}, && \text{for odd } m,\\ &= 2^{m-1}-2^{m/2-1}, && \text{for even } m \text{ such that } m/2 \text{ is even},\\ &= 2^{m-1}-2^{m/2}, && \text{for even } m \text{ such that } m/2 \text{ is odd}, \end{aligned}$$

and

$$\begin{aligned} mw[C_b(I)] &= 2^{m-1}-2^{(m-1)/2}, && \text{for odd } m,\\ &= 2^{m-1}-2^{m/2}, && \text{for even } m. \end{aligned}$$

Since $mw[\tilde{C}_1] = m-2$ and $mw[\tilde{C}_3] = m$ by Lemma 1, it follows from Theorem 2 that

$$\begin{aligned} mw[C(I)^{(b)}] = mw[C(I')^{(b)}] &\geq (m-2)2^{m-1}, && \text{for odd } m,\\ &\geq (m-2)(2^{m-1}-2^{m/2-1}), && \text{for even } m \text{ such that } m/2 \text{ is even},\\ &\geq (m-2)(2^{m-1}-2^{m/2}), && \text{for even } m \text{ such that } m/2 \text{ is odd}. \end{aligned}$$

ΔΔ

4. Binary Weight Enumerator for ERS_4

Hereafter we assume that

$$m \geq 3,$$

$$n = 2^m-1.$$

For $0 \leq i < j < n$, let

$$I_{i,j} = \{ i, i-1, \cdots, j\}.$$

Then it is known [5] that

$$RS_k = \{ v(f(X)) \mid f(X) \in P(I_{0,k-1}) \} , \tag{27}$$

$$RS_{k,e} = \{ v(f(X)) \mid f(X) \in P(I_{1,k}) \} , \tag{28}$$

and

$$ERS_k = \{ ev(f(X)) \mid f(X) \in P(I_{0,k-1}) \} . \tag{29}$$

For $0 \leq h < n-1$, $v[f(\alpha^h X)]$ is the vector obtained from $v[f(X)]$ by the h symbol cyclic shift, $ev[f(\alpha^h X)]$ is the vector obtained from $ev[f(X)]$ by the h symbol cyclic shift among the second to the last symbols, and

$$|v[f(\alpha^h X)]|_2 = |v[f(X)]|_2 , \tag{30}$$

$$|ev[f(\alpha^h X)]|_2 = |ev[f(X)]|_2 . \tag{31}$$

For $f(X) = a_0 + a_1 X + a_2 X^2 + a_3 X^3 \in P(I_{0,3})$, $ev[f(X)] \in ERS_4 - ERS_3$ if and only if $a_3 \neq 0$. The cyclic permutations on the second to the last symbols induce a permutation group on the codewords of ERS_4, which divides $ERS_4 - ERS_3$ into disjoint set of transitivity. Each set consists of $(2^m-1)/\nu$ codewords, where

$$\nu = (2^m-1, 3) ,$$

where (a,b) denotes the greatest common divisor of integers a and b. If m is odd, then

$$\nu = 1 , \tag{32}$$

and otherwise,

$$\nu = 3 . \tag{33}$$

Let $ev[a_0+a_1X+a_2X^2+\alpha^h X^3]$ for $0 \leq h < \nu$ represent each set of $(2^m-1)/\nu$ codewords of $ERS_4 - ERS_3$. Note that

$$\begin{aligned} &Tr(\delta_j a_0+\delta_j a_1 X+\delta_j a_2 X^2+\delta_j \alpha^h X^3) \\ &\qquad = Tr(\delta_j a_0+[\delta_j a_1+(\delta_j a_2)^{2^{m-1}}]X+\delta_j \alpha^h X^3) . \end{aligned} \tag{34}$$

On the weight of $ev[Tr(b_0+b_1X+b_3X^3)]$ where b_0, b_1 and b_3 are in $GF(2^m)$, the following theorem holds [6,7].

<u>Theorem 3</u>:

(1) For odd m and $0 \le i < n$,

$$|ev[Tr(b_0+\alpha^i b_1 X+\alpha^{3i}X^3)]|_2$$

$$= 2^{m-1} \quad , \text{ if } Tr(b_1) = 0 \, , \tag{35}$$

$$= 2^{m-1} \pm 2^{(m-1)/2} \, , \text{ if } Tr(b_1) = 1 \, . \tag{36}$$

(2) For even m and $0 \le i < n$,

$$|ev[Tr(b_0+\alpha^i b_1 X+\alpha^{3i}X^3)]|_2$$

$$= 2^{m-1} \pm 2^{m/2} \, , \text{ if } Tr^{(2)}(b_1) = 0 \, , \tag{37}$$

$$= 2^{m-1} \quad , \text{ if } Tr^{(2)}(b_1) \ne 0 \, , \tag{38}$$

(3) For even m, $0 \le i < n$ and $1 \le h \le 2$,

$$|ev[Tr(b_0+b_1 X+\alpha^{3i+h}X^3)]|_2$$

$$= 2^{m-1} \pm 2^{m/2-1} \, . \tag{39}$$

(4) If $Tr(b_0) \ne Tr(b_0')$, then

$$|ev[Tr(b_0+b_1X+b_3X^3)]|_2 + |ev[Tr(b_0'+b_1X+b_3X^3)]|_2$$

$$= 2^m \, . \tag{40}$$

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For $0 \le i \le m2^m$, let $N_i^{(k)}$ denote the number of codewords of weight i in ERS_k. For deriving the weight enumerator for ERS_4-ERS_3, there are two cases to be considered.

4.1 Case I: m is odd.

Suppose that m is odd. Then, $\nu = 1$. For $1 \le j \le m$, let δ_j be represented as

$$\delta_j = \alpha^{u_j} . \tag{41}$$

Since 2^m-1 and 3 are relatively prime, there is an integer μ such that $1 \le \mu < 2^m-1$ and

$$3\mu \equiv 1 \bmod (2^m-1) . \tag{42}$$

Then

$$\delta_j = \alpha^{3\mu u_j} . \tag{43}$$

Let $ev[a_0+a_1X+a_2X^2+X^3]$, denoted $\bar{v}$, be a representative codeword in ERS_4-ERS_3. Then the v_j component vector of $\bar{v}, \bar{v}_j$, is defined by

$$\bar{v}_j = ev[Tr(\delta_j a_0+\delta_j a_1 X+\delta_j a_2 X^2+\delta_j X^3)] \quad \text{for } 1 \le j \le m .$$

By (34) and (43), we have that

$$Tr(\delta_j a_0+\delta_j a_1 X+\delta_j a_2 X^2+\delta_j X^3)$$

$$= Tr(\alpha^{3\mu u_j} a_0+\alpha^{\mu u_j}(\alpha^{2\mu u_j} a_1+[\alpha^{\mu u_j} a_2]^{2^{m-1}})X+\alpha^{3\mu u_j} X^3) . \tag{44}$$

Since $Tr(X^2) = Tr(X^{2^{m-1}}) = Tr(X)$ for $X \in GF(2^m)$, it follows from (1) of Theorem 3 and (44) that if $Tr(\alpha^{2\mu u_j} a_1) = Tr(\alpha^{\mu u_j} a_2)$, then

$$|\bar{v}_j|_2 = 2^{m-1} , \tag{45}$$

and otherwise,

$$|\bar{v}_j|_2 = 2^{m-1} \pm 2^{(m-1)/2} . \tag{46}$$

Let $S_+(\bar{v})$ and $S_-(\bar{v})$ be defined as

$$S_+(\bar{v}) = \#\{\, i \mid |\bar{v}_j|_2 = 2^{m-1} + 2^{(m-1)/2} , 1 \le j \le m \,\} ,$$

$$S_-(\bar{v}) = \#\{\, i \mid |\bar{v}_j|_2 = 2^{m-1} - 2^{(m-1)/2} , 1 \le j \le m \,\} ,$$

Then it follows from (45) and (46) that

$$\#\{ i \mid |\bar{v}_j|_2 = 2^{m-1} , 1 \le j \le m \} = m - S_+(\bar{v}) - S_-(\bar{v}) .$$

Then we have that

$$|\bar{v}|_2 = m2^{m-1} + (S_+(\bar{v}) - S_-(\bar{v}))2^{(m-1)/2} . \qquad (47)$$

Suppose that $\{\delta_1^\mu, \delta_2^\mu, \cdots, \delta_m^\mu\}$ is linearly independent. It follows from (42) that μ is relatively prime to 2^m-1. If $\{ \delta_1, \delta_2, \cdots, \delta_m\}$ is a polynomial basis, then $\{\delta_1^\mu, \delta_2^\mu, \cdots, \delta_m^\mu\}$ is linearly independent. Since $\delta_j = \alpha^{3\mu u_j}$ and $\delta_j^\mu = \alpha^{\mu u_j}$, $\{\alpha^{i\mu u_1}, \alpha^{i\mu u_2}, \cdots, \alpha^{i\mu u_m}\}$ is linearly independent for $1 \le i \le 3$. Therefore, we have that

$$\{ (Tr(\alpha^{\mu u_1}a_2), Tr(\alpha^{\mu u_2}a_2), \cdots, Tr(\alpha^{\mu u_m}a_2)) \mid a_2 \in GF(2^m) \}$$

$$= \{ (Tr(\alpha^{2\mu u_1}a_1), Tr(\alpha^{2\mu u_2}a_1), \cdots, Tr(\alpha^{2\mu u_m}a_1)) \mid a_1 \in GF(2^m) \}$$

$$= \{ (Tr(\alpha^{3\mu u_1}a_0), Tr(\alpha^{3\mu u_2}a_0), \cdots, Tr(\alpha^{3\mu u_m}a_0)) \mid a_0 \in GF(2^m) \}$$

$$= \text{the set of all binary m-tuples.} \qquad (48)$$

It follows from (40) and (45) to (48) that for given nonnegative integers s_+ and s_- with $0 \le s_+ + s_- \le m$, the number of choices of (a_0, a_1, a_2) of $\bar{v}$ such that $S_+(\bar{v}) = s_+$ and $S_-(\bar{v}) = s_-$ is given by

$$\binom{m}{s_+}\binom{m-s_+}{s_-}2^{s_++s_-}4^{m-s_+-s_-} .$$

Since there are 2^m-1 choices of nonzero a_3, it follows from (47) and (48) that for $0 \le j \le m$,

$$N^{(4)}_{m2^{m-1}\pm j2^{(m-1)/2}} - N^{(3)}_{m2^{m-1}\pm j2^{(m-1)/2}}$$

$$= (2^m-1) \sum_{i=0}^{\lfloor (m-j)/2 \rfloor} \binom{m}{j+i} \binom{m-j-i}{i} 2^{2m-j-2i} , \tag{49}$$

$$N_i^{(4)} = N_i^{(3)} , \quad \text{for other } i , \tag{50}$$

where sign ± is to be taken in the same order.

4.2 Case II: m is even.

Suppose that m is even. Then, $m \geq 4$ and $\nu = 3$. For $1 \leq j \leq m$, let δ_j be represented as

$$\delta_j = \alpha^{3u_j+w_j}, \tag{51}$$

where $0 \leq u_j < (2^m-1)/3$ and $0 \leq w_j \leq 2$. For $f(x) \in P(I_{0,3}) - P(I_{0,2})$, let the coefficient of X^3 be represented as α^e, and let

$$e \equiv h, \text{ mod } 3, \ 0 \leq h \leq 2. \tag{52}$$

Let $ev[a_0 + a_1X + a_2X^2 + \alpha^hX^3]$, denoted $\bar{v}$, be a representative codeword. Then the δ_j component vector of $\bar{v}$, $\bar{v}_j$, is defined by

$$\bar{v}_j = ev[Tr(\delta_j a_0 + \delta_j a_1 X + \delta_j a_2 X^2 + \delta_j \alpha^h X^3)], \text{ for } 1 \leq j \leq m.$$

By (34), we have that

$$\bar{v}_j = ev[Tr(\alpha^{3u_j+w_j}a_0 + [\alpha^{3u_j+w_j}a_1 + (\alpha^{3u_j+w_j}a_2)^{2^{m-1}}]X + \alpha^{3u_j+w_j+h}X^3)]. \tag{53}$$

For $0 \leq h \leq 2$, let

$$J_h = \{ j \mid w_j+h \equiv 0 \ (\text{mod } 3), \ 1 \leq j \leq m\},$$

and

$$CJ_h = \{1,2,\cdots,m\} - J_h.$$

It follows from (3) of Theorem 3 and (53) that for $0 \leq h \leq 2$ and $j \in CJ_h$,

$$|\bar{v}_j|_2 = 2^{m-1} \pm 2^{m/2-1}. \tag{54}$$

For $0 \le h \le 2$ and $j \in J_h$, it follows from (53) that

$$\bar{v}_j = ev[Tr(\alpha^{3u_j}a_0 + \alpha^{u_j}[\alpha^{2u_j}a_1 + (\alpha^{2u_j}a_2^2)^{2^{m-2}}]X + \alpha^{3u_j}X^3)] ,$$

for $h = 0$, (55)

$$= ev[Tr(\alpha^{3u_j+2}a_0 + \alpha^{u_j+1}[\alpha^{2u_j+1}a_1 + (\alpha^{2u_j}a_2^2)^{2^{m-2}}]X + \alpha^{3(u_j+1)}X^3)] ,$$

for $h = 1$, (56)

$$= ev[Tr(\alpha^{3u_j+1}a_0 + \alpha^{u_j+1}[\alpha^{2u_j}a_1 + (\alpha^{2u_j-2}a_2^2)^{2^{m-2}}]X + \alpha^{3(u_j+1)}X^3)] ,$$

for $h = 2$. (57)

Since $Tr^{(2)}(X^{2^{m-2}}) = Tr^{(2)}(X)$ for even m and X in $GF(2^m)$, it follows from (2) of Theorem 3 and (55) to (57) that if either $j \in J_0$ and $Tr^{(2)}(\alpha^{2u_j}a_1) = Tr^{(2)}(\alpha^{2u_j}a_2^2)$, or $j \in J_1$ and $Tr^{(2)}(\alpha^{2u_j+1}a_1) = Tr^{(2)}(\alpha^{2u_j}a_2^2)$, or $j \in J_2$ and $Tr^{(2)}(\alpha^{2u_j}a_1) = Tr^{(2)}(\alpha^{2u_j-2}a_2^2)$, then

$$|\bar{v}_j|_2 = 2^{m-1} \pm 2^{m/2}, \quad (58)$$

and otherwise,

$$|\bar{v}_j|_2 = 2^{m-1}. \quad (59)$$

Suppose that for $0 \le h \le 2$, $\{ \alpha^{2u_j} \mid j \in J_h\}$ is linearly independent over $GF(2^2)$. This condition holds for a primitive polynomial basis.

For $0 \le h \le 2$, let $\{ u_j \mid j \in J_h\}$ be represented by $\{u_{h1}, u_{h2}, \cdots, u_{hj_h}\}$, where $j_h = \#J_h$. Since $\{ a^2 \mid a \in GF(2^m)\} = \{ \alpha^i a \mid a \in GF(2^m)\} = GF(2^m)$ for an integer i, we have that

$$\{(Tr^{(2)}(\alpha^{2u_{01}}a_1), Tr^{(2)}(\alpha^{2u_{02}}a_1), \cdots, Tr^{(2)}(\alpha^{2u_{0j_0}}a_1)) \mid a_1 \in GF(2^m)\}$$

$$=\{(Tr^{(2)}(\alpha^{2u_{01}}a_2^2), Tr^{(2)}(\alpha^{2u_{02}}a_2^2), \cdots, Tr^{(2)}(\alpha^{2u_{0j_0}}a_2^2)) \mid a_2 \in GF(2^m)\}$$

$=$ the set of all j_0-tuples over $GF(2^2)$, (60)

$$\{(Tr^{(2)}(\alpha^{2u_{11}+1}a_1), Tr^{(2)}(\alpha^{2u_{12}+1}a_1), \cdots, Tr^{(2)}(\alpha^{2u_{1j_1}+1}a_1)) \mid a_1 \varepsilon GF(2^m)\}$$

$$= \{(Tr^{(2)}(\alpha^{2u_{11}}a_2^2), Tr^{(2)}(\alpha^{2u_{12}}a_2^2), \cdots, Tr^{(2)}(\alpha^{2u_{1j_1}}a_2^2)) \mid a_2 \varepsilon GF(2^m)\}$$

$=$ the set of all j_1-tuples over $GF(2^2)$, (61)

$$\{(Tr^{(2)}(\alpha^{2u_{21}}a_1), Tr^{(2)}(\alpha^{2u_{22}}a_1), \cdots, Tr^{(2)}(\alpha^{2u_{2j_2}}a_1)) \mid a_1 \varepsilon GF(2^m)\}$$

$$=\{(Tr^{(2)}(\alpha^{2u_{21}-2}a_2^2), Tr^{(2)}(\alpha^{2u_{22}-2}a_2^2), \cdots, Tr^{(2)}(\alpha^{2u_{2j_2}-2}a_2^2)) \mid a_2 \varepsilon GF(2^m)\}$$

$=$ the set of all j_2-tuples over $GF(2^2)$. (62)

For any given j_0-tuple $(b_1, b_2, \cdots, b_{j_0})$ over $GF(2^2)$, the number of a_1 in $GF(2^m)$ such that $Tr^{(2)}(\alpha^{2u_{0j}}a_1) = b_j$ for $1 \le j \le j_0$ is 2^{m-2j_0}. For other sets in (60) to (62), similar results hold. Since $\{\delta_1, \delta_2, \cdots, \delta_m\}$ is linearly independent, we have that

$$\{Tr(\delta_1 a_0), Tr(\delta_2 a_0), \cdots, Tr(\delta_m a_0) \mid a_0 \varepsilon GF(2^m)\}$$

$=$ the set of all binary m-tuples. (63)

Let $S_+(\bar{v})$, $S_-(\bar{v})$ and $T_+(\bar{v})$ be defined as

$$S_+(\bar{v}) = \#\{ i \mid |\bar{v}_j|_2 = 2^{m-1} + 2^{m/2}, j \varepsilon J_h\}, \quad (64)$$

$$S_-(\bar{v}) = \#\{ i \mid |\bar{v}_j|_2 = 2^{m-1} - 2^{m/2}, j \varepsilon J_h\}, \quad (65)$$

$$T_+(\bar{v}) = \#\{ i \mid |\bar{v}_j|_2 = 2^{m-1} + 2^{m/2-1}, j \varepsilon CJ_h\}. \quad (66)$$

Then it follows from (54) and (59) that

$$\#\{ i \mid |\bar{v}_j|_2 = 2^{m-1} - 2^{m/2-1}, 1 \le j \le m\} = m - j_h - T_+(\bar{v}), \quad (67)$$

$$\# \{ i \mid |\bar{v}_j|_2 = 2^{m-1}, 1 \le j \le m\} = j_h - S_+(\bar{v}) - S_-(\bar{v}) . \quad (68)$$

Then it follows from (13), (2) and (3) of Theorem 3 and (64) to (68) that

$$|\bar{v}_j|_2 = m2^{m-1} + (2S_+(\bar{v}) - 2S_-(\bar{v}) + 2T_+(\bar{v}) - m + j_h)2^{m/2-1}. \quad (69)$$

It follows from (4) of Theorem 3 and (54) to (63) that for given nonnegative integers s_+, s_- and t_+ with $0 \le s_+ + s_- \le j_h$ and $0 \le t_+ \le m - j_h$, the number of choices of (a_0,a_1,a_2) of $\bar{v}$ such that $s_+ = S_+(\bar{v})$, $s_- = S_-(\bar{v})$ and $t_+ = T_+(\bar{v})$ is given by

$$\binom{j_h}{s_+}\binom{j_h-s_+}{s_-}\binom{m-j_h}{t_+}2^{2(s_++s_-)}2^{4^{j_h-s_+-s_-}}2^{2m-4j_h}. \quad (70)$$

For $0 \le h \le 2$ and integer j with $-2m \le j \le 2m$, let $D_{h,j}$ be defined by

$$D_{h,j} = \{ (s_+,s_-,t_+) \mid 0 \le s_+ \le j_h, 0 \le s_- \le j_h, 0 \le s_+ + s_- \le j_h,$$
$$0 \le t_+ \le m - j_h, \ 2(s_+ - s_- + t_+) = m + j - j_h\} . \quad (71)$$

Since there are $(2^m-1)/3$ choices of nonzero α^e satisfying (52), it follows from (69), (70) and (71) that for $-2m \le j \le 2m$,

$$N^{(4)}_{m2^{m-1}+j2^{m/2-1}} - N^{(3)}_{m2^{m-1}+j2^{m/2-1}}$$
$$= (2^m-1)/3 \sum_{h=0}^{2} \sum_{(s_+,s_-,t_+)\in D_{h,j}} \binom{j_h}{s_+}\binom{j_h-s_+}{s_-}\binom{m-j_h}{t_+}24^{j_h-s_+-s_-}4^{m+s_++s_--2j_h},$$

and

$$N_i^{(4)} = N_i^{(3)} , \text{ for other } i. \quad (72)$$

4.3 Binary Weight Enumerator for ERS_3

Let $\bar{v} = ev[a_0+a_1X+a_2X^2]$, and $\bar{v}_j = ev[\delta_j a_0 + \delta_j a_1 X + \delta_j a_2 X^2]$. If $a_1 = a_2 = 0$, then

$$|\bar{v}|_2 = |ev[a_0]|_2 = 2^m|a_0|_2, \quad (73)$$

where $|a_0|_2$ denotes the weight of the binary representation of a_0 in $GF(2^m)$. For $0 \le j \le m$,

$$N^{(1)}_{j2^m} = \binom{m}{j} , \tag{74}$$

$$N^{(1)}_i = 0 , \quad \text{for other } i. \tag{75}$$

Suppose that either $a_1 \neq 0$ or $a_2 \neq 0$. There are $2^m(2^{2m}-1)$ combinations of such (a_0, a_1, a_2). Note that

$$\begin{aligned} &Tr(\delta_j a_0+\delta_j a_1 X+\delta_j a_2 X^2) \\ &\quad = Tr(\delta_j a_0 + [\delta_j a_1 + (\delta_j a_2)^{2^{m-1}}]X) . \end{aligned} \tag{76}$$

For each j with $1 \le j \le m$, $\delta_j a_1+(\delta_j a_2)^{2^{m-1}} = 0$ if and only if $a_2 = a_1^2\delta_j$. There are $m2^{m-1}(2^m-1)$ combinations of (a_0, a_1, a_2) such that $a_2 = a_1^2\delta_j$ and $Tr(\delta_j a_0) = 0$ (or 1). If $\delta_j a_1 + (\delta_j a_2)^{2^{m-1}} = 0$ and $Tr(\delta_j a_0) = 0$ (or 1), then

$$|v_j|_2 = |ev[Tr(\delta_j a_0)]|_2 = 0 \quad (\text{or } 2^m) . \tag{77}$$

If $\delta_j a_1 + (\delta_j a_2)^{2^{m-1}} \neq 0$, then

$$|v_j|_2 = |ev[Tr(\delta_j a_0+[\delta_j a_1+(\delta_j a_2)^{2^{m-1}}]X)]|_2 = 2^{m-1} . \tag{78}$$

Therefore, we have that

$$N^{(3)}_{(m+1)2^{m-1}} - N^{(1)}_{(m+1)2^{m-1}} = m2^{m-1}(2^m-1) , \tag{79}$$

$$N^{(3)}_{m2^{m-1}} - N^{(1)}_{m2^{m-1}} = 2^m(2^m-1)(2^m+1-m) , \tag{80}$$

$$N^{(3)}_{(m-1)2^{m-1}} - N^{(1)}_{(m-1)2^{m-1}} = m2^{m-1}(2^m-1) , \tag{81}$$

$$N^{(3)}_i = N^{(1)}_i , \quad \text{for other } i . \tag{82}$$

Note that the binary weight enumerator for ERS_3 is independent of the choice of basis.

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