Annual Status Report

submitted to

NASA

Goddard Space Flight Center

Greenbelt, Maryland

Error Control Techniques for Satellite and Space Communications

Grant Number NAG5-557

June 1, 1985 - May 31, 1987

Principal Investigator:

Daniel J. Costello, Jr.

Department of Electrical Engineering

University of Notre Dame

Notre Dame, IN 46556

(NASA-CR-180586) ERROR CONTROL TECHNIQUES FOR SATELLITE AND SPACE COMMUNICATIONS Annual Status Report, 1 Jun. 1985 - 31 May 1987 (Notre Dame Univ.) 79 p Avail: NTIS HC A05/MF A01 CSCL 17B G3/32

N87-22078

Unclas 0071465

June, 1987

Summary of Progress

During the period December 1, 1986 - May 31, 1987, progress was made in the following areas:

1) Concatenated Codes Using Bandwidth Efficient Trellis Inner Codes.

A paper summarizing our work on the performance of bandwidth efficient trellis inner codes using two-dimensional MPSK signal constellations in a NASA concatenated coding system has been submitted for publication to the *IEEE Transactions on Communications* [1]. These coding schemes achieve an effective information rate (information bits transmitted per unit bandwidth) at least double that currently used in NASA's TDRS system. They also provide large coding gains, which make them desirable for use in high speed satellite communication systems. Implementations of an inner coding scheme similar to those proposed in [1] have been demonstrated at speeds up to 120 Mbps [2].

Using a soft decision Viterbi decoder for the inner code, we have shown in [1] that coding gains of 4dB @ 10^{-6} and 6dB @ 10^{-9} can be achieved without bandwidth expansion using 16-state trellis coded 8 PSK modulation as the inner code and a length 255 Reed-Solomon (RS) outer code. With a modified Viterbi decoder for the inner code which passes some erasure symbols to the outer decoder, a decoded bit error rate (BER) of 10^{-16}, with an erasure rate of 10^{-3}, can be achieved at an $E_b/N_0 = 5.5dB$ and only 10 % bandwidth expansion. This paper [1] is included as Appendix A of this report.

We have also continued our work on trellis coded modulation using multi-dimensional signal sets. A paper on this subject has been submitted to the *IEEE Transactions on Information Theory* [3]. A copy of this paper was included in our previous report. A talk on this subject was also presented at the 1987 Conference on Information Sciences and Systems [4], and the Principal Investigator has been invited to give a lecture on multi-dimensional trellis coded modulation at the IEEE Information

Theory Workshop, in Bellagio, Italy [5].

We are currently preparing two additional papers in this area for publication [6,7]. One summarizes our work on multi-dimensional 16 PSK code constructions, while the other investigates the performance of multi-dimensional trellis coded modulation as inner codes in a concatenated coding system. Copies of these papers will be included in a future report.

2) Bounds on the Minimum Free Euclidean Distance of Bandwidth Efficient Trellis Codes.

On a more theoretical note, a paper summarizing our work on lower bounds on the minimum free Euclidean distance of bandwidth efficient trellis codes has been submitted to the *IEEE Transactions on Information Theory* [8], and is included as Appendix B of this report. This paper proves achievable lower bounds on free distance for trellis codes and establishes the existence of good trellis coded modulation (TCM) schemes for a variety of signal constellations, including MPSK. These bounds can be used as a means of comparing the performance of various TCM schemes, particularly at large constraint lengths.

A weakness of this bound is that it is not tight at small constraint lengths, and we are currently working on a method of tightening the bound. Two additional papers are being planned, one on a tightened bound and the other on the basic structural properties of TCM schemes.

3) Performance Analysis of Bandwidth Efficient Trellis Codes on Non-Uniform Channels

Communication over real world channels, such as mobile satellite channels, suffers from non-uniform disturbances that cannot be described by additive white Gaussian noise. One important such disturbance is the multiplicative noise arising from multipath reception at the receiver [9]. Communication channels suffering from multipath are known as Rayleigh/Rician fading channels.

We have begun an investigation of the performance of TCM schemes on fading channels. Our preliminary results indicate that bandwidth efficient trellis coding is feasible on such channels, but that the important design parameter is no longer the minimum free Euclidean distance. Two new parameters, the “effective length” and the “minimum product distance” are more important. Intuitively, a good code in a nonuniform noise environment should try to spread the “distance” between two codewords over all the branches over which the codewords differ. This is the same idea used in diversity transmission, which is usually employed on fading channels. A paper summarizing our preliminary results is included as Appendix C [10] of this report.

References

[1] R. H. Deng and D. J. Costello, Jr., "High Rate Concatenated Coding Systems Using Bandwidth Efficient Trellis Inner Codes," *IEEE Trans. Commun.*, submitted for publication, April 1987.

[2] T. Fujino et.al., "A 120 Mbits/s Coded 8 PSK Modem with Soft-Decision Viterbi Decoder," *IEEE Int. Conf. on Commun. Conf. Record*, pp. 1774-1780, Toronto, Canada, June 1986.

[3] A. LaFanchere, R. H. Deng, and D. J. Costello, Jr., "Multidimensional Trellis Coded Phase Modulation Using Unit-Memory and Partial-Unit-Memory Convolutional Codes," *IEEE Trans. Inform. Th.*, submitted for publication, January 1987.

[4] D. J. Costello, Jr. and R. H. Deng, "Multi-Dimensional Unit-Memory and Partial-Unit-Memory Trellis Coded Modulation," 1987 Conf. on Inform. Sciences and Systems, Baltimore, MD, March 1987.

[5] D. J. Costello, Jr., "High Rate Concatenated Coding Systems with Multi-Dimensional Trellis Coded MPSK Modulation as Inner Codes," IEEE Information Theory Workshop, Bellagio, ITALY, June 1987.

[6] R. H. Deng and D. J. Costello, Jr., "Multi-Dimensional Trellis Coded 16 PSK Modulation," *IEEE Trans. Inform. Th.*, submitted for publication, June 1987.

[7] R. H. Deng and D. J. Costello, Jr., "High Rate Concatenated Coding Systems using Multi-Dimensional Bandwidth Efficient Trellis Inner Codes," *IEEE Trans. Commun.*, submitted for publication, June 1987.

[8] M. Rouanne and D. J. Costello, Jr., "A Gilbert Lower Bound on the Minimum Free Euclidean Distance of Trellis Coded Modulation," *IEEE Trans. Inform. Th.*, submitted for publication, May 1987.

[9] J. Hagenauer and E. Lutz, "Forward Error Correction Coding for Fading Compensation in Mobile Satellite Channels," *IEEE Journal*

Selected Areas Commun., Vol. SAC-5, No. 2, pp. 215-225, February 1987.

[10] C. Schlegel and D. J. Costello, Jr., "Bandwidth Efficient Coding on a Fading Channel," International Conf. on Systems Engr., Dayton, OH, September 1987.

Appendix A

High Rate Concatenated Coding Systems Using
Bandwidth Efficient Trellis Inner Codes

High Rate Concatenated Coding Systems Using Bandwidth Efficient Trellis Inner Codes*

Robert H. Deng
Daniel J. Costello, Jr.

Department of Electrical and Computer Engineering
University of Notre Dame
Notre Dame, IN 46556

May 18, 1987

Abstract

High rate concatenated coding systems with bandwidth efficient trellis inner codes and Reed-Solomon (RS) outer codes are investigated for application in high speed satellite communication systems. Two concatenated coding schemes are proposed. In Scheme I, the inner code is decoded with soft-decision Viterbi decoding and the outer RS code performs error-correction only decoding (decoding without side information). In scheme II, the inner code is decoded with a modified Viterbi algorithm which produces reliability information along with the decoded output. In this algorithm, path metrics are used to estimate the entire information sequence, while branch metrics are used to provide reliability information on the decoded sequence. This information is used to erase unreliable bits in the decoded output. An errors-and-erasures RS decoder is then used for the outer code. These two schemes have been proposed for high speed data communication on NASA satellite channels. The rates considered are at least double those used in current NASA systems, and our results indicate that high system reliability can still be achieved.

*This work was supported by NASA Grant NAG5-557.

I Introduction

To meet the ever-increasing demand for satellite communication and voice-band telephone services, the search for bandwidth and power-efficient modulation/coding systems has recently become a very active research area [1-9]. The basic idea of bandwidth efficient trellis codes, or trellis coded modulation (TCM), pioneered by Ungerboeck [1], is that by trellis coding onto an expanded modulation set (relative to that needed for uncoded transmission) and by designing the trellis codes to maximize the minimum free Euclidean distance between allowable code sequences, asymptotic (high signal-to-noise ratio) coding gains of $3 \sim 6dB$ compared with an uncoded system can be achieved without bandwidth expansion. The performance of TCM systems can be measured in terms of their asymptotic coding gain γ over an uncoded system with the same spectral efficiency. The asymptotic coding gain increases with the number of states in the trellis. For example, Ungerboeck's simplest scheme uses a 4-state, rate 2/3 trellis encoder with 8-PSK modulation and achieves a $\gamma = 3\ dB$ coding gain over uncoded QPSK modulation without bandwidth expansion. A more complex 16-state encoder achieves a $\gamma = 4.1\ dB$ coding gain.

Practically, we are more concerned with the real coding gains which can be achieved at moderate values of the decoded bit error rate (BER) than with the asymptotic coding gain. For satellite communication, decoded BER's in the range of $10^{-6} \sim 10^{-9}$ are reasonable requirements. Channel capacity results imply that real coding gains over uncoded QPSK of 8.7 dB at a BER of 10^{-6} and 15.5 dB at a BER of 10^{-9} can be achieved [1,2]. Ungerboeck's 4-state code achieves a 3 dB real coding gain over uncoded QPSK without bandwidth expansion both at 10^{-6} and 10^{-9} BER's, while his 16-state code achieves a 3.7 dB gain at 10^{-6} and a 3.9 dB gain at 10^{-9}. In this paper, we investigate the use of concatenated codes with

TCM inner codes and Reed-Solomon (RS) outer codes to achieve larger real coding gains over uncoded QPSK at BER's of $10^{-6} \sim 10^{-9}$ with little or no bandwidth expansion.

Concatenated coding has long been used as a practical means of achieving long block or constraint lengths (and thereby achieving large coding gains) [10-14]. One such system consists of Viterbi decoding of an (n,1,m) convolutional inner code with BPSK modulation concatenated with an outer RS code. However, the overall effective information transmission rate of this system is less than $1/n, n > 2$, bit per signal dimension [11, 12]. This system achieves increased power efficiency at the expense of decreased spectral efficiency.

In this paper, we present a bandwidth efficient trellis inner code/RS outer code concatenated coding system for use in high speed satellite communication systems. The goal of using bandwidth efficient trellis codes as inner codes is to achieve large coding gains over uncoded QPSK without bandwidth expansion. We will see in the following sections that such a system can provide coding gains from 4 to 8 dB at BER's of $10^{-6} \sim 10^{-9}$ with little or no bandwidth expansion (while the asymptotic coding gain can be as large as 20 dB), when uncoded QPSK is used as our reference system.

There are two types of bandwidth efficient trellis codes: lattice-type trellis coded modulation (e.g., QASK) [1,2,3,4,5] and constant-envelope-type trellis coded modulation (e.g., MPSK) [1,6,7,8,9]. The former is more power efficient than the latter under an average energy constraint. However, since constant-envelope signals mitigate the non-linear effects of TWT amplifiers and can be implemented for high speed modem operation [15], trellis coded MPSK modulation is the preferred modulation scheme for satellite channels. Hence, we will consider the case when trellis coded MPSK modulation is used as the inner code.

Fig. 1 shows the encoding-decoding block diagram of the concatenated coding system. Encoding is performed in two stages. An information sequence of Kb bits is divided into K symbols of b bits each, and each b-bit symbol is regarded as an element of $GF(2^b)$. These K symbols are used as the input to the RS encoder. The output of this encoder is an N-symbol codeword which is symbol-interleaved and then serially encoded by the trellis encoder. Decoding is accomplished in the reverse order. The output of the maximum-likelihood (Viterbi) trellis decoder contains bursty errors. The main purpose of the inner code is to shape the distribution of the errors on the inner channel, rather than to correct errors, particularly when the inner code rate is around the cutoff rate of the inner channel. When an RS outer code is used, this shaping compresses the random errors on the inner channel into symbol errors corresponding to the symbol size used by the outer code [13]. The use of bandwidth efficient trellis codes as inner codes has one important feature: because of the bandwidth efficient property of the code, it compensates for the bandwidth expansion introduced by the outer RS code, so that the overall system suffers no bandwidth expansion.

Two concatenated coding schemes are studied. Scheme I, a Viterbi decoded trellis inner code/RS outer code system, where the Viterbi decoder outputs are provided to the outer decoder without side information, is studied in Section II. Code performance results for several examples of scheme I are shown in figures 2-3 and tables 1-2. Asymptotic coding gains are also determined and presented in figures 4-6. A modified Viterbi decoding algorithm for trellis codes with output reliability information is introduced in Section III. Scheme II, a concatenated coding system with side information available to the outer decoder, is then studied in Section IV. In the modified algorithm, path metrics are used to estimate the entire information sequence, and branch metrics are used to provide reliability information on the decoded sequence. This is done by erasing a certain number of bits on the survivor of a comparison between the two most likely paths whose metrics are within some fixed threshold of each other. Code performance results for several examples of scheme II are shown in figures 8-9. Finally, in Section V, we summarize our results and draw some conclusions which are useful in the design of concatenated coding systems.

II Coding Scheme I - Without Side Information

A concatenated coding system is shown in Figure 1. As mentioned in Section I, we use bandwidth efficient trellis codes as inner codes and (N,K) RS codes with symbols over $GF(2^b)$ as outer codes. Decoding is accomplished in the reverse order of encoding. The inner decoder uses the Viterbi algorithm. The outputs of the inner decoder (without side information) are grouped into b-bit symbols. Because the lengths of the bursts of decoding errors made by the Viterbi decoder are widely distributed, symbol deinterleaving is used so that errors in the individual RS-symbols of one block (N symbols) are independent; otherwise, a very long block code would be required for efficient operation.

In this concatenated coding system, the bandwidth efficient inner trellis codes have two functions:

1. To compensate for the bandwidth expansion introduced by the outer RS code;

2. To compress the random errors on the inner channel into symbol errors which can be corrected by the outer RS code.

In the reminder of this section, we study the performance of Scheme I. We first calculate the decoded BER and determine the coding gain, at a given decoded BER, in terms of the "information energy-to-noise power density ratio", E_b/N_0. Then we derive an expression for the asymptotic coding gain for very large values of E_b/N_0. All coding gains are given with reference to uncoded QPSK modulation.

A. Decoded BER

Let the minimum distance of the (N,K) RS code be d_2. When an N-symbol block is received by the outer decoder, it performs errors-only decoding. That is, if the N-symbol block contains $t_2 = \lfloor (d_2 - 1)/2 \rfloor$ or fewer errors, the errors are corrected; otherwise, decoding fails. t_2 is called the error-correcting-capability of the (N,K) RS code [14]. The decoded bit error rate at the output of outer decoder is closely approximated by

$$P_b \simeq \frac{d_2}{2N} \sum_{i=t_2+1}^{N} \binom{N}{i} P_s^i (1 - P_s)^{N-i}, \tag{1}$$

where P_s is the symbol error probability into the outer decoder. In this subsection, we give several examples of Scheme I. For each example, we use an RS code with symbols over $GF(2^8)$ and $N = 255$ as the outer code. Rate R_1 trellis coded 2^L-ary PSK modulation, with $L = 3$ or 4, is used as inner code. Since 2^L-ary PSK modulation is two dimensional, the effective information rate of the inner code is given by

$$R_{eff}^{(1)} = \frac{L}{2} R_1 \text{ bits/dimension}, \tag{2}$$

and the overall effective information rate of the concatenated coding system is

$$R_{eff} = R_{eff}^{(1)} \frac{K}{N} = R_{eff}^{(1)} \ R_2 \text{ bits/ dimension}, \tag{3}$$

where $R_2 = K/N$ is the rate of the (N,K) RS code.

The symbol error probability P_s in (1) is obtained by computer simulation on an AWGN channel unless otherwise noted, and (1) is used to estimate the final decoded BER.

Example 2.1

Periodically time-varying trellis codes (PTVTC) of rate $R_1 = (2P + i)/3P,\ P \geq 2,\ 1 \leq i < P$, are used as inner codes with 8-PSK modulation [6]. These codes are especially designed for high data rate channels with low decoding complexity. This PTVTC/8-PSK system has an effective information rate $R_{eff}^{(1)} = R_1 3/2 = (2P+i)/2P > 1$ bits/dimension, so it is possible to achieve an overall effective information rate R_{eff} equal to 1 bit/ dimension or greater. Moreover, because of the periodic property of the code, its trellis structure is the same as the trellis structure of a lower-rate code, and as a result the complexity of the decoder is reduced significantly [6].

The PTVTC's are decoded by the Viterbi algorithm assuming no demodulator output quantization. Fig. 2.1 shows the decoded BER P_b vs. E_b/N_0 for concatenated codes with 4-state inner codes and 3 different RS outer codes, where the outer code rate R_2 is chosen such that $R_{eff} = 1$ bit/dimension. Fig. 2.2 gives similar results for 16-state inner codes. In Figs. 2.3 and 2.4 we show the required E_b/N_0 to achieve decoded BER's of 10^{-6} and 10^{-9} for 4-state and 16-state inner codes, respectively, as a function of R_{eff}.

From Figs. 2.1 - 2.4 we observe that, for a given R_{eff}, systems with lower inner code rates give better performance. Also note that at high effective code rates, e.g., $R_{eff} = 1$ bit/dimension, we still get large coding gains. For example, with a 16-state, $R_1 = 7/9$ PTVTC, the coding gain equals 3.92 dB at $P_b = 10^{-6}$ and 6.18 dB at $P_b = 10^{-9}$. Even for a 4-state, $R_1 = 7/9$ PTVTC, coding gains of 3.68 dB and 5.98 dB at $P_b = 10^{-6}$ and $P_b = 10^{-9}$, respectively, can be obtained. Note that as the number of trellis states increases from 4 to 16, only about 0.2 dB more coding gain is obtained. This appears to be a characteristic of coded modulation in general [1]. We will see this more clearly when we derive the asymptotic coding gain.

Example 2.2

Ungerboeck's 16-state, rate $R_1 = 2/3$ convolutional code with 8-PSK modulation is used as the inner code. This code has an effective information rate $R_{eff}^{(1)} = 1$ bit/dimension. Therefore, the overall effective information rate, R_{eff}, of the concatenated coding system is less than 1. Again, the inner code is decoded by the Viterbi algorithm without demodulator output quantization. The final decoded bit error probability P_b is shown in Fig. 3.1 for 2 different RS outer codes. Fig. 3.2 shows the required E_b/N_0 to achieve $P_b = 10^{-6}$ and 10^{-9} as a function of R_{eff}. At $P_b = 10^{-6}$ and 10^{-9}, coding gains of 4.96 dB and 7.25 dB, respectively, can be obtained with only 12.5% bandwidth expansion. This compares with 3.7 dB and 3.9 dB coding gains, respectively, with no bandwidth expansion, for the inner code alone.

Example 2.3

In this example, two kinds of trellis codes are used as inner codes. The first kind uses 8-PSK modulation twice per trellis interval (forming a 64-ary set), and this set is then coded with a $R_1 = 5/6$ trellis code [7,8]. Viewed more generally, code symbols are mapped onto a four-dimensional signal set (8-PSK × 8-PSK) with 5 information bits for every two 8-PSK signals. Thus the effective information rate is $R_{eff}^{(1)} = 5/4 = 1.25$ bits/dimension. The second kind uses rate $R_1 = 3/4$ convolutional codes with 16-PSK modulation [9]. The effective information rate is $R_{eff}^{(1)} = 3/2$ = 1.5 bits/dimension. This code is more bandwidth efficient than the R_1 = 5/6, 8-PSK code, but it is less power efficient.

Let d_f, E_s, and A be the normalized minimum free Euclidean distance, the energy per modulation symbol, and the number of nonzero information bits in the set of paths at distance d_f from the correct path, respectively. Then at high signal-to-noise ratios, the bit error probability at the Viterbi decoder output is approximated by

$$\epsilon \simeq \frac{A}{5}Q\left(\sqrt{\frac{d_f^2 E_s R_2}{2N_0}}\right) = \frac{A}{5}Q\left(\sqrt{\frac{1.25 d_f^2 E_b R_2}{N_0}}\right), \text{ for } R_1 = 5/6 \text{ and 8-PSK}, \quad (4.1)$$

and

$$\epsilon \simeq \frac{A}{3}Q\left(\sqrt{\frac{d_f^2 E_s R_2}{2N_0}}\right) = \frac{A}{3}Q\left(\sqrt{\frac{1.5 d_f^2 E_b R_2}{N_0}}\right), \text{ for } R_1 = 3/4 \text{ and 16-PSK}. \quad (4.2)$$

The RS-symbol (over $GF(2^b)$) error probability at the output of the Viterbi decoder is bounded by

$$P_s \leq b\epsilon. \quad (5)$$

The final decoded bit error probability P_b can be obtained from (4.1), (4.2), (5), and (1). Tables 1.1 and 1.2 list the coding gains over uncoded QPSK at $P_b = 10^{-6}$ and 10^{-9} for the $R_1 = 5/6$, 8-PSK and $R_1 = 3/4$, 16-PSK inner codes, respectively, where we have chosen the outer RS code rate R_2 such that the overall effective information rate $R_{eff} = 1$ bit/dimension. In both cases, the relatively poor performance of the 8-state and 16-state codes compared to the 4-state codes is due to a large number of minimum free distance paths.

B. Asymptotic coding gain

Having studied the performance of concatenated coding systems for small and medium values of E_b/N_0, it is interesting to look at how the systems perform at large values of E_b/N_0, i.e., the asymptotic coding gain. This also gives us some insight into concatenated coding system design.

When uncoded QPSK is used on an AWGN channel with coherent demodulation, the demodulator output BER is given by

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \simeq \frac{1}{2}e^{-E_b/N_0}, \text{ for large } E_b/N_0. \quad (6)$$

For our concatenated coding system, we assume rate R_1 trellis coded 2^L-ary PSK modulation as the inner code and an (N,K) RS code with symbols over $GF(2^b)$ as the outer code. Then

$$E_s = R_2 R_1 L E_b,$$

and from (2) and (3) it follows that

$$E_s = 2R_2 R_{eff}^{(1)} E_b = 2R_{eff} E_b. \tag{7}$$

The BER at the output of a Viterbi decoder without demodulator output quantization is approximated by

$$\begin{aligned} \epsilon &\simeq AQ\left(\sqrt{\frac{d_f^2 E_s}{2N_0}}\right) = AQ\left(\sqrt{\frac{d_f^2 R_{eff} E_b}{N_0}}\right) \\ &\simeq \frac{A}{2} e^{-\frac{d_f^2 R_{eff} E_b}{2N_0}}. \end{aligned} \tag{8}$$

From (8), (5), and (1), the final decoded BER P_b of the concatenated coding system is approximated by

$$\begin{aligned} P_b &\simeq \frac{d_2}{2N}\binom{N}{t_2+1} P_s^{t_2+1}(1-P_s)^{N-t_2-1} \\ &\simeq \frac{d_2}{2N}\binom{N}{t_2+1} P_s^{t_2+1} \\ &\simeq \frac{d_2}{N}\binom{N}{t_2+1}\left(\frac{Ab}{2}\right)^{t_2+1} e^{-\frac{d_f^2 R_{eff}(t_2+1)E_b}{2N_0}}. \end{aligned} \tag{9}$$

Comparing (6) and (9), we see that for a fixed E_b/N_0, the (negative) exponent with concatenated coding is larger by a factor of $d_f^2 R_{eff}(t_2+1)/2$ than the exponent without coding. Since the exponential term dominates the error probability expression for large E_b/N_0, we define the asymptotic coding gain (in decibels) as

$$\gamma \triangleq 10 \log_{10} \frac{d_f^2 R_{eff}(t_2+1)}{2}. \tag{10}$$

For RS codes,

$$t_2 = \frac{N-K}{2} = \frac{N(1-R_2)}{2}. \tag{11}$$

Substituting (11) into (10), the asymptotic coding gain can be rewritten as

$$\begin{aligned}\gamma &= 10\log_{10}\frac{d_f^2 R_{eff}}{2}\left(\frac{N(1-R_2)}{2}+1\right) \\ &= 10\log_{10}\frac{d_f^2 R_{eff}^{(1)}}{2} + 10\log_{10} R_2\left(\frac{N(1-R_2)}{2}+1\right).\end{aligned} \tag{12}$$

Note that in (12) the first term denotes the coding gain due to the bandwidth efficient inner code and the second term represents the coding gain contributed by the outer code. This implies that, for very large E_b/N_0, the inner and the outer codes in a concatenated coding system can be designed independently of each other.

Taking derivatives of γ with respect to R_2, the maximum coding gain is achieved at $R_2 = 1/2 + 1/N$ for a fixed inner code, and

$$\gamma_{max} = 10\log_{10}\frac{d_f^2 R_{eff}^{(1)}}{2} + 10\log_{10}\frac{1}{2}\left(\frac{N}{4}+1+\frac{1}{N}\right). \tag{13}$$

Figure 4 shows the asymptotic coding gain γ for Example 2.1. Figure 5 shows γ for Example 2.2, Figure 6.1 shows γ for Example 2.3 when $R_1 = 5/6$ coded 8-PSK is used as the inner code, and Figure 6.2 shows γ for Example 2.3 when $R_1 = 3/4$ coded 16-PSK is used as the inner code. The asymptotic coding gains shown in these figures are very large, even at high effective information rates. For example, in Figure 4 at $R_{eff} = 1$ bit/dimension, the asymptotic coding gain can be as large as 17 dB.

Before finishing this section, we can draw several conclusions from the above discussion.

1. For a given R_{eff}, there exists an $R_{eff}^{(1)}$ which optimizes system performance, or conversely, for a given $R_{eff}^{(1)}$, there exists an optimum value of R_{eff}.

2. System performance is very sensitive to R_2 at very low and very high values of R_2.

3. For small or moderate values of E_b/N_0, and for a fixed R_{eff}, a lower $R^{(1)}_{eff}$ is preferred as long as the outer code rate R_2 is not too high.

4. Because d_f increases slowly as the number of trellis states increases, choosing inner codes with a small number of trellis states increases the data transmission rate (by reducing the number of decoder computations) with only a slight sacrifice in system performance.

III A Modified Viterbi Algorithm

It has been shown that the channel capacity of a concatenated coding system with side information is an upper bound on the capacity without side information [13]. The difference between the two capacities is significant, especially on very noise channels. Therefore, it is advantageous if the inner decoder can provide some kind of reliability information (side information) about its estimated output which will aid the outer decoder. In scheme I, discussed in Section II, the inner code is decoded by the Viterbi algorithm and the output of the decoder is a sequence of "hard decisioned" binary digits, i.e., no side information is associated with the output. To provide side information, it becomes necessary to modify the conventional Viterbi algorithm so that soft decisioned outputs are made available to the outer decoder.

Several ideas have been proposed to provide some kind of side information with the convolutional decoder outputs. Zeoli [16] proposed a concatenated coding system that employs a long constraint length ($m = 31$) convolutional code obtained by annexing a tail to a (3,1,7) convolutional code. The longer code is then decoded by the same Viterbi decoder as the short code, with the exception that the information sequence along the best path to each state is treated as correct and used to "cancel" the effect of the longer tail from the encoded sequence. The tail provides excellent error-detection capability once the decoder starts to make mistakes. But the algorithm is subject to very serious error propagation and the decoder has to be reset frequently.

Another method of providing reliability information with the decoder output is to compute the a posteriori probability of each decoded symbol from the decoder being correct [12]. However, the a posteriori probability must be computed for each branch using a recursive method and real numbers must also be stored. This computation process slows down the decoder, so it is not suitable for high speed systems.

A third alternative is to use the "Viterbi decoding algorithm for convolutional codes with repeat request" [17] to extract reliability information from the inner decoder. When all the path metrics at some level of the trellis are below a predetermined threshold, the received sequence up to that level is erased. But this approach has two major drawbacks: 1) when a long received sequence is erased, this erasure information cannot be used by the outer decoder because the number of erasures may exceed the erasure correction capability of the outer code; 2) with high probability, most of the symbols in the received sequence can be decoded correctly and hence should not be erased.

In the following, we propose a decoding technique based on the Viterbi algorithm which erases only the information symbols that are "probably in error". This erasure procedure does not affect the decoder's selection of the most likely path. Therefore, the decoder is still maximum-likelihood. Furthermore, the algorithm is very simple to implement and capable of high speed operation.

As the name suggests, an (n,k,m) trellis code is best described in terms of its trellis diagram. In the decoding of trellis codes by the Viterbi algorithm, a first-event error is made at an arbitrary level j if the correct path is eliminated for the first time at level j in favor of the incorrect path. This is illustrated in Figure 7. The incorrect path must be a path that had previously diverged from the correct path. For the example shown in Figure 7, let

$$\mathbf{V}_j = (\cdots, V_{j-3}, \quad V_{j-2} \quad V_{j-1}, \quad V_j)$$

and

$$\mathbf{V}'_j = (\cdots, V'_{j-3}, \quad V'_{j-2}, \quad V'_{j-1}, \quad V'_j)$$

represent the correct path and the incorrect path, respectively, where $V_i(V'_i)$ is the symbol on the $i^{\underline{th}}$ branch of the correct (incorrect) path. Also let

$$\mathbf{U}_j = (\cdots, U_{j-3}, \quad U_{j-2}, \quad U_{j-1}, \quad U_j)$$

and

$$\mathbf{U}'_j = (\cdots, U'_{j-3}, \quad U'_{j-2}, \quad U'_{j-1}, \quad U'_j)$$

represent the information sequences associated with paths $\mathbf{V}_j$ and $\mathbf{V}'_j$, respectively, where $U_i(U'_i)$ is a binary k-tuple. Denote a branch metric by $\lambda(V_i)$ and a path matric by $\lambda(\mathbf{V}_j)$. From Figure 7 we observe that the bits $U'_{j-3}, \quad U'_{j-2}, \quad U'_{j-1}, \quad U'_j$ are unreliable and should be erased. This observation motivates us to propose the following decoding algorithm.

<u>Modified Viterbi Algorithm:</u>

For each state at level j, select the path $\mathbf{V}'_j = (\cdots, V'_{j-l+1}, \quad V'_{j-l+2}, \cdots, V'_{j-1}, \quad V'_j)$ $(l > 1)$ that has the largest metric $\lambda(\mathbf{V}'_j)$ and the path $\mathbf{V}_j = (\cdots, V_{j-l+1}, \quad V_{j-l+2}, \cdots, V_{j-1}, \quad V_j)$ that has the next largest metric $\lambda(\mathbf{V}_j)$. Because $\mathbf{V}'_j$ has the largest metric, $\mathbf{V}'_j$ survives at that state. Moreover, if

$$\lambda(\mathbf{V}'_j) \geq \lambda(\mathbf{V}_j) + T \quad (T > 0), \tag{14}$$

decoding continues as in the conventional Viterbi algorithm. On the other hand, if

$$\lambda(\mathbf{V}'_j) < \lambda(\mathbf{V}_j) + T, \tag{15}$$

then the bits $U'_{j-l+1}, \quad U'_{j-l+2}, \cdots, U'_{j-1}, \quad U'_j$ on the surviving path are erased and decoding continues.

This algorithm applied to any (n,k,m) trellis code. If, however, all k input bits are shifted into memory, there can be no parallel transitions in the trellis. Since the last km information bits associated with an error event of length l branches must be correct for the incorrect path to remerge with the correct path, decoding errors are confined to the first $l - m$ branches. In this case, when (15) is satisfied, only the bits $U'_{j-l+1}, \quad U'_{j-l+2}, \cdots, U'_{j-m}$ on the surviving path are erased.

Note that if $T = 0$, the algorithm reduces to the conventional Viterbi algorithm. As an illustration of the algorithm, in Figure 7, for some state at level j, suppose that $\lambda(\mathbf{V}'_j) > \lambda(\mathbf{V}_j)$, so the incorrect path $\mathbf{V}'_j$ survives. If (15) is satisfied, then the bits $U'_{j-3}, \quad U'_{j-2}, \quad U'_{j-1}, \quad U'_j$ are erased. Note that (15) is equivalent to

$$\lambda(V'_{j-3})+\lambda(V'_{j-2})+\lambda(V'_{j-1})+\lambda(V'_j) < \lambda(V_{j-3})+\lambda(V_{j-2})+\lambda(V_{j-1})+\lambda(V_j)+T. \tag{16}$$

This implies that, if the metrics of the two 4-branch sequences are too "close" in terms of the threshold T, the corresponding information sequence on the surviving path is not reliable, and therefore a tag should be attached to it which indicates its degree of reliability. The simplest way of doing this is to erase it.

From (16) we see that this erasure decision is made on the most recent l branches for an l-branch error event, and hence it is a "local" estimate. On the other hand, the path estimate is a "global" estimate. From this point of view, we see that the algorithm is constructed based on the following ideas:

1. By letting the path with the largest path metric survive, we are choosing the maximum-likelihood estimate.

2. By performing a "branch comparison" over the most recent l branches, we provide some reliability information on the maximum-likelihood path.

Because the reliability is estimated over only l branches, the reliability estimates are suboptimum. However, in feedback decoding of convolutional codes, decisions are made with a delay of only one constraint length [14]. Hence, we can say that the reliability estimates have a certain degree of precision. Ideally, the number of erased k-tuples, l, should equal the length of the error event. Practically, however, choosing l equal to the average error event length should be sufficient.

IV Coding Scheme II - With Side Information

The encoding process is the same as in Scheme I. Decoding is done in two steps. First, the inner trellis code is decoded by the modified Viterbi algorithm presented in the last section. The outputs of the inner decoder consist of binary digits as well as erased bits, and they are grouped into b-bit symbols, deinterleaved, and sent to the outer decoder. Symbols which contain any erased bits are considered as erasures by the outer decoder.

Let i be the number of erased symbols in an RS outer codeword of length N. The outer decoder declares an erasure (or raises a flag) for the entire block of N symbols if i is greater than the erasure-correction threshold T_{es}, where

$$T_{es} \leq (d_2 - 1) \tag{17}$$

and d_2 is the minimum distance of the outer (N,K) RS code. If i is less than T_{es}, the outer decoder starts errors-and-erasures decoding on the N-symbol block. Let $t(i)$ be the error-correction threshold for a given i, where

$$t(i) = \lfloor (d_2 - i - 1)/2 \rfloor. \tag{18}$$

If the syndrome of the N-symbol block corresponds to an error pattern of i erasures and $t(i)$ or fewer symbol errors, errors-and-erasures decoding is successful. The values of the erased symbols, and the values and the locations of the symbol errors, are determined based on the decoding algorithm. However, if more than $t(i)$ symbol errors are detected, then the outer decoder again declares an erasure (or raises a flag) for the entire N-symbol block.

Let P_s and P_e be the RS-symbol error probability and RS-symbol erasure probability at the input of the outer decoder, respectively. The probability of correct decoding of a block is given by

$$P_c = \sum_{i=0}^{T_{es}} \binom{N}{i} P_e^i \sum_{j=0}^{t(i)} \binom{N-i}{j} P_s^j (1 - P_s - P_e)^{N-i-j}, \tag{19}$$

where T_{es} and $t(i)$ are defined in (17) and (18). Let P_{be} and P_{bi} denote the probabilities of block erasure and incorrect decoding, respectively. Then

$$P_c + P_{be} + P_{bi} = 1, \tag{20}$$

and

$$\begin{aligned} P_{be} + P_{bi} &= 1 - P_c \\ &= \sum_{i=0}^{T_{es}} \binom{N}{8} P_e^i \sum_{j=t(i)+1}^{N-i} \binom{N-i}{j} P_s^j (1 - P_e - P_s)^{N-i-j} \\ &+ \sum_{i=T_{es}+1}^{N} \binom{N}{i} P_e^i (1 - P_e)^{N-i}. \end{aligned} \tag{21}$$

The probability of incorrect decoding of a block is not easy to determine, but it can be upper bounded by [18]

$$P_{bi} \leq \sum_{i=0}^{T_{es}} \binom{N}{i} P_e^i \sum_{j=d_2-i-t(i)}^{N-i} \binom{N-i}{j} P_s^j (1 - P_s - P_e)^{N-i-j}, \tag{22}$$

and the final decoded BER of the concatenated coding system is approximated by

$$P_b \cong \frac{d_2}{2N} P_{bi}. \tag{23}$$

In the following we consider several examples for Scheme II. The outer code is again an RS code of length $N = 255$. Bandwidth efficient inner trellis codes are decoded by the modified Viterbi algorithm. We use B to denote the number of erased branches in the modified Viterbi algorithm, and the erasure threshold is chosen to be

$$T = (T' d_f' E_b / N_0)^2, \tag{24}$$

where d_f' is the minimum free Euclidean distance of the inner code normalized by the signal energy at the demodulator output, and T' is a normalized threshold. The RS-symbol error probability P_s and the RS-symbol erasure probability P_e at the input to the outer decoder are determined by computer simulation. In decoding the outer code, fixed erasure-correction and error-correction thresholds $T_{es} \le (d_2 - 1)$ and $t \le \lfloor (d_2 - 1)/2 \rfloor$, which are independent of the number of erased symbols in an N-symbol block, are assumed, and (21) and (22) are modified as

$$\begin{aligned} P_{be} + P_{bi} &= \sum_{i=0}^{T_{es}} \binom{N}{i} P_e^j \sum_{j=min(t(i),t)+1}^{N-i} \binom{N-i}{j} P_s^j (1 - P_e - P_s)^{N-i-j} \\ &+ \sum_{i=T_{es}+1}^{N} \binom{N}{i} P_e^i (1 - P_e)^{N-i}, \end{aligned} \tag{25}$$

and

$$P_{bi} \le \sum_{i=0}^{T_{es}} \binom{N}{i} P_e^i \sum_{j=d_2-i-min(t(i),t)}^{N-i} \binom{N-i}{j} P_s^j (1 - P_s - P_e)^{N-i-j}. \tag{26}$$

Usually, the probability of block erasure P_{be} is much larger than the probability of block decoding error P_{bi}. Therefore, $P_{be} + P_{bi}$ is a tight bound on P_{be}. In the examples below we compute the sum of the probability of block erasure and block decoding error, $P_{be} + P_{bi}$, and the final decoded BER P_b.

Example 4.1:

As in Example 2.1, the PTVTC's of $R_1 = (2P+i)/3P, P \geq 2$, and $1 \leq i < P$ with 8-PSK modulation are used as inner codes. Only 16-state codes are considered. P_b and $P_{be} + P_{bi}$, with $R_1 = 7/9, 5/6, 8/9$, are shown as functions of t in Figures 8.1, 8.2, and 8.3, with $T_{es} = 10$, 8, and 9, respectively.

Example 4.2:

As in Example 2.2, Ungerboeck's $R_1 = 2/3$ coded 8-PSK with 16 trellis states is used as the inner code. P_b and $P_{be} + P_{bi}$ are shown in Figure 9 as a function of t with $T_{es} = 10$.

From the above examples, we see that for a given E_b/N_0, the decoded BER of Scheme II is significantly lower than that of Scheme I. Scheme II provides us with flexibility in the system design. Tradeoffs between the decoded BER and the probability of block erasure can be achieved by changing T', B, T_{es}, and t.

V Summary and Conclusions

In this paper we considered high rate concatenated coding systems with bandwidth efficient inner codes and RS outer codes for application to satellite communication systems. Specifically, we considered trellis inner codes with MPSK modulation which achieve effective information rates equal to or greater than 1 bit/dimension. These codes mitigate the non-linear effects of TWT amplifies and make use of soft decision Viterbi decoding to achieve significant coding gains on bandlimited channels.

Two types of concatenated coding systems were studied. Scheme I operates without side information, while Scheme II uses side information. The performance of Scheme I was studied by computer simulations, formula calculations, and by asymptotic coding gain derivations. Our results indicate that coding gains from 4 to 8 dB can be achieved at decoded BER's of 10^{-6}

to 10^{-9} with little or no bandwidth expansion. For example, with 16-state inner codes with 8-PSK modulation, coding gains of 6.18 dB at 10^{-9} and 3.92 dB at 10^{-6} without bandwidth expansion (Example 2.1, Figure 2.2) can be achieved. With only 12.5 % bandwidth expansion, coding gains of 7.25 dB at 10^{-9} and 4.96 dB at 10^{-6} (Example 2.2, Figure 3.1), are achievable. These 16-state decoders are capable of high speed operation.

In studying the performance of Scheme II, we first proposed a modified Viterbi algorithm for decoding trellis codes. Side information was provided to the outer decoder in the form of symbol erasures. The outer RS code was then decoded by an errors-and-erasures decoder. A significant improvement in decoded BER was obtained for Scheme II. For example, decoded BER's as low as 10^{-16}, with an erasure rate no more than 10^{-3}, can be achieved with a 16-state inner trellis code, 8-PSK modulation, and an overall effective information rate of 0.9 bits/dimension at an E_b/N_0 of 5.45 dB (Example 4.2, Figure 9). For systems where block erasures are allowed, such as ARQ systems, Scheme II is highly recommended.

References

[1] G. Ungerboeck, "Channel Coding with Multilevel/Phase Signals," *IEEE Trans. on Inform. Theory*, IT-28, pp. 55-67, January 1982.

[2] G. D. Forney, Jr., et. al., "Efficient Modulation for Band-Limited Channels," *IEEE J. Select. Areas Commun.*, SAC-2, pp. 632-647, September 1984.

[3] A. R. Calderbank and N.J.A. Sloane, "New Trellis Codes," *IEEE Trans. Inform. Theory*, to appear.

[4] L. F. Wei, "Trellis-Coded Modulation with Multi-Dimensional Constellations," *IEEE Trans. Inform. Theory*, to appear.

[5] G. D. Forney, Jr., "Coset Codes," *IEEE Trans. Inform. Theory*, to appear.

[6] F. Hemmati and R.J.F. Fang, "Low Complexity Coding Methods for High Data Rate Channels," Comsat Laboratories Technical Note, February 1984.

[7] A. LaFanchere, R. H. Deng, and D. J. Costello, Jr., "Trellis Coded Phase Modulation, Using Unit-Memory and Partial-Unit-Memory Convolutional Codes," submitted to *IEEE Trans. Inform. Theory*, January 1987.

[8] S. G. Wilson, "Rate 5/6 Trellis-Coded 8-PSK," *IEEE Trans. on Commun.*, COM-34, pp. 1045-1049, October 1986.

[9] S. G. Wilson, H. A. Sleeper, II, P. J. Schottler, and M. T. Lyons, "Rate 3/4 Convolutional Coding of 16-PSK: Code Design and Performance Study," *IEEE Trans. on Commun.*, COM-32, pp. 1308-1315, December 1984.

[10] G. D. Forney, Jr., *Concatenated Codes*, Cambridge, MA: MIT Press, 1966.

[11] J. P. Odenwalder, "Optimal Decoding of Convolutional Codes," Ph.D. Thesis, University of California, Los Angeles, 1970.

[12] L. N. Lee, "Concatenated Coding Systems Employing a Unit-Memory Convolutional Code and a Byte-Oriented Decoding Algorithm," *IEEE Trans. Commun.*, COM-25, pp. 1064-1074, October 1977.

[13] M. A. Herro, D. J. Costello, Jr., and L. Hu, "Capacity and Cutoff Rate Calculations for a Concatenated Coding System," *IEEE Trans. Inform. Theory*, to appear.

[14] S. Lin and D. J. Costello, Jr., *Error Control Coding: Fundamentals and Applications*, Englewood Cliffs, NJ: Prentice-Hall, 1983.

[15] T. Fujino, et.al., "A 120 M bits/s Coded 8-PSK Modem with Soft-Decision Viterbi Decoder," *IEEE Int. Conf. on Communications Conference Record*, June 1986, Toronto, Canada.

[16] G. W. Zeoli, "Coupled Decoding of Block-Convolutional Concatenated Codes," *IEEE Trans. on Commun.*, COM-21, pp. 219-226, March 1973.

[17] H. Yamamoto and K. Itoh, "Viterbi Decoding Algorithm for Convolutional Codes with Repeat Request," *IEEE Trans. Inform. Theory*, IT-26, pp. 540-546, September 1980.

[18] T. Kasami, S. Lin, and T. Takata, "A Cacaded Coding Scheme for Error Control," *IEEE Trans. Inform. Theory*, to appear.

Table 1. Coding gain over QPSK of Example 2.3 with R_1 = 5/6 coded 16-PSK inner code (formula calculation)

# of inner code states	R_{eff}, bits per dimension	Coding gain at 10^{-6} (E_b/N_0)dB	Coding gain at 10^{-9} (E_b/N_0)dB
2	1	2.85	5.08
4	1	5.07	7.13
8	1	3.65	6.28
16	1	4.15	6.35

Table 2. Coding gain over QPSK of Example 2.3 with R_1 = 3/4 coded 16-PSK inner code (formula calculation)

# of inner code states	R_{eff}, bits per dimension	Coding gain at 10^{-6} (E_b/N_0)dB	Coding gain at 10^{-9} (E_b/N_0)dB
2	1	2.05	4.18
4	1	3.60	5.77
8	1	3.22	5.45
16	1	3.22	5.47

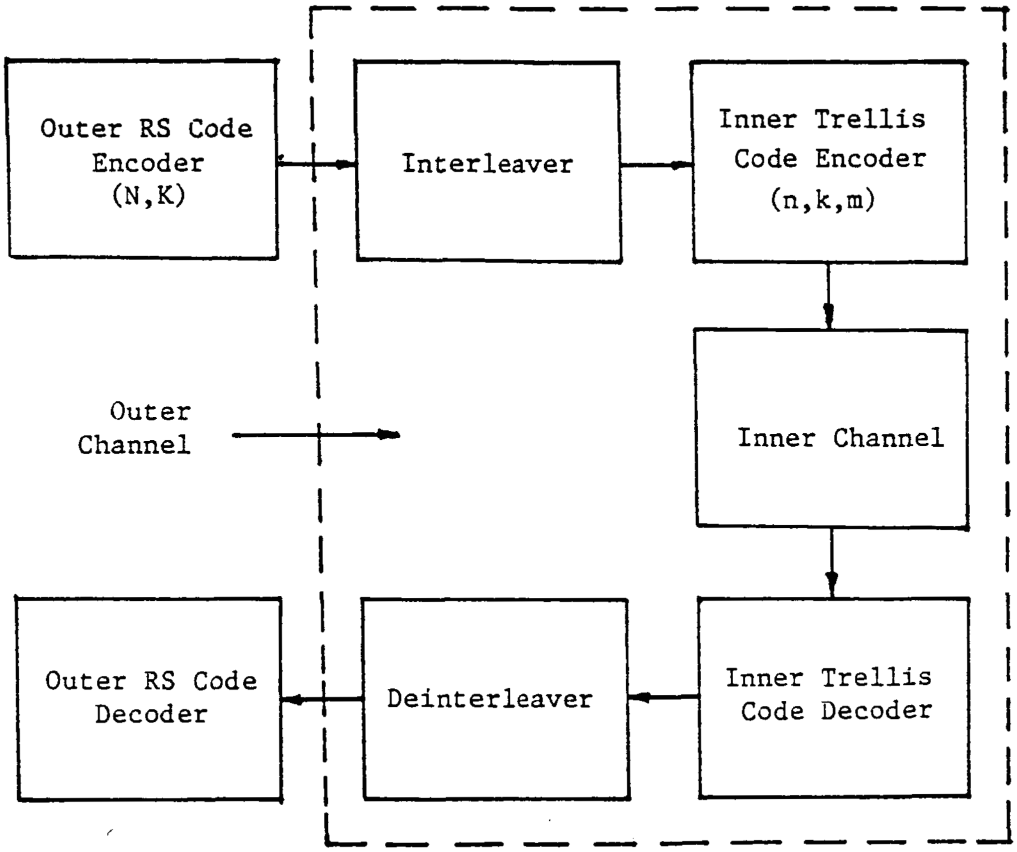

Fig. 1 A concatenated coding system

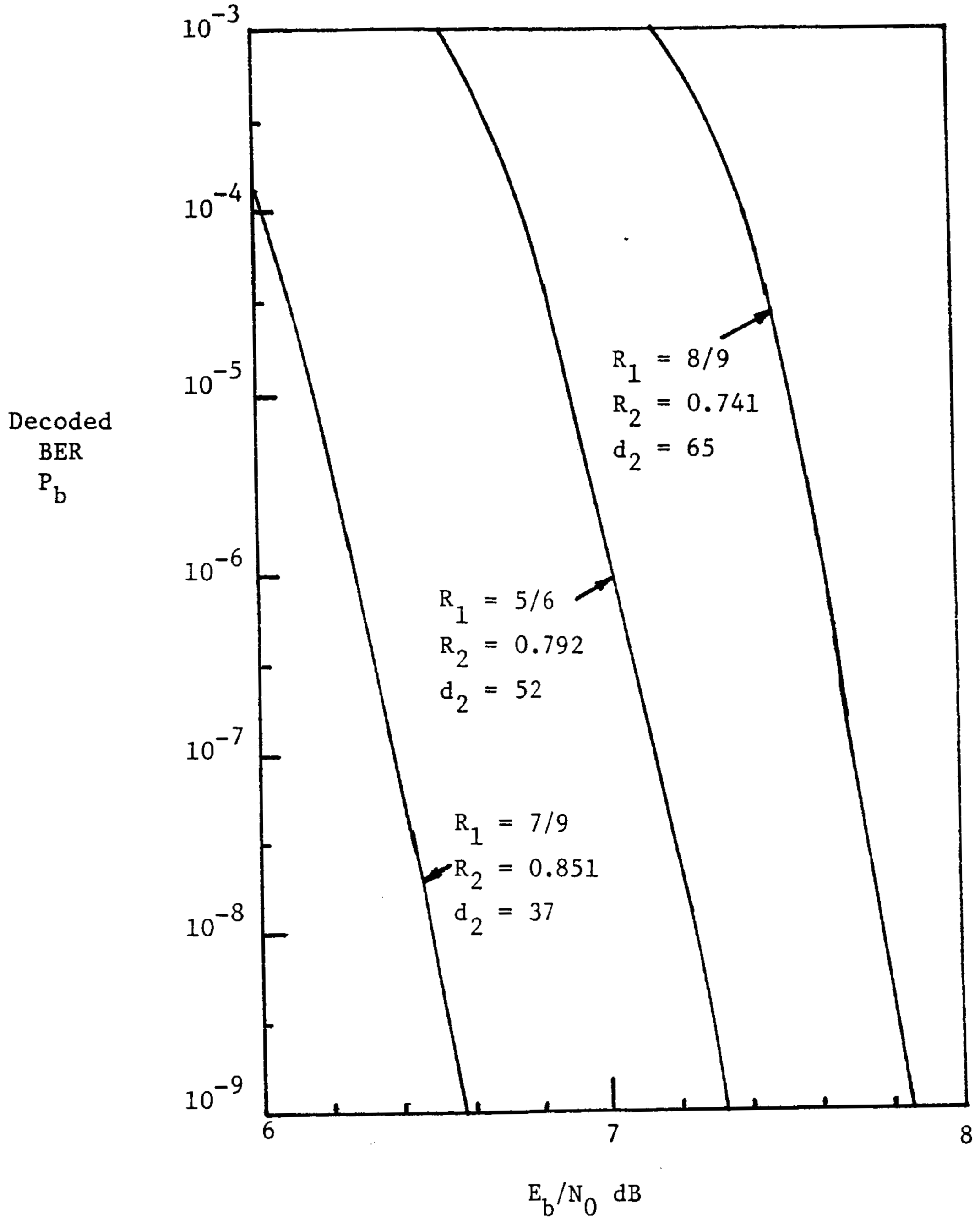

Fig. 2.1 Performance of Example 2.1 (simulation) with 4-state rate R_1 PTVTC inner code and R_{eff} = 1 bit/dimension.

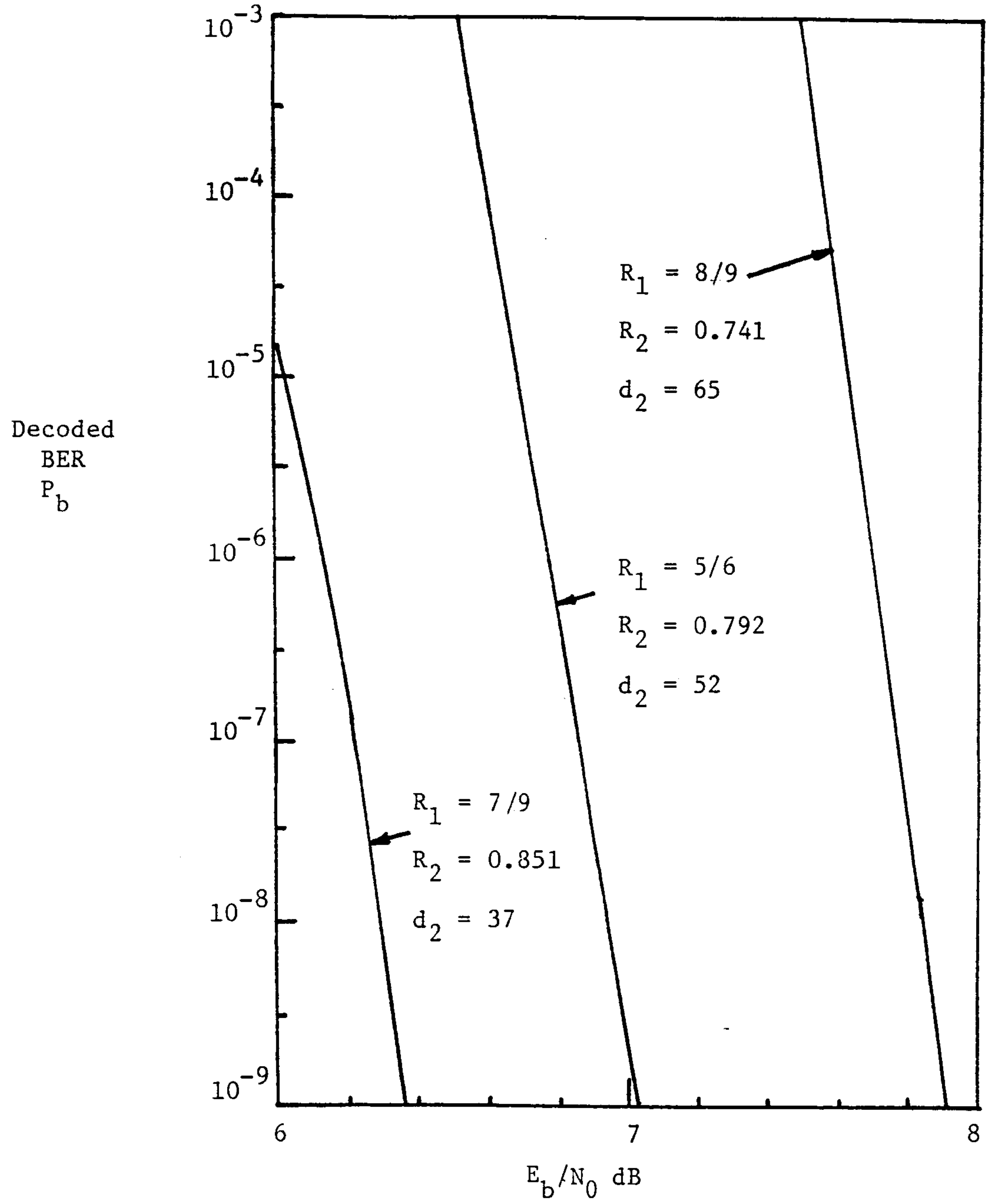

Fig. 2.2 Performance of Example 2.1 (simulation) with 16-state rate R_1 PTVTC inner code and $R_{eff} = 1$ bit/dimension.

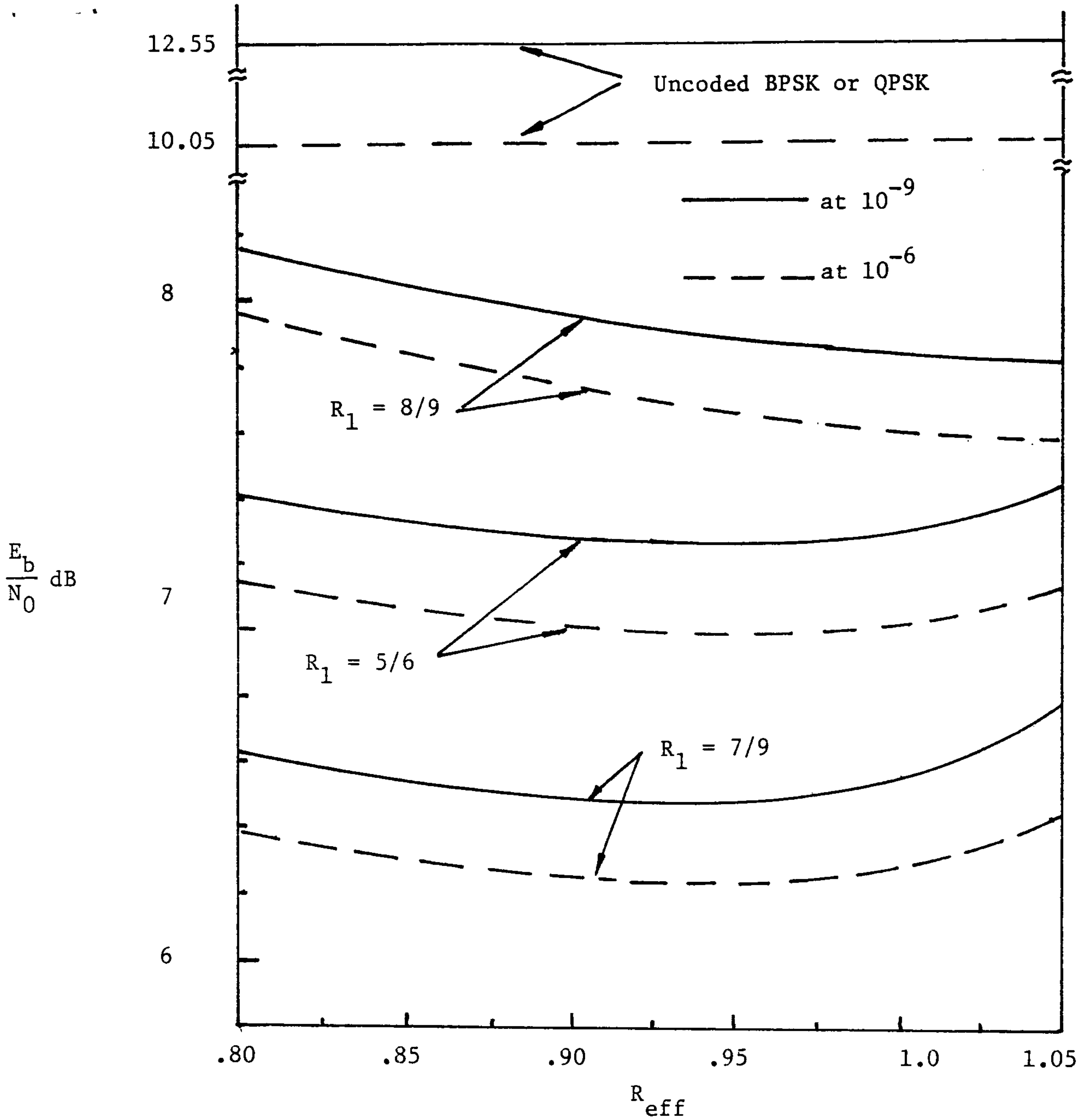

Fig. 2.3 E_b/N_0 required to achieve $P_b = 10^{-6}$ and 10^{-9} vs. R_{eff} for Example 2.1 with 4-state PTVTC inner codes (simulation).

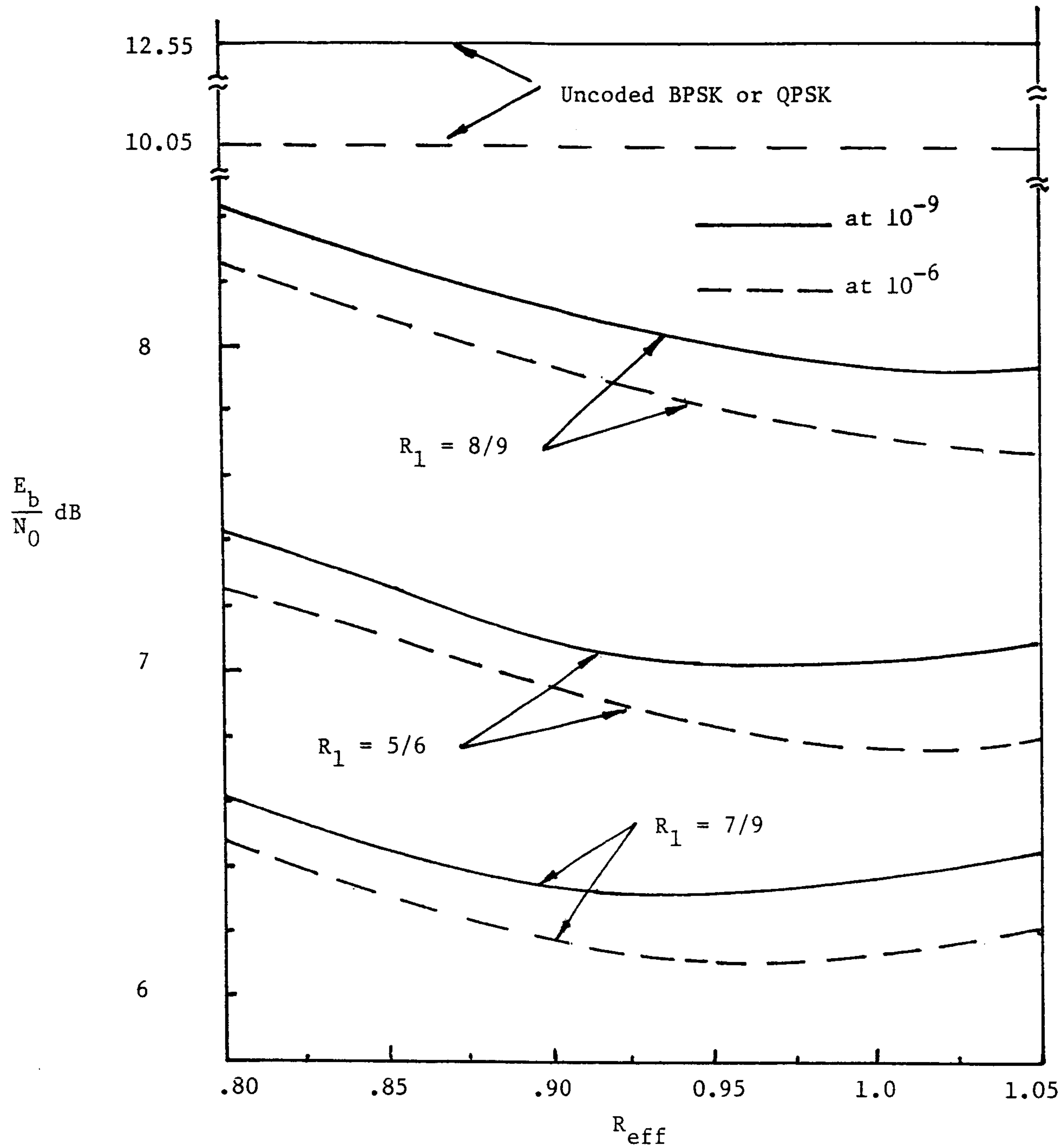

Fig. 2.4 E_b/N_0 required to achieve $P_b = 10^{-6}$ and 10^{-9} vs. R_{eff} for Example 2.1 with 16-state PTVTC inner codes (simulation).

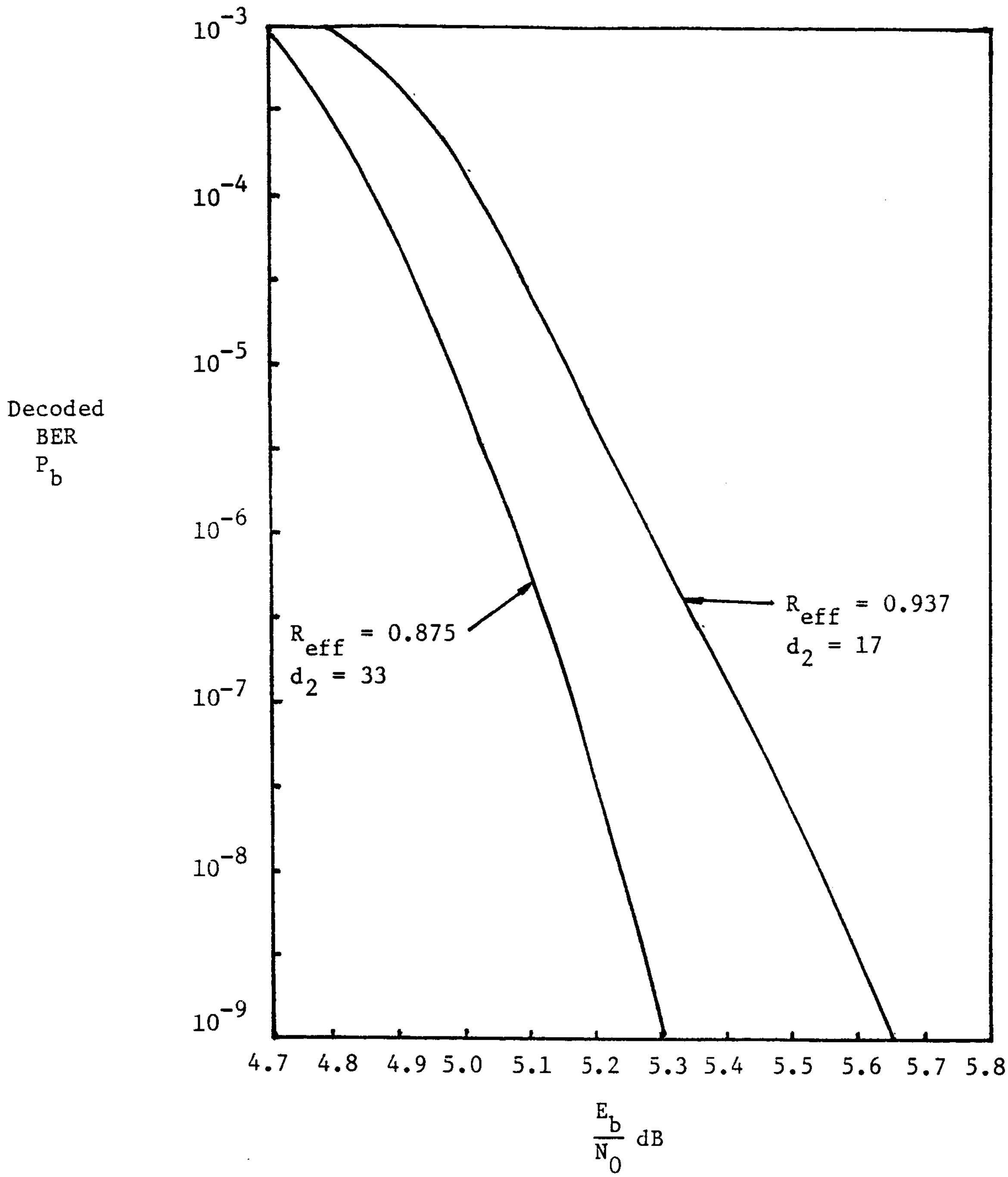

Fig. 3.1 Code performance of Example 2.2 (simulation) with Ungerboeck's 16-state $R_1 = 2/3$ coded 8-PSK inner code.

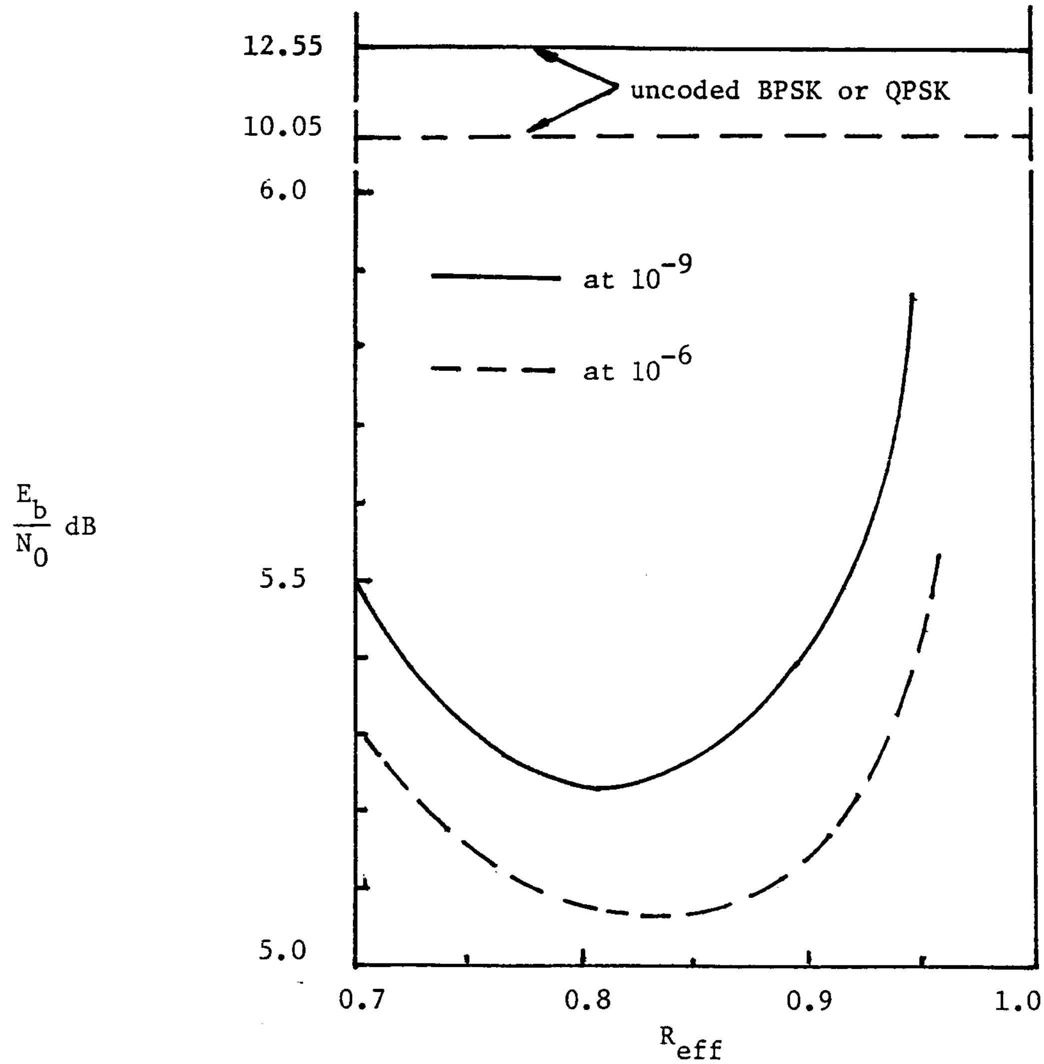

Fig. 3.2 E_b/N_0 required to achieve $P_b = 10^{-6}$ and 10^{-9} vs. R_{eff} for Example 2.2 with Ungerboeck's 16-state $R_1 = 2/3$ coded 8-PSK inner code (simulation).

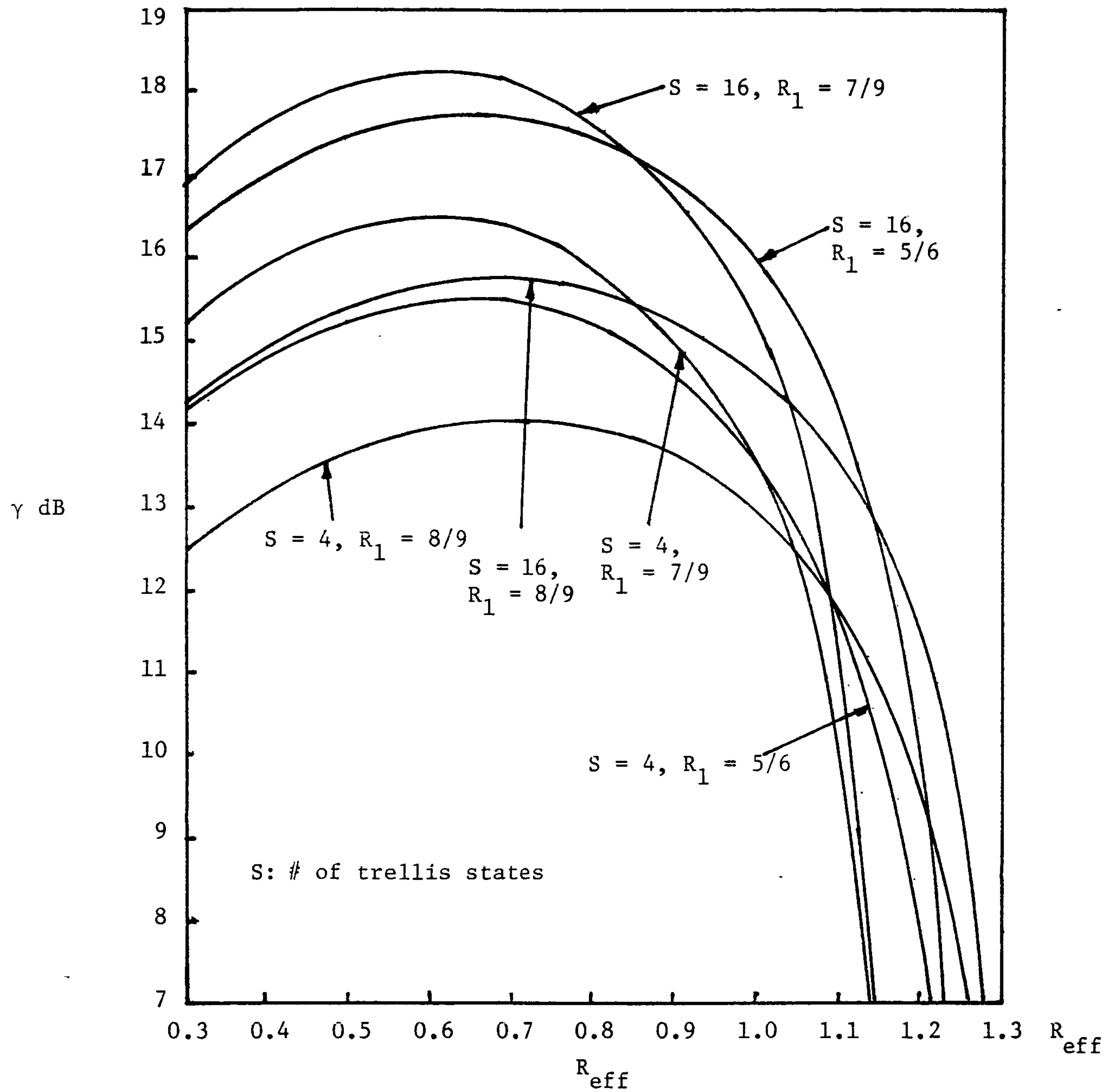

Fig. 4 Asymptotic coding gain of Example 2.1 with PTVTC inner codes and N = 255 RS outer codes.

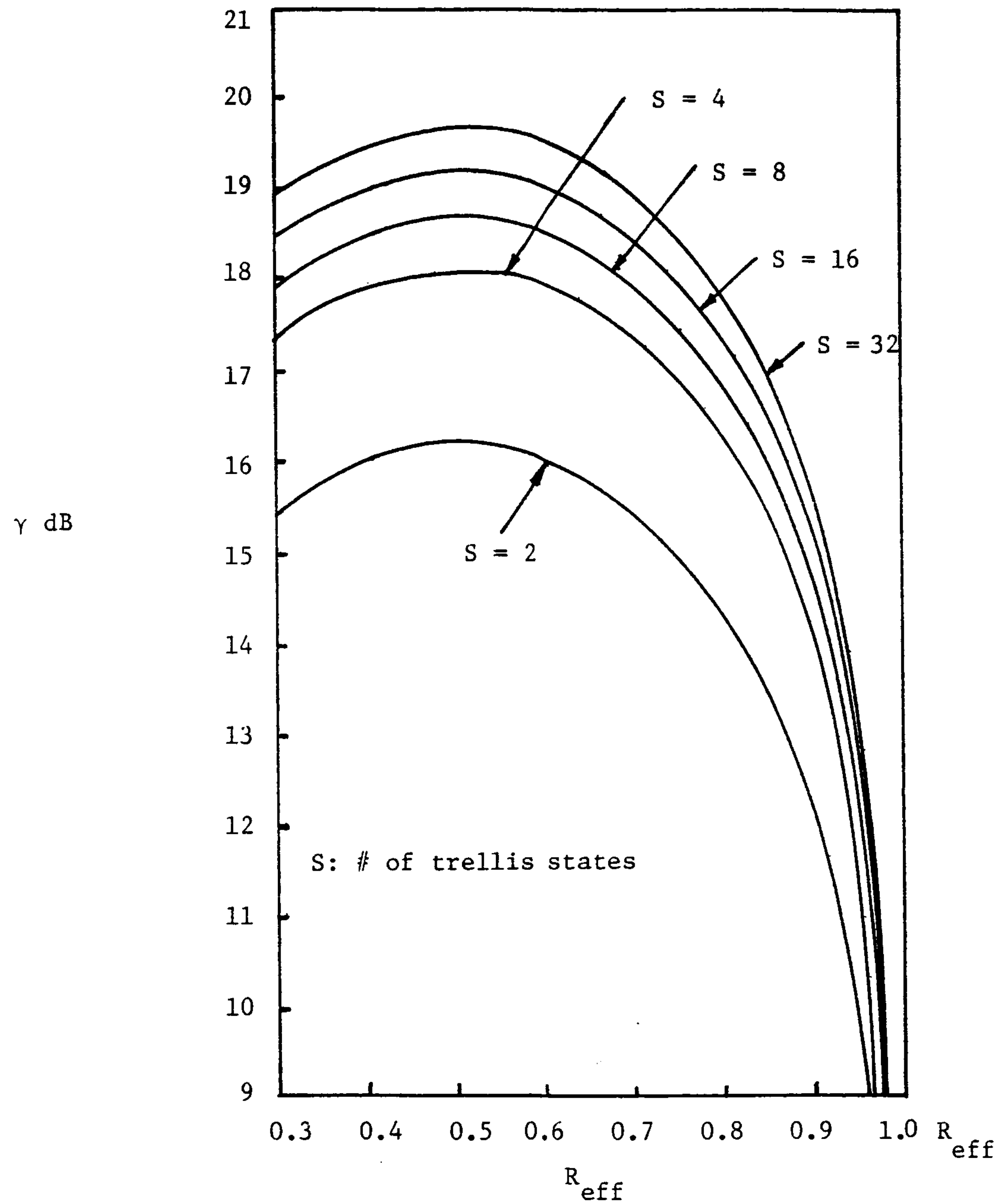

Fig. 5 Asymptotic coding gain of Example 2.2 with Ungerboeck's $R_1 = 2/3$ coded 8-PSK inner codes and N = 255 RS outer codes.

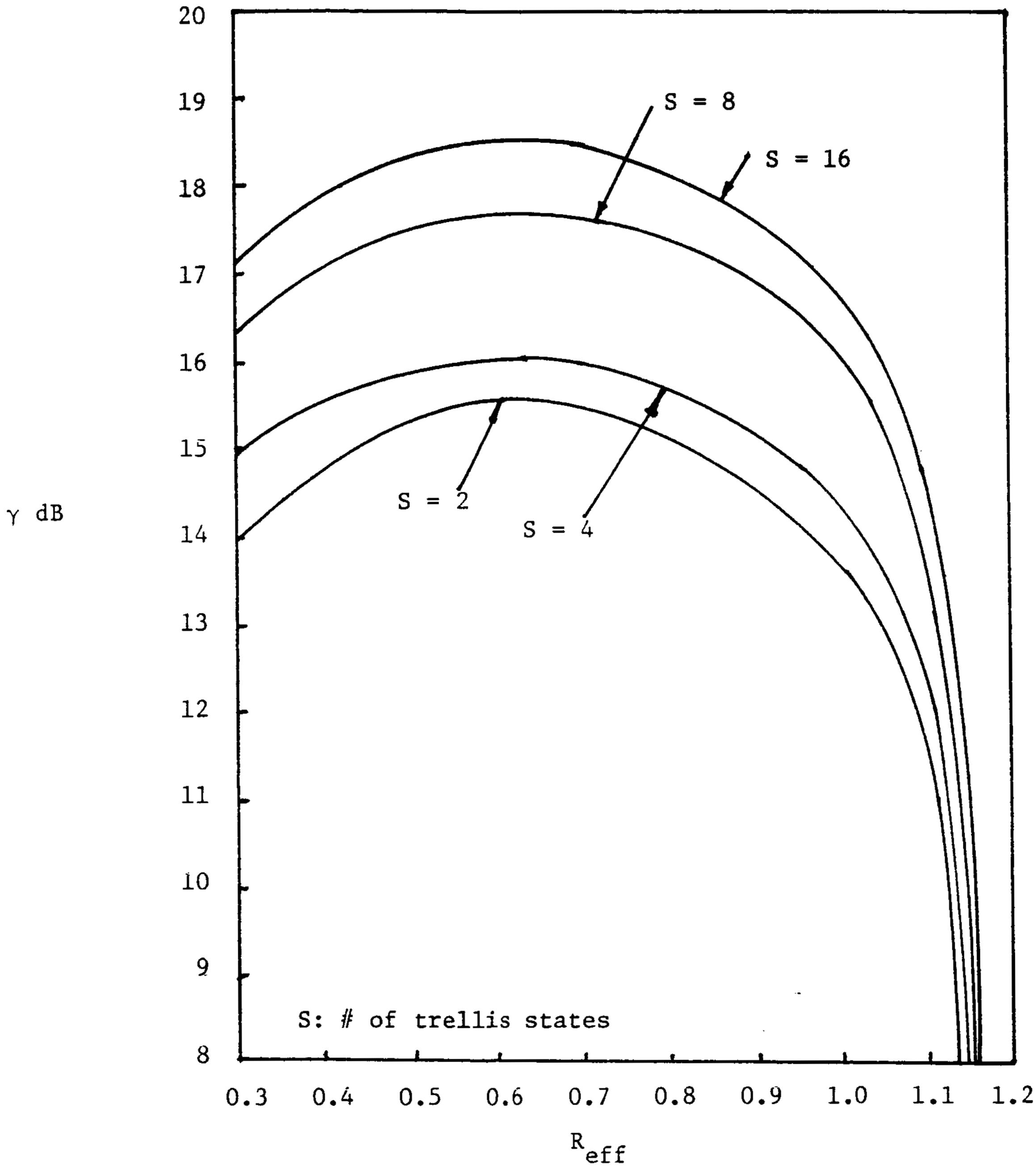

Fig. 6.1 Asymptotic coding gain of Example 2.3 with $R_1 = 5/6$ coded 8-PSK inner codes and $N = 255$ RS outer codes.

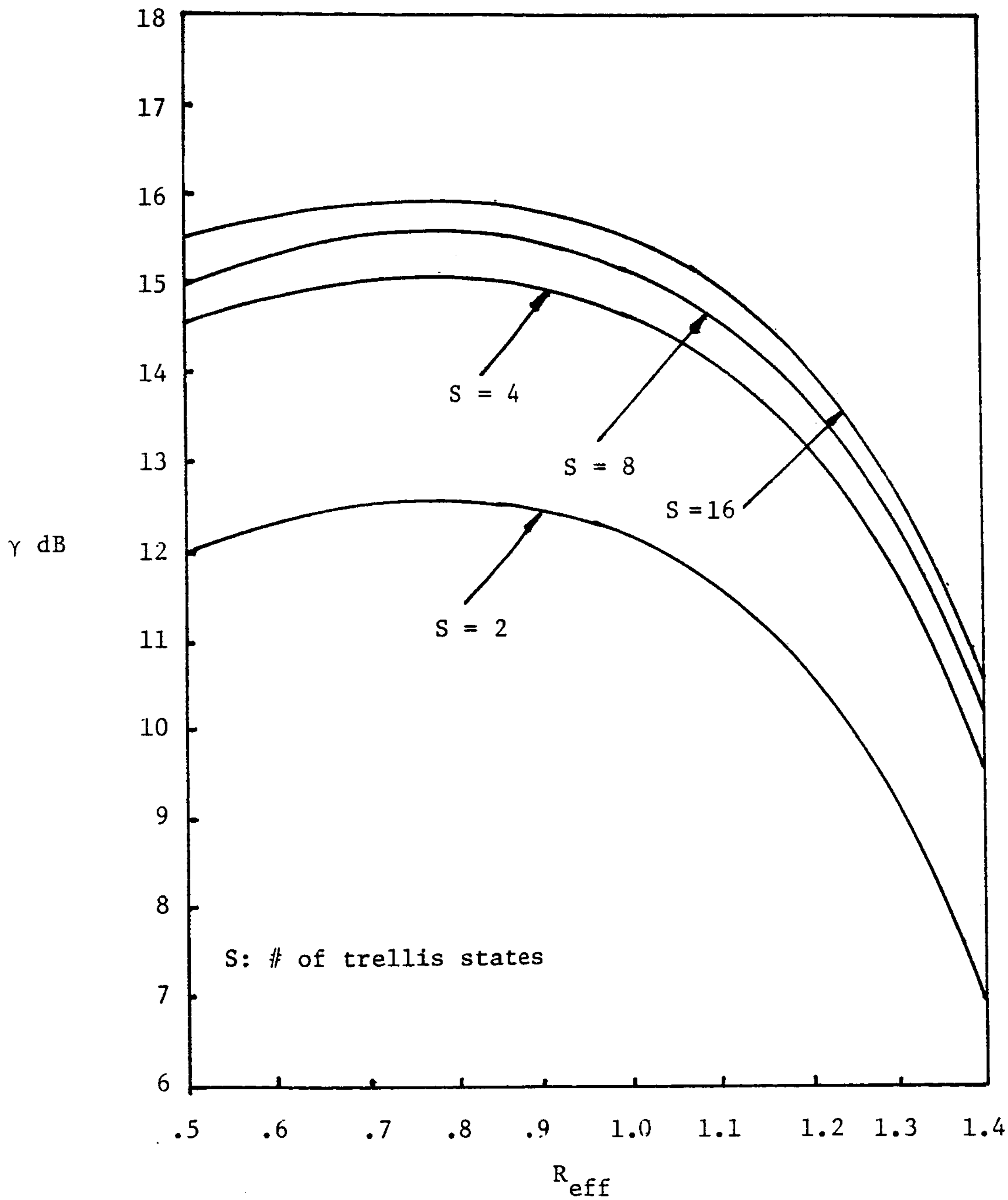

Fig. 6.2 Asymptotic coding gain of Example 2.3 with R_1 = 3/4 coded 16-PSK inner codes and N = 255 RS outer codes.

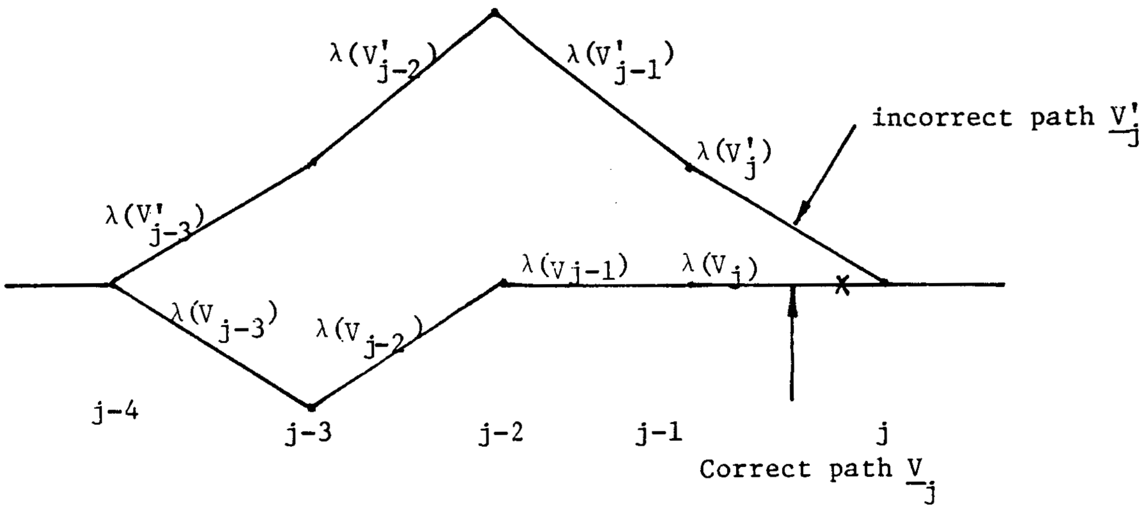

Fig. 7 A 4-branch first event error at level j in decoding a trellis code.

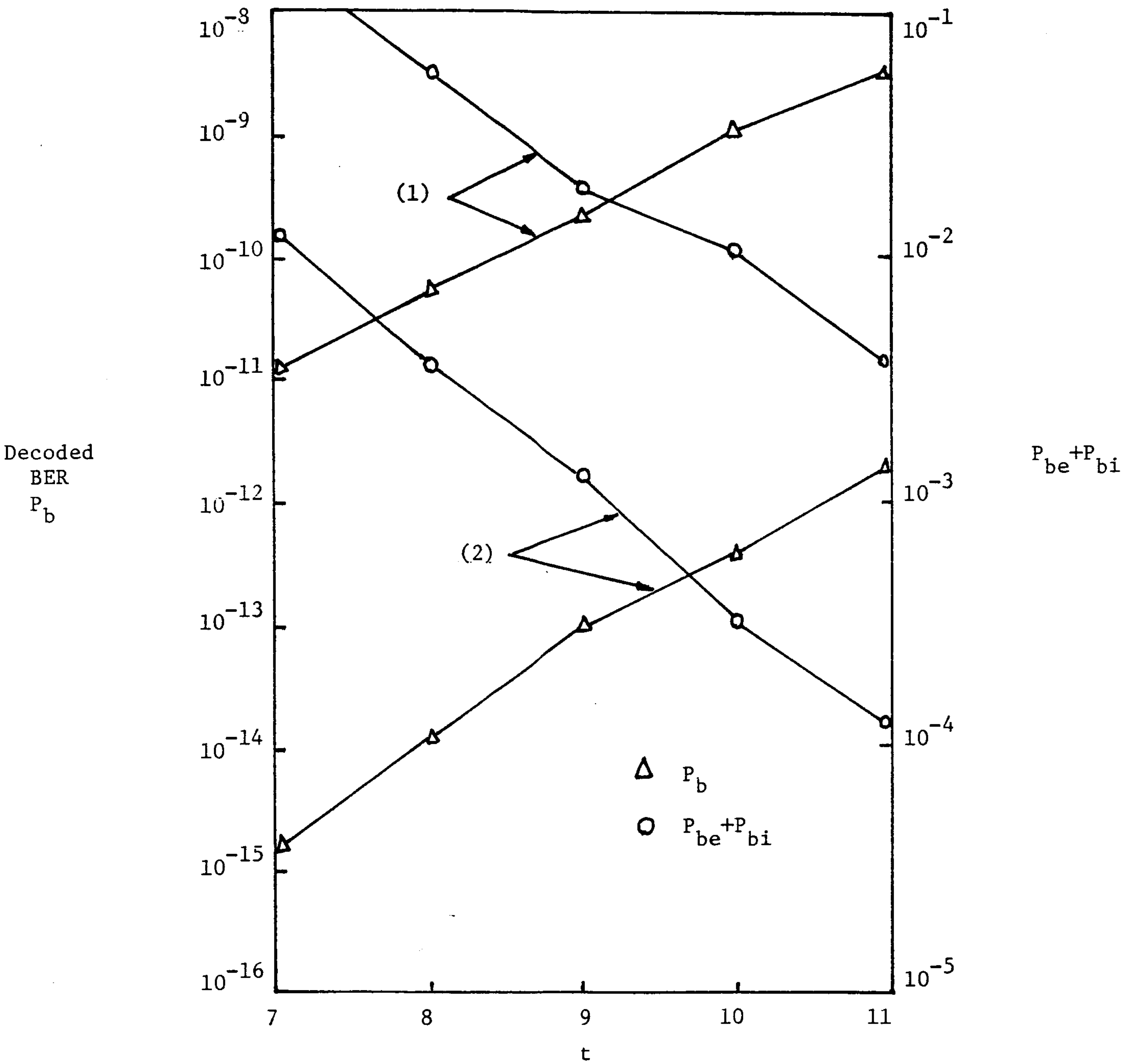

Fig. 8.1 Performance of Example 4.1 (simulation) with 16-state $R_1=7/9$ PTVTC 8-PSK inner code and B=7, $T'=0.2$, $T_{es}=10$, $d_2=37$, $R_{eff}=1$. (1) $E_b/N_0=6.1$ dB ($P_b=2.5\times10^{-6}$ in scheme 1); (2) $E_b/N_0=6.34$ dB ($P_b = 8.5\times10^{-9}$ in scheme 1).

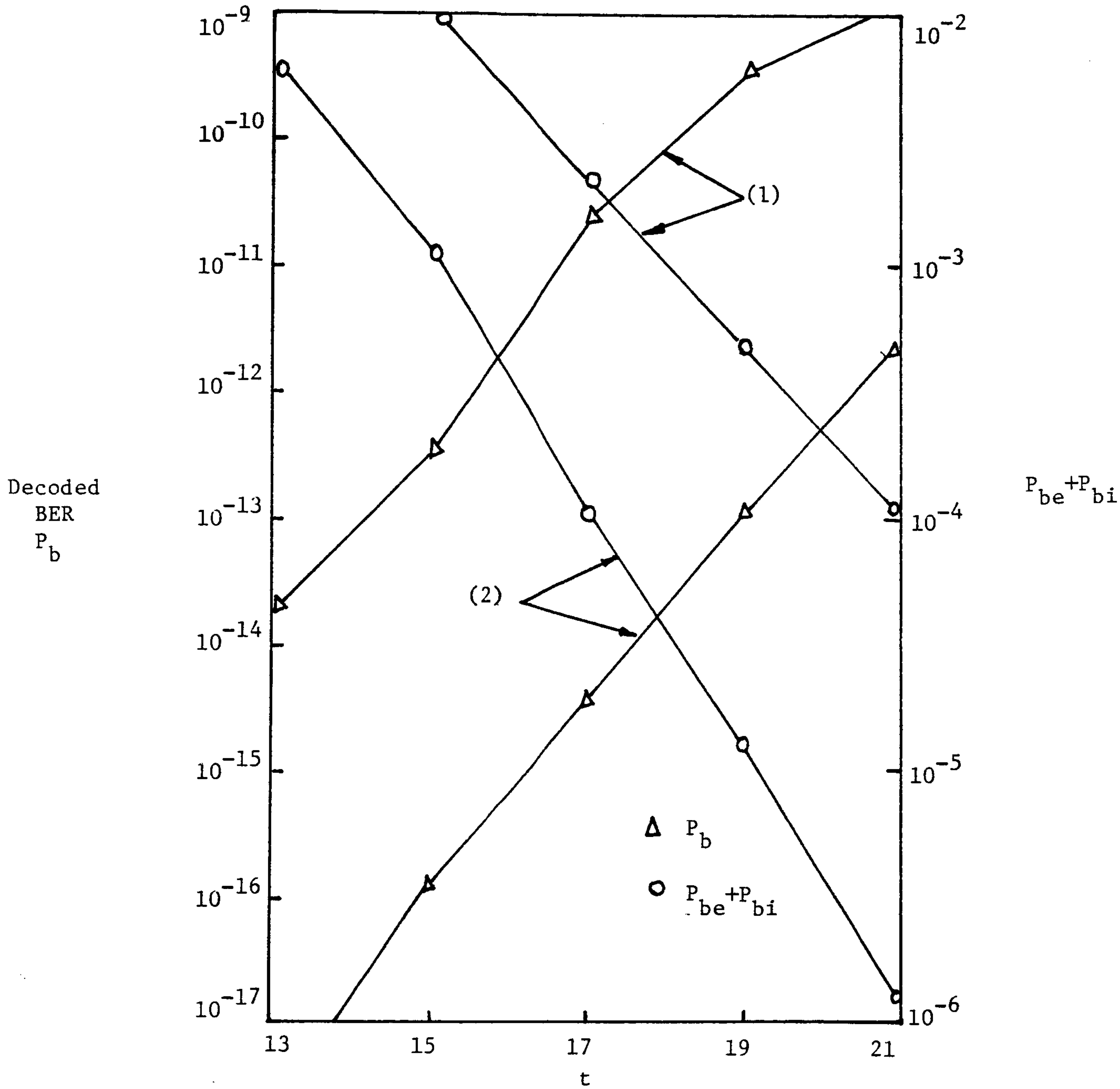

Fig. 8.2 Performance of Example 4.1 (simulation) with 16-state R_1=5/6 PTVTC 8-PSK inner code and B=5, T'=0.1, T_{es}=8, d_2=52, R_{eff}=1. (1) E_b/N_0=6.53 dB (P_b=4.2x10^{-7} in scheme 1); (2) E_b/N_0=6.77 dB (P_b=1.9x10^{-9} in scheme 1).

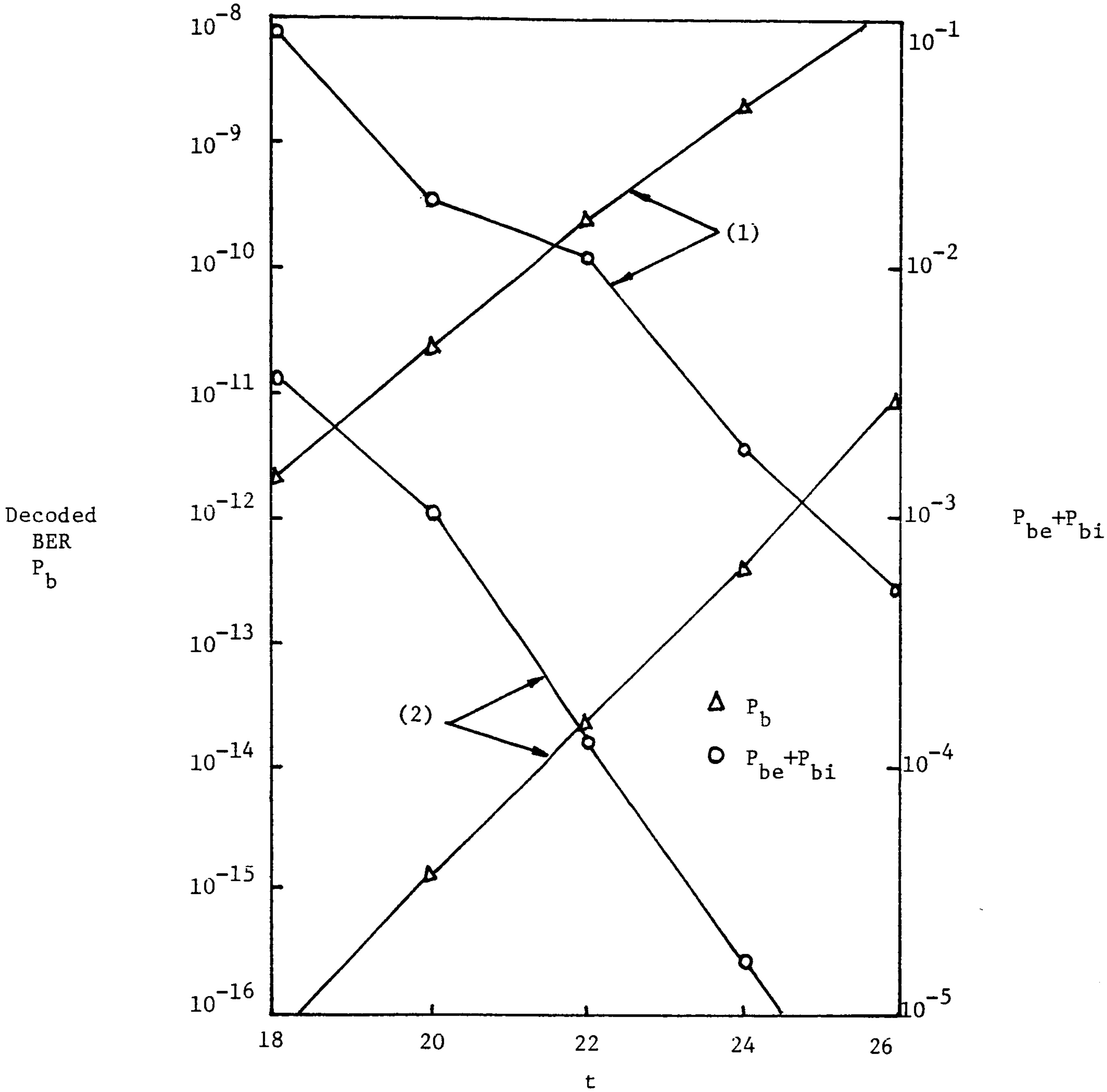

Fig. 8.3 Performance of Example 4.1 (simulation) with 16-state R_1=8/9 PTVTC 8-PSK inner code and B=8, T'=0.1, T_{es}=9, d_2=64, R_{eff}=1. (1) E_b/N_0=7.48 dB (P_b=4.3x10^{-6} in scheme 1); (2) E_b/N_0=7.68 dB (P_b=5,8x10^{-9} in scheme 1).

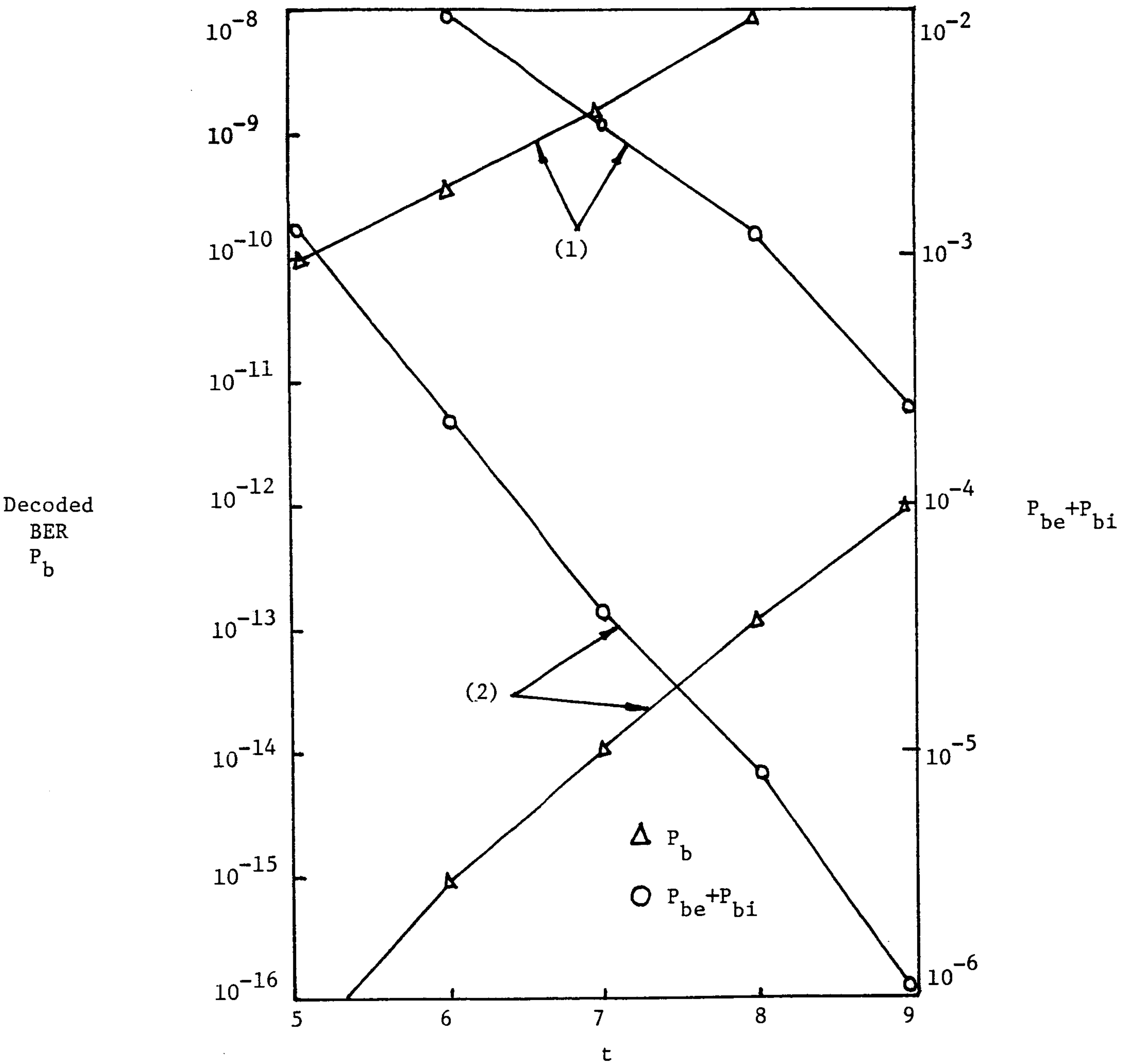

Fig. 9 Performance of Example 4.2 (simulation) with 16-state $R_1=2/3$ coded 8-PSK inner code and B=4, T'=0.3, $T_{es}=10$, $d_2=26$, $R_{eff}=0.9$. (1) $E_b/N_0=5.07$ dB ($P_b=1.11\times10^{-6}$ in scheme 1); (2) $E_b/N_0=5.45$ dB ($P_b=5.3\times10^{-10}$ in scheme 1).

Appendix B

A Lower Bound on the Minimum Euclidean Distance of Trellis Coded Modulation Schemes

A LOWER BOUND ON THE MINIMUM EUCLIDEAN DISTANCE OF TRELLIS CODED MODULATION SCHEMES

Marc Rouanne
Daniel J. Costello, Jr.
Dept. of Elec. & Comp. Engr.
Univ. of Notre Dame
Notre Dame, IN 46556

Submitted to
IEEE Transactions on Information Theory

May 1987

This work was supported by NASA grant NAG5-557 and NSF grant ECS84-14608.

ABSTRACT

Bandwidth efficient codes can be used to improve the performance of digital transmission on bandwidth limited channels such as the telephone channel. An important class of error correcting codes suitable for bandwidth limited channels is referred to as Trellis Coded Modulation (TCM). Such codes combine a binary convolutional encoder and a modulation scheme. Recently, many good TCM schemes have been found. In this paper, we derive a lower bound on the minimum free Euclidean distance d_{free} of TCM which guarantees the existence of good TCM codes of any complexity. We use the bound to compare trellis codes combined with PSK, PAM, or QASK modulation. We also compare the bound with known upper bounds. This random coding bound is exponentially tight for large constraint lengths and predicts the asymptotic performance of TCM when the complexity of the code becomes large. The bound can be used with any code rates and any modulation scheme and shows that the free distance increases linearly with the constraint length for large values of the constraint length.

I INTRODUCTION

The transmitter power and the channel bandwidth constrain the design of a digital communication system. On some channels the bandwidth constraint prevails over the power constraint. For example, on the telephone channel (a typical bandwidth limited channel with bandwidth $W = 3$ kHz), a large Signal-to-Noise Ratio (SNR) of around 28 dB permits the use of several signal amplitudes with a moderate error probability [Forney et. al., 1984]. On such channels, efficient coding systems transmit at more than one information bit per signal dimension. On satellite channels, the power constraint used to prevail over the bandwidth constraint. But the increasing need for bandwidth efficient satellite communication systems makes the satellite channel more bandwidth limited. Bandwidth efficient codes increase the power efficiency on bandwidth limited channels, i.e., for a given bandwidth and rate of information bits sent per transmission interval, a desired error probability can be achieved with less average signal energy. If Additive White Gaussian Noise (AWGN) perturbs the channel, the error probability of an uncoded system depends on the minimum Euclidean distance between signals. We will define the minimum Euclidean distance more precisely later. Distributing a given average energy into more signals without coding reduces the distances between signals and increases the probability of error. A bandwidth efficient coding scheme lowers this increased error probability to that of an uncoded scheme of larger average energy by introducing redundancy between successive symbols. In other words, a bandwidth efficient scheme transmits with the same rate and error probability as an uncoded system with larger average energy. Alternately, a bandwidth efficient scheme can transmit more information bits per transmission interval than an uncoded scheme with the same average energy and error probability. However, the tradeoff is increased complexity. TCM is an application of Shannon's information theory, and cutoff rate and channel capacity calculations show that 6 dB to 9 dB can be gained over an uncoded system at an error probability of 10^{-5} to 10^{-6} [Ungerboeck, 1982].

II DEFINITIONS

A typical trellis coded modulation (TCM) scheme includes a binary encoder followed by a modulator (Fig.1). During every signaling interval $[tT,(t+1)T]$ of length T, k input bits $u_i^{(t)}$, $i = 1, \ldots, k$, enter the encoder, and n output bits $v_j^{(t)}$, $j = 1, \ldots, n$, leave the encoder. k is the information stream width. For any t, there are 2^k possible input blocks $u^{(t)} \triangleq (u_1^{(t)}, \ldots, u_k^{(t)})$ and 2^n possible output blocks $v^{(t)} \triangleq (v_1^{(t)}, \ldots, v_n^{(t)})$. $R_b \triangleq k/n$ is the rate of the binary encoder. The memory order of the encoder is ν, and every output $v_j^{(t)}$ depends on the current information block (k bits) and on the ν previous blocks (νk bits). However $v_j^{(t)}$, $j = 1, \ldots, n$, may not depend on all the νk past information bits, and the number of memory registers ν_i that store one input stream may depend on i, so that $0 \le \nu_i \le \nu$. We call $\nu_0 \triangleq \sum_{i=1}^{k} \nu_i$ the total memory of the TCM scheme. In this paper, we restrict our study to the case where $\nu_i = \nu$, $i = 1, \ldots, k$, so that the constraint length is $\nu_0 = k\nu$, i.e., we assume that the encoder contains k memory registers of equal length ν. The encoder of a TCM scheme is a finite state machine with 2^{ν_0} states, where the contents of the ν_0 binary registers specifies the encoder state. 2^k branches corresponding to all possible input blocks enter and leave each state. The modulator maps each of the 2^n binary n-tuples leaving the binary encoder onto a signal constellation. A signal constellation S is a finite set of M signal symbols y_s, $s = 1, \ldots, M$, and can be generated from a set of D orthogonal signals (signal space representation). In our case $M = 2^n$, and the signal constellation has D dimensions. We define the rate of a TCM scheme as $R \triangleq k/D$, the number of information bits transmitted per dimension. The codewords in a TCM scheme can be represented by a trellis whose branches are labeled with channel signals, and TCM schemes are sometimes called trellis codes. A (k, ν) trellis code is defined by a trellis with $2^{k\nu}$ states and 2^k branches leaving each state and entering 2^k different states, and a time-varying mapping of signals in S onto branches. The complexity of the decoder is a function of $2^{(\nu_0 + k)} = 2^{k(\nu + 1)}$. (Note that certain authors define the constraint length as $n(\nu + 1)$ [Lin-Costello, 1983].)

Binary and quaternary phase shift keying (BPSK, QPSK) send one information bit per dimension ($R = 1$). Coding on bandwidth efficient systems requires signal constellations with more than 2 signals per dimension, for example 8-PSK, to send more than one information bit per dimension ($R > 1$). These higher alphabet modulation schemes can be used to generate redundancy in the transmitted message to improve the reliability of forward error correction (FEC) coding. In conventional FEC coding systems, the encoder is designed to maximize the minimum Hamming distance between binary codewords (n-tuples $v^{(t)}$). Then these binary codewords are mapped onto a signal constellation disregarding the geometry of the constellation. Nonetheless, for AWGN channels, the channel error probability between channel signals y_t and y_t' in S depends on the Euclidean distance $d_e(y_t, y_t')$ between signals y_t and y_t' in the signal constellation S. The Euclidean distance can be made proportional to the Hamming distance only for BPSK and QPSK modulation. This is why for TCM schemes that use signal constellations other than BPSK or QPSK, the encoder and the modulator are selected jointly to maximize the minimum Euclidean distance between codewords rather than the minimum Hamming distance.

The signal error probability P_s of a trellis code used on an AWGN channel and decoded with a maximum-likelihood sequence estimator (Viterbi algorithm) depends on the Euclidean distance between channel signals [Viterbi, 1971]. P_s is upperbounded by:

$$P_s \leq \frac{1}{2} \sum_{d = d_{free}}^{\infty} A(d)\, e^{-d^2 E_s/N_0} \ , \qquad (1)$$

where the sum is taken over all possible distances d between distinct paths. d_{free} is the minimum Euclidean distance between distinct paths, $A(d)$ is the path multiplicity of distance d, i.e., the number of error paths at distance d from the correct path, E_s is the average signal energy of the constellation, N_0 is the one-sided power spectral density of the Gaussian noise, and E_s/N_0 is the SNR of the channel. Equation (1) means that the performance of a TCM scheme depends on the distribution of distances between encoder output sequences corresponding to distinct encoder input sequences. For large values of SNR, the sum of (1) is equivalent to 1/2 $A(d_{free})\, e^{-d_{free}^2 E_s/N_0}$, whose behavior is mostly determined by the exponent, or simply by d_{free}^2. Therefore, for large SNR, a small error probability P_s corresponds to a large d_{free}^2. We call $d_{\min}$ the minimum Euclidean distance between signals in a constellation. Then the asymptotic coding gain (for large SNR) of a TCM scheme over an uncoded system is given by $G \triangleq 10 \log_{10} \dfrac{d_{free}}{d_{\min}}$, where d_{free} is the minimum Euclidean distance of the TCM scheme, and $d_{\min}$ is the minimum Euclidean distance of the uncoded scheme.

In this paper we derive a lower bound on the minimum Euclidean distance of TCM schemes. The bound is similar to the bound of Costello [Costello, 1974] and Forney [Forney, 1974] on the free distance of convolutional codes. The bound guarantees the existence of codes that perform better than the average. We use this bound to compare different modulation and coding schemes, such as those proposed by Ungerboeck [Ungerboeck, 1982], Calderbank and Sloane [Calderbank-Sloane, 1985], and Lafanechere, Deng, and Costello [Lafanechere-Deng-Costello, 1987]. We also compare it with Calderbank, Mazo, Shapiro, and Wei's upper bound [Calderbank-Mazo-Shapiro, 1983 and Calderbank-Mazo-Wei, 1985]. The main advantage of using a bound to compare schemes over searching for good codes is that we do not use exhaustive methods but rather optimize the bound and its parameters. This reduces the complexity of finding good TCM schemes, although we cannot guarantee that we find the optimum schemes, because there may exist codes that perform better than the bound. However the best known codes behave in accordance with the bound. The bound can also be used to predict the performance of new schemes and, if promising, then actual codes can be constructed.

An efficient TCM scheme assigns channel signals to branches to achieve maximum d_{free} and minimum error probability P_s when using maximum likelihood decoding. The lower bound on d_{free} uses a random coding argument based on the moment generating function $\overline{e^{\alpha\rho d^2} \sum_{\underline{y}} \left[\sum_{\underline{y}'} e^{-\alpha d_e^2(\underline{y},\underline{y}')} \right]^{\rho}}$, where the overbar indicates an average over the ensemble of all code choices (for a given trellis), the first sum is over all choices of paths $\underline{y}$, the second sum is over all paths $\underline{y}'$ diverging from $\underline{y}$ at time 1, α and ρ are arbitrary constants, d is the lower bound on d_{free}, and $d_e(\underline{y},\underline{y}')$ is the Euclidean distance between the channel paths $\underline{y}$ and $\underline{y}'$. The lower bound depends on ν, k, and the distance distribution of the signal constellation.

III DERIVATION OF THE BOUND

For simplicity, we restrict our notation to TCM schemes. However, the derivation, except for configuration counting, applies to any Coded Modulation scheme. We call a topological trellis a trellis with no signal assignments on the branches, i.e., a trellis with no labels. A topological path $\underline{Y}$ through a topological trellis is a sequence of consecutive branches with no labels. A channel path $\underline{y}$ is a path through a trellis labeled with channel signals on the branches. To a topological path $\underline{Y}$ corresponds an infinite set of possible channel paths $\underline{y}$, but to each channel path $\underline{y}$ corresponds one and only one topological path $\underline{Y}$. A channel path $\underline{y}$ is defined by a topological path $\underline{Y}$ and a sequence of labels: $\underline{y} \triangleq (\ldots , y_j, y_{j+1}, \ldots , y_{j+t}, \ldots)$, where y_{j+t} is a branch in $\underline{Y}$ labeled with a channel signal in S and $j+t$ is a time index, so that y_{j+t} is the output signal during time interval $j+t$. We will sometimes call y_t a signal although it is a labeled branch. The context should make clear when we mean a labeled branch or a signal point. A code c is the set of all labeled trellis branches for all t. If the code is time-varying, labels depends on t. A code c generates a unique set of infinite channel paths $\underline{y}$, each path being a set of consecutive branches in c. To construct a code, one only needs to label trellis branches with channel signals. The labeling is the same for every signal interval if the code is time-invariant. We consider the set C of time-varying trellis codes of a given input width k and a given memory length $\nu_1 = \ldots = \nu_k = \nu$. The input width k and the memory length ν fully determine the trellis topology, i.e., the number of states, the number of branches to and from each state, and the states linked by a branch. Note that 2^k branches leave each state and enter 2^k different states.

We now proceed to derive the bound. The proof of the bound depends on first stating precisely what must be proved.

Statement 1:

Given a set C of codes c, there exists a code $c_0 \in C$ for which $d_{free}(c_0) > d$ if the probability, over all codes c in C, that $d_{free}(c) \leq d$ is less than 1.

We can rewrite the previous statement as:

$$\underset{c \in C}{Prob}\, (d_{free}(c) \leq d) < 1 \quad \Rightarrow \quad \exists\, c_0 \in C \; s.t. \; d_{free}(c_0) > d. \tag{2}$$

By definition $d_{free}(c) = \min_{(\underline{y},\underline{y}')} \left[d_e\ (\underline{y},\underline{y}') \right]$, where $(\underline{y},\underline{y}')$ varies over all pairs of distinct paths $\underline{y}$ and $\underline{y}'$ in c diverging from the same state at time instant 1. Such a pair $(\underline{y},\underline{y}')$ of paths is usually called an error event in the convolutional code literature. We call $e_{\underline{y}}\ (c)$ the set of paths $\underline{y}' \in c$ such that $(\underline{y},\underline{y}')$ is an error event starting at time 1, i.e., $\underline{y}$ and $\underline{y}'$ are merged at time 1 and may remerge later or may never remerge. The notation $e_{\underline{y}}\ (c)$ stands for "the set of error paths in c given a correct path $\underline{y}$". The free distance of a code c is:

$$d_{free}(c) = \min_{\begin{cases}(\underline{y},\underline{y}') \\ \underline{y} \in c \\ \underline{y}' \in e_{\underline{y}}\ (c)\end{cases}} \left[d_e\ (\underline{y},\underline{y}') \right]. \tag{3}$$

Thus, if for a code c and a positive real number d, $d_{free}(c) \leq d$, at least two distinct paths $\underline{y}_0 \in c$ and $\underline{y}'_0 \in e_{\underline{y}_0}\ (c)$ exist such that $d_e\ (\underline{y}_0,\underline{y}_0') < d$. Thus for any positive real number α,

$$e^{\alpha d^2}\, e^{-\alpha d_e^2(\underline{y}_0,\underline{y}_0')} \geq 1. \tag{4}$$

We upperbound the left side of (4) by summing over all $\underline{y}' \in e_{\underline{y}_0}\ (c)$:

$$\forall\ \alpha \geq 0, \qquad e^{\alpha d^2} \sum_{\underline{y}' \in e_{\underline{y}_0}\ (c)} e^{-\alpha d_e^2(\underline{y}_0,\underline{y}')} \geq 1, \tag{5}$$

$$\Rightarrow\ \forall\ \alpha,\ \rho \geq 0, \qquad e^{\alpha\rho d^2} \left[\sum_{\underline{y}' \in e_{\underline{y}_0}\ (c)} e^{-\alpha d_e^2(\underline{y}_0,\underline{y}')} \right]^{\rho} \geq 1. \tag{6}$$

The previous equation holds for a given correct path $\underline{y}_0$. We now upperbound the left side of (6) by summing over all possible correct paths $\underline{y}$ in c:

$$\Rightarrow\ \forall\ \alpha,\ \rho \geq 0, \qquad e^{\alpha\rho d^2} \sum_{\underline{y} \in c} \left[\sum_{\underline{y}' \in e_{\underline{y}}\ (c)} e^{-\alpha d_e^2(\underline{y},\underline{y}')} \right]^{\rho} \geq 1. \tag{7}$$

We define a function $T_{\alpha,\rho}(c)$ of a code c by:

$$T_{\alpha,\rho}(c) \triangleq \sum_{\underline{y} \in c} \left[\sum_{\underline{y}' \in e_{\underline{y}}\ (c)} e^{-\alpha d_e^2(\underline{y},\underline{y}')} \right]^{\rho}, \tag{8}$$

where α and ρ are two positive parameters. Then (7) and (8) imply that

$$\forall\ c \in C \text{ and } \forall\ \alpha,\ \rho,\ d \geq 0\,, \qquad e^{\alpha\rho d^2}\, T_{\alpha,\rho}(c) \geq \begin{cases} 1 \text{ if } d_{free}(c) \leq d \\ 0 \text{ if } d_{free}(c) > d\,. \end{cases} \tag{9}$$

Let us define an indicator function on the set C as follows:

$$\forall\ d \geq 0,\ \ X_d\,(c) \triangleq \begin{cases} 1 \text{ iff } d_{free}(c) \leq d \\ 0 \text{ iff } d_{free}(c) > d\ . \end{cases} \tag{10}$$

Then for any positive real number d,

$$\underset{c\,\in\,C}{Prob}\ (d_{free}(c) \leq d) = \sum_{c\,\in\,C} p(c)\,X_d\,(c)\ , \tag{11}$$

where $p(c)$ is the probability of the code c among the set C of time-varying codes associated with a given trellis and a given signal constellation. Clearly, from (9)and (10),

$$\forall\ c \in\ C \text{ and } \forall\ \alpha, \rho, d \geq 0\ ,\ \ X_d\,(c)\ \leq\ e^{\alpha\rho d^2}\,T_{\alpha,\rho}(c). \tag{12}$$

If (12) were an equality, then (11) and (12) would give an exact formulation of $\underset{c\,\in\,C}{Prob}\ (d_{free}(c) \leq d)$. So the tightness of (12) will determine the tightness of the bound. Combining (11) and (12) yields

$$\underset{c\,\in\,C}{Prob}\ (d_{free}(c) \leq d) \leq\ e^{\alpha\rho d^2} \sum_{c\,\in\,C} p(c)\ \ T_{\alpha,\rho}(c), \tag{13}$$

where α and ρ are positive parameters that will be used later to optimize the bound. We now give a weaker but more explicit version of statement 1.

Statement 2:

For any set C of codes c,

$$e^{\alpha\rho d^2} \sum_{c\,\in\,C} p(c)\,T_{\alpha,\rho}(c)\ <\ 1\ \ \Rightarrow\ \ \exists\, c_0 \in\ C\ s.t.\ \ d_{free}(c_0) > d. \tag{14}$$

In the following, we upper bound the left side of (14) to obtain an even weaker, but very explicit statement. The derivation of a random coding bound is based on switching the order of summations [Shannon, 1962]. The derivation starts with an average over all codes and switches summations to obtain an average over all the output sequences or channel paths. We explain Shannon's concept with a formal example. Let c be a code in C, $\underline{y}$ a channel path of the code c, and $f(c,\underline{y})$ a function of the code c and of one of its channel paths $\underline{y}$ ($\underline{y}$ is often called the correct path). According to our notation, $c \in C$ and $\underline{y} \in c$. Then switching the summations gives:

$$\sum_{c \in C} \sum_{\underline{y} \in c} f(c,\underline{y}) = \sum_{\underline{y} \in \bigcup_{c \in C} c} \sum_{c \in C \mid \underline{y}} f(c,\underline{y}) , \tag{15}$$

where $\underline{y} \in \bigcup_{c \in C} c$ denotes the set of all channel paths belonging to at least one code c in C and $c \in C \mid \underline{y}$ denotes the set of all codes in C that contain the channel path $\underline{y}$. To simplify further, we will replace the sum over $\underline{y} \in \bigcup_{c \in C} c$ with a sum over $\underline{y}$ (not to be confused with the more restrictive sum over $\underline{y} \in c$) and the sum over $c \in C \mid \underline{y}$ by a sum over $c \mid \underline{y}$ (not to be confused with a sum over $c \in C$).

Let us now introduce a function

$$G(\underline{y}, c) \triangleq \sum_{\underline{y}' \in e_{\underline{y}}(c)} e^{-\alpha d_e^2(\underline{y},\underline{y}')} . \tag{16}$$

$G(\underline{y},c)$ is the distance generating function of the code C given that the correct path is $\underline{y}$. For binary linear codes and regular codes [Calderbank-Sloane, 1985], $G(\underline{y},c)$ does not depend on the correct path $\underline{y}$. From the definition of $G(\underline{y},c)$,

$$\sum_{c \in C} p(c) \; T_{\alpha,\rho}(c) = \sum_{c \in C} p(c) \sum_{\underline{y} \in c} G(\underline{y},c)^{\rho} . \tag{17}$$

Applying Shannon's summation switching,

$$\sum_{c \in C} p(c) \; T_{\alpha,\rho}(c) = \sum_{\underline{y}} \sum_{c \mid \underline{y}} p(c) \; G(\underline{y},c)^{\rho}. \tag{18}$$

Since, for all codes $c \mid \underline{y}$, $p(c) = p(c,\underline{y}) = p(c \mid \underline{y})p(\underline{y})$,

$$\sum_{\underline{y}} \sum_{c \mid \underline{y}} p(c) \; G(\underline{y},c)^{\rho} = \sum_{\underline{y}} p(\underline{y}) \sum_{c \mid \underline{y}} p(c \mid \underline{y}) \; G(\underline{y},c)^{\rho}. \tag{19}$$

If we use the moment inequality $\sum_i p_i \, a_i^{\rho} \le \left[\sum_i p_i \, a_i \right]^{\rho}$ for $\rho \le 1$, then

$$\sum_{c \mid \underline{y}} p(c \mid \underline{y}) \; G(\underline{y},c)^{\rho} \le \left(\sum_{c \mid \underline{y}} p(c \mid \underline{y}) \; G(\underline{y},c) \right)^{\rho} . \tag{20}$$

We now switch the order of summations over the codes $c \mid \underline{y}$ and the paths $\underline{y}'$:

$$\sum_{c \mid \underline{y}} p(c \mid \underline{y})\, G(\underline{y},c) \;=\; \sum_{c \mid \underline{y}} p(c \mid \underline{y}) \sum_{\underline{y}' \in e_{\underline{y}}(c)} e^{-\alpha d_\epsilon^2(\underline{y},\underline{y}')}$$

$$= \sum_{\underline{y}' \in e_{\underline{y}}} \sum_{c \mid \underline{y},\underline{y}'} p(c \mid \underline{y})\; e^{-\alpha d_\epsilon^2(\underline{y},\underline{y}')}, \tag{21}$$

where $e_{\underline{y}} \triangleq \bigcup_{c \in C} e_{\underline{y}}(c)$. Since, for all codes $c \mid \underline{y},\underline{y}'$, $p(c \mid \underline{y}) = p(c,\underline{y}' \mid \underline{y}) = p(c \mid \underline{y},\underline{y}')\, p(\underline{y}' \mid \underline{y})$, (21) becomes:

$$\sum_{c \mid \underline{y}} p(c \mid \underline{y})\, G(\underline{y},c) = \sum_{\underline{y}' \in e_{\underline{y}}} p(\underline{y}' \mid \underline{y})\, e^{-\alpha d_\epsilon^2(\underline{y},\underline{y}')} \sum_{c \mid \underline{y},\underline{y}'} p(c \mid \underline{y},\underline{y}'),$$

and since $\sum_{c \mid \underline{y},\underline{y}'} p(c \mid \underline{y},\underline{y}') = 1$,

$$\sum_{c \mid \underline{y}} p(c \mid \underline{y})\, G(\underline{y},c) = \sum_{\underline{y}' \in e_{\underline{y}}} p(\underline{y}' \mid \underline{y})\; e^{-\alpha d_\epsilon^2(\underline{y},\underline{y}')}, \tag{22}$$

where we have eliminated all the branches that do not belong to $\underline{y}$ or to $\underline{y}'$, and hence do not contribute to the error event. When C represents the set of all time varying trellis codes associated with a particular topological trellis, the probability of choosing a signal on one branch during time interval $[tT,(t+1)T]$ does not depend on the signals chosen during previous or future intervals. Then if t represents a time index, $p(\underline{y}' \mid \underline{y}) = \prod_{t=1}^{\infty} p(y_t' | y_t)$, where y_t' is the signal on the t^{th} branch of the path $\underline{y}'$, and $p(y_t' | y_t)$ is the probability of having y_t' on that branch given y_t. Normally, all channel signals are equiprobable and can be chosen independently, so that for $y_t \neq y_t'$ $p(y_t' | y_t) = p(y_t') = 1/M$, where M is the number of signals in the signal constellation S. However, for the derivation of the bound channel signals do not have to be equiprobable. Finally, from (18), (19), (20), and (22)

$$\sum_{c \in C} p(c)\;\; T_{\alpha,\rho}(c) \;\le\; \sum_{\underline{y}} p(\underline{y}) \left[\sum_{\underline{y}' \in e_{\underline{y}}} \prod_{t=1}^{\infty} p(y_t' | y_t)\;\; e^{-\alpha d_\epsilon^2(y_t,y_t')} \right]^{\rho}. \tag{23}$$

Equation (23) holds for any trellis and does not depend explicitly on the encoder input width k or on the constraint length ν_0. The next step, what Forney calls configuration counting, consists of regrouping the paths $\underline{y}$ (respectively $\underline{y}'$) that correspond to the same topological paths $\underline{Y}$ (respectively $\underline{Y}'$).

Configuration counting:

Given $\underline{y}$, configuration counting regroups error events $(\underline{y},\underline{y}')$ of equal length τ, and then regroups channel paths $\underline{y}'$ that correspond to identical topological paths $\underline{Y}'$. By definition, an error event $(\underline{y},\underline{y}')$ of length τ starting at time 1 and corresponding to a topological error event $(\underline{Y},\underline{Y}')$ satisfies

$$\begin{cases} Y_t \neq Y_t' & \text{for } 1 \leq t \leq \tau \\ Y_t = Y_t' & \text{for } \tau < t \, , \end{cases}$$

which means that the branches are different for τ intervals, but the signals can be the same. In other words, $y_t \neq y_t'$ for $1 \leq t \leq \tau$, but y_t and y_t' can still be labeled with the same signal point. Then, if $(\underline{y},\underline{y}')$ is an error event of length τ starting at time 1,

$$\prod_{t=1}^{\infty} p(y_t'|y_t) \; e^{-\alpha d_\varepsilon^2(y_t,y_t')} = \prod_{t=1}^{\tau} p(y_t'|y_t) \; e^{-\alpha d_\varepsilon^2(y_t,y_t')}. \tag{24}$$

An error event lasts at least $\nu + 1$ information blocks. The input block of width k that feeds the encoder at time 1 differs for paths $\underline{y}$ and $\underline{y}'$. This different input block is shifted into memory, and it takes at least ν time units to clear the memory of that difference. So two paths that diverge at time 1, cannot remerge before time $\nu + 1$. We define $e_{\underline{y},\tau}$ as the set of paths $\underline{y}'$ in $e_{\underline{y}}$ such that the error event $(\underline{y},\underline{y}')$ lasts τ branches. Then, $e_{\underline{y}} = \bigcup_{\tau=\nu+1}^{\infty} e_{\underline{y},\tau}$. Hence, combining (23) and (24),

$$\sum_{c \in C} p(c) \;\; T_{\alpha,\rho}(c) \leq \sum_{\underline{y}} p(\underline{y}) \left[\sum_{\tau=\nu+1}^{\infty} \; \sum_{\underline{y}' \in e_{\underline{y},\tau}} \; \prod_{t=1}^{\tau} p(y_t'|y_t) \; e^{-\alpha d_\varepsilon^2(y_t,y_t')} \right]^{\rho}, \tag{25}$$

where we regrouped error events of length τ in $e_{\underline{y}}$, starting with the shortest error events of length $\nu + 1$. We now regroup paths $\underline{y}'$ in $e_{\underline{y},\tau}$ that refer to the same topological path $\underline{Y}'$, but with different signals on the branches. We note by $\underline{y}' \rightarrow \underline{Y}'$ the fact that $\underline{Y}'$ is the topological path associated with $\underline{y}'$, i.e., that $\underline{y}'$ is a labeled version of $\underline{Y}'$. We define $E_{\underline{y},\tau}$ as the set of topological paths $\underline{Y}'$ associated with elements of $e_{\underline{y},\tau}$: $E_{\underline{y},\tau} \triangleq \{ \underline{Y}' \mid \underline{y}' \rightarrow \underline{Y}', \underline{y}' \in e_{\underline{y},\tau} \}$. Therefore,

$$\sum_{\underline{y}' \in e_{\underline{y},\tau}} \; \prod_{t=1}^{\tau} p(y_t'|y_t) \; e^{-\alpha d_\varepsilon^2(y_t,y_t')} = \sum_{\underline{Y}' \in E_{\underline{y},\tau}} \; \sum_{\underline{y}' | \underline{y}' \rightarrow \underline{Y}'} \; \prod_{t=1}^{\tau} p(y_t'|y_t) \; e^{-\alpha d_\varepsilon^2(y_t,y_t')}. \tag{26}$$

The choice of a signal on a branch y_t' does not depend on the signal labeling the branch y_t, since different branches of a code are labeled independently. Therefore $p(y_t'|y_t) = p(y_t')$, and all topological paths of equal length τ generate the same sum over all the channel assignments:

$$\sum_{\underline{y}' \mid \underline{y}' \to \underline{Y}'} \prod_{t=1}^{\tau} p(y_t' \mid y_t)\; e^{-\alpha d_e^2(y_t, y_t')} = \sum_{\underline{y}' \in S^\tau} \prod_{t=1}^{\tau} p(y_t')\; e^{-\alpha d_e^2(y_t, y_t')}. \tag{27}$$

We call $C_{\underline{y},\tau}$ the cardinality of $E_{\underline{y},\tau}$. $C_{\underline{y},\tau}$ is the number of topological error paths of length τ diverging from $\underline{Y}$ at time 1. $C_{\underline{y},\tau}$ does not depend on the correct path $\underline{y}$, so that $C_{\underline{y},\tau} = C_\tau$. Hence, (26) becomes

$$\sum_{\underline{y}' \in e_{\underline{y},\tau}} \prod_{t=1}^{\tau} p(y_t' \mid y_t)\; e^{-\alpha d_e^2(y_t, y_t')} = C_\tau \sum_{\underline{y}' \in S^\tau} \prod_{t=1}^{\tau} p(y_t')\; e^{-\alpha d_e^2(y_t, y_t')}, \tag{28}$$

and (25) gives

$$\sum_{c \in C} p(c)\; T_{\alpha,\rho}(c) \le \sum_{\underline{y}} p(\underline{y}) \left[\sum_{\tau=\nu+1}^{\infty} C_\tau \sum_{\underline{y}' \in S^\tau} \prod_{t=1}^{\tau} p(y_t')\; e^{-\alpha d_e^2(y_t, y_t')} \right]^\rho. \tag{29}$$

Using the inequality $\left[\sum_i a_i\right]^\rho \le \sum_i a_i^\rho$ for $\rho \le 1$, we get

$$\sum_{c \in C} p(c)\; T_{\alpha,\rho}(c) \le \sum_{\tau=\nu+1}^{\infty} \sum_{\underline{y}} p(\underline{y})\; C_\tau^\rho \left[\sum_{\underline{y}' \in S^\tau} \prod_{t=1}^{\tau} p(y_t')\; e^{-\alpha d_e^2(y_t, y_t')} \right]^\rho. \tag{30}$$

We now write $p(\underline{y}) = \prod_{l=1}^{\infty} p(y_l)$:

$$\sum_{c \in C} p(c)\; T_{\alpha,\rho}(c) \le \sum_{\tau=\nu+1}^{\infty} C_\tau^\rho \sum_{\underline{y}} \prod_{l=1}^{\infty} p(y_l) \left[\prod_{t=1}^{\tau} \sum_{y' \in S} p(y')\; e^{-\alpha d_e^2(y_t, y')} \right]^\rho \tag{31}$$

$$= \sum_{\tau=\nu+1}^{\infty} C_\tau^\rho \sum_{\underline{y}} \prod_{l=1}^{\infty} p(y_l) \prod_{t=1}^{\tau} \left[\sum_{y' \in S} p(y')\; e^{-\alpha d_e^2(y_t, y')} \right]^\rho. \tag{32}$$

Now let us introduce the notation:

$$H(y_t, S) \triangleq \sum_{y' \in S} p(y')\; e^{-\alpha d_e^2(y_t, y')}. \tag{33}$$

$H(y_t,S)$ gives the position of the signal y_t relative to the constellation S. Hence,

$$\sum_{\underline{y}} \prod_{l=1}^{\infty} p(y_l) \prod_{t=1}^{\tau} \left[\sum_{y' \in S} p(y') \, e^{-\alpha d_t^2(y_t,y')} \right]^{\rho} = \sum_{\underline{y}} \prod_{l=1}^{\infty} p(y_l) \prod_{t=1}^{\tau} H(y_t,S)^{\rho} \,. \quad (34)$$

Now we eliminate the branches of $\underline{y}$ that belong to $\underline{y}'$ in the same way that we eliminated the branches of $\underline{y}'$ that belonged to $\underline{y}$. We calculate the right side of (34):

$$\sum_{\underline{y}} \prod_{l=1}^{\infty} p(y_l) \prod_{t=1}^{\tau} H(y_t,S)^{\rho} = \sum_{\underline{y}} \prod_{t=1}^{\tau} p(y_t) \; H(y_t,S)^{\rho} \prod_{\begin{cases} l=1 \\ l \neq t \end{cases}}^{\infty} p(y_l) \quad (35)$$

$$= \prod_{t=1}^{\tau} \left[\sum_{y \in S} p(y) \; H(y,S)^{\rho} \right] \prod_{l \neq t} \left[\sum_{y \in S} p(y) \right]. \quad (36)$$

Since $\sum_{y \in S} p(y) = 1$,

$$\sum_{\underline{y}} \prod_{l=1}^{\infty} p(y_l) \prod_{t=1}^{\tau} H(y_t,S)^{\rho} = \prod_{t=1}^{\tau} \sum_{y \in S} p(y) \; H(y,S)^{\rho} \quad (37)$$

$$= \left[\sum_{y \in S} p(y) \; H(y,S)^{\rho} \right]^{\tau}. \quad (38)$$

Returning to equation (32):

$$\sum_{c \in C} p(c) \; T_{\alpha,\rho}(c) \le \sum_{\tau=\nu+1}^{\text{if}} C_{\tau}^{\rho} \left[\sum_{y \in S} p(y) \; H(y,S)^{\rho} \right]^{\tau}. \quad (39)$$

The cardinality C_τ of $E_{\underline{y},\tau}$ is $C_\tau = \prod_{j=1}^{k} 2^{[\tau-\nu_j]_+}$, where $[x]_+ = x$ if $x \ge 0$ and zero otherwise [Calderbank-Mazo-Wei, 1985]. Hence for any $\tau \ge \nu + 1$, $C_\tau = 2^{k\tau - k\nu}$, where $\nu_1 = \ . \ . \ . \ = \nu_k = \nu$.

According to Forney's notation [Forney, 1974], we define

$$E(\alpha,\rho) \triangleq -\ln \left[\sum_{y \in S} p(y) \left[\sum_{y' \in S} p(y') \, e^{-\alpha d_t^2(y,y')} \right]^{\rho} \right]^{1/\rho} , \quad (40)$$

where y and y' belong to the signal constellation S, and $p(y)$ is the probability of signal y.

A simpler form of (39), using the definition of $E(\alpha,\rho)$, is

$$\sum_{c \in C} p(c)\ T_{\alpha,\rho}(c) \le \sum_{\tau=\nu+1}^{\infty} C_\tau^{\rho}\ e^{-\tau\rho E(\alpha,\rho)} \tag{41}$$

$$\le \sum_{\tau=\nu+1}^{\infty} 2^{k\tau\rho - k\nu\rho}\ e^{-\tau\rho E(\alpha,\rho)}. \tag{42}$$

We can now write a third statement that follows from statement 2 and (42):

Statement 3:

For any set C of trellis codes of input width k and memory length ν,

$$e^{\alpha\rho d^2} \sum_{\tau=\nu+1}^{\infty} 2^{k\tau\rho - k\nu\rho}\ e^{-\tau\rho E(\alpha,\rho)} < 1 \quad \Rightarrow \quad \exists\ c_0 \in C \ s.t.\ d_{free}(c_0) > d. \tag{43}$$

Statement 3 is as tight as statement 2 when $\rho = 1$. We calculate the largest d such that inequality (43) holds. If $E(\alpha,\rho) > k\ln 2$, the left side of (43) can be written as

$$e^{\alpha\rho d^2 - k\nu\rho \ln 2}\ \frac{e^{-(\nu+1)\rho[E(\alpha,\rho) - k\ln 2]}}{1 - e^{-\rho[E(\alpha,\rho) - k\ln 2]}} = e^{-\rho[\nu E(\alpha,\rho) - \alpha d^2]}\ \frac{1}{e^{\rho[E(\alpha,\rho) - k\ln 2]} - 1}.$$

The previous expression will be smaller than 1 (i.e., (43) is satisfied) if and only if

$$d^2 < \nu\,\frac{E(\alpha,\rho)}{\alpha} - \frac{\theta[E(\alpha,\rho)]}{\alpha}, \tag{44}$$

where

$$\theta\Big[E(\alpha,\rho)\Big] = \ln\Big[e^{\rho[E(\alpha,\rho) - k\ln 2]} - 1\Big]^{1/\rho}. \tag{45}$$

This concludes the derivation of the bound. The bound can now be put in the form of Costello's bound [Costello, 1974].

Theorem:

Given a signal constellation S, there exists a (k, ν) trellis code with minimum free Euclidean distance d_{free} such that:

$$d^2_{free} > \max_{\begin{cases} E(\alpha,\rho) > k\ln 2 \\ 0 \le \alpha \\ 0 \le \rho \le 1 \\ p(y) \end{cases}} \left\{ \nu \frac{E(\alpha\,,\rho)}{\alpha} - \frac{\theta[\,E(\alpha\,,\rho)\,]}{\alpha} \right\}, \tag{46}$$

where k is the information block size, $\nu_0 = k\nu$ is the constraint length, S is the signal constellation, α and ρ are parameters which optimize the bound, y and y' are signal points from the signal constellation S, $d_e(y,y')$ is the Euclidean distance between signals y and y', $p(y)$ is a probability distribution on S, $E(\alpha,\rho) = -\ln\left\{\sum_{y\in S} p(y) \left[\sum_{y'\in S} p(y')\, e^{-\alpha d_e^2(y,y')}\right]^{\rho}\right\}^{1/\rho}$, and $\theta\left[E(\alpha,\rho)\right] = \ln\left[e^{\rho[E(\alpha,\rho)-k\ln 2]} - 1\right]^{1/\rho}$.

IV APPLICATION OF THE BOUND

Symbols y that label branches of the trellis can represent subsets of the constellation when the trellis contains parallel transitions. The subset y is composed of the signals assigned to a parallel transition. The encoder outputs sequences of subsets rather than sequences of signals. This enables us to study the effects of uncoded bits, or, equivalently, the effects of mapping signals onto a trellis with parallel transitions using Ungerboeck's "set partitioning" method [Ungerboeck, 1982]. In that case, if $d_{\min}$ is the minimum distance between points in the same subset, and $d_{free}^{(subsets)}$ the free Euclidean distance between sequences of subsets, then the overall minimum Euclidean distance of the TCM scheme is $d_{free} = \min(d_{free}^{(subsets)}, d_{\min})$. This shows that parallel transitions limit the maximum achievable d_{free} of a TCM scheme.

An evaluation of the bound corresponds to a trellis characterized by k and $\nu_0 = k\nu$ and represents the average free distance of all the corresponding trellis codes (all labelings of the trellis). However, since the bound is a random coding bound, it does not represent the best free distance achievable with a particular trellis, because optimum trellis codes may perform better than the average.

Like the lower bounds on the Hamming d_{free} [Costello, 1974 and Forney, 1974], this lower bound on the Euclidean d_{free} is exponentially tight and varies linearly with ν_0 for large ν_0. Like any other "Gilbert type" bound, it is not expected to be tight for small ν_0.

The bound depends implicitly on the constraint length $\nu_0 = k\nu$ and on the information rate per dimension R. Although only certain discrete values of ν_0 and R are possible, we treat $d^2_{free}(\nu_0)$ and $\frac{d_{free}}{\nu_0}(R)$ as continuous functions.

V USING THE BOUND TO EVALUATE TCM SCHEMES

1) Variations of d_{free} with ν_0:

Bounds on the free Hamming distance of binary convolutional codes have usually been expressed as functions of ν_0. We compare TCM schemes with different constellations at a given rate R and at a given average energy E_{avg}, where E_{avg} is the average signal energy of the constellation used in the TCM scheme. The lower bound on d_{free} increases with the constraint length ν_0, which means that any d_{free} can be obtained with a large constraint length. However the decoding complexity increases exponentially with the constraint length, so that practically only small constraint lengths will be implementable .

* The effect of increasing the number of signals on d_{free} (same rate R, same dimensionality D, same average energy E_{avg}):

Increasing the number M of signals increases d_{free} because it provides more flexibility in the choice of signals assigned to trellis branches (flexibility in the mapping). Signals with greater distance can be chosen on the branches of error paths $\underline{y}'$, given a correct path $\underline{y}$, which increases d_{free}. Note that increasing M, while keeping k constant, lowers the binary code rate R_b, which is consistant with the fact that d_{free} increases. Fig.2 shows PAM schemes with different numbers of signals (M = 4, 8, 16, 128). Increasing the number of signals provides more gain for small constraint lengths ν_0 than for large constraint lengths. In Fig.2, going from 4-PAM to 8-PAM or even 16-PAM provides most of the gain and higher order PAM schemes do not improve d_{free} significantly. Note that the gain in d_{free} may not be worth the added modulation complexity.

* The effect of increasing the signal set dimensionality on d_{free} (same rate R, same number of signals M, same average energy E_{avg}):

Higher dimensional schemes can be obtained by using a basic D-dimensional constellation for several transmission intervals [Wei, 1985] and [Lafanechere-Deng-Costello, 1987]. Fig.3 shows L-dimensional schemes for which the constellation is constructed from L uses of 4-PAM (1-dimensional). This construction has the advantage of retaining the distance properties of 1- or 2-dimensional constellations so that the only parameter of the TCM scheme that varies is D. For a given constraint length ν_0, higher L-dimensional schemes yield a larger d_{free}, and as ν_0 increases the curves diverge and the gain goes to infinity. However, the error coefficient increases with dimensionality [Forney et. al., 1984], which may cancel the gain from the increased d_{free} at moderate decoded bit error rates.

* The effect of changing the modulation scheme on d_{free} (same dimensionality D, same number of signals M, same average energy E_{avg}, same rate R):

Changing the modulation scheme affects d_{free}. For example, rectangular constellations yield a larger d_{free} than constant envelope constellations. Fig.4 shows M-PSK vs. M-QASK. This suggests that a d_{free} penalty is associated with constant envelope modulation. Note that this penalty increases with the constraint length. Furthermore, the bound gives a means of optimizing some parameters associated with a specific type of constellation without searching exhaustively for the best constellation.

* The effect of uncoded bits or parallel transitions on d_{free} when the signal constellation is "mapped by set partitioning":

Uncoded bits generate parallel transitions in the trellis and limit d_{free}, since d_{free} cannot be larger than the minimum distance between parallel transitions (the minimum distance within the subsets created by "set partitioning"). Fig.5 shows that for 8-PSK, 1 uncoded bit increases d_{free} for small constraint lengths. For small ν_0, this agrees with the best known codes constructed [Ungerboeck, 1982]. For large ν_0, however, we see that 1 uncoded bit limits the achievable d_{free}. The bound shows whether uncoded bits combined with mapping by set partitioning will increase d_{free} for small constraint lengths ν_0.

2) Comparison of the lower bound on d_{free} with upper bounds [Calderbank-Mazo-Wei, 1985 and Pottie-Taylor, 1986] and known codes [Ungerboeck, 1982]:

Upper bounds are expected to be tighter than the lower bound for small constraint lengths, and the lower bound is expected to be tighter for large constraint lengths (this is analogous to the way bounds on binary convolutional codes behave). The lower bound is a "Chernoff type" bound and therefore is exponentially tight when ν_0 is large. Fig.6 indeed shows that upper bounds are tighter than the lower bound for small constraint lengths. But the lower bound becomes tighter as ν_0 increases. The slope of the lower bound gives a precise indication of the asymptotic rate of increase in d_{free}. Finally, the lower bound guarantees the existence of codes that can achieve a certain d_{free}.

3) Asymptotic behavior: $\dfrac{d_{free}}{\nu_0}(R)$

For a given constellation, the achievable d_{free} / ν_0 is larger for small R than for large R. Graphs of $\dfrac{d_{free}}{\nu_0}(R)$ give a means of comparing the asymptotic behavior of d_{free} for different constellations. Several examples are shown in Fig.7. The largest achievable rate R depends on the constellation and its dimensionality : it is $(R_b \log_2 M) / D$ for a code rate R_b. These largest achievable rates appear clearly in Fig.7. Rectangular lattices constructed from the same 1-dimensional constellation (for example 4-PAM in Fig.7) yield the same graph, which means that these rectangular lattices have the same asymptotic behavior. In other words the curves $d_{free}(\nu_0)$ are parallel for large ν_0. The lower bounds are compared with an asymptotic upper bound [Calderbank-Mazo-Wei, 1985] in Fig.7.

CONCLUSION

We have obtained a random coding lower bound on d_{free}. It takes into account the actual Euclidean distances between points of the channel signal constellation. As with any "Gilbert" bound [Gilbert, 1952] it is most useful as an asymptotic bound. It provides a means of comparing the asymptotic performance of different modulation schemes and shows that the free Euclidean distance increases linearly for large constraint lengths. However, even for short constraint lengths, the bound behaves similarly to the best known codes and can be used as a preliminary calculation when designing TCM schemes to help select the most promising schemes.

REFERENCES

[Calderbank-Sloane, 1985] A.R. Calderbank and N.J.A. Sloane, "New Trellis Codes", IEEE Trans. Inform. Theory, vol. IT-33, pp. 177-195, March 1987.

[Calderbank-Mazo-Shapiro, 1983] A. R. Calderbank, J. E. Mazo, and H. M. Shapiro, "Upper Bounds on the Minimum Distance of Trellis Codes", Bell Syst. Tech. J., vol. 62, pp. 2617-2646, Oct. 1983, part I.

[Calderbank-Mazo-Wei, 1985] A. R. Calderbank, J. E. Mazo, and V. K. Wei, "Asymptotic Upper Bounds on the Minimum Distance of Trellis Codes", IEEE Trans. Commun., vol. COM-33, no. 4, pp. 305-310, April 1985.

[Chernoff, 1952] H. Chernoff, "A Measure of Asymptotic Efficiency for Tests of a Hypothesis Based on a Sum of Observations," Ann. Math. Stat., vol. 23, pp. 493-507, 1952.

[Costello, 1974] D. J. Costello, Jr. "Free Distance Bounds for Convolutional Codes", IEEE Trans. Inform. Theory, vol. IT-20, pp. 356-365, May 1974.

[Forney, 1974] G. D. Forney, Jr. "Convolutional Codes II: Maximum Likelihood Decoding", Information and Control, vol. 25, pp. 222-266, July 1974.

[Forney-Gallager-Lang-Longstaff-Qureshi, 1984] G. D. Forney, Jr., R. G. Gallager, G. R. Lang, F. M. Longstaff, and S. U. Qureshi, "Efficient Modulation for Band-limited Channels", IEEE J. on Selected Areas in Communication, vol. SAC-2, no. 5, pp. 632-647, Sept. 1984.

[Gallager, 1965] R. G. Gallager, "A simple Derivation of the Coding Theorem and Some Applications," IEEE Trans. Inform. Theory, vol. IT-11, pp. 3-18, Jan. 1965.

[Gilbert, 1952] E. N. Gilbert, "A comparison of Signalling Alphabets," Bell Syst. Tech. J., vol. 31, pp. 504-522, 1952.

[Lafanechere-Deng-Costello, 1987] A. Lafanechere, R.H. Deng, and D.J. Costello, Jr., "Multidimensional Trellis Coded Phase Modulation Using Unit-Memory and Partial-Unit- Memory Convolutional Codes", Submitted to IEEE Transactions on Information Theory, January 1987.

[Lin-Costello, 1983] S.Lin and D.J. Costello, Jr., Error Control Coding, Prentice-Hall, 1983.

[Pottie-Taylor, 1986] G.J. Pottie and D.P. Taylor, "Two Sphere-Packing Bounds on the Free Distance of Trellis Codes", submitted to IEEE Trans. Inform. Theory , 1986.

[Rouanne-Costello, 1986a] M. Rouanne and D.J. Costello, Jr., "Comparison of Trellis Coded Modulation Schemes Using a Lower Bound on the Free Euclidean Distance", Proc. Allerton Conf. on Commun., Cont., and Comp., pp. 1037-1046, Monticello, IL, October 1986.

[Rouanne-Costello, 1986b] M. Rouanne and D.J. Costello, Jr., "A Lower Bound on the Minimum Euclidean Distance of Trellis Codes", Presented at the Conference "Trois Journées sur le Codage", Paris, November 24-26, 1986.

[Shannon, 1962] C.E. Shannon, The Mathematical Theory of Communication" , University of Illinois Press, Urbana, Illinois, 1962.

[Ungerboeck, 1982] G. Ungerboeck, "Channel Coding with Multilevel Phase Signals", IEEE Trans. Inform. Theory, vol. IT-28, pp. 55-67, Jan. 1982.

[Viterbi, 1971] A.J. Viterbi, "Convolutional Codes and their Performance in Communication Systems", IEEE Trans. Commun. , vol. Com-19, No. 5, pp. 751-772, October 1971.

[Wei, 1985] L.F. Wei, "Trellis-Coded Modulation with Multi-Dimensional Constellations", IEEE Trans. Inform. Theory , to appear.

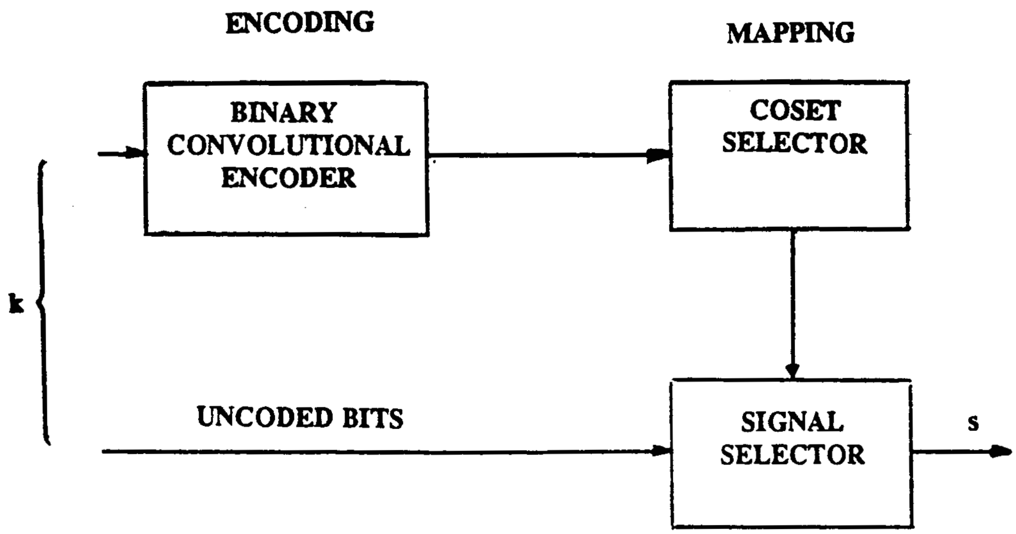

Figure 1: Trellis Coded Modulation Scheme.

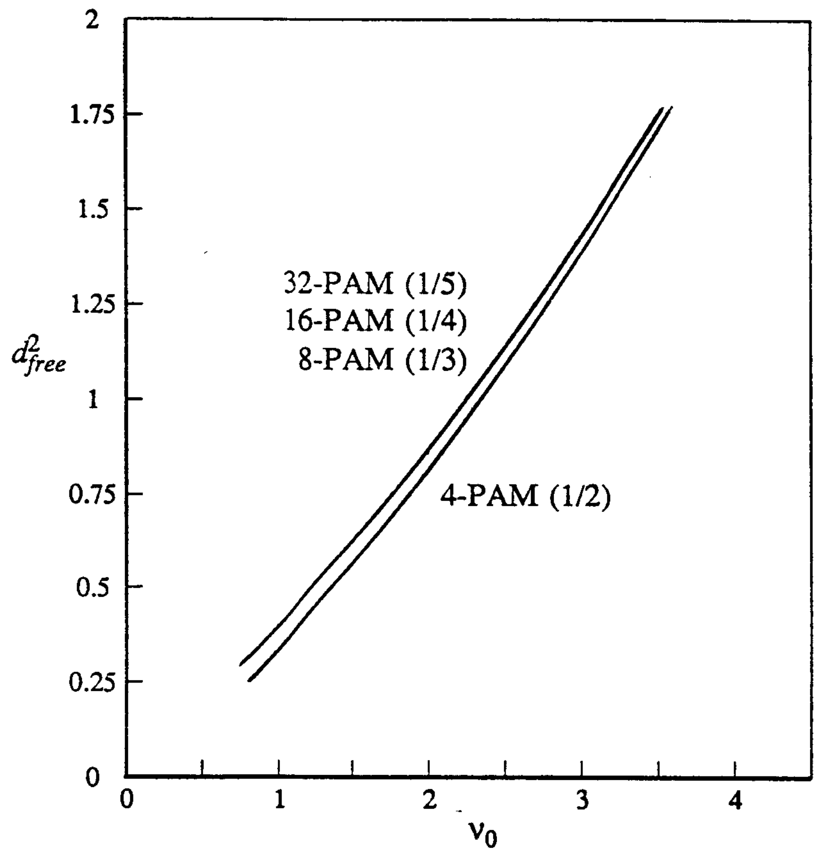

Figure 2: PAM comparison, $R = 1$ *bit/dim*, $E_{avg} = 1$ *J/dim.*

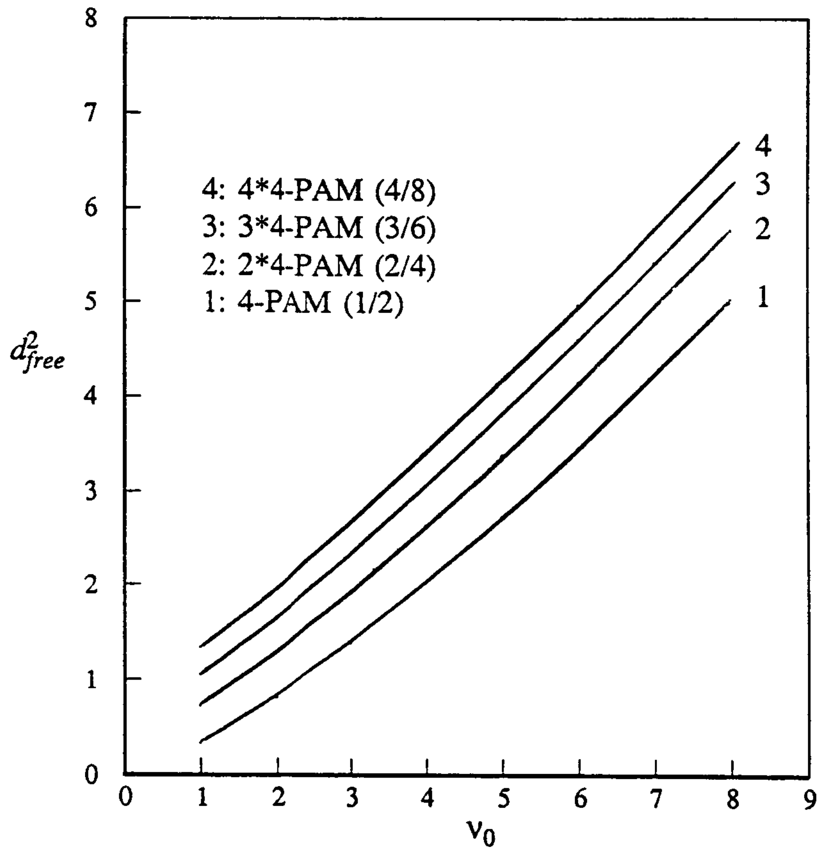

Figure 3: Comparison of schemes with different dimensionalities, $R = 1$ *bit/dim*, $E_{avg} = 1$ *J/dim*.

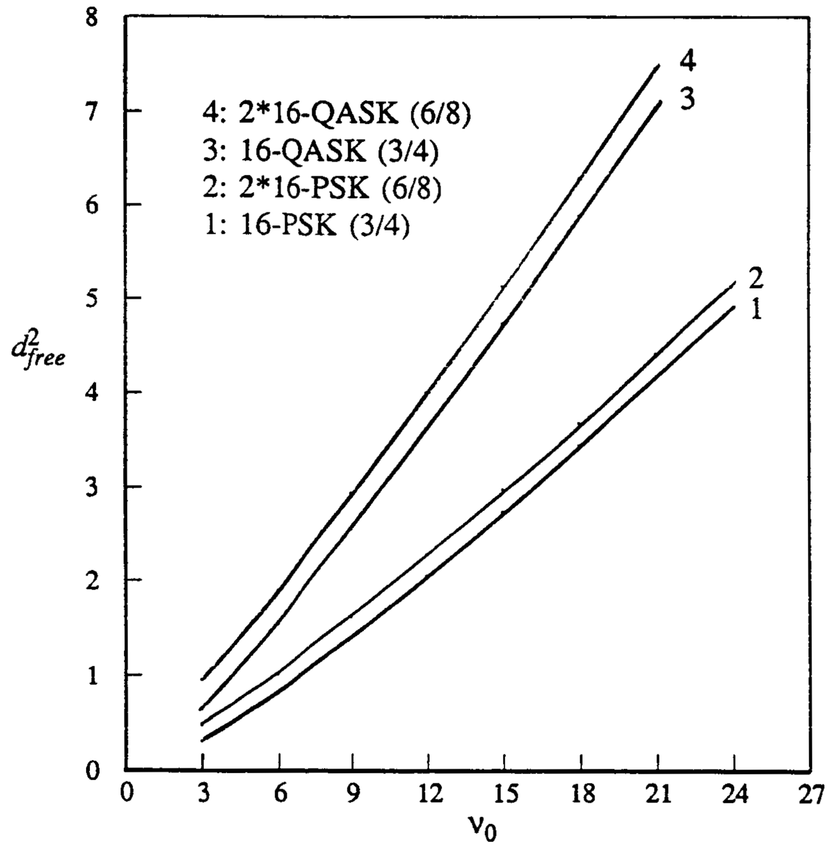

Figure 4: Comparison of M-PSK, M-QASK, L*M-PSK, and L*M-QASK schemes, $R = 1.5$ *bits/dim*, $E_{avg} = 1$ *J/dim*.

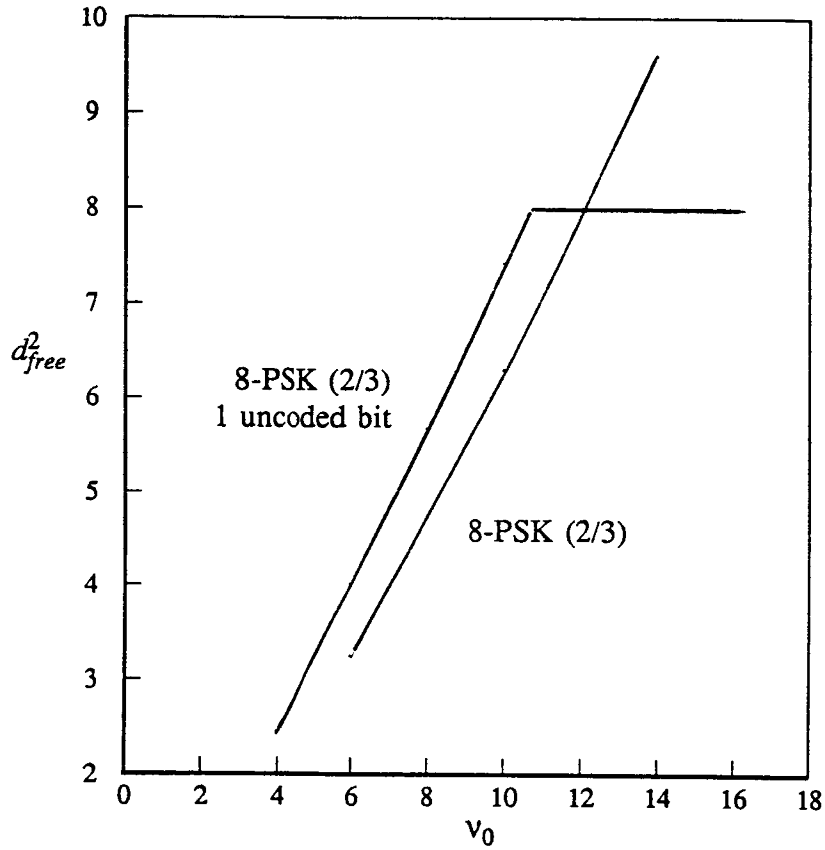

Figure 5: Trellis coding and uncoded bits (Ungerboeck) for 8-PSK, $R = 1$ *bit/dim*, $E_{avg} = 1$ *J/dim*.

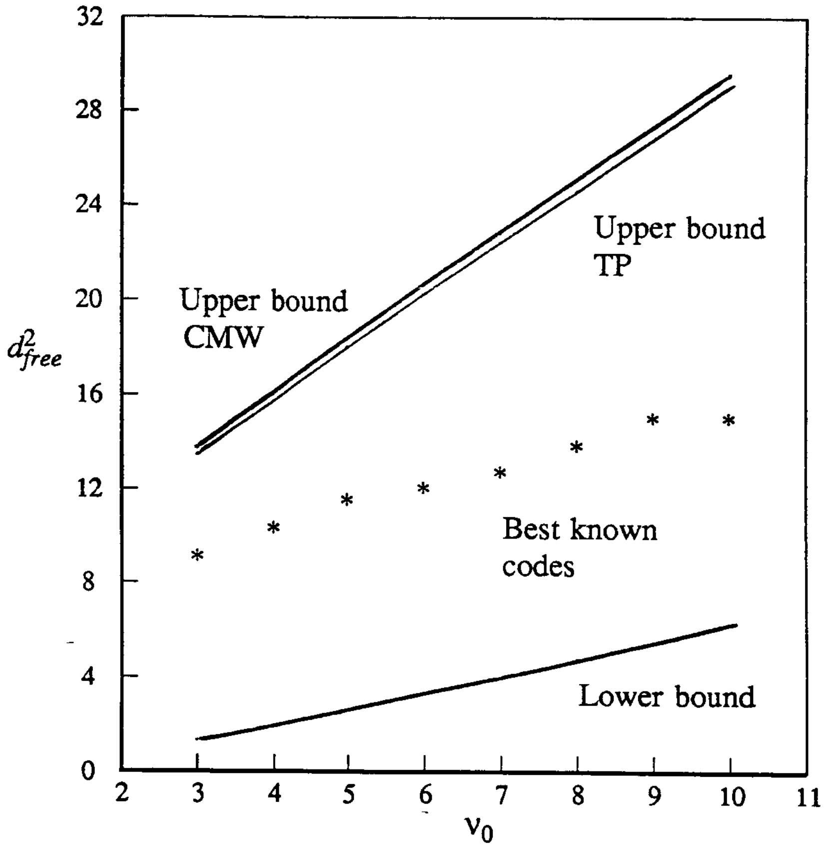

Figure 6: Comparison of the lower bound on d^2_{free} with upper bounds (CMW and PT) and with the best known codes for 8-PSK, $R = 1.5$ *bits/dim*, $E_{avg} = 1$ *J/dim.*

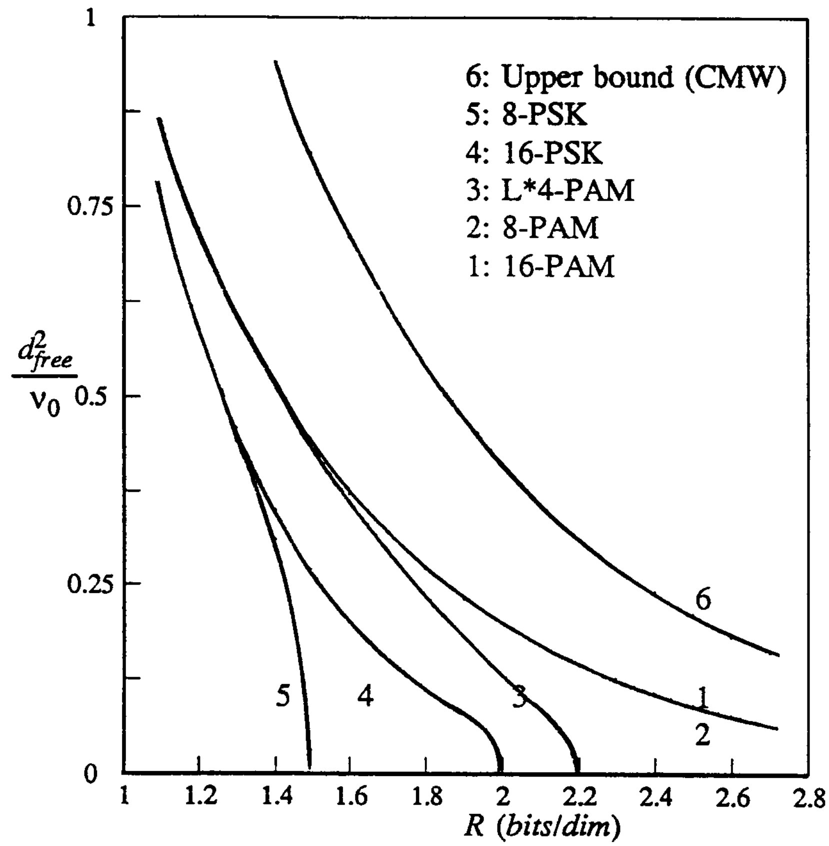

Figure 7: Asymptotic bounds vs. rate, $E_{avg} = 1\ J/dim$.

Appendix C

Bandwidth Efficient Coding on a Fading Channel

BANDWIDTH EFFICIENT CODING ON A FADING CHANNEL

Christian Schlegel and Daniel J. Costello, Jr.
Department of Electrical and Computer Engineering
University of Notre Dame
Notre Dame, Indiana 46556

INTRODUCTION

Communication over real world channels, particularly mobile satellite channels, suffers from nonuniform disturbances that cannot be described by additive white Gaussian noise. One important such disturbance is the multiplicative noise arising from multipath reception at the receiver (see Hagenauer [1]). Communication channels suffering from multipath reception are known as Rayleigh/Rician fading channels. If $s_i(t)$ is a signal that is sent over a fading channel, then the received signal $r_i(t)$ is given by

$$r_i(t) = as_i(t) + n(t), \tag{1}$$

where $n(t)$ is additive white Gaussian noise and a is the fading amplitude whose probability distribution is given by

$$p(a) = 2a(1+K)e^{-K-a^2(1+K)}I_0\left(2a\sqrt{K(1+K)}\right), \tag{2}$$

where I_0 denotes the modified spherical Bessel function. K is the ratio of the signal energy received on the direct path to the energy received via reflected paths (compare [1], [2]).

Due to the increasing density of communication systems most digital channels are power as well as bandwidth limited. Ungerboeck [3] has introduced bandwidth efficient trellis coding which provides a coding gain of up to 6dB on additive white Gaussian noise channels. In this paper we look at the performance of these bandwidth efficient trellis codes on channels plagued by fading. We will limit our attention to the Rayleigh fading channel, i.e., $K = 0$ in (2).

CUTOFF RATE FOR RAYLEIGH CHANNELS

Since a great variety of signal constellations can be used on the Gaussian channel, it is instructive to look at the cutoff-rates of these constellations on a fading channel. The cutoff rate R_0 is the exponent in the random coding bound, $P_B \leq 2^{-NR_0}$, which gives a bound on the average block error probability of all codes of length N. The importance of R_0 transcends its use in the error probability bound, as shown by Massey [4]. We assume that the decoder is given not only the received signal $r_i(t)$, but may be given an estimate of the fading amplitude a during each signal interval. We consider two cases: complete and no side information, i.e., the receiver either knows the exact value of a or has no information about a. We further assume a slowly varying fading amplitude a, which remains essentially constant over a signaling interval. Also, the signals $s_i(t)$ are interleaved so that the amplitude fades affecting successive symbols are independant.

Without Side Information
If the decoder has no information about a, R_0 is given by

$$R_0 = -\log_2 \min_\lambda \left(\sum_{k=1}^{A} \sum_{l=1}^{A} p_k p_l e^{\lambda(\underline{x}_k^2 - \underline{x}_l^2) + \lambda^2 N_0 (\underline{x}_k - \underline{x}_l)^2} (1 - \sqrt{\pi}\lambda c \operatorname{erfc}(\lambda c) e^{\lambda c}) \right), \tag{3}$$

where A is the signal alphabet size, $\underline{x}_l$ and $\underline{x}_k$ are the signals, p_k and p_l are the probabilities with which they are chosen, N_0 is the one sided spectral noise power density, λ is the Chernoff parameter and $c = \underline{x}_k^2 - \underline{x}_k \underline{x}_l$. Equation (3) is too messy to interpret in the sense that a design criterion can be deduced from it. A computer evaluation shows interesting differences between different constellations. It is most noteworthy that the rectangular signal constellations have a significantly poorer R_0 than the constant envelope signal sets, while they have a slightly better R_0 on the Gaussian channel.

With Side Information
Here we consider a decoder that is given perfect knowledge of the fading amplitude. Such information can partly be extracted from the received power of a pilot tone, which is needed anyway in order to synchronize the phase. In this case R_0 is given by

$$R_0 = -\log_2 \left(\sum_{k=1}^{A} \sum_{l=1}^{A} p_k p_l \frac{1}{1 + \frac{(\underline{x}_k - \underline{x}_l)^2}{4N_0}} \right). \tag{4}$$

This equation is considerably less complex and provides a key to code design for fading channels, as we shall see in the next section.

TRANSFER FUNCTION BOUND

Any decoding error in a trellis code is characterized by a set of incorrect branches which the decoder chooses over the correct branches. With each incorrect path we may associate a sequence of incorrect trellis states $\hat{S}_k$, while the sequence of correct states is S_k. Any error event of length l can then be described by l state pairs, $(S_0, \hat{S}_0), \ldots, (S_l, \hat{S}_l)$, with $S_0 = \hat{S}_0$, $S_l = \hat{S}_l$, and $S_k \neq \hat{S}_k$ for $0 < k < l$, i.e., the incorrect path must not touch the correct path. Associated with these paths are two symbol sequences $\underline{a} = [a_0, a_1, \ldots, a_l]$ and $\hat{\underline{a}} = [\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_l]$. The probability of an error event may be upper bounded by the Chernoff bound on the two codewords which the paths generate. We may therefore write

$$\Pr[(S_0, \ldots, S_l) \to (\hat{S}_0, \ldots, \hat{S}_l)] \leq C(\underline{a}, \hat{\underline{a}}) = \prod_{j=0}^{l-1} C(a_j, \hat{a}_j) \tag{5}$$

where $C(a_j, \hat{a}_j)$ is the Chernoff bound on the two signals a_j and $\hat{a}_j$. The last equality is a consequence of the interleaving process. The event error probability is now upper bounded by the sum of the probability of all incorrect paths, i.e.,

$$P_E \leq \sum_{\substack{\text{error} \\ \text{paths } k}} \prod_{j=0}^{d_k-1} C(a_{jk}, \hat{a}_{jk}), \tag{6}$$

where d_k is the length of path k. Let us now contrast the Chernoff bound for the Gaussian channel with that for the fading channel. In the Gaussian case

$$\prod_{j=0}^{d_k-1} C(a_{jk}, \hat{a}_{jk}) = e^{\frac{1}{4N_0} \sum_{j=0}^{d_k-1} (a_{jk}-\hat{a}_{jk})^2}, \tag{7}$$

i.e., the error probability is determined by the total Euclidean distance of each error path. In the fading case with side information we have

$$\prod_{j=0}^{d_k-1} C(a_{jk}, \hat{a}_{jk}) = \prod_{j=0}^{d_k-1} \frac{1}{1 + \frac{(a_{jk}-\hat{a}_{jk})^2}{4N_0}} \approx \frac{(4N_0)^{d_k'}}{\prod_{\substack{j=0 \\ a_{jk} \neq \hat{a}_{jk}}}^{d_k-1} (a_{jk} - \hat{a}_{jk})^2}, \tag{8}$$

where d_k' equals d_k, the length of the path less the number of *matches*, i.e., the number of branches where $a_{jk} = \hat{a}_{jk}$. We call d_k' the *effective length* of path k and $d' = \min(d_k)$ the *effective length* of the code. The approximation is valid for SNR values > 6dB, which is usually the case for transmission over a fading channel. The signal to noise ratio, or SNR, is defined in the usual way as $\text{SNR} = \log_{10}(E_b/N_0)$, where E_b is the average signal energy of the signal set in question, i.e., $E_b = \sum_{i=1}^{A} p_i x_i^2$. It becomes clear that the error performance in the fading case is dominated by the paths with the shortest effective length, i.e., $d_k' = d'$ and among those by the one having the smallest product in the denominator, i.e., with the smallest *product distance.* While preparing this paper, it has come to our attention that Simon & Divsalar [5] have come up with a similar interpretation.

CODES FOR THE FADING CHANNEL

In this section we examine the performance on a fading channel of bandwidth efficient codes that were designed for the Gaussian channel and of some new codes designed for a fading channel. From equation (8) it is seen that parallel transitions in the code ($d' = 1$) are most detrimental to code performance and should be avoided. We now concentrate on trellis codes without parallel transitions. Ungerboeck [3] gives a list of optimal trellis codes based on convolutional codes. His list is given in table 1, together with their effective lengths d' and minimum product distance *mpd.* Table 2 gives a list of codes that will outperform the Ungerboeck codes on a fading channel. Codes in the two tables have to be compared on the basis of equation (8), i.e., the dominant parameter is d' since it determines the slope of the error bound. Among codes with equal d', the one with larger *mpd* is superior. For example, for the $\nu = 6$ code the coding gain at an error probability of 10^{-5} is 8 dB. All the codes use an 8PSK signal set. We have seen in section II that, unless perfect side information is available, constant amplitude schemes are preferable. It also seems unlikely that larger signal sets that 8PSK will be used in the near future due to the problem of synchronizing the phase properly on fading channels.

ν	$H(D)^0$	$H(D)^1$	$H(D)^2$	d'	mpd
2	5	2	-	1	2
4	23	04	16	3	2.34
5	45	16	34	2	8
6	105	036	074	2	8
7	203	014	016	3	2.34
8	405	250	176	3	16
9	1007	164	260	3	16

Table 1: Ungerboeck's 8PSK codes

ν	$H(D)^0$	$H(D)^1$	$H(D)^2$	d'	mpd
2	5	2	1	2	1.17
4	37	04	16	3	2.34
5	77	14	16	3	2.34
6	177	041	116	4	1.37
7	-	-	-	-	-
8	-	-	-	-	-
9	-	-	-	-	-

Table 2: New codes

CONCLUSIONS

We conclude that bandwidth efficient coding is feasible on fading channels, but the important design parameter is no longer the Euclidean distance but two parameters, the effective length and the minimum product distance. Intuitively a good code in a nonuniform noise environment should try to spread the "distance" between two codewords evenly over all the branches over which the codewords differ. The same basic idea is used in diversity transmission, which is usually employed to improve performance on fading channels.

REFERENCES

[1] J. Hagenauer and E. Lutz, "Forward Error Correction Coding for Fading Compensation in Mobile Satellite Channels," *IEEE Journal Selected Areas Commun.*, Vol. SAC-5, No.2, pp. 215–225, February 1987.

[2] J. Hagenauer et al., "The Maritime Satellite Communication Channel," *IEEE Journal Selected Areas Commun.*, Vol. SAC-6, May 1987.

[3] G. Ungerboeck, "Channel Coding with Multilevel/Phase Signals," *IEEE Trans. Inform. Theory*, Vol. IT-28, No.1, pp. 55–67, January, 1982.

[4] J. L. Massey, "Coding and Modulation in Digital Communications," *Proc. Int. Zürich Sem. Digital Commun.,*, Zürich, Switzerland March 1974, pp. E2(1)–E2(4).

[5] M. K. Simon and D. Divsalar, "The Performance of Trellis Coded Multilevel DPSK on a Fading Mobile Satellite Channel" To be presented at the IEEE 1987 International Conference on Communications, Seattle, WA June 7–10.

[6] M. K. Simon and D. Divsalar, "Trellis Coded Modulation for 4800-9600 bits/s Transmission Over a Fading Mobile Satellite Channel," *IEEE Journal Selected Areas Commun.*, Vol. SAC-5, No.2, pp. 162–177, February 1987.