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A Quasi-Analytical Method for Non-iterative Computation of Nonlinear Controls

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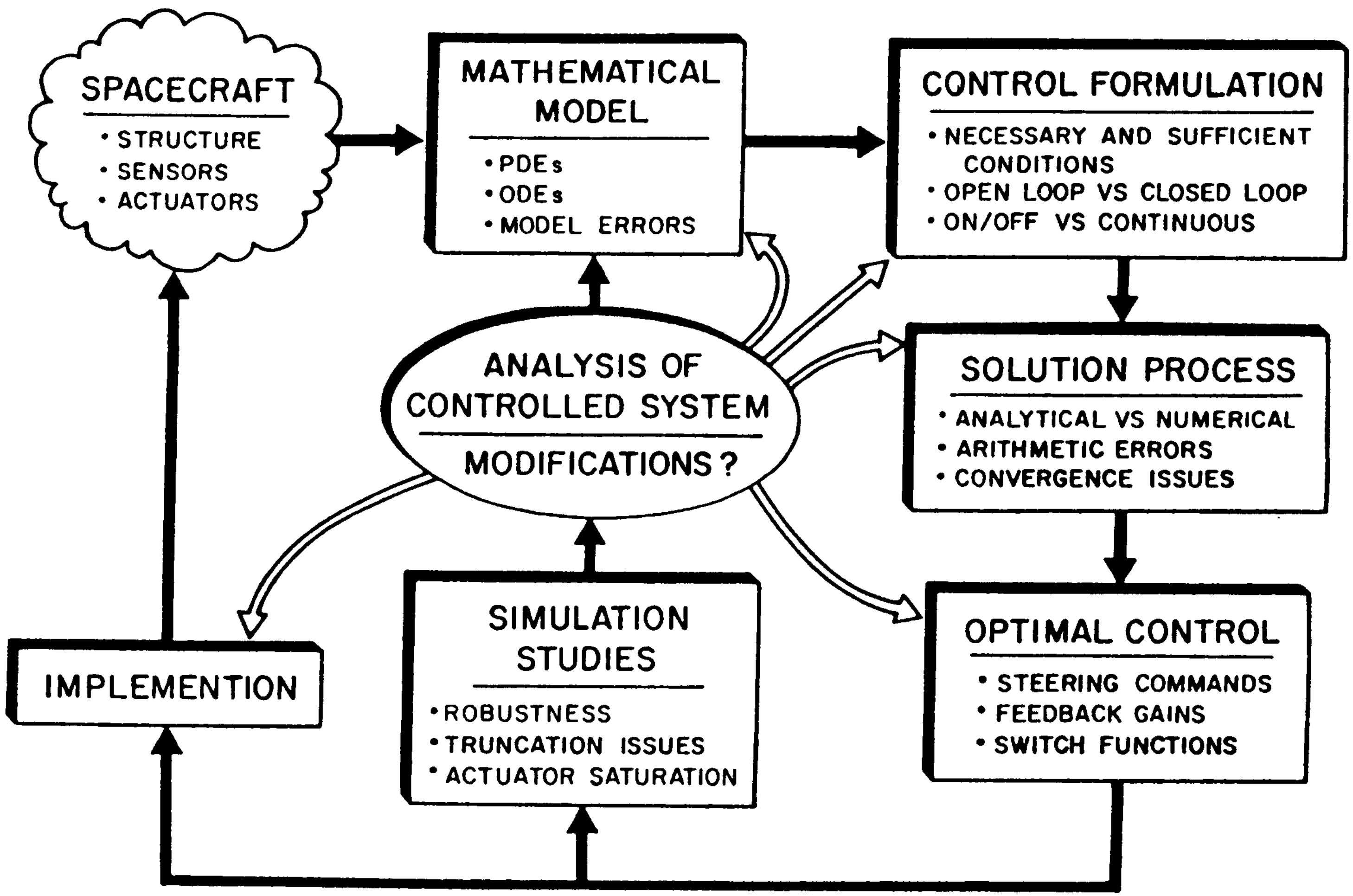
*Coupling of Spacecraft Structural Modeling with Dynamics/Controls Analysis, Design, and Implementation**



* the above figure is from our book:
Junkins, J. L. and Turner, J. D. , *Optimal Spacecraft Rotational Maneuvers*, Elsevier, 1986.

PRELIMINARIES

Consider a dynamical system described by

$$\dot{z} = Fz + Du + \epsilon\, g(z,u,t) \qquad (1)$$

where

- z *is an* $n \times 1$ *state vector*
- u *is an* $m \times 1$ *control vector*
- F *&* D *are constant matrices*
- z, u, *and the nonlinear terms* $g(z,u,t)$ *are continuous & differentiable*

We seek an optimal control $u^*(t)$ *and corresponding optimal trajectory* $z^*(t)$ *which minimize the quadratic performance measure*

$$J = 1/2\,[z^T S z]_{t_f} + 1/2 \int_0^{t_f} (z^T Q z + u^T R u)\,dt \qquad (2)$$

The necessary conditions involve the Hamiltonian $H(z, u, p, t)$

$$H = 1/2\,(z^T Q z + u^T R u) + p^T (Fz + Du + \epsilon g); \qquad (3)$$

These are: $\frac{\partial H}{\partial u} = 0$, $\quad \frac{\partial H}{\partial p} = \dot{z}$, $\quad -\frac{\partial H}{\partial z} = \dot{p}$

plus boundary conditions: $z(0) = z_0$, $p(t_f) = Sz(t_f)$ *or* $z(t_f) = z_f$.

Two Point Boundary Value Problem

Pontryagin Necessary Conditions:

$$\dot{z} = Fz + Du + \epsilon g \;, \qquad z(o) = z_0$$

$$\dot{p} = -Qz - F^T p - \epsilon \left[\frac{\partial g}{\partial z} \right]^T, \quad p(t_f) = Sz(t_f) \,, \text{for } z(t_f) \text{ "free"}$$

$0 = Ru + D^T p$, ... ***from which the optimal control is*** $u^* = -R^{-1} D^T p$.

The state/co-state coupled system can be written as a 2n order system:

$$\dot{x} = A\,x + \epsilon h(x,t) \tag{4}$$

where

$$x^T = [z^T \;\; p^T], \quad A = \begin{bmatrix} F & -DR^{-1}D^T \\ -Q & -F^T \end{bmatrix}, \quad h(x,t) = \begin{Bmatrix} g \\ -\frac{\partial g^T}{\partial z} \end{Bmatrix}$$

Due to the nonlinear terms in $h(x,t)$, exact analytical solutions of Eq. (4) are most often impossible. A variety of iterative techniques are available; they are often expensive due to initial ignorance of a "good starting estimate"(of $p(t_0)$ <u>or</u> $p(t_f)$) required for reliable convergence. We seek to avoid iteration through use of a perturbation method & "quasi-analytical" integration. >>>

The Asymptotic Expansion of the Necessary Conditions

We seek a power series solution of the usual form

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots + \epsilon^k x_k(t) + \dots \tag{5}$$

Substitution of the power series into the state/co-state system of Eq. (4), and equating like powers of ϵ, leads to the <u>sequence</u> of linear systems:

$$\begin{aligned} \dot{x}_0 &= A\, x_0 && \longrightarrow x_0(t), \\ \dot{x}_1 &= A\, x_1 + g_1(t, x_0(t)) && \longrightarrow x_1(t) \\ &\vdots \\ \dot{x}_k &= A\, x_k + g_k(t, x_0(t), x_1(t), \dots, x_{k-1}(t)) && \longrightarrow x_k(t) \end{aligned} \tag{6}$$

with the boundary conditions

$$x_0(0) = \begin{Bmatrix} Z(0) \\ p_0(0) \end{Bmatrix}, \; x_0(t_f) = \begin{Bmatrix} Z(t_f) \\ p_0(t_f) \end{Bmatrix}; \; \dots ; \; x_k(0) = \begin{Bmatrix} 0 \\ p_k(0) \end{Bmatrix}, \; x_k(t_f) = \begin{Bmatrix} 0 \\ p_k(t_f) \end{Bmatrix}$$

where, at least formally, the sequence of solutions is given by

$$x_k(t) = e^{At} \left[x_k(0) + \int_0^t e^{-A\tau} g_k(\tau, x_0(\tau), x_1(\tau), \dots, x_{k-1}(\tau))\, d\tau \right], \quad k = 1, 2, 3, \dots \tag{7}$$

But... how do we make efficient algorithms? Does convergence occur in the "real world"? Can the above be implemented in a way which automates the algebra usually associated with perturbation methods? What about secular terms? Does this approach apply to systems of non-trivial dimensions & "messy" nonlinear terms? We have made some progress in answering these questions.

Van Loan's Identity for Integrating Forced Linear Systems

Consider

$$\dot{x}_k = A\,x_k + g_k, \quad x_k(t) = e^{At}x_k(0) + e^{At}\int_0^t e^{-A\tau} g_k(\tau)\,d\tau \qquad ,k = 1,2,\ldots \quad (8)$$

For the special case that $u(t)$ can be represented as Fourier series, the Fourier series can be re-written as a matrix exponential

$$g_k(t) = b_{ok} + \sum_{r=1}^{N} b_{rk}\cos(\omega_r t) + a_{rk}\sin(\omega_r t) = G_k\, e^{\Omega t} c \qquad (9)$$

where

b_{ok}, b_{rk}, a_{rk} *are 2n x 1 vectors of Fourier coefficients*

$G_k = [\, b_o\, b_1\, a_1\, b_2\, a_2 \ldots b_r a_r \ldots b_N a_N \,]_k$, *a 2n x (2N + 1) constant matrix*

$c = \{1\ 1\ 0\ 1\ 0\ 1\ 0 \ldots\ldots 1\ 0\}^T$, *a 2N x 1 selection vector*

$$\Omega_r = \begin{bmatrix} 0 & -\omega_r \\ \omega_r & 0 \end{bmatrix}, \qquad \omega_r = r(2\Pi/(t_f - t_o))$$

$$\Omega = diag\,[\,0,\ \Omega_1,\ \Omega_2, \ldots,\ \Omega_r, \ldots,\ \Omega_N]$$

Substituting Eq. (9) into Eq. (8),

$$x_k(t) = e^{At}x_k(0) + [\int_0^t e^{-A\tau}G_k\, e\ d\tau]c = e^{At}x_k(0) + [\Psi_k]c$$

Van Loan has established the interesting & useful identity which permits computation of the forced response using a matrix exponential (via, for example, Ward's Pade' algorithm):

$$e^{\begin{bmatrix} A & G_k \\ 0 & \Omega \end{bmatrix} t} = \left[\begin{array}{c|c} e^{At} & \Psi_k \\ \hline 0 & e^{\Omega t} \end{array}\right] \qquad (10)$$

For large N, we can use superposition & keep the order of the matrix exponentials small .... thus the response to a relatively arbitrary $g_k(t)$ can be calculated via matrix exponentials.

Control Rate Smoothing & State Vector Augmentation

We choose to minimize

$$J = 1/2\{ z(t_f)^T S z(t_f) + u(t_f)^T S_0 u(t_f) + \dot{u}(t_f)^T S_1 \dot{u}(t_f) \}$$

$$+ 1/2 \int_0^{t_f} \{ z^T Q z + u^T R_0 u + \dot{u}^T R_1 \dot{u} + \ddot{u}^T R_2 \ddot{u} \} \, dt$$

Subject to: $\dot{z} = Az + Du$. *This can be converted to standard form via the definitions:*

$$\tilde{z} = \begin{Bmatrix} z \\ u \\ \dot{u} \end{Bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & D & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \quad \begin{matrix} \tilde{S} = \text{block diag}[S, S_0, S_1] \\ \tilde{Q} = \text{block diag}[Q, R_0, R_1] \\ \tilde{R} = R_2 \;, \; \tilde{u} = \ddot{u} \end{matrix}$$

So we can equivalently minimize

$$J = 1/2 \; \tilde{z}(t_f)^T \tilde{S} \tilde{z}(t_f) + 1/2 \int_0^{t_f} \{ \tilde{z}^T \tilde{Q} \tilde{z} + \tilde{u}^T \tilde{R} \tilde{u} \} \, dt$$

Subject to: $\dot{\tilde{z}} = \tilde{A}\tilde{z} + \tilde{D}\tilde{u}$. *The necessary conditions have the identical form as those developed in the foregoing. Penalizing the control derivatives has been found most constructive in frequency-shaping the torque profiles to decrease excitation of the poorly modeled higher frequency modes.*

Case 1 Optimal Detumble/Attitude Aquisition

STATE DYNAMICS

Euler (quaternion) parameters

$$\begin{Bmatrix} \dot\beta_0 \\ \dot\beta_1 \\ \dot\beta_2 \\ \dot\beta_3 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} \qquad \text{or} \quad \dot\beta = \frac{1}{2} (\omega) \beta$$

Euler's Equations

$$\begin{Bmatrix} \dot\omega_1 \\ \dot\omega_2 \\ \dot\omega_3 \end{Bmatrix} = \begin{Bmatrix} -I_1 \omega_2\omega_3 + u_1/I_1 \\ -I_2 \omega_3\omega_1 + u_2/I_2 \\ -I_3 \omega_1\omega_2 + u_3/I_3 \end{Bmatrix}, \quad \begin{matrix} I_1 = (I_3-I_2)/I_1 \\ I_2 = (I_1-I_3)/I_2 \\ I_3 = (I_2-I_1)/I_3 \end{matrix} \qquad \text{or} \quad \dot\omega = f(\omega, u)$$

BOUNDARY CONDITIONS $(t_f = 2 \text{ sec})$

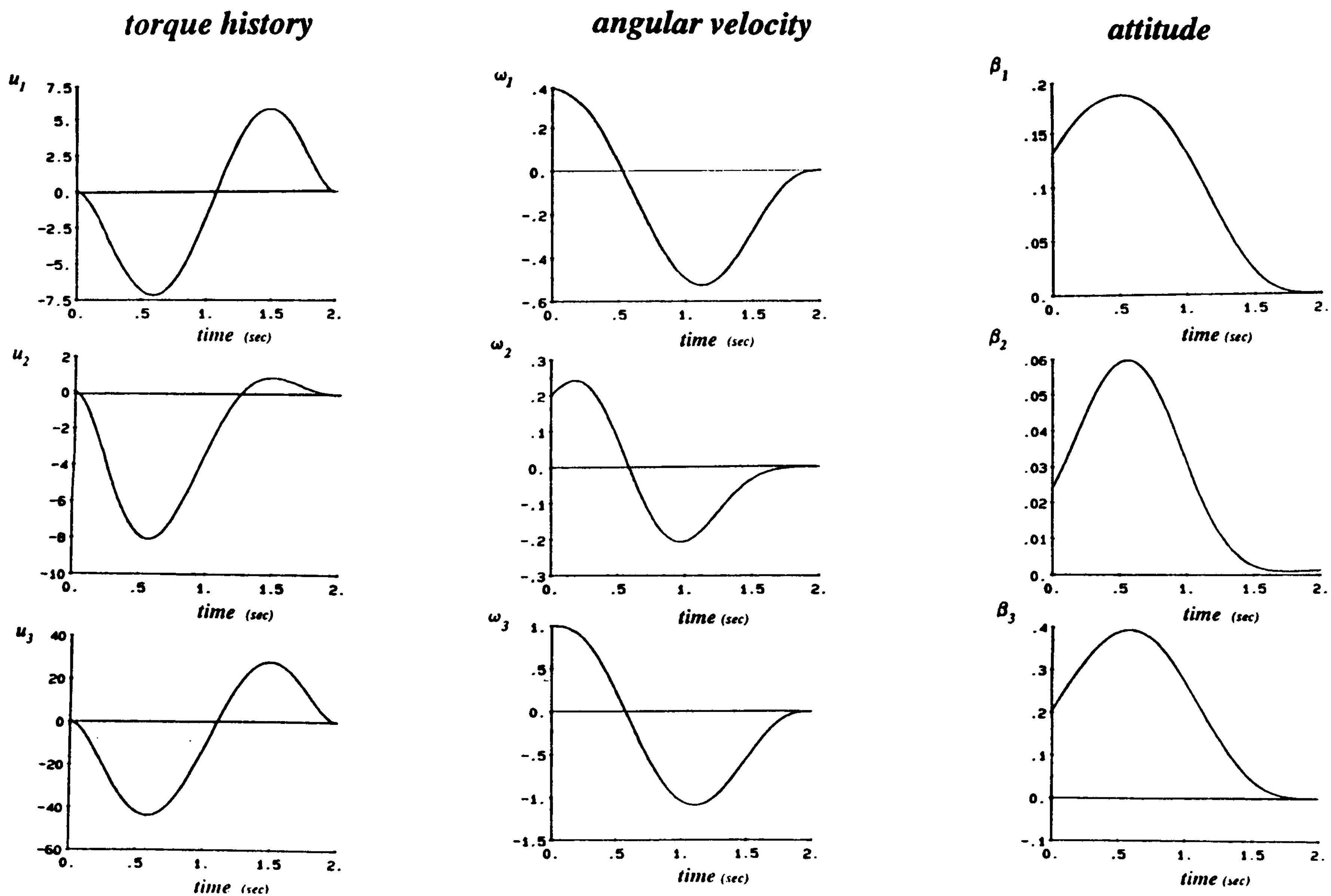
$$\beta(0) = \begin{Bmatrix} .9699665 \\ .1318887 \\ .0238626 \\ .2029798 \end{Bmatrix}, \quad \omega(0) = \begin{Bmatrix} .4 \text{ r/s} \\ .2 \\ 1.0 \end{Bmatrix} \quad \beta(t_f) = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \omega(t_f) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Case 1 Numerical Results for the TPBVP Solution

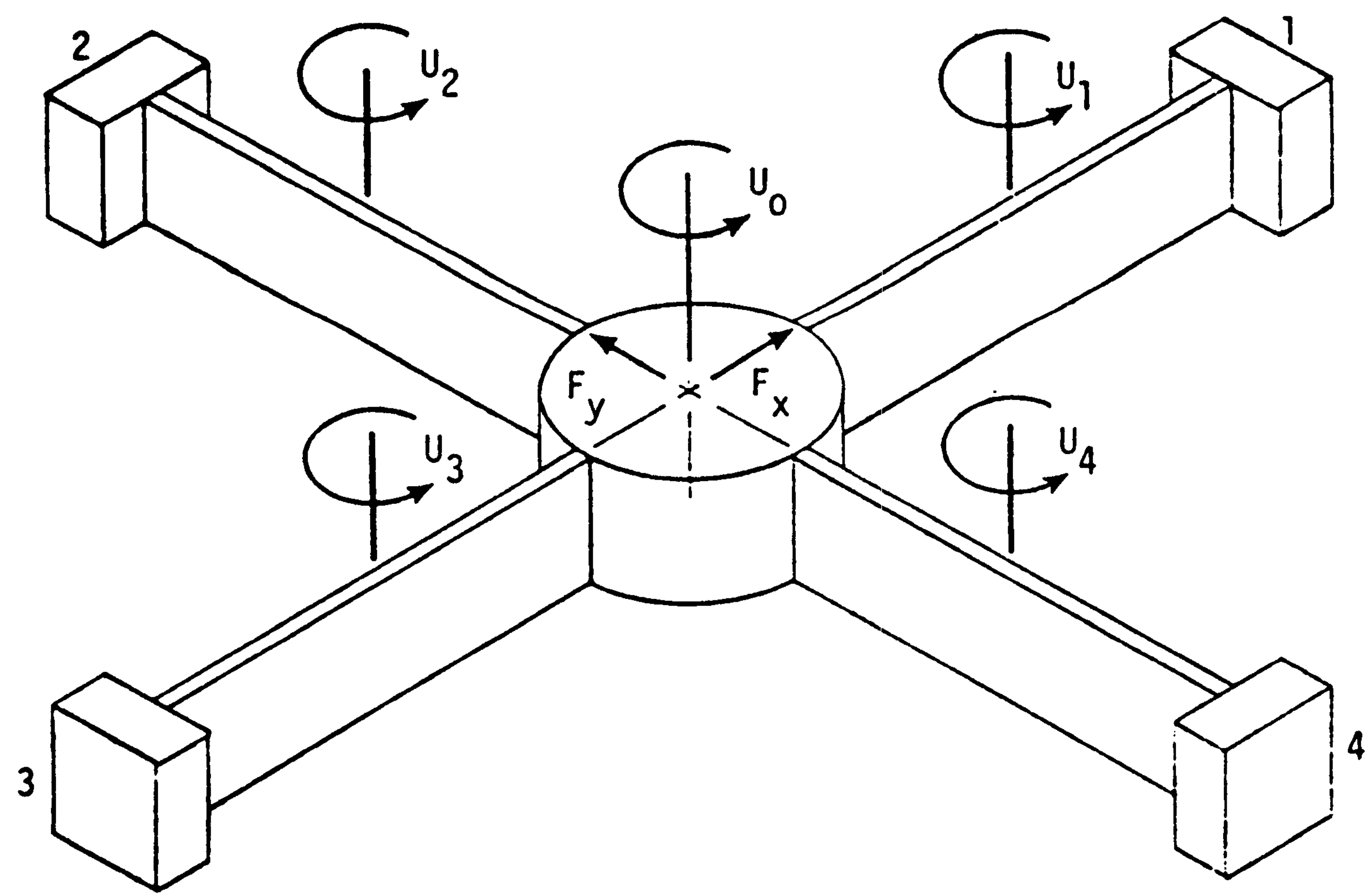
FINAL STATE ERRORS

	LINEAR SOLUTION		FIRST ORDER		SECOND ORDER	
$\Delta\beta_0$	.01999	$\approx 10^\circ$	.00091	$\approx .5^\circ$	7×10^{-7}	$\approx .05^\circ$
$\Delta\beta_1$	-.08232		-.00866		.00058	
$\Delta\beta_2$	.18033		-.00944		.00094	
$\Delta\beta_3$	.01734		-.00412		-.00034	
$\Delta\omega_1$	.01914		-.01525		.00109	
$\Delta\omega_2$	.43150		-.00292		.00295	
$\Delta\omega_3$	.00461		-.00071		.00015	

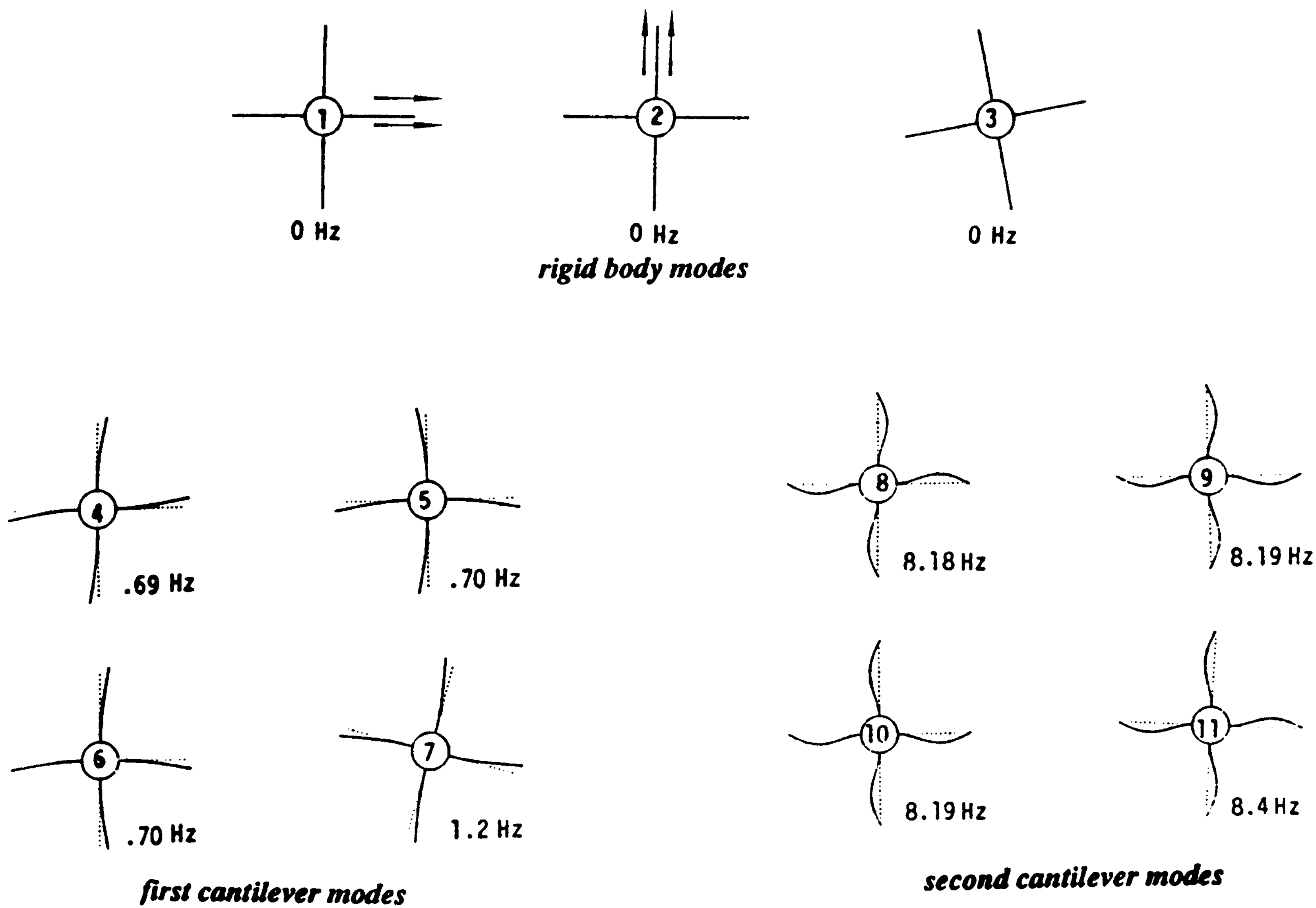
Case 1 Optimal Detumble/Attitude Acquisition Maneuver



The Draper/RPL Slewing Experimental Configuration



Draper/RPL Configuration: First Eleven in-plane Vibration Modes

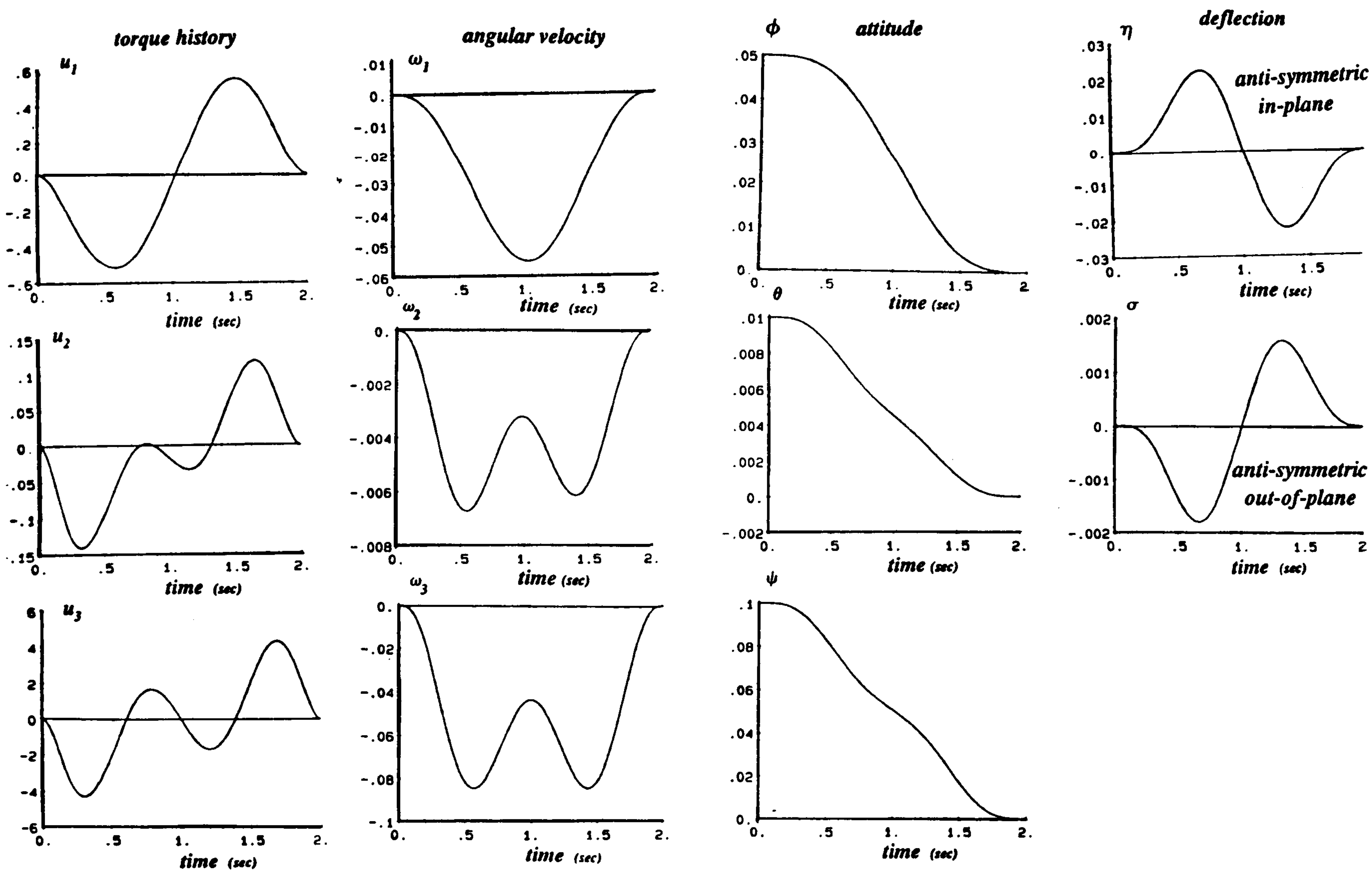


Case 2 Numerical Results for the TPBVP Solution (small angle flexible body maneuver)

FINAL STATE ERRORS

	LINEAR SOLUTION	FIRST ORDER	SECOND ORDER
ϕ	-0.828E-5	0.276E-4	0.184E-5
θ	0.104E-2	0.667E-5	-0.878E-6
ψ	0.668E-3	-0.270E-4	0.130E-5
ω_1	-0.450E-3	0.150E-3	0.342E-5
ω_2	0.317E-2	0.293E-4	-0.474E-5
ω_3	0.597E-3	-0.396E-5	0.135E-6
η_1 (in plane)	-0.960E-3	-0.120E-5	-0.217E-8
σ_1 (out-of-plane)	0.317E-2	0.293E-4	-0.474E-5

Case 2 Optimal Maneuver with Vibration Suppression/Arrest



Concluding Remarks

A novel optimal control solution process has been developed for a general class of nonlinear dynamical systems

The method combines control theory, perturbation methods, and Van Loan's recent matrix exponential results

All controlled response integrations are accomplished via matrix exponentials (using Ward's Pade algorithm) and recursions developed herein

A variety of applications support the practical utility of this method; nonlinear rigid body optimal maneuvers are routinely solved; flexible body dynamical systems of order ≥ 40 have been solved

The method fails occasionally due to poor convergence of the perturbation expansion or numerical difficulties associated with computing the matrix exponential

The method is attractive because it appears to be a good candidate for semi-automation; <u>no initial guess is required</u>, and it usually converges at 2nd or 3rd order in <u>minutes</u> of machine time

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