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FOR CONTROL SYSTEMS WITH DELAYS

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**ON NON-CONVERGENCE OF ADJOINT SEMIGROUPS  
FOR CONTROL SYSTEMS WITH DELAYS<sup>†</sup>**

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**ABSTRACT**

It is shown that the adjoints of a spline based approximation scheme for delay equations do not converge strongly.

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## 1. INTRODUCTION

The primary goal of this paper is to illustrate, by a simple problem, the necessity of conducting a careful analysis of numerical schemes (that were developed for open-loop simulation) when these schemes are to be applied to an optimization based control design problem. It is our feeling that many distributed parameter control systems (viscoelastic structures, fluid flow control, etc.) are such that "standard finite element/finite difference" schemes might lead to numerical difficulties in certain control design methods, if care is not taken to ensure that these approximation schemes have convergence properties essential for the intended application.

We shall concentrate on a non-convergence result for an optimal control problem governed by a delay differential equation. Although we feel that the technical details required to prove non-convergence are interesting, we hope that this proof is not viewed as the major contribution of the paper. Indeed, we hope that the reader is motivated to think about similar problems for more complex distributed parameter systems.

During the past fifteen years considerable attention has been devoted to the construction of finite dimensional approximations of distributed parameter systems. Much of this work is based on algorithms first developed primarily for simulation. However, it is not clear that finite dimensional models developed for (open-loop) simulations will also be suitable for certain optimization based control design techniques. Moreover, there may be several reasons that a numerical scheme developed specifically for simulation does not perform well when applied to a control design problem.

In this note, we concentrate on the use of the so-called spline scheme developed by Banks and Kappel [2] as an approximation method for calculating

optimal feedback gains for control systems governed by functional differential equations. This problem is simple enough to be addressed in a short paper and yet it still illustrates how a specific numerical scheme when combined with a particular control design approach can lead to numerical difficulties.

In order to motivate the paper and to describe the main technical contribution of the paper, we first review some known results.

Let  $H$  and  $U$  be real Hilbert spaces and  $S(t): H \rightarrow H$  denote a  $C_0$ -semigroup of bounded linear operators with generator  $A$ . We assume that  $B: U \rightarrow H$ ,  $Q: H \rightarrow H$  and  $R: U \rightarrow U$  are bounded linear operators with  $Q$  and  $R$  self-adjoint, non-negative and  $R$  satisfies  $R \geq mI > 0$ . The (LQ) optimal control problem is to minimize

$$(1.1) \quad J(u) = \int_0^T [\langle Qz(s), z(s) \rangle + \langle Ru(s), u(s) \rangle] ds$$

where  $z(t)$  is defined by

$$(1.2) \quad z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)ds$$

for  $0 \leq t < T$  and given  $z_0 \in H$ . If  $T = +\infty$ , then one has the linear quadratic regulator problem.

Assume now that there exists a sequence of  $C_0$ -semigroups  $S^N(t)$  on  $H$  and positive constants  $M, \beta$  such that

$$(1.3) \quad \|S^N(t)\| \leq Me^{\beta t}, \quad t \geq 0, \quad N \geq 1$$

and

$$(1.4) \quad S^N(t) \rightarrow S(t) \quad \text{strongly as } N \rightarrow +\infty$$

and the convergence in (1.4) is uniform in  $t$  on bounded intervals. Denote by  $A^N$  the generator of  $S^N(t)$  and also assume the existence of bounded linear operators  $B^N: U \rightarrow H$ ,  $Q^N: H \rightarrow H$  with  $Q^N$  self-adjoint and  $Q^N \geq 0$  satisfying

$$(1.5) \quad \|B^N - B\| \rightarrow 0, \|Q^N - Q\| \rightarrow 0.$$

Note the assumption of uniform convergence (1.5) is stronger than required by Gibson (see page 113 in [4]). It is well known that the optimal control (if it exists) is given by state feedback

$$(1.6) \quad u^*(t) = -R^{-1}B^* \Pi(t)z^*(t)$$

where the bounded self-adjoint operator  $\Pi(\cdot)$  satisfies the Riccati (integral) equation

$$(1.7) \quad \Pi(t)z = \int_t^T S^*(s-t)[Q - \Pi(s)BR^{-1}B^* \Pi(s)]S(s-t)ds.$$

Let  $\Pi^N(t)$  be the solution to the "approximating" Riccati equation

$$(1.8) \quad \Pi^N(t)z = \int_t^T S^{N*}(s-t)[Q^N - \Pi^N(s)B^N R^{-1} B^{N*} \Pi^N(s)]S^N(s-t)ds$$

and observe that (1.8) would determine the sub-optimal gains if one used the approximating system  $(A^N, B^N)$  with weights  $Q^N$ .

The following theorem is a direct consequent of Gibson's (more general) results (Theorem 6.1 and Theorem 6.2 in [4]).

**Theorem 1.1:** If conditions (1.1) - (1.5) hold, then for  $0 < t < T$ ,  $\Pi^N(t)$  converges weakly to  $\Pi(t)$  and the convergence is uniform on  $[0, T]$ . If in addition

$$(1.9) \quad S^{N^*}(t) \rightarrow S^*(t) \quad \text{strongly,}$$

then  $\Pi^N(t)$  converges strongly to  $\Pi(t)$  and the convergence is uniform for  $t \in [0, T]$ .

**Corollary 1.2:** Let  $K(t) = R^{-1}B^* \Pi(t)$  and  $K^N(t) = R^{-1}B^{N^*} \Pi^N(t)$  denote the feedback gain operators. Assume that  $U = \mathbb{R}^m$  (i.e., is finite dimensional). If  $\Pi^N(t)$  converges strongly to  $\Pi(t)$ , then as  $N \rightarrow +\infty$ ,

$$(1.10) \quad \|K^N(t) - K(t)\| \rightarrow 0.$$

The main point to be emphasized is that if the control space is finite dimensional, then uniform convergence is assured provided the numerical scheme is stable and consistent and (1.9) holds. If one is concerned only with simulation, then stability and consistency is sufficient for most numerical problems. Moreover, it can be shown that many standard numerical schemes developed for simulation of self-adjoint partial differential equations do satisfy (1.9). Therefore, it is not surprising that until Gibson "needed" (1.9) to establish the uniform convergence of optimal gain operators, the

question of whether a numerical scheme satisfied (1.9) received little attention. Indeed, even after Gibson published his result it was still not obvious that condition (1.9) was anything more than a technical assumption needed in Gibson's proof.

In [3] Banks, Ito, and Rosen applied a convergent spline based scheme to an optimal control problem governed by a delay differential equation. The numerical results in [3] seemed to show that  $K^N(t)$  did not converge uniformly to  $K(t)$  and these numerical results have often been used as evidence that (1.9) did not hold for this particular scheme. Moreover, several new schemes have since been generated specifically to ensure (1.3) - (1.4) and (1.9) are valid. Still, it was not known if condition (1.9) held for the spline scheme used in [3]. We shall provide a proof that (1.9) fails for this scheme. We also show that this spline scheme is stable and consistent to  $A^*$  on a dense subset of  $D(A^*)$ .

## 2. SPLINE APPROXIMATIONS OF HEREDITARY SYSTEMS

Consider the delay differential equation

$$(2.1) \quad \dot{x}(t) = A_0 x(t) + A_1 x(t - r) + \int_{-r}^0 A(s)x(t + s)ds$$

with initial data

$$(2.2) \quad x(0) = \eta; \quad x(s) = \phi(s), \quad -r \leq s < 0,$$

where  $x(t) \in \mathbb{R}^n$  and the elements of  $A(\cdot)$  are square integrable on

$[-r, 0]$ . It is well known that (e.g., [1]) for  $\eta \in \mathbb{R}^n$  and  $\phi \in L^2(-r, 0; \mathbb{R}^n)$  there exists a unique solution of (2.1) - (2.2)  $x: [-r, +\infty) \rightarrow \mathbb{R}^n$  such that  $x \in W^{1,2}(0, T; \mathbb{R}^n)$  for all  $T > 0$ . If one defines the solution map  $S(t)$ ,  $t \geq 0$  on the product space  $Z = \mathbb{R}^n \times L^2(-r, 0; \mathbb{R}^n)$  by

$$(2.3) \quad S(t)(\eta, \phi(\cdot)) = (x(t), x(t+\cdot))$$

where  $x$  is the solution to (2.1) - (2.2), then  $\{S(t)\}_{t \geq 0}$  is a strongly continuous semigroup (i.e.,  $C_0$ -semigroup) on  $Z$ . The infinitesimal generator  $A$  is the operator defined on the domain

$$(2.4) \quad D(A) = \{(\eta, \phi(\cdot)) \in Z \mid \phi(\cdot) \in W^{1,2}(-r, 0; \mathbb{R}^n), \phi(0) = \eta\}$$

by

$$(2.5) \quad A(\eta, \phi(\cdot)) = (A_0 \eta + A_1 \phi(-r) + \int_{-r}^0 A(s) \phi(s) ds, \dot{\phi}(\cdot)).$$

The adjoint operator  $A^*$  generates the adjoint semigroup  $S^*(t)$  and it is easy to show that (see [4,7])

$$(2.6) \quad D(A^*) = \{(\xi, \psi) \in Z \mid \psi \in W^{1,2}(-r, 0; \mathbb{R}^n), \psi(-r) = A_1^T \xi\}$$

and for  $(\xi, \psi) \in D(A^*)$

$$(2.7) \quad A^*(\xi, \psi) = (\psi(0) + A_0^T \xi, [A^T(\cdot) \xi - \dot{\psi}(\cdot)]).$$



As in [2], we define the linear spline based approximation for  $S(t)$ .

Let  $\{B_i^N\}_{i=0}^N$  denote the usual linear B-splines defined on the interval  $[-r, 0]$  by

$$(2.8) \quad B_i^N(s) = \begin{cases} \frac{N}{r} (s - \tau_{i+1}^N), & s \in [\tau_{i+1}^N, \tau_i^N] \\ \frac{N}{r} (\tau_{i-1}^N - s), & s \in [\tau_i^N, \tau_{i-1}^N] \\ 0, & \text{otherwise,} \end{cases}$$

where  $\tau_i^N = -ir/N$ ,  $i = 0, 1, \dots, N$ ,  $\tau_{N+1}^N = -r$  and  $\tau_{-1}^N = 0$ . For each  $N = 1, 2, \dots$  let  $Z^N$  denote the linear subspace of  $Z$  defined by

$$(2.9) \quad Z^N = \{z \in Z \mid z = \sum_{k=0}^N a_k (B_k^N(0), B_k^N(\cdot)), a_k \in \mathbb{R}^n\}$$

and let  $P^N$  denote the orthogonal projection of  $Z$  onto  $Z^N$ . This subspace can be identified with  $\mathbb{R}^{n(N+1)}$  by the prolongation  $i^N: \mathbb{R}^{n(N+1)} \rightarrow Z$  defined by

$$(2.10) \quad i^N a = (a_0, \sum_{k=0}^N a_k B_k^N(\cdot))$$

where  $a = (a_0^T, a_1^T, \dots, a_N^T)^T \in \mathbb{R}^{n(N+1)}$ . The space  $\mathbb{R}^{n(N+1)}$  is normed with the induced inner product

$$(2.11) \quad \langle a, b \rangle_N = a^T Q^N b,$$

where  $a, b \in \mathbb{R}^{n(N+1)}$  and  $Q^N$  is defined by

$$(2.12) \quad Q^N = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots \\ & & & 0 \end{bmatrix} + \frac{r}{N} \begin{bmatrix} \frac{1}{3}I & \frac{1}{6}I & \cdot & \cdot & \cdot & 0 \\ \frac{1}{6}I & \frac{2}{3}I & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \frac{2}{3}I & \cdot \\ & & & \cdot & \frac{1}{6}I & \frac{1}{3}I \\ & & & & & \frac{1}{3}I \end{bmatrix} \cdot$$

The adjoint operator  $[i^N]^*$ :  $Z \rightarrow \mathbb{R}^{n(N+1)}$  is given by

$$(2.13) \quad [i^N]^*(\eta, \phi(\cdot)) = [Q^N]^{-1} \begin{bmatrix} \eta + \phi_0^N \\ \phi_1^N \\ \vdots \\ \phi_N^N \end{bmatrix},$$

where  $\phi_i^N = \int_{-r}^0 \phi(s) B_i^N(s) ds$ . Moreover, it is easy to show that

$$(2.14) \quad \begin{cases} [i^N]^* i^N = I & \text{the identity on } \mathbb{R}^{n(N+1)} \\ i^N [i^N]^* = P^N \end{cases}$$

and for  $z, w \in Z^N$

$$(2.15) \quad \langle z, w \rangle_Z = \langle [i^N]^* z, [i^N]^* w \rangle_N.$$

In order to construct the standard Galerkin approximation of  $A$ , we note that  $Z^N \subseteq D(A)$  and define  $A^N$  by  $A^N = P^N A P^N$ . Observe that  $A^N$  (and hence  $[A^N]^*$ ) is continuous and although  $P^N Z \subseteq D(A)$ ,  $P^N$  does not map all of  $Z$  into  $D(A^*)$ . It is shown in [2] that

$$(2.16) \quad A^N = i^N [Q^N]^{-1} H^N [i^N]^*$$

where

$$(2.17) \quad H^N = \begin{bmatrix} A_0^N & & & & \\ & A_1^N & \cdots & & \\ & & & \cdots & \\ & & 0 & & \\ & & & & A_N^N \end{bmatrix} + \frac{1}{2} \begin{bmatrix} I & -I & \cdot & & 0 \\ I & \cdot & 0 & \cdot & \\ \cdot & \cdot & \cdot & 0 & \cdot \\ 0 & & \cdot & I & -I \end{bmatrix}$$

and

$$A_0^N = A_0 + \int_{-r}^0 A(s)B_0^N(s)ds$$

$$A_k^N = \int_{-r}^0 A(s)B_k^N(s)ds, \quad 1 \leq k \leq N-1$$

$$A_N^N = A_1 + \int_{-r}^0 A(s)B_N^N(s)ds.$$

If  $S^N(t)$  denotes the  $C_0$ -semigroup generated by  $A^N$ , then it is shown in [2] that for each  $z \in Z$ ,

$$(2.18) \quad \|S^N(t)z - S(t)z\| \rightarrow 0,$$

where the convergence is uniform in  $t$  on bounded intervals. The convergence (2.18) was established by proving that the Galerkin approximations  $A^N$  satisfy the Trotter-Kato Theorem [6] and hence provided a stable and consistent approximation scheme for  $A$ . We shall prove that the above convergence statement does not hold for the sequence of adjoint semigroups  $[S^N]^*(t)$ . Moreover, this convergence fails even though the adjoint operators  $[A^N]^*$  are stable and consistent to  $A^*$  on a dense subset of  $Z$  (i.e., there is a dense set  $C \subseteq D(A^*) \subseteq Z$  such that  $[A^N]^*z \rightarrow A^*z$  for all  $z \in C$ ).

### 3. CONVERGENCE OF THE ADJOINT GENERATORS

We shall follow the approach given in [5] and consider the  $n(N+1) \times n$  matrix

$$\gamma_+^N = [Q^N]^{-1}(0 \dots 0, I)^T.$$

Define the operators  $\delta_+^N: \mathbb{R}^1 \rightarrow Z^N$  by

$$\delta_{+x}^N = (\phi_x(0), \phi_x(\cdot))$$

where

$$\phi_x(\cdot) = [B_0^N(\cdot)I, B_1^N(\cdot)I, \dots, B_N^N(\cdot)I] \gamma_{+x}^N.$$

It follows that for all  $(\eta, \phi) = z \in Z$  and  $x \in \mathbb{R}^1$

$$\begin{aligned} [[i^N]^* \delta_{+x}^N]^T Q^N [i^N]^* P^N z &= [\gamma_{+x}^N]^T Q^N [i^N]^* z \\ &= x^T [B_0^N(-r)I, \dots, B_N^N(-r)I] [i^N]^* z, \end{aligned}$$

and (2.14) - (2.15) implies

$$(3.1) \quad \langle \delta_{+x}^N, z \rangle_Z = x^T [\phi^N(-r)],$$

where  $P^N z = P^N(\eta, \phi) = (\phi^N(0), \phi^N(\cdot)) \in Z^N$ . Furthermore, if  $\lambda_{\min}^N$  denotes the smallest eigenvalue of  $Q^N$ , then  $(r/6N) \leq \lambda_{\min}^N$  and it follows from (2.11) that

$$(3.2) \quad \|\delta_{+x}^N\| \leq (6N/r)^{1/2} \quad \text{for all } N.$$

We also need the following representation.

**Lemma 3.1:** The operators  $[A^N]^*$ :  $Z \rightarrow Z^N$  are given by

$$[A^N]^*(\xi, \psi) = P^N(\psi^N(0) + A_0^T \psi^N(0), [A^T(\cdot)\psi^N(0) - \dot{\psi}^N(\cdot)]) \\ + \delta_+^N(A_1^T \psi^N(0) - \psi^N(-r)),$$

where  $P^N(\xi, \psi) = (\psi^N(0), \psi^N(\cdot))$ .

Proof: Assume that  $z = (\eta, \phi)$  and  $w = (\xi, \psi)$  belong to  $Z$  and let  $P_z^N = (\phi^N(0), \phi^N(\cdot))$  and  $P_w^N = (\psi^N(0), \psi^N(\cdot))$  denote the orthogonal projections of  $z$  and  $w$ , respectively. The identity (3.1) implies that

$$\langle (\psi^N(0) + A_0^T \psi^N(0), [A^T(\cdot)\psi^N(0) - \dot{\psi}^N(\cdot)]), P_z^N \rangle_Z \\ + \langle \delta_+^N(A_1^T \psi^N(0) - \psi^N(-r)), z \rangle_Z \\ = [\psi^N(0)]^T [\phi^N(0)] + [\psi^N(0)]^T A_0 [\phi^N(0)] - \int_{-r}^0 \langle \dot{\psi}^N(s), \phi^N(s) \rangle ds \\ + [\psi^N(0)]^T \int_{-r}^0 A(s) \psi^N(s) ds + [\psi^N(0)]^T A_1 [\phi^N(-r)] \\ - [\psi^N(-r)]^T [\phi^N(-r)].$$

Integrating by parts, the boundary terms cancel with the first and last term in the sum. Therefore, it follows that if  $[A^N]^*$  is defined as above, then

$$\langle [A^N]^* w, z \rangle = \langle P^N w, A P^N z \rangle = \langle w, A^N z \rangle$$

and this completes the proof.

It should be noted that  $[A^N]^* w = P^N A^* P^N w$  if and only if  $A_1^T \psi^N(0) - \psi^N(-r) = 0$ , i.e., if and only if  $P^N w \in D(A^*)$ . Also, if

$$D = D(A) \cap D(A^*)$$

and

$$C = \{(\xi, \psi) \in D \mid \psi \in C^2(-r, 0; \mathbb{R}^n)\}$$

then  $D$  and  $C$  are dense in  $Z$ . Moreover, we have the following convergence result.

**Lemma 3.2:** If  $C$  and  $D$  are defined as above, then

(a)  $[A^N]^* w \rightarrow A^* w$  for all  $w \in C$

and

(b) for all  $\lambda \in \mathbb{R}$ ,  $(\lambda I - A^*)D$  is not dense in  $Z$ .

Proof: Let  $w \in C \subseteq D(A) \cap D(A^*)$ . Note that  $w = (\psi(0), \psi(\cdot))$  where  $\psi(-r) = A_1^T \psi(0)$  and  $P^N w = (\psi^N(0), \psi^N(\cdot)) \in D(A)$ . It follows from (2.7) and Lemma 3.1 that

$$\|[A^N]^* w - A^* w\| \leq \|P^N A^* w - A^* w\|$$

$$+ \|(\psi^N(0) + A_0^T \psi^N(0), [A^T(\cdot)\psi^N(0) - \dot{\psi}^N(\cdot)]) - (\psi(0) + A_0^T \psi(0), [A_0^T(\cdot)\psi(0) - \dot{\psi}(\cdot)])\|$$

$$\begin{aligned}
 & + \|\delta_+^N [A_1^T \psi^N(0) - \psi^N(-r)]\| \\
 & = F_1^N + F_2^N + F_3^N.
 \end{aligned}$$

Since  $w = (\psi(0), \psi(\cdot)) \in C \subseteq D(A) \cap D(A^*)$  and  $\psi(\cdot) \in C^2(-r, 0; \mathbb{R}^n)$ , it follows from standard estimates on interpolating splines (see equations (4.1) - (4.3) in [2]) that

$$|\psi^N(0) - \psi(0)| \leq O(1/N^2)$$

$$|\psi^N(s) - \psi(s)| \leq O(1/N), \quad -r \leq s \leq 0$$

and

$$\|\dot{\psi}^N(\cdot) - \dot{\psi}(\cdot)\| \leq O(1/N).$$

The first term  $F_1^N \rightarrow 0$  since  $w \in D(A^*)$  and  $p_z^N \rightarrow z$  for all  $z \in Z$ . The second term is estimated by

$$\begin{aligned}
 F_2^N & \leq |\psi^N(0) + A_0^T \psi^N(0) - \psi(0) - A_0^T \psi(0)| + \|A^T(\cdot)(\psi^N(0) - \psi(0))\| \\
 & \quad + \|\dot{\psi}^N(\cdot) - \dot{\psi}(\cdot)\| \\
 & \leq (1 + |A_0| + \|A(\cdot)\|) |\psi^N(0) - \psi(0)| + O(1/N).
 \end{aligned}$$

Therefore,  $F_2^N \rightarrow 0$  as  $N \rightarrow \infty$ . Applying (3.2), the last term is estimated by

$$F_3^N \leq (6N/r)^{1/2} |A_1^T \psi(0) - \psi^N(-r)|$$

$$\leq (6N/r)^{1/2} (|A_1| |\psi^N(0) - \psi(0)| + |\psi(-r) - \psi^N(-r)|),$$

and hence  $F_3^N \rightarrow 0$  which establishes part (a).

Turning to part (b), let  $w = (\psi(0), \psi) \in D$ ,  $\xi, x \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$ . A straightforward calculation yields that

$$\begin{aligned} & \langle (\xi, e^{\lambda \cdot} x), [\lambda I - A^*](\psi(0), \psi(\cdot)) \rangle \\ &= \psi^T(0) [(\lambda - 1)I - A_0] \xi + \psi^T(0) [I - e^{-\lambda r} A_1 - \int_{-r}^0 e^{\lambda s} A^T(s) ds] x. \end{aligned}$$

If  $(\lambda - 1) \in \sigma(A_0)$  and  $e^{\lambda r} \in \sigma(A_1 + \int_{-r}^0 e^{(\lambda+r)s} A^T(s) ds)$ , then there exist  $\xi \neq 0$  and  $x \neq 0$  such that

$$\langle (\xi, e^{\lambda \cdot} x), [\lambda I - A^*](\psi(0), \psi(\cdot)) \rangle = 0.$$

If  $(\lambda - 1) \notin \sigma(A_0)$ , let  $x \neq 0$  and define  $\xi$  by

$$\xi = -[(\lambda - 1)I - A_0]^{-1} (I - e^{-\lambda r} A_1 - \int_{-r}^0 e^{\lambda s} A^T(s) ds) x,$$

or if  $e^{\lambda r} \notin \sigma(A_1 + \int_{-r}^0 e^{(\lambda+r)s} A^T(s) ds)$ , let  $\xi \neq 0$  and define  $x$  by

$$x = -[I - e^{-\lambda r} A_1 - \int_{-r}^0 e^{\lambda s} A^T(s) ds]^{-1} [(\lambda - 1)I - A_0] \xi.$$

In any case, there always exists an element  $z = (\xi, \psi(\cdot)) = (\xi, e^{\lambda \cdot} x) \neq 0$ , such that

$$\langle z, (\lambda I - A^*) w \rangle = 0$$



for all  $w = (\psi(0), \psi(\cdot)) \in D$ . Consequently,  $(\lambda I - A^*)D$  is not dense in  $Z$ .

Since  $\|[S^N]^*(t)\| = \|S^N(t)\| \leq e^{\omega t}$  for some  $\omega$  independent on  $N$  (see [2]), the existence of a set  $E \subseteq D(A^*)$  and a  $\lambda \in \mathbb{R}$  such that  $(\lambda I - A^*)E$  is dense in  $Z$  and  $[A^N]^*w \rightarrow A^*w$  for  $w \in E$  would imply strong convergence of the semigroups  $[S^N]^*(t)$  to  $S^*(t)$  ([6], III, Th. 4.5). Although  $[A^N]^*w \rightarrow A^*w$  on the dense set  $C$ ,  $(\lambda I - A^*)C$  is not dense in  $Z$ . We shall establish that there does not exist a set  $E \subseteq D(A^*)$  with the above properties. In fact, we shall show that  $[S^N]^*(t)$  does not converge strongly to  $S^*(t)$ .

#### 4. CALCULATION OF $P^N([A^N]^*)^{-1}P^N$

In this section, we present several technical lemmas that will be needed in Section 5. The proof of Lemma 4.1 is straightforward and hence omitted. Lemma 4.3 can be found in [4,7].

**Lemma 4.1:** The operator  $[A^N]^*$  can be decomposed as

$$(4.1) \quad [A^N]^* = i^N [Q^N]^{-1} [H^N]^T [i^N]^*.$$

**Lemma 4.2:** Assume that  $H^N$  is invertible. If  $z = (\eta, \phi(\cdot)) \in Z$ , then  $P^N([A^N]^*)^{-1}P^N z$  exists and

$$(4.2) \quad P^N([A^N]^*)^{-1}P^N z = i^N a^N,$$

where  $a^N$  is the solution of

$$(4.3) \quad [H^N]^T a^N = \begin{bmatrix} \eta + \phi_0^N \\ \phi_1^N \\ \vdots \\ \phi_N^N \end{bmatrix} .$$

Proof: Consider the equation  $A^{N*} w = P^N z$  for  $w \in Z^N$ . By Lemma 4.1 and from (2.14)

$$i^N (Q^N)^{-1} H^{NT} i^{N*} w = i^N i^{N*} z .$$

Multiplying with  $i^{N*}$  and from (2.13) and (2.14)

$$H^{NT} i^{N*} w = \begin{bmatrix} \eta + \phi_0^N \\ \phi_1^N \\ \vdots \\ \phi_N^N \end{bmatrix} .$$

The lemma now follows from the fact that  $w = P^N w = i^N i^{N*} w = i^N a^N$ .

**Lemma 4.3:** If  $\Delta = A_0 + A_1 + \int_{-r}^0 A(s)ds$ , then  $0 \in \rho(A^*)$  if and only if  $\Delta$  is invertible and  $(\xi, \psi(\cdot)) = (A^*)^{-1}(\eta, \phi(\cdot)) \in D(A^*)$  is given  
by

$$(4.4) \quad \xi = [\Delta^T]^{-1} \left( \eta + \int_{-r}^0 \phi(s) ds \right)$$

$$(4.5) \quad \psi(s) = (A_1^T + \int_{-r}^s A^T(\theta) d\theta) \xi - \int_{-r}^s \phi(\theta) d\theta .$$

Consider equation (4.3). It follows from (2.17) that (4.3) is equivalent to the system

$$(4.6) \quad \frac{1}{2} (a_{k+1}^N - a_{k-1}^N) + [A_k^N]^T a_0^N = \phi_k^N \quad 1 \leq k \leq N-1$$

$$(4.7) \quad -\frac{1}{2} (a_N^N + a_{N-1}^N) + [A_N^N]^T a_0^N = \phi_N^N$$

$$(4.8) \quad \frac{1}{2} (a_0^N + a_1^N) + [A_0^N]^T a_0^N = \eta + \phi_0^N.$$

A straightforward induction argument yields

$$(4.9) \quad a_{2k}^N = a_0^N - 2 \sum_{i=1}^k ([A_{2i-1}^N]^T a_0^N - \phi_{2i-1}^N), \text{ if } 2k \leq N$$

and

$$(4.10) \quad a_{2k+1}^N = a_1^N - 2 \sum_{i=1}^k ([A_{2i}^N]^T a_0^N - \phi_{2i}^N), \text{ if } 2k+1 \leq N.$$

Thus, it follows from (4.7) that

$$(4.11) \quad -\frac{1}{2} (a_0^N - a_1^N) + \sum_{k=1}^N [A_k^N]^T a_0^N = \sum_{k=1}^N \phi_k^N.$$

Moreover, (4.8) and (4.11) imply that

$$\sum_{k=0}^N [A_k^N]^T a_0^N = \eta + \sum_{j=0}^N \phi_j^N = \eta + \int_{-r}^0 \phi(s) ds$$

where

$$\sum_{k=0}^N [A_k^N]^T = [A_0 + A_1 + \int_{-r}^0 A(s) ds]^T = \Delta^T.$$

If one assumes that  $0 \in \rho(A^*)$ , then by Lemma 4.3 it follows that  $\Delta^T$  is

invertible and

$$(4.12) \quad a_0^N = \Delta^{-T}(\eta + \int_{-r}^0 \phi(s)ds).$$

Observe that (4.12) implies that  $a_0^N$  is independent of  $N$  and by (4.4) it follows that  $a_0^N = \xi$  where  $(\xi, \psi(\cdot)) = [A^*]^{-1}(\eta, \phi(\cdot))$ . Equation (4.8) yields the identity

$$(4.13) \quad a_1^N = -a_0^N + 2 \sum_{k=1}^N ([A_k^N]^T a_0^N - \phi_k^N)$$

and hence it now follows from (4.10) that

$$(4.14) \quad \begin{aligned} a_{2k+1}^N &= -a_0^N + 2 \sum_{k=1}^N ([A_k^N]^T a_0^N - \phi_k^N) - 2 \sum_{i=1}^k ([A_{2i}^N]^T a_0^N - \phi_{2i}^N) \\ &= -a_0^N + 2 \sum_{j=2k+1}^N ([A_j^N]^T a_0^N - \phi_j^N) + 2 \sum_{i=1}^k ([A_{2i-1}^N]^T a_0^N - \phi_{2i-1}^N). \end{aligned}$$

Note that for  $1 \leq k \leq N-1$

$$|\phi_k^N|^2 = \left| \int_{-r}^0 \phi(s) B_k^N(s) ds \right|^2 \leq \left( \frac{2r}{3N} \right) \int_{\tau_{k+1}^N}^{\tau_{k-1}^N} |\phi(s)|^2 ds$$

and similarly

$$\begin{aligned} |\phi_0^N|^2 &\leq \left( \frac{r}{3N} \right) \int_{\tau_1^N}^0 |\phi(s)|^2 ds \\ |\phi_N^N|^2 &\leq \left( \frac{r}{3N} \right) \int_{-r}^{\tau_{N-1}^N} |\phi(s)|^2 ds. \end{aligned}$$

Therefore, (4.9) and (4.10) imply

$$|a_{2k}^N| \leq |a_0^N| (1 + 2|A_1|) + 2\sqrt{r/3} (\|A\|_{L_2} |a_0^N| + \|\phi\|_{L_2})$$

and

$$|a_{2k+1}^N| \leq |a_1^N| + 2|A_1| |a_0^N| + 2\sqrt{r/3} (\|A\|_{L_2} |a_0^N| + \|\phi\|_{L_2}),$$

respectively. The identity (4.12) yields the estimate

$$|a_0^N| \leq |\Delta^{-T}| (1 + \sqrt{r}) \|(n, \phi(\cdot))\|_Z$$

and (4.13) leads to the bound

$$\begin{aligned} |a_1^N| &= |-(1 + 2[A_0^N]^T) a_0^N + 2(n + \phi_0^N)| \\ &\leq |a_0^N| + 2|n| + 2\sqrt{r/3N} (\|A\|_{L_2} |\phi_0^N| + \|\phi\|_{L_2}). \end{aligned}$$

Combining these estimates one has that for  $0 \leq k \leq N$

$$|a_k^N| \leq M \|(n, \phi(\cdot))\|_Z$$

where  $M \geq 0$  is independent of  $N$ . An application of Lemma 4.2 yields the estimate

$$\|P^N [A^{N*}]^{-1} P^N z\|_Z = \|i^N a^N\| \leq (\sqrt{1+r}) M \|z\|_Z$$

for all  $z \in Z$ . We summarize these results in the following theorem.

**Theorem 4.4:** If  $0 \in \rho(A)$ , then

$$P^N [A^{N*}]^{-1} P^N (\eta, \phi(\cdot)) = i^N a^N$$

where  $a^N = \text{col}([a_0^N]^T, [a_1^N]^T, \dots, [a_N^N]^T)$  is given by

$$a_0^N = \Delta^{-T} (\eta + \int_{-r}^0 \phi(s) ds),$$

and for  $0 < 2k < N$

$$a_{2k}^N = a_0^N - 2 \sum_{i=1}^k ([A_{2i-1}^N]^T a_0^N - \phi_{2i-1}^N)$$

while for  $1 < 2k+1 < N$

$$a_{2k+1}^N = -a_0^N + 2 \sum_{j=2k+1}^N ([A_j^N]^T a_0^N - \phi_j^N) + 2 \sum_{i=1}^k ([A_{2i-1}^N]^T a_0^N - \phi_{2i-1}^N).$$

Moreover,  $\|P^N [A^{N*}]^{-1} P^N\|_{L(Z)}$  is uniformly bounded in  $N$ .

Let  $P_{AVE}^N$  denote the "averaging" orthogonal projection on  $Z$  defined by

$$P_{AVE}^N (\eta, \phi(\cdot)) = (\eta, \sum_{i=1}^N \frac{N}{r} \left( \int_{\tau_i^N}^{\tau_{i-1}^N} \phi(s) ds \right) \chi_{[\tau_i^N, \tau_{i-1}^N)}),$$

where  $\chi_I$  denotes the characteristic function for the interval  $I$  (see [1] for details).

**Corollary 4.5:** There exist a constant  $K > 0$  (independent of  $N$  )  
such that for all  $z \in Z$

$$\|P_{AVE}^N (P^N [A^{N*}]^{-1} P^N z) - [A^*]^{-1} z\| \leq \left(\frac{K}{N}\right) \|z\|.$$

Proof: A direct application of Theorem 4.4 yields the identities

$$\frac{1}{2} (a_{2k+1}^N + a_{2k}^N) = \sum_{i=2k+1}^N ([A_i^N]^T \xi - \phi_i^N), \quad 0 \leq 2k \leq N-1$$

and

$$\frac{1}{2} (a_{2k-1}^N + a_{2k}^N) = \sum_{i=2k}^N ([A_i^N]^T \xi - \phi_i^N), \quad 2 \leq 2k \leq N.$$

On the other hand, since  $(\xi, \psi(\cdot)) = [A^*]^{-1}(\eta, \phi(\cdot))$ . Lemma 4.3 implies that

$$\xi = a_0^N = \Delta^{-T} \left( \eta + \int_{-r}^0 \phi(s) ds \right)$$

and

$$\psi(s) = (A_1^T + \int_{-r}^s A^T(\theta) d\theta) \xi - \int_{-r}^s \phi(\theta) d\theta.$$

Therefore, if  $s_j^N = (\tau_j^N + \tau_{j-1}^N)/2$ , then

$$\begin{aligned} & |(a_{2k+1}^N + a_{2k}^N)/2 - \psi(s_{2k+1}^N)| \\ &= \left| \int_{\tau_{2k+1}^N}^{s_{2k+1}^N} (B_{2k+1}^N(s) - 1)(A^T(s)\xi - \phi(s)) ds \right. \\ & \quad \left. + \int_{s_{2k+1}^N}^{\tau_{2k}^N} (B_{2k+1}^N(s)(A^T(s)\xi - \phi(s)) ds \right| \end{aligned}$$

$$\leq (r/\sqrt{12} N) \left\{ \left( \int_{\tau_{2k+1}^N}^{\tau_{2k}^N} |A(s)|^2 ds \right)^{1/2} |\xi| + \left( \int_{\tau_{2k+1}^N}^{\tau_{2k}^N} |\phi(s)|^2 ds \right)^{1/2} \right\},$$

and similarly,

$$\begin{aligned} & |(a_{2k}^N + a_{2k-1}^N)/2 - \psi(s_{2k}^N)| \\ & \leq (r/\sqrt{12} N) \left\{ \left( \int_{\tau_{2k}^N}^{\tau_{2k-1}^N} |A(s)|^2 ds \right)^{1/2} |\xi| + \left( \int_{\tau_{2k}^N}^{\tau_{2k-1}^N} |\phi(s)|^2 ds \right)^{1/2} \right\}. \end{aligned}$$

It follows that

$$\begin{aligned} & \| P_{AVE}^N i^N a - (\xi, \sum_{j=1}^N \psi(s_j^N) \chi_{[\tau_j^N, \tau_{j-1}^N)}) \| \\ & \leq (r/\sqrt{12} N) (\|A\|_{L_2} |\xi| + \|\phi\|_{L_2}). \end{aligned}$$

Since  $\psi \in W^{1,2}(-r, 0; \mathbb{R}^1)$  and

$$\dot{\psi}(s) = A^T(s)\xi - \phi(s) = A^T(s)\Delta^{-T}(\eta + \int_{-r}^0 \phi(\theta)d\theta) - \phi(s),$$

there exists a constant  $\hat{K}$  such that

$$\| \psi - \sum_{j=1}^N \psi(s_j^N) \chi_{[\tau_j^N, \tau_{j-1}^N)} \|_{L_2} \leq (\hat{K}/N) \|(\eta, \phi(\cdot))\|_Z.$$

This estimate combined with the previous inequality establishes the proof.

We turn now to providing a proof that the approximating adjoint semigroups constructed above do not converge strongly to the adjoint semigroup generated by  $A^*$ .



**5. NON-STRONG CONVERGENCE**

Let  $A$ ,  $A^*$ ,  $A^N$ , and  $A^{N*}$  be defined by (2.4) - (2.5), (2.6) - (2.7), (2.16) and (4.1), respectively. The corresponding semigroups will be denoted by  $S(t)$ ,  $S^*(t)$ ,  $S^N(t)$ , and  $S^{N*}(t)$ . Recall that for  $z \in Z$  (see [2])

$$(5.1) \quad \langle A^N z, z \rangle = \langle AP^N z, P^N z \rangle \leq \omega \|P^N z\|^2 \leq \omega \|z\|^2$$

where  $\omega = (1 + 2|A_0| + |A_1| + 2\|A\|_{L_2})/2$  and for  $t \geq 0$

$$(5.2) \quad \|S^N(t)\| \leq e^{\omega t}, \quad \|S^{N*}(t)\| \leq e^{\omega t}.$$

The following result is a special case of Theorem 4.2 of Chapter 3 in [6].

**Theorem 5.1:** The following are equivalent:

(a) For every  $z \in Z$  and  $\lambda \in \mathcal{C}$  with  $\text{Re } \lambda > \omega$

$$(\lambda I - A^{N*})^{-1} z \rightarrow (\lambda I - A^*)^{-1} z, \text{ as } N \rightarrow \infty.$$

(b) For every  $z \in Z$  and  $t \geq 0$

$$S^{N*}(t)z \rightarrow S^*(t)z, \text{ as } N \rightarrow +\infty,$$

the convergence being uniform in  $t$  on bounded intervals.

We shall also need the following technical lemmas.

**Lemma 5.2:** Suppose the condition (b) of Theorem 5.1 holds,  $\lambda \in \rho(A^*)$  and  $\|P^N(\lambda I - A^{N*})^{-1}P^N\|$  is uniformly bounded in  $N$ . Then for every  $z \in Z$

$$P^N(\lambda I - A^{N*})^{-1}P^N z \rightarrow (\lambda I - A^*)^{-1}z.$$

Proof: From Theorem 5.1, for  $\lambda_0 > \omega$  and  $z \in Z$

$$P^N(\lambda_0 I - A^{N*})^{-1}P^N z \rightarrow (\lambda_0 I - A^*)^{-1}z, \text{ as } N \rightarrow \infty.$$

Note that for  $z \in Z$

$$\begin{aligned} & P^N(\lambda I - A^{N*})^{-1}P^N z - P^N(\lambda_0 I - A^{N*})^{-1}P^N z \\ &= (\lambda_0 - \lambda)P^N(\lambda I - A^{N*})^{-1}P^N(\lambda_0 I - A^{N*})^{-1}P^N z, \end{aligned}$$

and similarly

$$(\lambda I - A^*)^{-1}(I + (\lambda - \lambda_0)(\lambda_0 I - A^*)^{-1}) = (\lambda_0 I - A^*)^{-1}.$$

Hence, if  $w = z + (\lambda - \lambda_0)(\lambda_0 I - A^*)^{-1}z$ , then

$$\begin{aligned} & P^N(\lambda I - A^{N*})^{-1}P^N w - (\lambda I - A^*)^{-1}w = \\ &= (\lambda_0 - \lambda)P^N(\lambda I - A^{N*})^{-1}P^N((\lambda_0 I - A^{N*})^{-1}P^N z - (\lambda_0 I - A^*)^{-1}z) \\ &+ P^N(\lambda_0 I - A^{N*})^{-1}P^N z - (\lambda_0 I - A^*)^{-1}z. \end{aligned}$$

This implies that for all  $z \in Z$

$$P^N(\lambda I - A^{N*})^{-1}P^N w \rightarrow (\lambda I - A^*)^{-1}w.$$

But since  $I + (\lambda - \lambda_0)(\lambda_0 I - A^*)^{-1}$  is onto, the above statement holds for all  $w \in Z$ .

We turn now to a special case where  $\tilde{A}_0 = A_0 + \alpha I$ ,  $\alpha \in \mathbb{R}^1$ ,  $\tilde{A}(s) \equiv 0$ ,  $-r \leq s \leq 0$ ,  $\tilde{A} = \tilde{A}_0 + A_1$  and denote by  $\tilde{A}^*$ ,  $\tilde{A}^{N*}$  and  $\tilde{S}^*(t)$ ,  $\tilde{S}^{N*}(t)$  the corresponding infinitesimal generators and semigroups.

**Lemma 5.3:** If statement (b) of Theorem 5.1 holds for  $S^{N*}(t)$ , then it holds for  $\tilde{S}^{N*}(t)$ .

Proof: Note that

$$\tilde{A}^* = A^* + E \quad \text{and} \quad \tilde{A}^{N*} = A^{N*} + P^N E P^N$$

where  $E: Z \rightarrow Z$  is the bounded linear operator defined by

$$E(\eta, \phi(\cdot)) = (\alpha\eta, -A^T(\cdot)\eta).$$

It follows from (5.2) that  $\|\tilde{S}^{N*}(t)\| \leq e^{\tilde{\omega}t}$  where  $\tilde{\omega} = \omega + |\alpha| + \|A\|_{L_2}$ .

Consequently, if  $\lambda > \tilde{\omega}$ , then

$$(5.3) \quad (\lambda I - \tilde{A}^{N*})^{-1} = (\lambda I - A^{N*})^{-1} + (\lambda I - A^{N*})^{-1}P^N E P^N (\lambda I - \tilde{A}^{N*})^{-1}$$

and

$$(5.4) \quad (\lambda I - \tilde{A}^*)^{-1} = (\lambda I - A^*)^{-1} + (\lambda I - A^*)^{-1} E (\lambda I - \tilde{A}^*)^{-1}.$$

Theorem 5.1 implies that for all  $z \in Z$

$$(\lambda I - A^{N*})^{-1} z \rightarrow (\lambda I - A^*)^{-1} z$$

and since the ranks of  $E$  and  $E^*$  are finite it follows that

$$\|P_{EP}^N - E\| \rightarrow 0.$$

It now follows from (5.3) - (5.4) that

$$(\lambda I - \tilde{A}^{N*})^{-1} z \rightarrow (\lambda I - \tilde{A}^*)^{-1} z$$

for all  $z \in Z$  and this completes the proof.

By Lemma 5.3, without loss of generality, one can assume that  $A(\cdot) = 0$  and  $\Delta = A_0 + A_1$  is invertible in what follows. We will show that there exists an element  $z \in Z$  such that  $P^N (A^{N*})^{-1} P^N z$  does not converge to  $(A^*)^{-1} z$ . First we consider the case when  $A_1$  is not the identity. From Lemma 4.3, if  $(\xi, \psi(\cdot)) = [A^*]^{-1}(\eta, 0)$  where  $0 \neq \eta \in \mathbb{R}^n$ , then

$$\xi = \Delta^{-T} \eta \quad \text{and} \quad \psi(s) \equiv A_1^T \xi, \quad -r \leq s \leq 0.$$

Applying Theorem 4.4 we obtain

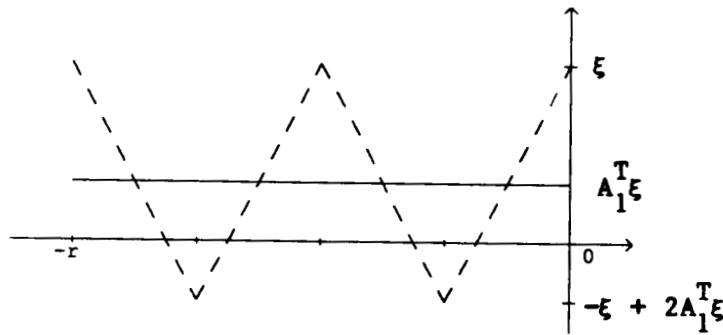
$$P^N[A^{N*}]^{-1}P^N(\eta, 0) = i^N a^N$$

where

$$a_{2k}^N = \xi, \quad 0 \leq 2k \leq N$$

$$a_{2k+1}^N = -\xi + 2A_1^T \xi, \quad 0 \leq 2k + 1 \leq N.$$

For illustration, we have the following picture for the case  $N = 4$  and  $n = 1$



where the solid line stands for  $(A^*)^{-1}(\eta, 0)$  and the dashed line for  $i^N a^N$ .

Since  $a_{2k}^N - A_1^T \xi = A_1^T \xi - a_{2k+1}^N = \xi - A_1^T \xi = \epsilon \in \mathbb{R}^n$  (independent of  $N$ ), it is easy to show that

$$(5.5) \quad \|P^N(A^{N*})^{-1}P^N(\eta, 0) - (A^*)^{-1}(\eta, 0)\|_Z^2 = 2N \int_0^{r/2N} \left| \frac{2N}{r} s \epsilon \right|^2 ds = \frac{r}{3} |\epsilon|^2 \neq 0.$$

Next we consider the case  $A_1 = I$ . Let  $\phi(s) = x \neq 0$  (constant vector in  $\mathbb{R}^n$ ). Then, from Lemma 4.3  $(\xi, \psi(\cdot)) = (A^*)^{-1}(0, \phi(\cdot))$  is given by

$$(5.6) \quad \xi = r\Delta^{-T}x \quad \text{and} \quad \psi(s) = \xi - (s + r)x, \quad -r \leq s \leq 0.$$

And, from Theorem 4.4,  $P^N(A^{N*})^{-1}P^N(0, \phi(\cdot)) = i^N a^N$ , where

$$a_0^N = \xi$$

$$(5.7) \quad a_{2k}^N = \xi + \sum_{\substack{j \text{ odd} \\ 1 \leq j \leq 2k-1}} \left(\frac{r}{N}\right)_x = \xi + \frac{2k}{N} rx, \quad 0 \leq 2k \leq N$$

$$a_{2k+1}^N = \xi - \frac{2N-1}{N} rx + 2 \sum_{\substack{j \text{ even} \\ 2 \leq j \leq 2k}} \left(\frac{r}{N}\right)_x = \xi - \left(\frac{2N-2k-1}{N}\right)rx, \quad 0 \leq 2k < N.$$

Since  $(a_{2k}^N + a_{2k+1}^N)/2 = \psi((\tau_{2k}^N + \tau_{2k+1}^N)/2)$  and  $(a_{2k-1}^N + a_{2k}^N)/2 = \psi((\tau_{2k-1}^N + \tau_{2k}^N)/2)$ , it follows from (5.6) and (5.7) that

$$(5.8) \quad \|P^N(A^{N*})^{-1}P^N(0, \phi(\cdot)) - (A^*)^{-1}(0, \phi(\cdot))\|_Z^2 = 2N \int_0^{\frac{r}{2N}} |2Nsx|^2 ds = \frac{r^2}{3} |x|^2 \neq 0.$$

Now we may state the main theorem.

**Theorem 5.4:** There exists an element  $z \in Z$  and  $t > 0$  such that

$$S^{N*}(t)z \text{ does not converge to } S^*(t)z.$$

Proof: If for every  $z \in Z$  and  $t > 0$ ,  $S^{N*}(t)z$  converges to  $S^*(t)z$ , then it follows from Theorem 4.4 and Lemma 5.2 that for every  $z \in Z$   $P^N(A^{N*})^{-1}P^N z + (A^*)^{-1}z$ , where by Lemma 5.3 one can assume that  $A(\cdot) = 0$  and  $\Delta = A_0 + A_1$  is invertible. This contradicts the facts (5.5) and (5.8).

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