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## **STRESS-INTENSITY FACTOR CALCULATIONS USING THE BOUNDARY FORCE METHOD**

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## SUMMARY

The Boundary Force Method (BFM) was formulated for the three fundamental problems of elasticity: the stress boundary value problem, the displacement boundary value problem, and the mixed boundary value problem. Because the BFM is a form of an indirect boundary element method, only the boundaries of the region of interest are modeled. The elasticity solution for the stress distribution due to concentrated forces and a moment applied at an arbitrary point in a cracked infinite plate is used as the fundamental solution. Thus, unlike other boundary element methods, here the crack face need not be modeled as part of the boundary. The other boundaries are divided into a finite number of straight-line elements, and at the center of each element, concentrated forces and a moment are applied. This set of unknown forces and moments are calculated to satisfy the prescribed boundary conditions of the problem.

The formulation of the BFM is described and the accuracy of the method is established by analyzing a center-cracked specimen subjected to mixed boundary conditions and a three-hole cracked configuration subjected to traction boundary conditions. The results obtained are in good agreement with accepted numerical solutions. The method is then used to generate stress-intensity solutions for two common cracked configurations: an edge crack emanating from a semi-elliptical notch, and an edge crack emanating from a V-notch.

The BFM is a versatile technique that can be used to obtain very accurate stress-intensity factors for complex crack configurations subjected to stress, displacement, or mixed boundary conditions. The method requires a minimal amount of modeling effort and, therefore, stress-intensity factor analyses for a range of crack lengths can be performed with ease.

## INTRODUCTION

Accurate stress-intensity factors are important in the prediction of crack growth rates and fracture strengths. These predictions are essential in the fail-safe design of aircraft structural components. Thus, in the field of fracture mechanics, one of the major research activities is the development of new techniques to obtain accurate stress-intensity factors for arbitrarily shaped plates with cracks.

At present, for two-dimensional fracture mechanics analyses, the three most popular numerical techniques for the computation of stress-intensity factors are the boundary element method (BEM), the finite element method (FEM), and the collocation method. In a recent paper [1], a more general numerical method, the Boundary Force Method (BFM), was introduced. The BFM is a form of an indirect BEM. There are two major differences between the present BFM and the other BEM [2-7]: the BFM does not require the modeling of the crack faces and the BFM includes the moment as an unknown. These two features were incorporated in the BFM through the use of Erdogan's analytical solution [8] for a vertical and a horizontal force and a moment in an infinite plate with a crack as the fundamental solution. The stress-free conditions on the crack faces, therefore, are exactly satisfied and only the boundaries of the analysis domain need to be modeled in this method. The effect of adding the moment unknown to the BFM will be investigated and the convergence rates will be compared with BEM using forces only as the unknowns.

Unlike in the BFM, with the FEM the entire region of interest has to be discretized. Thus, in fracture mechanics where a fine mesh is needed in regions with high stress gradients (crack tip), the FEM can be cumbersome to use. Therefore, for configurations with cracks and notches, the number of elements

required to accurately model these problems can be extremely large. Moreover, in crack-growth-rate predictions, stress-intensity factors must be found for various crack lengths. Thus, in the FEM, a new mesh must be generated for each crack length to accurately model the region near the crack tip and a large amount of time is needed for modeling.

In the collocation method, only the boundaries of the region of interest need to be modeled as in the BFM. However, unlike the BFM, the basic stress functions are different for each class of problems. Therefore, in the collocation method, a large amount of time can be spent in developing and formulating new stress functions.

The purpose of this paper is to present the formulation of the BFM for displacement and mixed boundary value problems. The following section briefly describes the formulation of the BFM for the stress boundary value problem, the displacement boundary value problem, and the mixed boundary value problem. Complete details on these formulation can be found in [9]. To evaluate the BFM for displacement and mixed boundary value problems, an end-clamped-tension specimen with a central crack was analyzed. And to further validate the BFM for stress boundary value problems, a three-hole cracked specimen was also analyzed. The solutions for these configurations are available in the literature for comparison. Also, stress-intensity factor solutions were obtained for the following two engineering problems for which no stress-intensity factor solutions are available: an edge crack emanating from a semi-elliptical notch and an edge crack emanating from a V-notch.

#### LIST OF SYMBOLS

- a                    crack length or one-half crack length
- b                    half height of notch

C	resultant couple
E	Young's modulus
[F]	influence coefficient matrix
F	stress-intensity correction factor
$F_x, F_y$	resultant forces in the x- and y-directions, respectively
H, W	height and width of rectangular plates, respectively
K	stress-intensity factor
$L_i$	length of element i
M	concentrated moment
n	number of subdivision per element
N	number of elements
$p_i, q_i, m_i$	unit loads and moment on the $i^{\text{th}}$ element
P, Q	concentrated forces in the y- and x-directions, respectively
{P}	vector of unknown forces and moments
{R}	externally applied load vector
r	notch radius
S	remote applied stress
u, v	displacements in the x- and y-directions, respectively
x, y	Cartesian coordinates
z	complex variable, $z = x + iy$
$z_0$	location of forces and moment, $z_0 = x_0 + iy_0$
$\kappa$	material constant: $= 3-4\nu$ for plane strain $= (3-\nu)/(1+\nu)$ for plane stress
$\lambda_x, \lambda_y$	projection of sub-arc onto the x- and y-axes, respectively
$\mu$	shear modulus
$\nu$	Poisson's ratio
$\theta$	rotation of element
$\sigma_x, \sigma_y, \sigma_{xy}$	Cartesian stresses

$\sigma_n, \sigma_{nt}$  normal and shear stresses on the boundary  
 $\phi^*, \Omega^*, \phi_0$  complex stress functions

#### FORMULATION OF THE BOUNDARY FORCE METHOD

In this section, first the formulation of the BFM for a stress boundary value problem is presented. Next the formulation for the displacement and the mixed boundary value problems is presented.

As mentioned earlier the fundamental solution used in the BFM is the solution for a horizontal force, a vertical force, and a moment acting at an arbitrary point in an infinite plate [8]. The details of this fundamental solution are presented in Appendix A. The BFM utilizes this solution in conjunction with the superposition technique to satisfy the boundary conditions on the boundaries of a finite cracked plate. The method is explained below using the stress boundary value formulation.

#### Stress Boundary Value Problems

For the first fundamental problem of elasticity, where only tractions are applied on the boundaries, consider a centrally cracked rectangular plate subjected to tractions with the coordinate system shown in Figure 1. For ease of illustration, the applied traction at the ends is assumed to be symmetric about the x- and y-axes. Therefore, only one quarter of the region needs to be modeled.

The first step in the BFM is to discretize the boundary by enclosing the region of interest with a finite number of elements. Since only one quadrant of the plate in Figure 1 needs to be modeled the region of interest is  $0 \leq x \leq W/2$  and  $0 \leq y \leq H/2$ , where W is the width of the plate and H is the height of the plate. For illustration, the boundaries AB and BC (see Figure 1) are divided into a total of N equal-sized elements. The solution to this problem is obtained by the superposition of the concentrated forces and moments in an

infinite plate as shown in Figure 2 for  $N = 4$ . The dashed lines in Figure 2 correspond to an imaginary boundary on the infinite plate traced from Figure 1.

The concentrated forces  $P_i$ ,  $Q_i$ , and the moment  $M_i$  acting on the  $i^{\text{th}}$  element create resultant forces  $R_{x_j}$  and  $R_{y_j}$  in the x- and y-directions, and a resultant couple  $C_j$  on the  $j^{\text{th}}$  element. The forces and moment were applied on the outward normal at a distance  $\delta_i$  from the center of each element on the boundaries. This offset was used to facilitate the computation of stress on the boundaries without encountering singularities. The value of  $\delta_i$  was chosen to be one-quarter of the element size. The resultant forces and couple on element  $j$  due to the forces and moments on all the elements are

$$R_{x_j} = \sum_{i=1}^N (F_{x_j p_i} P_i + F_{x_j q_i} Q_i + F_{x_j m_i} M_i) \quad (1)$$

$$R_{y_j} = \sum_{i=1}^N (F_{y_j p_i} P_i + F_{y_j q_i} Q_i + F_{y_j m_i} M_i) \quad (2)$$

$$C_j = \sum_{i=1}^N (C_{j p_i} P_i + C_{j q_i} Q_i + C_{j m_i} M_i) \quad (3)$$

where  $F_{x_j p_i}$  = force in the x-direction on the  $j^{\text{th}}$  element due to unit loads in the y-direction at four symmetric points: unit load  $p_i$  at  $(x_{0_i}, y_{0_i})$  and  $(-x_{0_i}, y_{0_i})$  and unit load  $-p_i$  at  $(x_{0_i}, -y_{0_i})$  and  $(-x_{0_i}, -y_{0_i})$ . Point  $(x_{0_i}, y_{0_i})$  is the load point of  $p_i$  on element  $i$ .

$F_{y_j p_i}$  = force in the y-direction on the  $j^{\text{th}}$  element due to unit loads at the four symmetric points.

$C_{j p_i}$  = resultant moment on the  $j^{\text{th}}$  element due to unit loads at the four symmetric points.

Similar definitions exist for  $F_{x_j q_i}$ ,  $F_{y_j q_i}$ ,  $C_{j q_i}$ ,  $F_{x_j m_i}$ ,  $F_{y_j m_i}$  and  $C_{j m_i}$ . The resultant forces and couples are assumed to act at the center of the corresponding element. For each element there are 3 equations (degrees of freedom) and, therefore, for a any number of element N on the boundary the resulting system of equations is

$$[F]_{3N \times 3N} \{P\}_{3N \times 1} = \{R\}_{3N \times 1}$$

where [F] is the influence coefficient matrix containing the  $F_{x_j p_i}$ ,  $F_{y_j p_i}$ ,  $C_{j m_i}$ , etc. terms. Here

$$C_{j p_i} = \sum_{i=1}^N \sum_{k=1}^n (F_{y_j p_i}^k \lambda_{x_j}^k + F_{y_j q_i}^k \lambda_{y_j}^k)$$

n = number of subdivisions on each element,  $F_{y_j p_i}$  and  $F_{y_j q_i}$  are defined above and superscript k refers to the  $k^{\text{th}}$  subdivision.

$\lambda_{x_j}^k$  = x-distance from the center of the  $k^{\text{th}}$  subdivision to the center of the  $j^{\text{th}}$  element and

$\lambda_{y_j}^k$  = y-distance from the center of the  $k^{\text{th}}$  subdivision to the center of the  $j^{\text{th}}$  element.

Once the influence coefficient matrix is computed and the applied loads have been determined, the unknown forces and moments along the boundaries can be found.



To determine the applied loads, again consider the simple example shown in Figure 1. Because the boundary AB is a free boundary, the resultant forces and moments due to the externally applied loads are zero for elements on boundary AB. Thus,

$$\begin{aligned} R_{x_j} &= 0 \\ R_{y_j} &= 0 \quad \text{for element } j \text{ on boundary AB} \\ C_j &= 0 \end{aligned} \tag{4}$$

For the  $j^{\text{th}}$  element on boundary BC the resultant forces and moment created by the externally applied loads are

$$\begin{aligned}
R_{x_j} &= 0 \\
R_{y_j} &= S_j \times L_j \quad \text{for element } j \text{ on boundary BC} \\
C_j &= 0
\end{aligned} \tag{5}$$

where  $S_j$  is the applied traction per unit thickness on the  $j^{\text{th}}$  element and  $L_j$  is the length of the  $j^{\text{th}}$  element

Replacing the left-hand side of equations (1), (2), and (3) by equations (4) and (5) results in a  $3N$  set of linear algebraic simultaneous equations. The unknowns are  $P_i$ ,  $Q_i$ , and  $M_i$  ( $i = 1$  to  $N$ ). The solution of this set of equations leads to the determination of  $P_i$ ,  $Q_i$ , and  $M_i$ . The stresses and displacements at any point can then be found using equations (A2) and (A3), respectively, given in Appendix A. The stress-intensity factor is found using equation (A6).

#### Displacement Boundary Value Problems

For the second fundamental problem of elasticity, where only displacements are prescribed on the boundaries, consider the centrally cracked rectangular plate subjected to prescribed displacements (see Figure 3).

Let the boundaries AB and BC be divided in  $N$  elements. Consider a representative element  $j$  on boundary AB. The displacements and rotation of the  $j^{\text{th}}$  element created by  $P_i$ ,  $Q_i$ , and  $M_i$  ( $i = 1$  to  $N$ ) are

$$2\mu u_j = \sum_{i=1}^N (D_{x_j p_i} P_i + D_{x_j q_i} Q_i + D_{x_j m_i} M_i) \tag{6}$$

$$2\mu v_j = \sum_{i=1}^N (D_{y_j p_i} P_i + D_{y_j q_i} Q_i + D_{y_j m_i} M_i) \tag{7}$$

$$2\mu\theta_j = \sum_{i=1}^N (G_{jp_i} P_i + G_{jq_i} Q_i + G_{jm_i} M_i) \quad (8)$$

where  $D_{x_j p_i}$  = x-displacement of the  $j^{\text{th}}$  element due to unit loads acting in the y-direction on the  $i^{\text{th}}$  element at four symmetric points: unit load  $p_i$  at  $(x_{0_i}, y_{0_i})$  and  $(-x_{0_i}, y_{0_i})$  and unit load  $-p_i$  at  $(x_{0_i}, -y_{0_i})$  and  $(-x_{0_i}, -y_{0_i})$ .

$D_{y_j p_i}$  = y-displacement of the  $j^{\text{th}}$  element due to unit loads at the four symmetric points.

$G_{jp_i}$  = rotation of the  $j^{\text{th}}$  element due to unit loads at the four symmetric points.

The rotation  $G_{jp_i}$  of the  $j^{\text{th}}$  element due to a unit load  $p_i$  acting on  $i^{\text{th}}$  element is obtained from the following expression:

$$G_{jp_i} = \frac{D_{x_j p_i}^{z_j} + D_{x_j p_i}^{z_{j+1}}}{L_y^j} + \frac{D_{y_j p_i}^{z_j} + D_{y_j p_i}^{z_{j+1}}}{L_x^j} \quad (9)$$

where  $z_j$  and  $z_{j+1}$  are the end points of the  $j^{\text{th}}$  element and

$L_y^j$  = y-projection of the  $j^{\text{th}}$  element

$L_x^j$  = x-projection of the  $j^{\text{th}}$  element

Similar definitions are used for  $D_{x_j q_i}$ ,  $D_{x_j m_i}$ ,  $D_{y_j q_i}$ ,  $D_{y_j m_i}$ ,  $G_{jq_i}$ , and  $G_{jm_i}$ .

Replacing the left-hand side of equations (6), (7), and (8) by the corresponding prescribed displacements results in a 3N set of linear algebraic

simultaneous equations. The unknowns are  $P_i$ ,  $Q_i$  and  $M_i$  ( $i = 1$  to  $N$ ). The solution of this set of equations determines  $P_i$ ,  $Q_i$ , and  $M_i$ , and then the stresses and displacements at any point can then be found using equation (A2) and (A3), respectively. The stress-intensity factor is found using equations (A6).

#### Mixed Boundary Value Problems

For the third fundamental problem of elasticity, where tractions are applied on a portion of the boundary and displacements are prescribed on the rest of the boundary, consider the centrally cracked rectangular plate shown in Figure 4. An example of such a problem is the end clamped tension specimen with a central crack, where the vertical ends of a rectangular plate are clamped and given an applied displacement. Because of symmetry, only the upper right-hand quadrant of the rectangular plate needs to be modeled.

Let the boundaries AB and BC be divided into  $N$  elements. As in the two previous cases, an influence coefficient matrix must be determined. In this case, the element  $j$  on boundary AB, the boundary conditions are the stress-free boundary conditions; thus, equations (1) through (3) apply. For element  $j$  on boundary BC, the boundary conditions are in terms of displacements; thus, equations (6) through (8) apply.

Again, this results in a  $3N$  set of linear algebraic simultaneous equations. The unknowns are  $P_i$ ,  $Q_i$ , and  $M_i$  ( $i = 1$  to  $N$ ). The solution of this set of equations determines the unknown. The stresses and displacements at any point can then be found using equations (A2) and (A3). The stress-intensity factor is found using equation (A6).

## RESULTS AND DISCUSSION

In this section, first, the accuracy of the BFM, where both forces and moments are used as unknown, is compared with other boundary methods where only forces are used as the unknowns. The convergence characteristics of the BFM are studied by analyzing a center-cracked tension specimen with a large crack. The accuracy of the BFM for displacement boundary value problems is investigated by analyzing a end-clamped tension specimen with a center crack subjected to mixed boundary conditions. Also, to further validate the BFM for stress boundary value problem, a three-hole cracked specimen subjected to uniform tension was analyzed. Accurate solutions for these two configuration are available in the literature for comparison. Finally, the method is used to obtain stress-intensity factor solutions for complex crack configurations for which no solutions are available, an edge crack emanating from a semi-elliptical notch, and an edge crack emanating from a V-notch.

### Accuracy and Convergence Studies

Previous work on boundary element methods has shown that improved accuracy is obtained when the boundary conditions were satisfied in terms of resultant forces instead of stresses [6]. In the BFM, an additional degree of freedom was added to each element so that the boundary conditions are satisfied in terms of resultant forces and moments. To illustrate the further improvements of this technique over previous work, where only resultant forces are used, a center-crack tension specimen with a large crack ( $2a/W = 0.8$ ) was analyzed with the BFM using resultant forces only and using both resultant forces and moments. In Figure 5, the relative error is plotted against the number of degrees of freedom for either the "force" method or the "force and moment" method. The relative error is defined as:

$$\text{Relative Error} = \frac{K_{\text{computed}} - K_{\text{ref.}}}{K_{\text{ref.}}}$$

where  $K_{\text{ref.}}$  is obtain from [10]. Figure 5 shows that the accuracy and the rate of convergence of the solution improved significantly when resultant forces and moments are used instead of resultant forces alone.

The results in Figure 5 were obtained with equal size elements on the boundaries. As shown in Figure 6, a further improvement in convergence was obtained with a graduated element distribution on the boundaries. This graduated element distribution was generated using the radial-line method described in Appendix B. A graduated boundary mesh was used in all the problems analyzed in the following sections.

#### Comparison with Existing Solution

Because solutions for displacement boundary value problems with cracks are not available for comparison, a mixed boundary value problem was used to demonstrate the applicability of the BFM to problems with prescribed displacements. Since the solution for a end clamped tension specimen with a central crack is available in the literature [11], this configuration was used to verify the BFM for a mixed boundary value problem. Next, to further validate the BFM for stress boundary problem, the method is used to analyze a three-hole cracked specimen subjected to uniaxial tension and compared to a solution obtained using the finite element method [12].

End clamped tension specimen. - The results for a end clamped tension specimen with a center crack are shown in Figure 7. Because of symmetry, only the upper right quadrant was modeled. The crack tip was chosen as the origin of the radial lines. Zero tractions are prescribed on the vertical boundary ( $x = W/2$ ) and a constant normal displacement  $v_0$  is prescribed on the horizontal

boundary ( $y = H/2$ ), as shown in Figure 7. The height-to-width ratios ( $H/W$ ) considered were 0.5, 1.0, and 1.5. The stress-intensity correction factors obtained for several crack-length-to-width ( $2a/W$ ) ratios are presented in Figure 7 and Table I, together with the results obtained by Isida [11] using the collocation technique. The agreement between the present results and the results obtained by Isida [11] is within 1 percent.

Three hole cracked specimen. - Stringers are widely used in aircraft structures as stiffening members. To simulate the effect of a stringer on a propagating crack, the three hole crack specimen shown in Figure 8 was developed [12].

For cracks with  $a/W < 0.1$ , the crack tip was chosen as the origin of the radial line for all boundaries. However, for cracks with  $a/W > 0.1$ , the crack tip was chosen as the origin of the radial line for all boundaries except the boundary of the circular hole from which the crack emanates. For this boundary, the center of the hole was chosen as the origin of the radial lines to ensure that a sufficient number of elements were used to model this boundary.

The stress-intensity correction factors obtained for several crack lengths are presented in Figure 8 and Table II, together with the results obtained by Newman [12] using the finite element method. The agreement between the present results and those obtained by finite element method [12] are within 1 percent for intermediate crack lengths ( $a/W = 0.125$  to  $0.300$ ). However, for short or long crack lengths, the present results are 2 to 3 percent higher than those obtained by Newman [12]. Based on convergence studies, Newman estimated that for short and long crack lengths his results were 2 to 3 percent lower than the "correct" solution. Thus, the BFM yielded a more accurate solution than those used in [12].

## Solutions for Cracks Emanating from Notches

In this section, two practical engineering configurations for which stress-intensity factor solutions are needed, and are not available in the literature, are analyzed. The configurations considered are an edge crack emanating from a semi-elliptical notch and an edge crack emanating from a V-notch, both subjected to uniaxial tension. For each configuration, several crack-length-to-notch-radius ratios were considered.

Edge crack emanating from semi-elliptical notch. - The stress-intensity correction factors obtained for three values of notch-depth-to-width ratio ( $r/W = 0.25, 0.125$  and  $0.0625$ ) and for various crack lengths are presented in Figure 9 and Table III. The aspect ratio for all values of  $r/W$  is  $b/r = 0.25$ . The solution for a single edge crack without a notch is also shown in Figure 9. Even for a relatively short crack, the results showed that the stress-intensity correction factors approached that of a single edge crack without the notch. In contrast, Figure 10 shows the stress-intensity correction factors for an edge crack emanating from a semi-circular notch [1] (aspect ratio  $b/r = 1.0$ ). In this case, the stress-intensity correction factors approached that of a single edge crack more gradually, that is, for a longer crack. Thus, the solution for a single edge crack can be used even for relatively short cracks emanating from a semi-elliptical notch having a small aspect ratio, whereas the solution for single edge crack can only be used for relatively longer cracks emanating from a semi-circular notch.

Edge crack emanating from a V-notch. - The stress-intensity correction factors obtained for three values of the notch-depth-to-width ratio ( $r/W = 0.25, 0.125,$  and  $0.0625$ ) are presented in Table IV. As for the semi-elliptical notch,



the aspect ratio considered here is 0.25. The results showed trends similar to the previous case. Again, even for relatively short cracks, the stress-intensity correction factors approached that of a single edge crack without a notch.

#### CONCLUDING REMARKS

The Boundary Force Method was formulated for the three fundamental problems of elasticity: the stress boundary value problem, the displacement boundary value problem, and the mixed boundary value problem.

The accuracy of the BFM was established by comparisons to crack configurations for which exact or accurate numerical stress-intensity factor solutions are available in literature. These crack configurations included mixed boundary value problems and stress boundary value problems. The method yielded stress-intensity correction factors which were in good agreement with those in the literature for the end clamped tension specimen with a center crack and the three-hole cracked tension specimen.

Two complex crack configurations for which no solutions are available were also analyzed: an edge crack emanating from a semi-elliptical notch and an edge crack emanating from a V-notch. For each configuration, several crack length-to-notch-radius ratios were analyzed and stress-intensity correction factors were presented. Because only the boundaries are modeled for each configuration, the stress-intensity factors were obtained for several crack lengths with very little increase in modeling effort.

The BFM is a versatile technique that can be used to obtain very accurate stress-intensity factors for complex crack configurations subjected to stress, displacement, or mixed boundary conditions. The method requires a minimal amount of modeling effort and, therefore, stress-intensity factor analyses for a range of crack lengths can be performed with ease.

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## APPENDIX A - FUNDAMENTAL SOLUTION

The BFM formulation uses the elasticity solution for concentrated forces and a moment in an infinite plate with a crack. Such a solution was formulated by Erdogan [8] for linear, isotropic and homogeneous materials. For completeness, the solution is presented below.

### Stress Functions

Consider an infinite plate with a crack subjected to concentrated forces  $Q$  and  $P$  and a moment  $M$  at an arbitrary point  $z_0 = x_0 + iy_0$  as shown in Figure A.1. The complex variable stress functions [8] are

$$\begin{aligned} \phi^*(z) &= -\frac{T}{z - z_0} + \phi_0(z) \\ \Omega^*(z) &= \frac{\kappa T}{z - \bar{z}_0} + \frac{\bar{T}(\bar{z}_0 - z_0) + im}{(z - \bar{z}_0)^2} + \phi_0(z) \\ \phi_0(z) &= \frac{1}{2\pi\sqrt{z^2 - a^2}} \left\{ \frac{T}{z - z_0} [I(z) - I(z_0)] \right. \\ &\quad - \frac{\kappa T}{z - \bar{z}_0} [I(z) - I(\bar{z}_0)] \\ &\quad \left. - [\bar{T}(\bar{z}_0 - z_0) + im] \left[ \frac{I(z) - I(\bar{z}_0)}{(z - \bar{z}_0)^2} - \frac{J(\bar{z}_0)}{z - \bar{z}_0} \right] \right\} \end{aligned} \quad (A.1)$$

where

$$m = \frac{M}{2\pi}$$

$$I(z) = \pi[\sqrt{z^2 - a^2} - z]$$

$$I(z_0) = \pi[\sqrt{z_0^2 - a^2} - z_0]$$

$$J(\bar{z}_0) = \pi\left[\frac{\bar{z}_0}{\sqrt{\bar{z}_0^2 - a^2}} - 1\right]$$

$$T = \frac{Q + iP}{2\pi(1 + \kappa)}$$

### Stresses

The stresses at any point  $z = x + iy$  are obtained from the stress functions as

$$\begin{aligned}\sigma_x + \sigma_y &= 2[\phi^*(z) + \phi^*(z)] \\ \sigma_y - \sigma_x + i2\sigma_{xy} &= 2[(\bar{z} - z)\phi^{*'}(z) - \phi^*(z) + \bar{\Omega}^*(z)]\end{aligned}\quad (A.2)$$

The barred quantities are the complex conjugates and the primed quantities represent the derivatives with respect to  $z$ .

### Displacements, Forces and Moments

The displacements  $u$  and  $v$  at any point  $z = x + iy$  are obtained from the stress functions as

$$2\mu(u + iv) = \kappa \int_0^z \phi^*(z) dz - \int_0^{\bar{z}} \bar{\Omega}^*(\bar{z}) d\bar{z} - (z - \bar{z})\overline{\phi^*(z)}\quad (A.3)$$

where  $\mu$  is the shear modulus. The resultant forces  $F_x$ ,  $F_y$  and the resultant moment  $M_0$  across the arc  $z_1$  to  $z_2$  (see Figure A.1) due to the concentrated forces  $P$  and  $Q$  and moment  $M$  can be obtained either by integrating the stresses

in equation (A.2) from  $z_1$  to  $z_2$ , or by using the stress functions in the following equations:

$$F_x + iF_y = -i \left[ \int \phi^*(z) dz + \int \bar{\Omega}^*(\bar{z}) d\bar{z} + (z - \bar{z}) \overline{\phi^*(z)} \right] \Big|_{z_1}^{z_2} \quad (\text{A.4})$$

$$M_0 = \text{Re} \left[ \int \int \{\phi(z) + \bar{\Omega}(z)\} dz d\bar{z} - z \int \{\phi(z) + \bar{\Omega}(z)\} dz + z(z - \bar{z})\phi(z) \right] \Big|_{z_1}^{z_2} \quad (\text{A.5})$$

### Stress-Intensity Factor

The stress-intensity factors for concentrated forces  $P$  and  $Q$  and moment  $M$  applied at an arbitrary point  $z_0 = x_0 + iy_0$  in an infinite plate with a crack are

$$\begin{aligned} K &= K_I + iK_{II} = 2\sqrt{2\pi} \lim_{z \rightarrow a} [\sqrt{(z - a)} \phi^*(z)] \\ &= \frac{1}{2\sqrt{\pi a}} \frac{1}{(1 + \kappa)} \left\{ (Q + iP) \left[ \left( \frac{a + z_0}{\sqrt{z^2 - a^2}} - 1 \right) - \kappa \left( \frac{a + \bar{z}_0}{z^2 - a^2} - 1 \right) \right] \right. \\ &\quad \left. + \frac{a[(Q - iP)(\bar{z}_0 - z_0) + i(1 + \kappa)M]}{(\bar{z}_0 - a)\sqrt{\bar{z}_0^2 - a^2}} \right\} \end{aligned} \quad (\text{A.6})$$

## APPENDIX B - RADIAL LINE METHOD

This appendix describes a systematic procedure, called the radial-line method, for modeling the boundaries of a crack configuration using a finite number of elements. In the BFM, the accuracy of the stress-intensity factors depends on how well the boundary conditions are approximated. Because the boundaries near the crack tip are subjected to higher stress gradients than the boundaries far from the crack tip, smaller elements are needed to accurately model regions near the crack tip. Therefore, a method that will automatically generate smaller line elements on the boundaries near the crack tip and larger line elements on the boundaries away from the crack tip was developed. With the radial-line method of modeling, the number of degrees of freedom needed is significantly reduced without sacrificing accuracy. This procedure can be best described using the following example.

Consider a center-crack tension specimen with a crack of length  $2a$ , as shown in Figure B.1. Because of symmetry, only one quarter of the plate needs to be modeled. The boundary of this quadrant can be divided into two sections: Boundary 1 - the vertical line A to B and Boundary 2 - the horizontal line B to C. Because the stress-intensity factor is the quantity of interest here, let the crack tip at  $x = a$  be the point from which all radial lines originate. To determine the element sizes on Boundary 1, first the distance between the origin of the radial lines (the crack tip) and the starting point  $z_1$  (point A) is computed. Label this distance  $r_1$ . The size of the first element ( $z_1$  to  $z_2$ ) is chosen to be a fraction of the distance  $r_1$ . Thus, the distance from  $z_1$  to  $z_2$  is set equal to  $r_1/DF$  where DF is referred to as the dividing factor. Because the factor DF is assumed to be known, the distance to point  $z_2$  can be determined. Next,  $r_2$ , the distance from the origin of the radial lines (the

crack tip) to  $z_2$ , is computed. The size of the second element,  $z_2$  to  $z_3$ , is equal to  $r_2/DF$ . The dividing factor  $DF$  is assumed to be identical throughout the entire modeling. This procedure has to be modified near the end points, such as point B in Figure B.1. If the end point of the last element exceeds the end point of the boundary (point B), the end point of the boundary is assigned as the end point of the last element on that boundary. The same procedure is repeated for Boundary 2 where the starting point of Boundary 2 is the end point of Boundary 1.

In the radial-line method, the size of the elements or the distribution density of the elements is determined by the choice of the origin of the radial line and the value of the dividing factor  $DF$ . Both can be chosen arbitrarily. In all the problems investigated in this paper, the origin of the radial-line was chosen to be the crack tip since the stress-intensity factor is the quantity of interest. A  $DF$  value of 10 was used in all problems.

The most significant advantage of the radial-line method can be demonstrated in the case of a very small crack emanating from a semi-circular notch (see Figure 10). Because of the high stress gradient near the crack tip, small elements are needed on portions of the semi-circular boundary nearest the crack tip. If equal size elements are used, a large number of elements are needed on this curved boundary. However, the portion of the curved boundary that is away from the crack tip does not have high stress gradients and so small size elements on that portion of the boundary are unnecessary. The radial-line method generates small elements near the crack tip and larger elements away from the crack tip. This type of modeling will significantly reduce the number of degrees of freedom necessary to model the curve boundary without sacrificing accuracy.

Table I - Stress-intensity correction factors for end-clamped tension specimen with a center crack.

H/W	2a/W	F	
		Present	Collocation[11].
0.5	0.1	1.0247	1.0300
	0.2	0.8915	0.8970
	0.3	0.7604	0.7600
	0.4	0.6595	0.6570
	0.5	0.5863	0.5900
	0.6	0.5319	0.5370
1.0	0.1	1.0514	1.0500
	0.2	1.0074	1.0100
	0.3	0.9453	0.9500
	0.4	0.8763	0.8800
	0.5	0.8089	0.8120
	0.6	0.7484	0.7520
	0.7	0.6975	0.7020
	0.8	0.6601	0.6650
1.5	0.1	1.0326	1.0370
	0.2	1.0146	1.0200
	0.3	0.9872	0.9900
	0.4	0.9534	0.9550
	0.5	0.9167	0.9200
	0.6	0.8810	0.8830



Table II - Stress-intensity correction factors for a three hole crack specimen.

a/W	F	
	Present Results	FEM [12]
.025	2.2655	2.2221
.050	1.7125	1.6930
.075	1.4829	1.4663
.100	1.3510	1.3372
.125	1.2504	1.2406
.150	1.1558	1.1487
.175	1.0506	1.0483
.200	0.9341	0.9347
.225	0.8101	0.8122
.250	0.6918	0.6939
.275	0.5999	0.5992
.300	0.5508	0.5455
.325	0.5491	0.5403
.350	0.5892	0.5773
.375	0.6551	0.6417
.400	0.7323	0.7183
.425	0.8119	0.7972
.450	0.8901	0.8747
.475	0.9682	0.9523
.500	1.0524	1.0348
.525	1.1507	1.1305
.550	1.2774	1.2557
.575	1.4609	1.4374
.600	1.7723	1.7296

Table III - Stress-intensity correction factors for an edge crack emanating from a semi-elliptical notch.

a/r	F(r/W = .25)	F(r/w = .125)	F(r/W = .0625)
1.04	-----	1.1741	1.0995
1.05	1.4861	-----	-----
1.06	-----	1.2112	1.1301
1.08	-----	1.2291	1.1434
1.10	1.5674	1.2397	1.1504
1.15	-----	1.2558	1.1583
1.20	1.6614	1.2676	1.1626
1.30	-----	1.2905	1.1688
1.50	1.9794	1.3407	1.1825
1.75	2.3375	1.4137	1.2033
2.00	2.8180	1.4985	1.2248
3.00	-----	1.9787	1.3403
4.00	-----	2.8182	1.4985
6.00	-----	-----	1.9791
8.00	-----	-----	2.8180

Table IV - Stress-intensity correction factors for an edge crack emanating from a V-notch.

a/r	F(r/W = .25)	F(r/W = .125)	F(r/W = .0625)
1.04	-----	1.1932	1.1160
1.05	1.4950	-----	-----
1.06	-----	1.2106	1.1300
1.08	-----	1.2228	1.1387
1.10	1.5575	1.2322	1.1450
1.15	-----	1.2501	1.1548
1.20	1.6565	1.2644	1.1609
1.30	-----	1.2898	1.1693
1.50	1.9797	1.3410	1.1834
1.75	2.3383	1.4137	1.2027
2.00	2.8200	1.4982	1.2248
3.00	-----	1.9792	1.3400
4.00	-----	2.8199	1.4981
6.00	-----	-----	1.9782
8.00	-----	-----	2.8204

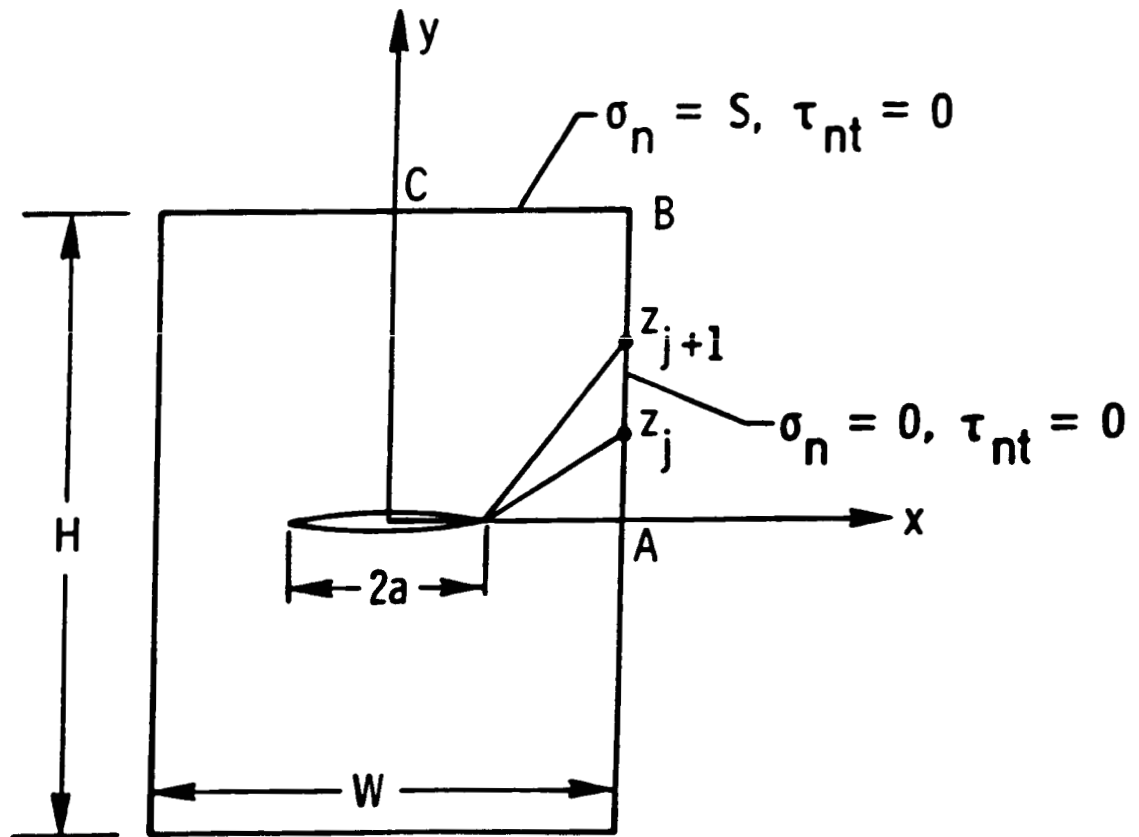


Figure 1 - Stress boundary value problem.

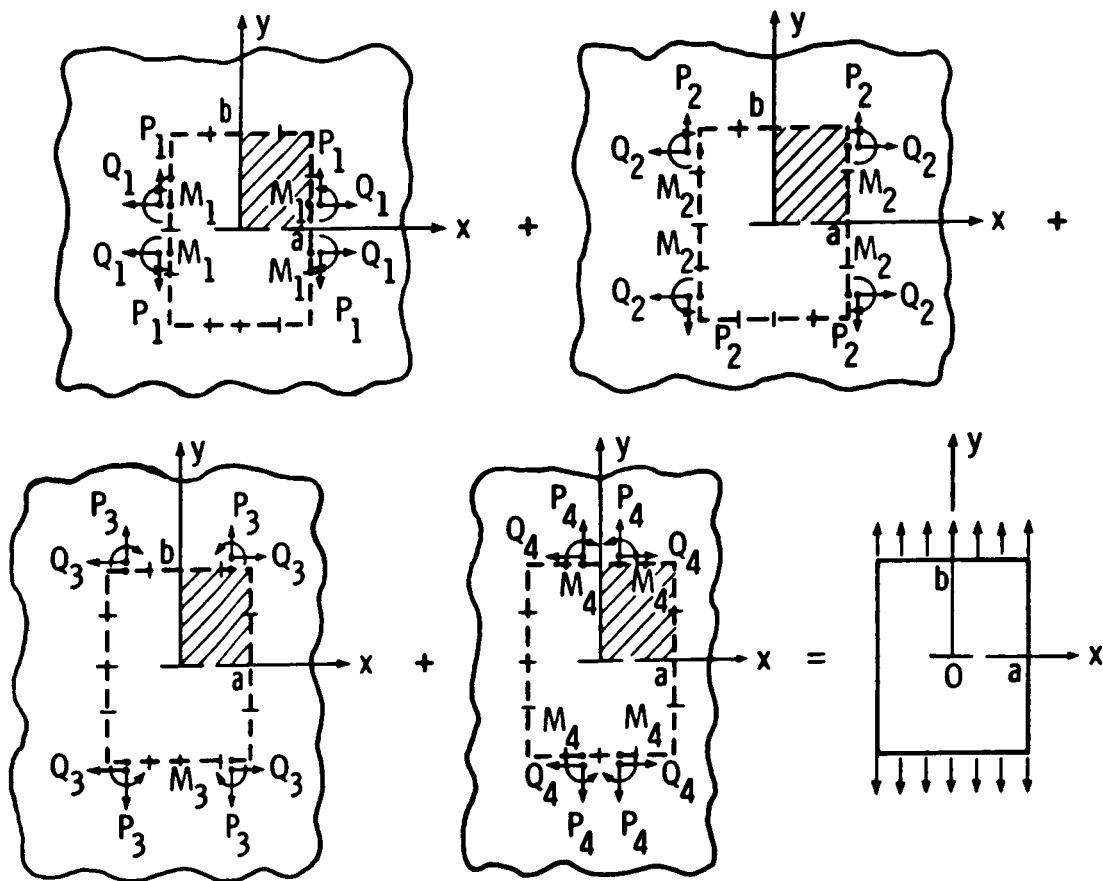


Figure 2 - Superposition of unknown forces  $P_i$ ,  $Q_i$  and moment  $M_i$  on elements  $i$ ,  $i = 1$  to 4.

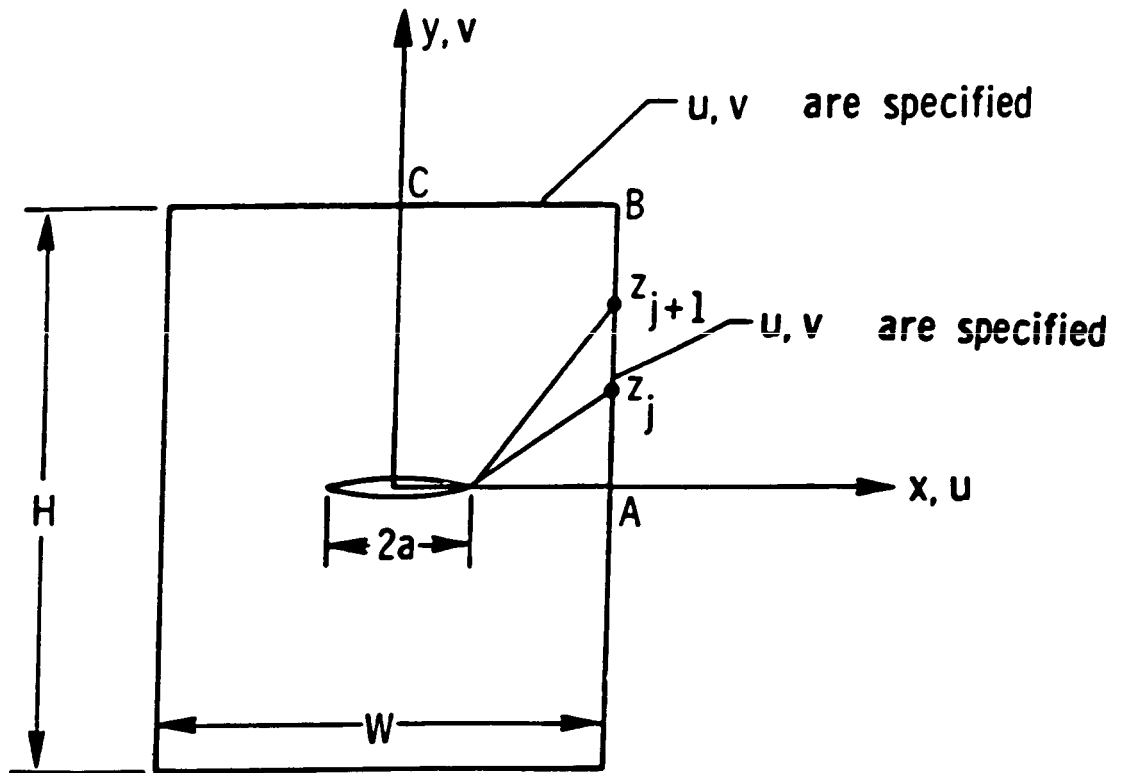


Figure 3 - Displacement boundary value problem.

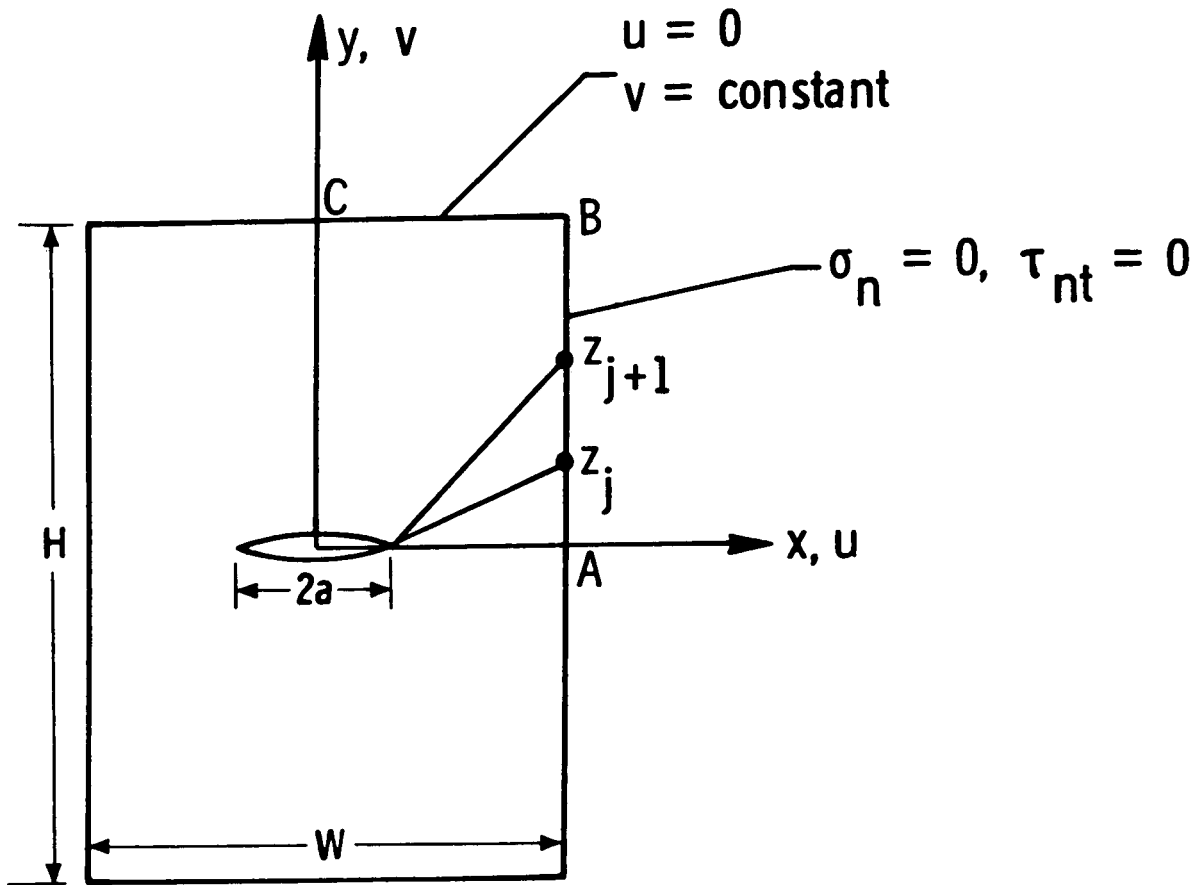


Figure 4 - Mixed boundary value problem.

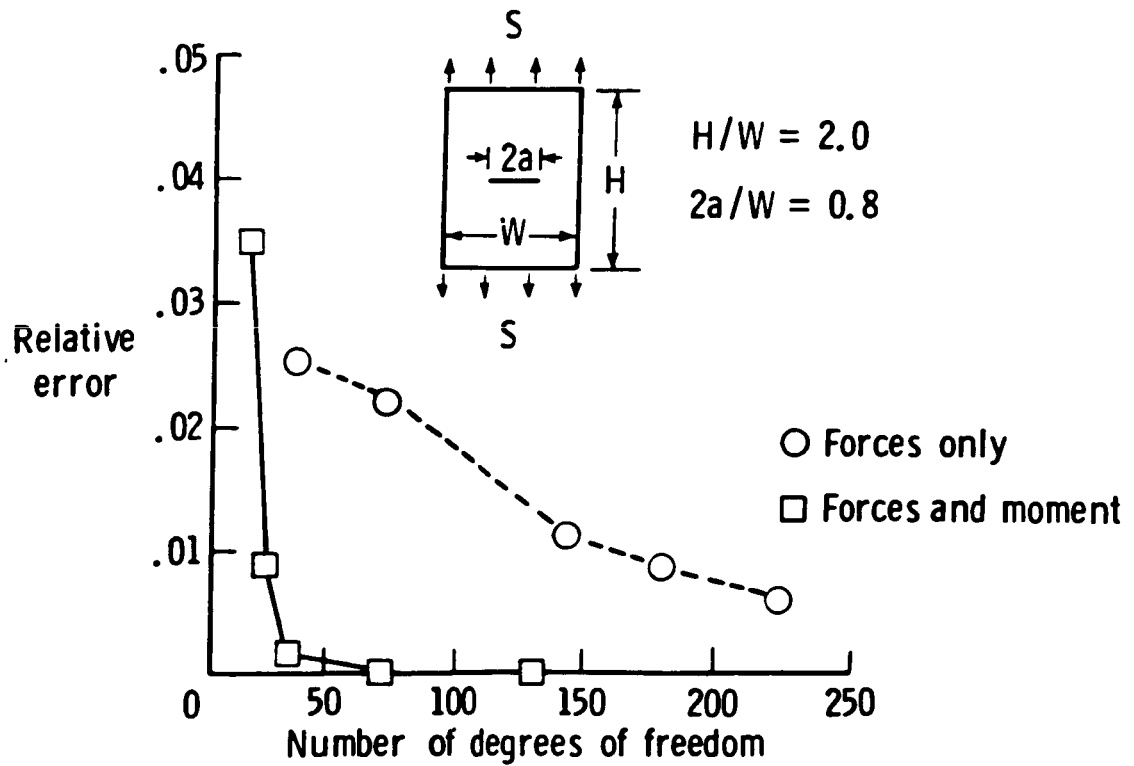


Figure 5 - Convergence of relative error using either "force" method or "force and moment" method.



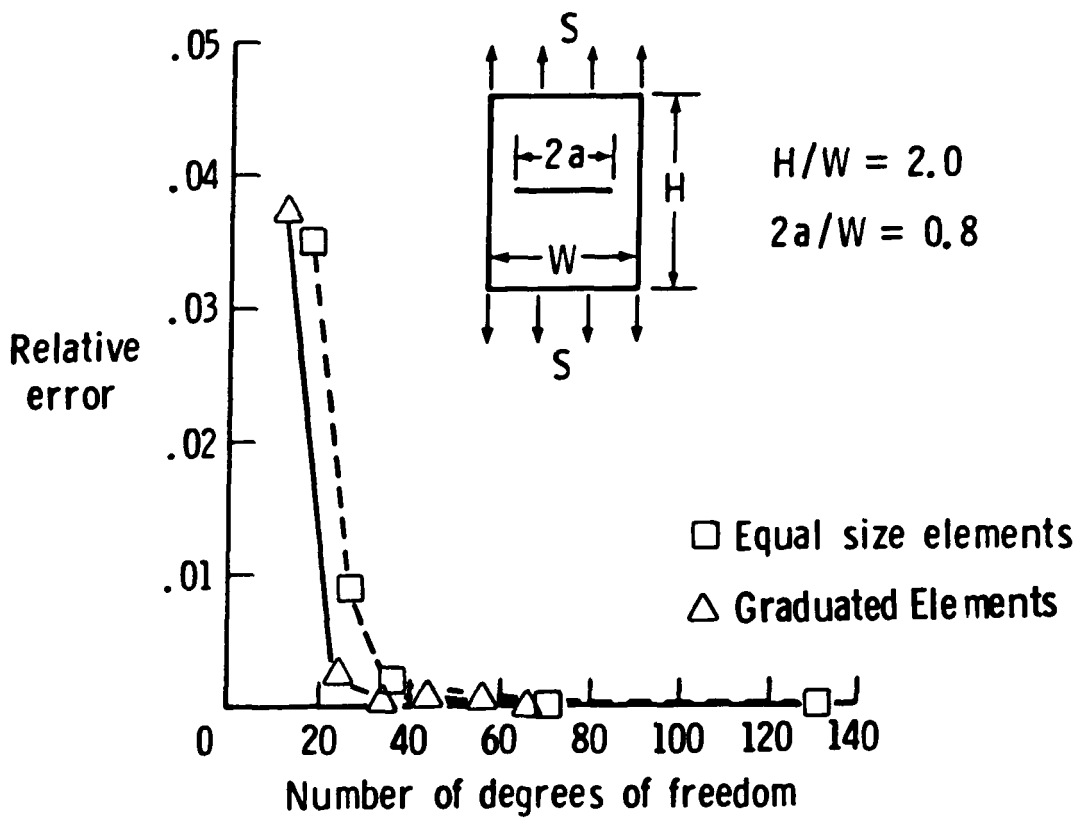


Figure 6 - Comparison of convergence rate for equal size elements and graduated elements discretization.

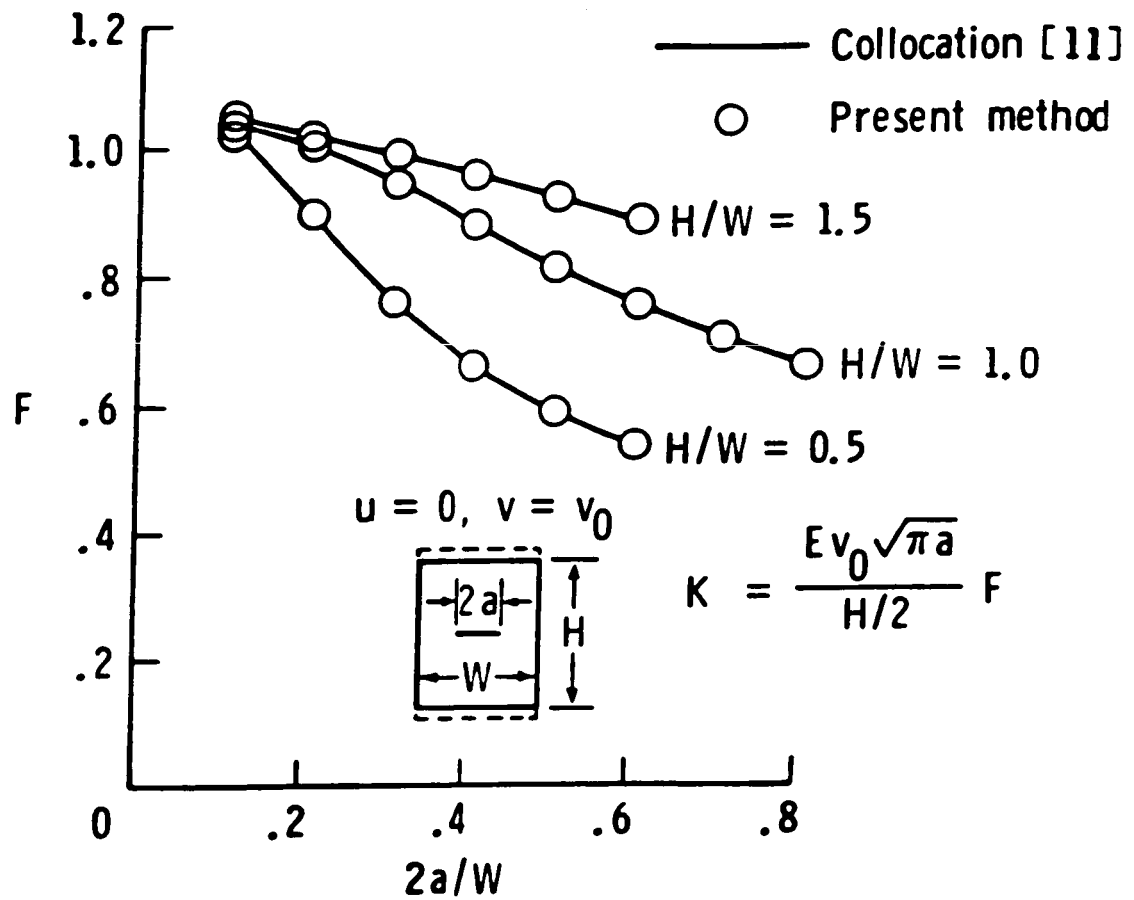


Figure 7 - Stress-intensity correction factors for a end clamped tension specimen with a center crack.

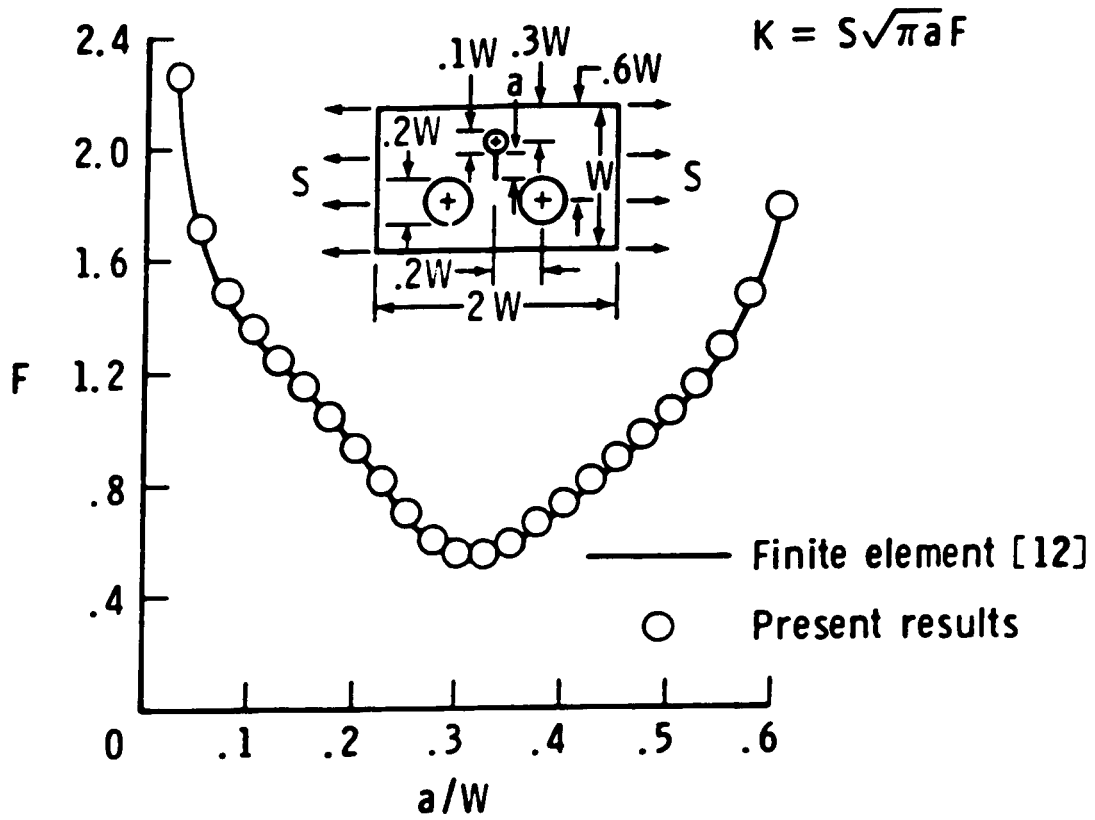


Figure 8 - Stress-intensity correction factors for a three hole crack specimen.

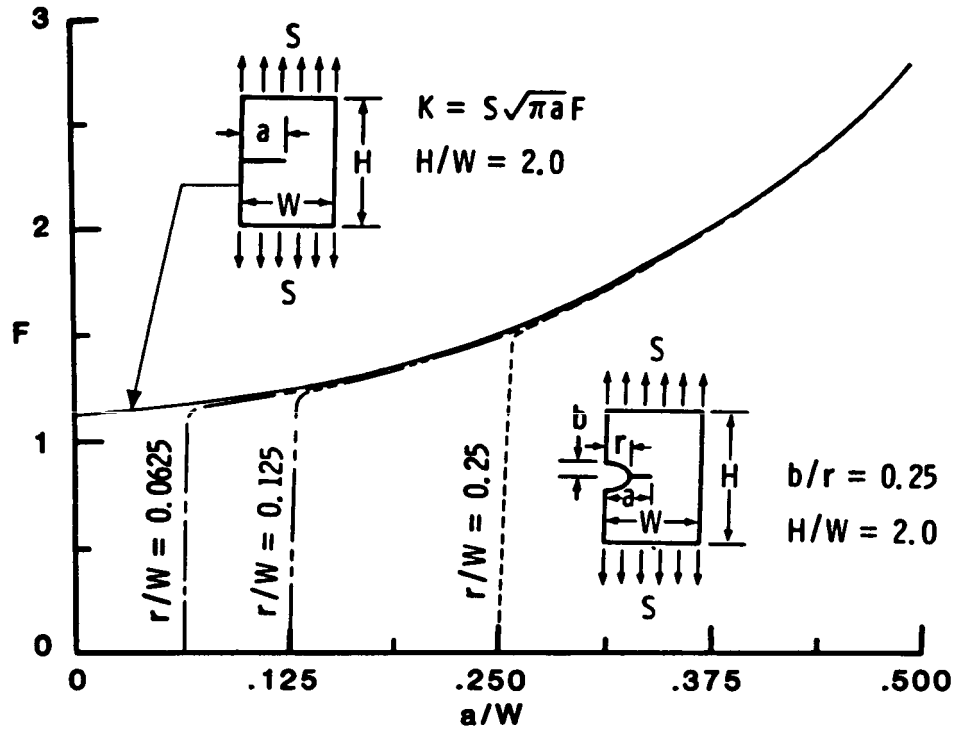


Figure 9 - Stress-intensity correction factors for a single edge-crack specimen and an edge crack emanating from a semi-elliptical notch.

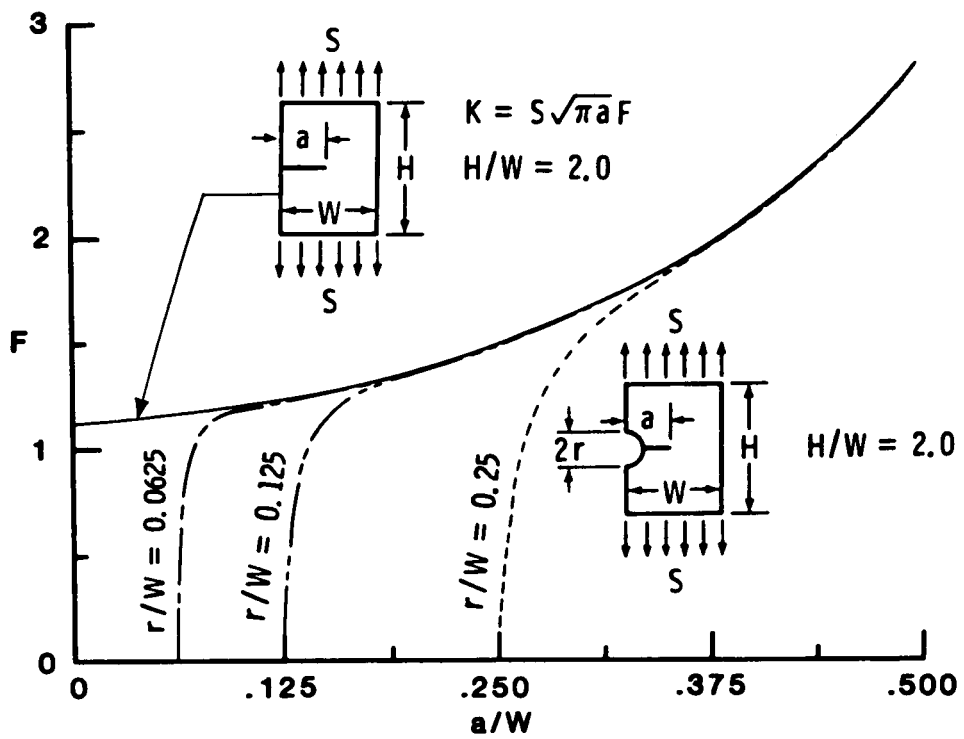


Figure 10 - Stress-intensity correction factors for a single edge-crack specimen and an edge crack emanating from a semi-circular notch.

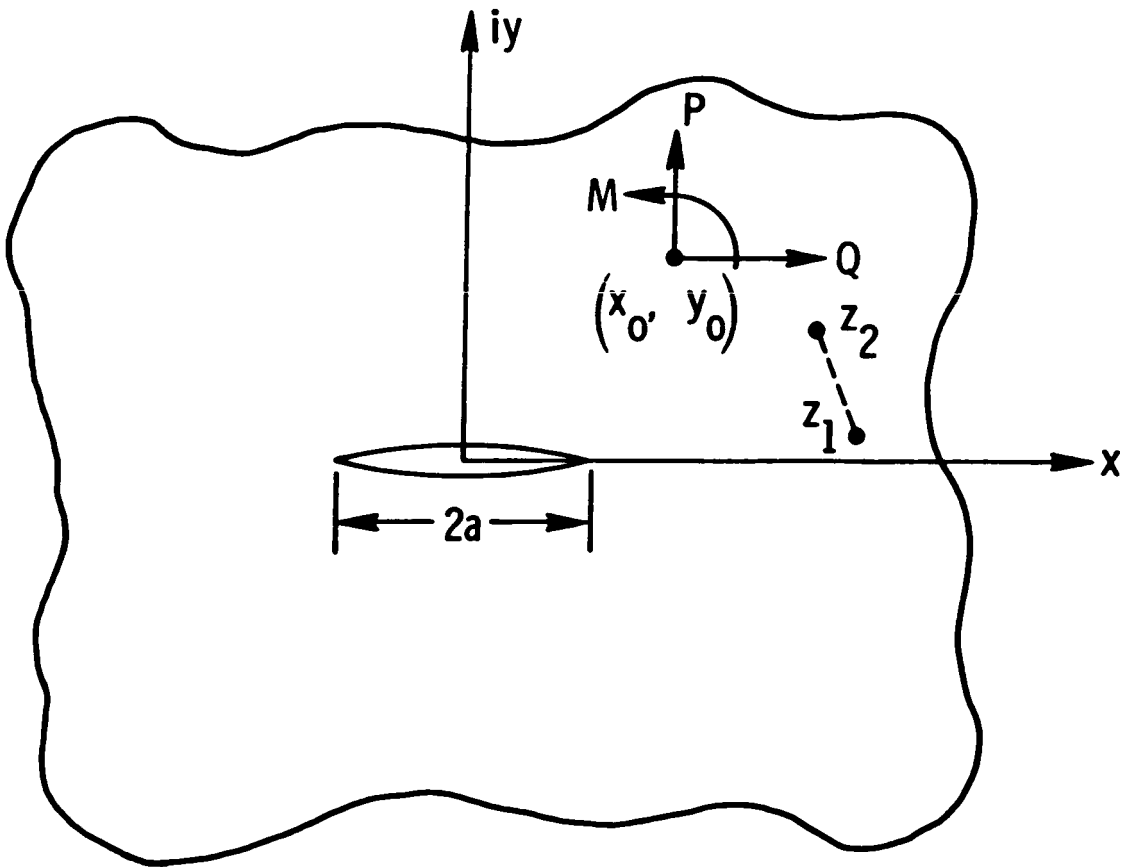


Figure A.1 - Concentrated loads and moment in a infinite plate with a crack.

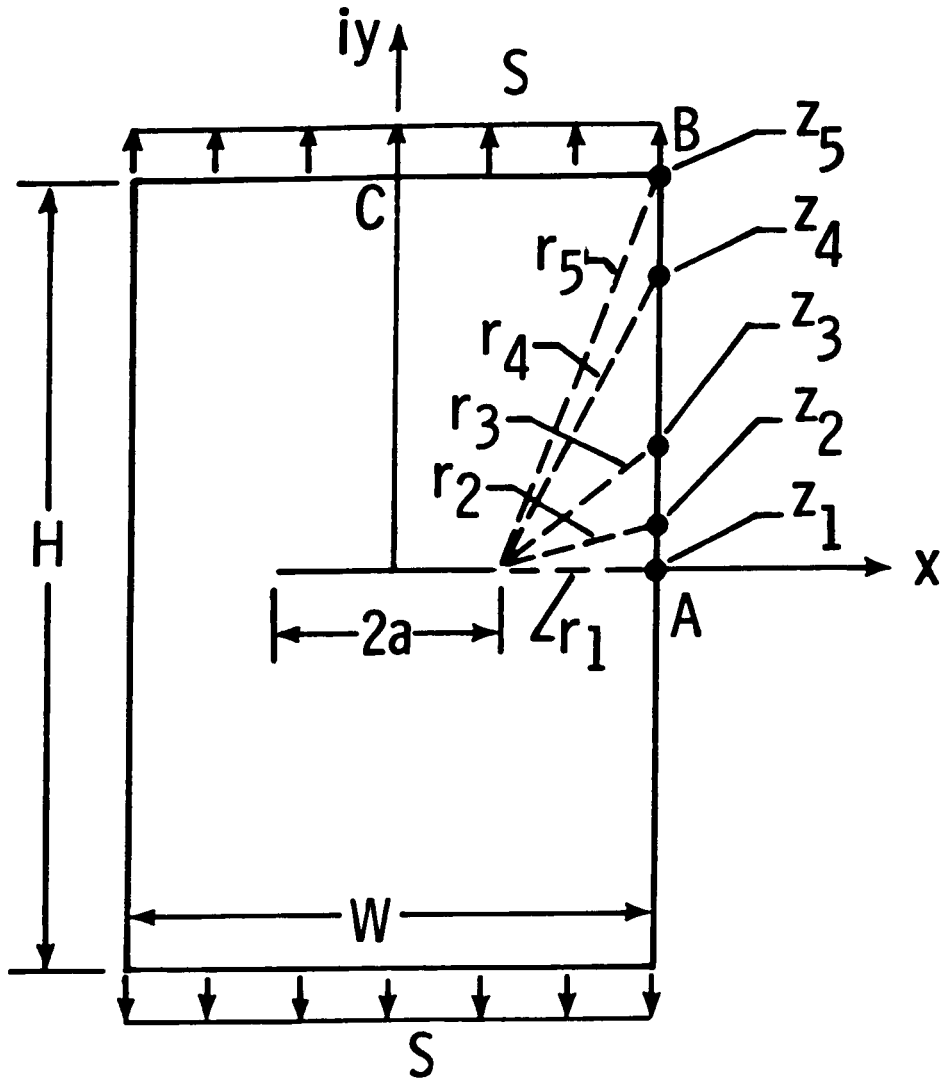


Figure B.1 - Radial-line method for generating mesh points.

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16. Abstract The Boundary Force Method (BFM) was formulated for the three fundamental problems of elasticity: the stress boundary value problem, the displacement boundary value problem, and the mixed boundary value problem. Because the BFM is a form of an indirect boundary element method, only the boundaries of the region of interest are modeled. The elasticity solution for the stress distribution due to concentrated forces and a moment applied at an arbitrary point in a cracked infinite plate is used as the fundamental solution. Thus, unlike other boundary element methods, here the crack face need not be modeled as part of the boundary.  The formulation of the BFM is described and the accuracy of the method is established by analyzing a center-cracked specimen subjected to mixed boundary conditions and a three-hole cracked configuration subjected to traction boundary conditions. The results obtained are in good agreement with accepted numerical solutions. The method is then used to generate stress-intensity solutions for two common cracked configurations: an edge crack emanating from a semi-elliptical notch, and an edge crack emanating from a V-notch.  The BFM is a versatile technique that can be used to obtain very accurate stress-intensity factors for complex crack configurations subjected to stress, displacement, or mixed boundary conditions. The method requires a minimal amount of modeling effort.					
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