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SUMMARY

A set of higher-order boundary-layer equations are derived that are valid for three-dimensional compressible flows. The equations are written in a generalized curvilinear coordinate system, in which the surface coordinates are nonorthogonal, whereas the third axis is restricted to be normal to the surface. Also, higher-order viscous terms which are retained depend on the surface curvature of the body. Thus, the equations are suitable for the calculation of the boundary layer about arbitrary vehicles. As a starting point, the Navier-Stokes equations are derived in a tensorian notation. Then by means of an order-of-magnitude analysis, the boundary-layer equations are developed. To provide an interface between the analytical partial-differentiation notation and the compact tensor notation, a brief review of the most essential theorems of the tensor analysis related to the equations of fluid dynamics is given. Many useful quantities, such as the contravariant and the covariant metrics and the physical velocity components, are written in both notations.

1. INTRODUCTION

For the numerical prediction of the viscous flow about three-dimensional aerodynamic configurations, two distinct approaches are being developed. In the first approach, the Reynolds-averaged Navier-Stokes equations are solved for the entire flow field. In the second approach, an inviscid-flow method is used for the prediction of the outer flow, whereas near the walls the boundary-layer equations are solved. The two solutions are then connected by means of a viscous-inviscid interaction technique.

While the Navier-Stokes approach is more general, the viscous-inviscid interaction approach may be more computationally efficient. In addition, Navier-Stokes algorithms have been developed which incorporate boundary-layer solvers in order to improve accuracy near the walls. This is called the Fortified Navier-Stokes approach(ref. 1).

A simple and computationally efficient algorithm for solving the unsteady three-dimensional boundary-layer equations has been developed recently by Van Dalsem and Steger(ref. 2). In the related

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boundary-layer equations, the surface curvature was neglected. As this poses restrictions in the use of this algorithm for those bodies with large curvature, it is necessary to develop a more general set of equations which includes the curvature effect.

In this work a set of higher-order boundary-layer equations are derived for a nearly-general curvilinear coordinate system. In this system, the surface coordinates can be nonorthogonal, whereas the third axis is restricted to be normal to the surface. Components of both the covariant and of the contravariant vectors appear in the equations because of the nonorthogonality of the surface coordinates. To obtain a compact set of equations, the theory of tensor analysis is used to derive the equations. The surface coordinates will be denoted by $(x^1, x^2) = (\xi, \eta)$, and the normal coordinate by $x^3 = \zeta$.

These equations should be easy to use for the derivation of a fast algorithm, which will be used in the Fortified Navier-Stokes approach.

Similar derivations of the boundary-layer equations in curvilinear coordinates have been described in the literature, but they are generally first-order in approximation. The majority of these are based on the classical equations of Squire(ref.3), which are valid for an incompressible fluid. Robert(ref.4) has treated the case of the compressible fluids in a higher approximation, that accounts for the effect of the curvature of the wall. The equations he derived are characterized by the use of the "shifters" technique that he borrowed from elasticity theory. Thus the various geometrical quantities above the wall are expressed as functions of their projected values on the wall and the vertical distance. Hirschell & Kordulla(ref.5) provide a comprehensive review of this topic and of the various methods that are used for the inclusion of curvature effects.

As a starting point, the Navier-Stokes equations wil be derived in a fully nonorthogonal coordinate system using the compact tensor notation. Then the boundary-layer equations will be developed, by means of an order-of-magnitude analysis. For brevity, most of the essentials of the tensor analysis and of the theory of transformations in curvilinear coordinates are reviewed without proofs in the following section. For a thorough review, see the books of Aris(ref.6), of Hinchey (ref.7), and of Borisenko & Tarapov(ref.8) .

2. BRIEF REVIEW OF TENSOR ANALYSIS

Throughout this work it will be assumed that the geometric and physical quantities of the flow are referenced to a Cartesian coordinate system $y^i(x,y,z)$. The curvilinear coordinate system into which the flow field will be transformed will be designated by $x^i(\xi,\eta,\zeta)$. Initially, the system will be assumed to be fully non-orthogonal. The various expressions will be given both in compact tensorian form as well as in the analytical form (using the letters x, y, z, ξ, η, ζ).

Many of the advantages of the tensor method derive from the simplifying nature of the tensor notation and of the summation convention. For example, if the product, c, of two matrices a and b is considered

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \tag{2.1}$$

then in the tensorian summation convention the elements of the matrix c can be calculated by

$$c_{ij} = a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j} \tag{2.2}$$

In the example of equation (2.1), for the definition of the elements of the 3x3 matrices a, b, c, it was necessary to use two indexes. These entities are called tensors of second order. A typical example is the stress tensor. A vector is a quantity specified by three numbers, namely its components with respect to a given base, whereas a scalar is a quantity whose specification requires just one number, in any coordinate system. Scalars and vectors are considered as special cases of the object called a tensor of order n. In this case the order is zero for a scalar (no index is used) and one for a vector (one index is required). Commonly-used tensors of higher order are the permutation index ε_{ijk} , and the curvature tensor, R_{ijkl} .

In addition to the number of indexes, distinction must be made between indexes appearing as superscripts, e.g., u^i , or subscripts, e.g., g_{ij} . Superscripted indexes are called contravariant tensors. Subscripted indexes, covariant tensors. The case of a mixed tensor appears quite often in applications, a typical example being the Kronecker delta : $\delta^i{}_j$.

A complete definition of a tensor includes the transformation law of its components. Thus, an entity specified by 3^{m+n} components,

$$A^{i1\ldots im}_{j1\ldots jn}$$

is called a tensor of order m+n if by transformation from coordinate $O(y^1, y^2, y^3)$ to $O(x^1, x^2, x^3)$, it becomes

$$A(x)^{p1\ldots pm}_{q1\ldots qn} = \frac{\partial x^{p1}}{\partial y^{i1}} \cdots \frac{\partial x^{pm}}{\partial y^{im}} \frac{\partial y^{j1}}{\partial x^{q1}} \cdots \frac{\partial y^{jn}}{\partial x^{qn}} A(y)^{i1\ldots im}_{j1\ldots jn} \qquad (2.3)$$

More precisely, it may be called an absolute tensor of contravariant order m and covariant order n .

The partial derivatives $\partial x^i/\partial y^j$ and their inversions must be known for the calculation of the elements of a tensor in another coordinate system. The transformation operations are easy if the metric coefficients are used (see the next section).

2.1 Bases and Metric Coefficients

A curvilinear coordinate system x^j (ξ, η, ζ) can be defined in relation to the basic Cartesian system $y^i(x, y, z)$

$$\xi = f_1(x,y,z), \quad \eta = f_2(x,y,z), \quad \zeta = f_3(x,y,z)$$

Then, through each point in a given region of space there will pass the surfaces ξ = constant, η = constant, ζ = constant, which are known as the coordinate surfaces. Any two of these constant surfaces intersect in a space curve. The three curves that pass through a point are known as the coordinate lines of this point. If **r** is the radius vector from an arbitrary origin to a point, the change in **r** caused by infinitesimal displacements along the three coordinate curves is

$$dr = \frac{\partial \mathbf{r}}{\partial \xi} d\xi + \frac{\partial \mathbf{r}}{\partial \eta} d\eta + \frac{\partial \mathbf{r}}{\partial \zeta} d\zeta \qquad (2.4)$$

The set of vectors

$$\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial \xi}, \qquad \mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial \eta}, \qquad \mathbf{e}_3 = \frac{\partial \mathbf{r}}{\partial \zeta}$$

are known as the covariant base vectors of the curvilinear coordinate system. The vector $\mathbf{e}_1$ is tangent to the coordinate curve (ξ) and points in the direction of increasing ξ. The base $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is said to be local, since it generally varies from point to point. In general, these vectors are neither perpendicular nor of unit length. Analytically, they are defined by the relations

$$\mathbf{e}_1 = \mathbf{i}\, x_\xi + \mathbf{j} y_\xi + \mathbf{k}\, z_\xi$$

$$\mathbf{e}_2 = \mathbf{i}\, x_\eta + \mathbf{j} y_\eta + \mathbf{k}\, z_\eta$$

$$\mathbf{e}_3 = \mathbf{i}\, x_\zeta + \mathbf{j}\, y_\zeta + \mathbf{k}\, z_\zeta \qquad (2.5)$$

where (i, j, k) are the unit vectors of the Cartesian system.

Since the set of the covariant base vectors are noncoplanar, they define a parallelepiped whose volume is given by

$$V = \mathbf{e}_1 .(\mathbf{e}_2 \times \mathbf{e}_3) = \mathbf{e}_2 .(\mathbf{e}_3 \times \mathbf{e}_1) = \mathbf{e}_3 .(\mathbf{e}_1 \times \mathbf{e}_2)$$

A triplet of vectors known as the contravariant base vectors is defined by the relations

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \mathrm{x} \mathbf{e}_3}{V} \quad , \quad \mathbf{e}^2 = \frac{\mathbf{e}_3 \mathrm{x} \mathbf{e}_1}{V} \quad , \quad \mathbf{e}^3 = \frac{\mathbf{e}_1 \mathrm{x} \mathbf{e}_2}{V} \tag{2.6}$$

It is clear from equation (2.6), that $\mathbf{e}^1$ is normal to the plane defined by the covariant base vectors ($\mathbf{e}_2$, $\mathbf{e}_3$), etc. Analytically the contravariant base vectors are defined by

$$\mathbf{e}^1 = \mathbf{i}\,\xi_x + \mathbf{j}\,\xi_y + \mathbf{k}\,\xi_z$$

$$\mathbf{e}^2 = \mathbf{i}\,\eta_x + \mathbf{j}\,\eta_y + \mathbf{k}\,\eta_z$$

$$\mathbf{e}^3 = \mathbf{i}\,\zeta_x + \mathbf{j}\,\zeta_y + \mathbf{k}\,\zeta_z \tag{2.7}$$

An increment of arc length along a space curve is given by

$$(\,ds\,)^2 = (\,d\mathbf{r}\,)^2 = d\mathbf{r}.d\mathbf{r} = \mathbf{e}_i\,.\mathbf{e}_j\,dx^i\,dx^j \tag{2.8}$$

The nine dot-products $\mathbf{e}_i.\mathbf{e}_j$ form a symmetric tensor called the covariant metric tensor $g_{ij} = g_{ji} = \mathbf{e}_i\,.\mathbf{e}_j$. Thus the length element is given by

$$(\,ds\,)^2 = g_{ij}\,dx^i\,dx^j \tag{2.8a}$$

From the classical theory of partial derivatives, it is known that a volume element is transformed by the equation

$$d\xi\,d\eta\,d\eta = \frac{\partial(\xi,\eta,\zeta)}{\partial(x,y,z)}\,dx\,dy\,dz = J\,dx\,dy\,dz \tag{2.9}$$

where J is the *Jacobian*.

In several textbooks on tensors, the volume element alternatively is given by

$$dV = dx\,dy\,dz = \sqrt{g}\;d\xi\,d\eta\,d\zeta \tag{2.10}$$

where g is the determinant of the metric tensor:

$$g = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{2.11}$$

Thus, the Jacobian of the transformation is related to the quantity g by the relation ,

$$J = \frac{1}{\sqrt{g}} \tag{2.12}$$

The contravariant metric tensor is defined in a matter similar to the covariant: $g^{ij} = g^{ji} = \mathbf{e}^i . \mathbf{e}^j$. Its elements can be estimated by using equation (2.7). Alternatively, they are connected to the elements of the covariant metric tensor with the relations

$$g^{il} = \frac{G^{il}}{g} = \frac{1}{g}(g_{jm}\, g_{kn} - g_{jn}\, g_{km}) \tag{2.13}$$

(i, j, k) cyclic (l, m, n) cyclic

where G^{il} is the cofactor of g_{il} in the determinant g .

Other quantities that are derived from the metrics are the Christoffel symbols of the first kind

$$[\, i j , k \,] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right) \tag{2.14}$$

and of the second kind

$$\{ {}^{\;l}_{i\,j} \} = \{ {}^{\;l}_{j\,i} \} = \Gamma^{l}_{i j} = g^{kl} [\, i j , k \,] , \quad \text{or} \; = g^{kl} \frac{\partial y^n}{\partial x^k} . \frac{\partial^2 y^n}{\partial x^i \partial x^j} \tag{2.15}$$

The Christoffel symbols may be interpreted in terms of the variation of the base vectors with respect to the coordinates. They appear in the expressions of the covariant derivatives (§ 2.2).

In orthogonal curvilinear systems, such as the cylindrical or the spherical system, many of the metrics are equal to zero. More specifically, in the case of an orthogonal curvilinear system:

$$g_{ij} = \delta^i_j \quad , \quad g^{ij} = \delta^i_j$$

As this relation is simple, the following symbols are usually chosen for the nonzero g_{ij} :

$$h_1 = \sqrt{g_{11}} \quad , \quad h_2 = \sqrt{g_{22}} \quad , \quad h_3 = \sqrt{g_{33}}$$

Finally, in a Cartesian coordinate system (rectangular) the covariant and the contravariant base are identical.

In the next section the usefulness of the various symbols that have been reviewed here will be demonstated. The metrics and the Christoffel symbols of the second kind are given analytically in the appendix for the coordinate system x^i (ξ, η, ζ) .

2.2 Derivatives Based on Tensor Algebra

If A^{ik} and B^{ik} are the components of two second-order tensors, then the numbers C^{ik}, defined by

$$C^{ik} = A^{ik} + B^{ik} \tag{2.16}$$

are the components of a second-order tensor , called the sum of the tensors with components A^{ik} and B^{ik} . Addition of any number of tensors of arbitrary order is defined similarly. Tensors of different orders cannot be added. The tensors must have not only the same order but also the same structure, i.e., the same numbers of covariant and contravariant indices in the same places. Subtraction of tensors is defined similarly to the addition.

On the contrary, tensors of arbitrary order and structure can be multiplied. The general rule , for multiplication(outer product), is

$$C^{ijk}_{m} = A^{ij} B^{k}_{m} \qquad (2.17)$$

The operation of summing a tensor of order n (n>=0) over two of its indices is called contraction. For example, a possible contraction for the tensor A_k^{ij} is the operation

$$A^{ik}_{k} = A^{i1}_{1} + A^{i2}_{2} + A^{i3}_{3} \qquad (i=1,2,3)$$

The resulting tensor actually is a contravariant vector. Contraction can be performed only on pairs of indices in different positions, i.e. , one contracted index must be covariant and the other contravariant, in the case of generalized coordinates. Otherwise, the result of the contraction will not be a tensor.

The result of multiplying two or more tensors and then contracting the product with respect to indices that belong to different factors is called an *i*nner product. For example, the relation (2.17) if contracted, is written

$$C^{ik} = A^{ij} B^{k}_{j} \qquad (2.18)$$

Another useful class of operations is the raising or the lowering of the index. If A_i is a covariant vector, the contravariant vector $A^j = g^{ij}A_i$ is called its associated vector. This operation is known as raising the index. Its inverse is the lowering of the index by an inner product with g_{ij} : $B_j = g_{ij}B^i$. The operation is also applied to tensors of higher order.

It has been mentioned previously (§2.1) that in a generalized system of coordinates the base vectors are local, i.e., they are functions of the coordinates x^1, x^2 ,x^3. Then it follows that the differential of a vector includes not only a term that expresses the change of the components of the vector itself, but also a term required because the base of the coordinate system also varies from point to point. The name covariant differentiation is applied to this operation. The covariant derivative is denoted by $A_{,j}$ (suffixes have been suppressed in the vector A). The covariant derivative is a tensor. Standard formulas are included in some of the reference books for the calculation of the covariant derivatives of tensors of various orders. Here we are interested in formulas up to the second order

$$u_{i,k} = \frac{\partial u_i}{\partial x^k} - \{{}_{i}{}^{j}{}_{k}\}\, u_j \qquad (2.19a)$$

$$u^i_{,k} = \frac{\partial u^i}{\partial x^k} + \{{}_{j}{}^{i}{}_{k}\}\, u^j \qquad (2.19b)$$

$$T^{ik}_{,l} = \frac{\partial T^{ik}}{\partial x^l} + \{{}_{m}{}^{i}{}_{l}\}\, T^{mk} + \{{}_{m}{}^{k}{}_{l}\}\, T^{im} \qquad (2.19c)$$

$$T_{ik,l} = \frac{\partial T_{ik}}{\partial x^l} - \{{}_{i}{}^{m}{}_{l}\}\, T_{mk} - \{{}_{k}{}^{m}{}_{l}\}\, T_{im} \qquad (2.19d)$$

For the derivation of the flow equations, Ricci's theorem is very useful; it states that the covariant derivatives of the metric tensors vanish, i. e., the components of the metric tensors and the determinant g can be regarded as constants under covariant differentiation. Also, in the case of a zero-order tensor (scalar), the covariant derivative reduces to the partial derivative with respect to the coordinates

$$f_{,j} = \frac{\partial f}{\partial x^j} \qquad (2.20)$$

Thus, the covariant derivative of a scalar f is a covariant vector with components equal to the covariant components of *grad* f.

The divergence of a vector **A** = A^i is defined as the contraction of the covariant derivative of A^i

$$\mathrm{div}\mathbf{A} = A_{,i} = \frac{\partial A^i}{\partial x^i} + \{{}_{i}{}^{i}{}_{j}\}\, A^j \qquad (2.21)$$

The contracted Christoffel symbol may be expressed, after some manipulations, in terms of the determinant of the metric tensor

$$\{ {}_{i}{}^{i}{}_{j} \} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{i}} (\sqrt{g}) \tag{2.22}$$

Substituting the above expression in equation (2.21) , the following results are obtained for a contravariant or a covariant vector respectively

$$\operatorname{div} \mathbf{A} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{i}} (A^{i} \sqrt{g}) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{i}} (g^{ij} A_{j} \sqrt{g}) \tag{2.23}$$

Equation (2.22) may also be used for the estimation of the covariant derivative of a second-order tensor (equation 2.19c), for the case in which the operation of contraction has been applied:

$$T^{ik}_{,k} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{m}} (\sqrt{g}\, T^{im}) + \{ {}_{m}{}^{i}{}_{k} \} T^{mk} \tag{2.24}$$

This relation will be used for the calculation of the viscous forces in the momentum equations.

2.3 Flow Quantities Expressed in a Generalized Curvilinear System

In order to relate the velocity components of the flow in a generalized coordinate system to the corresponding velocity components in the reference Cartesian system, the relation between the differentials of the coordinates must be found. This is easily done by application of the chain rule

$$dx^{i} = \frac{\partial x^{i}}{\partial y^{j}} dy^{j} \tag{2.25}$$

Then, by taking the time derivative of the contravariant vector dx^i , the following expression is found

$$U^i = \frac{dx^i}{dt} = \frac{\partial x^i}{\partial y^j}\frac{dx^j}{dt} = \frac{\partial x^i}{\partial y^j} u^j \tag{2.26}$$

Thus, the contravariant velocity components, U^i , are connected to the Cartesian velocity components, u^i, by the components of the contravariant base vectors. Analytically, the following well-known equations are valid

$$U = u\,\xi_x + v\,\xi_y + w\,\xi_z$$

$$V = u\,\eta_x + v\,\eta_y + w\,\eta_z$$

$$W = u\,\zeta_x + v\,\zeta_y + w\,\zeta_z \tag{2.27}$$

Similarly, the acceleration and all higher derivatives are contravariant vectors. If the derivative of a scalar quantity is considered, the resulting vector is a covariant one. Indeed, by application of the chain-rule

$$\frac{\partial f}{\partial x^i} = \frac{\partial y^j}{\partial x^i}\frac{\partial f}{\partial y^j} \tag{2.28}$$

It is seen that the components of the covariant base vectors connect the two flow fields. The scalar f may be the pressure, the density or other variables. As an example, the ξ-derivative of the pressure, p, is given by

$$p_\xi = x_\xi p_x + y_\xi p_y + z_\xi p_z$$

The three Cartesian coordinates have all the physical dimension of length, but in general this does not happen in a curvilinear system. For example, in the standard cylindrical polars, two of the coordinates have the dimension of length, but the third, being an angle, has no dimensions. Thus, the contravariant velocity components will not all have the same physical components of velocity. The same problem appears in all the vectors which are related to the flow, whether covariant or contravariant. Even if there is dimensional agreement, the numerical values may not be correct, because the base vectors are not unitary.

Formulae have been derived that resolve this problem. If the physical components of a contravariant vector, A^i, are denoted by A(i), then the transformation law of physical components is

$$A(j) = A^j \sqrt{g_{jj}} \qquad (\text{no sum on } j) \qquad (2.29)$$

In case of a covariant vector, first the associated contravariant vector has to be constructed and then equation (2.29) to be applied

$$A(j) = A_i \, g^{ij} \sqrt{g_{jj}} \qquad (\text{no sum on } j) \qquad (2.30)$$

In addition, the transformation law of physical components between a curvilinear and a Cartesian system, is

$$A(j)_{\text{curv.}} = \sqrt{g_{jj}} \, \frac{\partial x^j}{\partial y^i} A(i)_{\text{cart.}} \qquad (\text{no sum on } j) \qquad (2.31)$$

3. NAVIER-STOKES EQUATIONS

As a starting point the momentum equations of Aris(ref.4), which are valid for a constant viscosity coefficient μ, will be used

$$\rho \frac{\partial u^i}{\partial t} + \rho u^j u^i_{,j} = \rho f^i + T^{ij}_{,j} \qquad (3.1)$$

where $u^i_{,j}$ is the covariant derivative of the velocity vector, f^i is the vector of the external forces and $T^{ij}_{,j}$ is the covariant derivative of the stress tensor. The stress tensor for a Newtonian fluid is calculated from the following equations

$$T^{ij} = (-p + \lambda u,_k^k)\, g^{ij} + 2\mu e^{ij} \tag{3.2}$$

$$e^{ij} = \mu(g^{mi} g^{jn} + g^{in} g^{jm}) e_{mn} \tag{3.3}$$

$$e_{mn} = \frac{1}{2} (u_{n,m} + u_{m,n}) \tag{3.4}$$

These equations, e^{ij} and e_{mn} denote the deformation tensor in contravariant and covariant expression, respectively, and λ is the bulk viscosity.

After some manipulations, the stress tensor is found to be

$$T^{ij} = (-p + \lambda u,_k^k)\, g^{ij} + \mu(g^{im} u,_m^j + g^{jn} u,_n^i) \tag{3.5}$$

For the calculation of the covariant derivatives, the following equations must be used

$$u,_j^i = \frac{\partial u^i}{\partial x^j} + \{{}_j{}^i{}_k\}\, u^k \tag{3.6}$$

$$T,_j^{ij} = J \frac{\partial}{\partial x^j} \left(\frac{T^{ij}}{J}\right) + T^{jk} \{{}_j{}^i{}_k\} \tag{3.7}$$

where J is the Jacobian and $\{{}_j{}^i{}_k\}$ is the second kind of the Christoffel symbol.

If equation (3.6) is applied to the stress tensor, the following expression is found for its derivative

$$T,_j^{ij} = g^{ij} \frac{\partial}{\partial x^j} \left(-p + \lambda J \frac{\partial}{\partial x^k} \left(\frac{u^k}{J} \right)\right) + J \frac{\partial}{\partial x^j} \left(\frac{\mu}{J} (g^{im} u,_m^j + g^{jn} u,_n^i)\right) +$$

$$\{{}_j{}^i{}_k\}\, \mu (g^{jm} u,_m^k + g^{kn} u,_n^j) \tag{3.8}$$

For the derivation of this equation, Ricci's lemma has been used, according to which the covariant derivatives of the metric tensors vanish. Also, the rule that the covariant derivative of a scalar is equal to the conventional partial derivative, has been applied.

In the absense of external forces, the momentum equations are given by

$$\rho\frac{\partial u^i}{\partial t} + \rho u^j\frac{\partial u^i}{\partial x^j} + \rho\left\{ {}_{j}{}^{i}{}_{k} \right\} u^j u^k = T^{ij}_{,j} \tag{3.9}$$

The continuity and the energy equations are easier to derive in generalized coordinates, because they can be expressed in divergence form. For example the continuity equation can be expressed as

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\bar{c}) = 0 \tag{3.10}$$

and the energy equation can be expressed as

$$\frac{\partial e}{\partial t} + \nabla.(e\bar{c}) + \nabla.(p\bar{c}) = \nabla.(T^{ij}u_j) - \nabla.\bar{q} \tag{3.11}$$

where c is the velocity vector, e is the internal energy and q is the heat transfer. In the dissipation term, the stress tensor is multiplied by the velocity vector expressed in the covariant form because their product must be a contravariant vector.

The heat transfer, in tensorian notation, may be written as

$$\bar{q} = -k\nabla T = -kg^{ij}\frac{\partial T}{\partial x^j} \tag{3.12}$$

Finally, the divergence of a vector a is expressed in tensorian notation as

$$\nabla.\bar{a} = a^i_{,i} = J\frac{\partial}{\partial x^i}\left(\frac{1}{J}a^i\right) \tag{3.13}$$

By substituting equations (3.12) and (3.13) in equations (3.10) and (3.11), the following expressions for the continuity and energy equations are obtained in the coordinate system $x^i = (\xi, \eta, \zeta)$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{J}\right) + \frac{\partial}{\partial \xi}\left(\rho\frac{U}{J}\right) + \frac{\partial}{\partial \eta}\left(\rho\frac{V}{J}\right) + \frac{\partial}{\partial \zeta}\left(\rho\frac{W}{J}\right) = 0 \tag{3.14}$$

$$\frac{\partial}{\partial t}\left(\frac{e}{J}\right) + \frac{\partial}{\partial \xi}\left(\left(\frac{e+p}{J}\right)U\right) + \frac{\partial}{\partial \eta}\left(\left(\frac{e+p}{J}\right)V\right) + \frac{\partial}{\partial \zeta}\left(\left(\frac{e+p}{J}\right)W\right) = \frac{\partial}{\partial x^i}\left(\frac{\tau^{ij}}{J} g_{jk} u^k\right) +$$

$$+ \frac{\partial}{\partial x^i}\left(\frac{k}{J} g^{ij} \frac{\partial T}{\partial x^j}\right) \tag{3.15}$$

In equation (3.15), the symbol τ^{ij} denotes the viscous stress tensor.

In the next section, equations (3.9), (3.14), (3.15) will be used for deriving the boundary layer equations.

4. BOUNDARY-LAYER EQUATIONS

The boundary-layer equations will be derived in a system of coordinates, where the surface coordinates will be denoted by $(x^1, x^2) = (\xi, \eta)$, and the normal one by $x^3 = \zeta$. In the appendix it is shown that in this system, the metric terms g^{13}, g^{23}, g_{13}, g_{23} are equal to zero. Also, the Jacobian is simplified to the form

$$\frac{1}{J} = \sqrt{g} = \sqrt{g_{33}(g_{11}g_{22} - g_{12}^2)}$$

For an order-of-magnitude analysis, the Navier-Stokes equations are expressed in a nondimensional form. So the freestream conditions, U_∞, ρ_∞, T_∞, a reference length L and the quantity $\rho_\infty U^2_\infty$ (for energy and pressure) will be used. For these nondimensional variables, all the terms that include the viscosity coefficients (μ,λ) are multiplied by the inverse of the Reynolds number, based on the freestream conditions.

If the nondimensional boundary-layer thickness is denoted by ε, the normal velocity will be assumed to be $W=O(\varepsilon)$ and the Reynolds number $1/Re = O(\varepsilon^2)$. The metrics and the majority of their derivatives that appear in the Christoffel symbols, will be assumed to be of the order of O(1). However, the normal derivatives of the metrics will be assumed to be $O(1) < g_{ij,3} < O(1/\delta)$, (i,j =1 or 2), so that a number of curvature-dependent terms will be included in the viscous part of the equations. Thus, it will be possible to obtain results of higher accuracy for bodies that have large curvature.

4.1 Estimation of the Viscous Terms of the Boundary-Layer Equations

In this section, the boundary-layer approximation will be applied to the viscous terms of the momentum equations. The viscous terms are

$$\tau^i = \frac{J}{Re}\frac{\partial}{\partial x^j}\left(\frac{\mu}{J}\left(g^{im}u^j_{,m} + g^{jn}u^i_{,n}\right)\right) + \frac{\mu}{Re}\left\{{i \atop j\ k}\right\}\left(g^{jm}u^k_{,m} + g^{kn}u^j_{,n}\right) =$$

$$= \tau^i_a + \tau^i_b \tag{4.1}$$

Because of the presence of the Reynolds number, any term that is of the order of $1/\varepsilon^2$, will be significant, so that the product of the term times the Reynolds number will be of the order of one. Thus, both, the partial derivative $\partial/\partial x^j$, and the covariant derivatives of equation (4.1) must be taken only along the normal direction (j,m,n = 3). Starting with the term $\tau_a{}^i$ for the ξ– momentum equation (i=1)

$$\tau^1_a = \frac{J}{Re}\frac{\partial}{\partial\eta}\left(\frac{\mu}{J}\left(g^{13}u^3_{,3} + g^{33}u^1_{,3}\right)\right) = \frac{J}{Re}\frac{\partial}{\partial\eta}\left(\frac{\mu}{J}g^{33}u^1_{,3}\right) \tag{4.2}$$

The covariant derivative of the velocity component $u^1 = U$, is given by

$$u,{}_3^1 = \frac{\partial U}{\partial \zeta} + \{{}_1{}^1{}_3\} U + \{{}_2{}^1{}_3\} V + \{{}_3{}^1{}_3\} W \approx \frac{\partial U}{\partial \zeta} + \{{}_3{}^1{}_1\} U + \{{}_2{}^1{}_3\} V \quad (4.2)$$

The Christoffel symbols that appear in equation (4.2) are given in the appendix . Each of them includes two of the normal derivatives of the following metric coefficients: g_{11} , g_{12} , g_{23}. For example

$$\{{}_1{}^1{}_3\} = \frac{1}{2g} g_{22} g_{33} g_{11,3} - \frac{1}{2g} g_{33}\, g_{21} g_{12,3} \quad (4.3)$$

Evidently, these derivatives depend on the curvature of the body along the ξ- and the η- direction. For small values of the curvature, along one of these directions, the corresponding terms will be eliminated.

For the η- momentum equation (i=2)

$$\tau_a^2 = \frac{J}{Re} \frac{\partial}{\partial \zeta} \left(\frac{\mu}{J} \left(g^{23} u,{}_3^3 + g^{33} u,{}_3^2\right)\right) = \frac{J}{Re} \frac{\partial}{\partial \zeta} \left(\frac{\mu}{J} g^{33} u,{}_3^2\right) \quad (4.4)$$

where

$$u,{}_3^2 = \frac{\partial V}{\partial \zeta} + \{{}_1{}^2{}_3\} U + \{{}_2{}^2{}_3\} V + \{{}_3{}^2{}_3\} W \approx \frac{\partial V}{\partial \zeta} + \{{}_1{}^2{}_3\} U + \{{}_2{}^2{}_3\} V \quad (4.5)$$

The $\tau_b{}^i$ -component of equation (4.1) is estimated in a similar manner. For the ξ-momentum equation(i=1)

$$\tau_b^1 = \frac{\mu}{Re} \{{}_3{}^1{}_k\} \left(g^{33} u,{}_3^k + g^{k3} u,{}_3^3\right) \approx \frac{\mu}{Re} \left(\{{}_3{}^1{}_1\} g^{33} u,{}_3^1 + \{{}_2{}^1{}_3\} g^{33} u,{}_3^2\right) \quad (4.6)$$

If the covariant derivatives are substituted in this equation, and terms including the products of the Christoffel symbols are ignored, as being smaller than $O(1/\varepsilon^2)$, the following expression is found

$$\tau_b^1 = \frac{\mu}{Re} g^{33} \{{}_3{}^1{}_1\} \frac{\partial U}{\partial \zeta} + \frac{\mu}{Re} g^{33} \{{}_2{}^1{}_3\} \frac{\partial V}{\partial \zeta} \tag{4.7}$$

For the η-momentum equation(i=2)

$$\tau_b^2 = \frac{\mu}{Re} \{{}_3{}^2{}_k\} (g^{33} u,_3^k + g^{k3} u,_3^3) \approx \frac{\mu}{Re} (\{{}_3{}^2{}_1\} g^{33} u,_3^1 + \{{}_2{}^2{}_3\} g^{33} u,_3^2) \tag{4.8}$$

Otherwise, after substituting the values of the covariant derivatives

$$\tau_b^2 = \frac{\mu}{Re} g^{33} \{{}_3{}^2{}_1\} \frac{\partial U}{\partial \zeta} + \frac{\mu}{Re} g^{33} \{{}_2{}^2{}_3\} \frac{\partial V}{\partial \zeta} \tag{4.9}$$

The Christoffel symbols that appear in equations (4.7) and (4.9) include the normal derivatives of the metrics along the surface coordinates. Thus, for bodies of small curvature, these terms may be totally eliminated.

Finally, the term that depends on the bulk viscosity, λ, has to be examined

$$\sigma^i = \frac{1}{Re} g^{ij} \frac{\partial}{\partial x^j} (\lambda J \frac{\partial}{\partial x^k} (\frac{u^k}{J})) = \frac{1}{Re} g^{ij} \frac{\partial}{\partial x^j} (\lambda J ((\frac{U}{J})_\xi + (\frac{V}{J})_\eta + (\frac{W}{J})_\zeta)) \tag{4.10}$$

Since the dilatation terms are of the order of 1, no contribution from this term is expected. Furthermore, the term would become of the order of ε for j=3. However, the metric g^{ij} then becomes equal to zero, for i=1,2. Thus the bulk viscosity term will not appear in the boundary-layer equations.

In summary, the viscous terms of the boundary layer equations are

$$\tau^1 = \frac{J}{Re} \frac{\partial}{\partial \zeta} (\frac{\mu}{J} g^{33} (\frac{\partial U}{\partial \zeta} + \{{}_1{}^1{}_3\} U + \{{}_2{}^1{}_3\} V)) + \frac{\mu}{Re} g^{33} \{{}_3{}^1{}_1\} \frac{\partial U}{\partial \zeta} + \frac{\mu}{Re} g^{33} \{{}_2{}^1{}_3\} \frac{\partial V}{\partial \zeta} \tag{4.11}$$

$$\tau^2 = \frac{J}{Re}\frac{\partial}{\partial\zeta}\left(\frac{\mu}{J} g^{33}\left(\frac{\partial V}{\partial\zeta} + \left\{ {}_{1}{}^{2}{}_{3} \right\} U + \left\{ {}_{2}{}^{2}{}_{3} \right\} V\right)\right) + \frac{\mu}{Re} g^{33} \left\{ {}_{3}{}^{2}{}_{1} \right\} \frac{\partial U}{\partial\zeta} + \frac{\mu}{Re} g^{33} \left\{ {}_{2}{}^{2}{}_{3} \right\} \frac{\partial V}{\partial\zeta} \quad (4.12)$$

4.2 Estimation of the Viscous Terms in the Energy Equation

The dissipation term of the energy equation in a dimensional form is

$$D = \frac{\partial}{\partial x^i}\left(\tau^{ij} g_{jk} u^k\right) = \frac{\partial}{\partial x^i}\left(\mu\left(g^{im} u^j_{,m} + g^{jn} u^i_{,n}\right) g_{jk} u^k\right) \quad (4.13)$$

If the term is nondimensionalized, the Reynolds number appears in the denominator. Thus again only the normal derivatives will contribute , i. e., $i,m,n = 3$. Next, the following approximate expression is found

$$D = \frac{\partial}{\partial\zeta}\left(\mu\left(g^{33} u^j_{,3} + g^{j3} u^3_{,3}\right)\left(g_{j1} u^1 + g_{j2} u^2\right)\right) =$$

$$\frac{\partial}{\partial\zeta}\left(\mu\left(g^{33} u^1_{,3} + g^{13} u^3_{,3}\right)\left(g_{11} u^1 + g_{12} u^2\right) + \mu\left(g^{33} u^2_{,3} + g^{32} u^3_{,3}\right)\left(g_{21} u^1 + g_{22} u^2\right)\right) =$$

$$\frac{\partial}{\partial\zeta}\left(\mu g^{33} u^1_{,3}\left(g_{11} U + g_{12} V\right) + \mu g^{33} u^2_{,3}\left(g_{12} U + g_{22} V\right)\right) \quad (4.14)$$

The covariant derivatives of the velocity components are given by equations (4.2) and (4.5). Substituting them in the equation (4.14) and neglecting higher-order terms, it is found that

$$D = \frac{\partial}{\partial\zeta}\left(\mu g^{33}\left(g_{11} U \frac{\partial U}{\partial\zeta} + g_{12} V \frac{\partial U}{\partial\zeta} + g_{12} U \frac{\partial V}{\partial\zeta} + g_{22} V \frac{\partial V}{\partial\zeta}\right)\right) =$$

$$\frac{\partial}{\partial\zeta}\left[\mu\frac{g_{11}}{2g_{33}}\frac{\partial}{\partial\zeta}(U^2)+\mu\frac{g_{22}}{2g_{33}}\frac{\partial}{\partial\zeta}(V^2)+\mu\frac{g_{12}}{g_{33}}\frac{\partial}{\partial\zeta}(UV)\right] \tag{4.15}$$

This is the final expression for the dissipation term. For its calculation, the metric relation : $g^{33} = g_{33}$ has been used.

4.3 Heat Transfer Term in Energy Equation

The heat transfer term of the energy equation, in a nondimensional form, is given by

$$H = \frac{\partial}{\partial x^i}\left(\frac{k}{J}g^{ij}\frac{\partial T}{\partial x^j}\right) = \frac{\partial}{\partial\xi}\left(\frac{k}{J}g^{1j}\frac{\partial T}{\partial x^j}\right) + \frac{\partial}{\partial\eta}\left(\frac{k}{J}g^{2j}\frac{\partial T}{\partial x^j}\right) + \frac{\partial}{\partial\zeta}\left(\frac{k}{J}g^{3j}\frac{\partial T}{\partial x^j}\right) \tag{4.16}$$

In the light of the approximation of the thin thermal layer , we retain only the normal-derivative term. If, the summation of the dummy index j is then applied , it is found that

$$H = \frac{\partial}{\partial\zeta}\left(\frac{k}{J}g^{33}\frac{\partial T}{\partial\zeta}\right) = \frac{\partial}{\partial\zeta}\left(\frac{\mu g^{33}}{J\,Pr(\gamma-1)}\frac{\partial a^2}{\partial\zeta}\right) \tag{4.16a}$$

where a is the local sonic velocity and the heat transfer coefficient has been substituted by means of the Reynolds analogy ($Pr = \mu c_p/k$).

4.4 Pressure and Other Terms in the Momentum Equations

The pressure terms are given by the equation

$$P^i = -g^{ij}\frac{\partial p}{\partial x^j} = -g^{i1}\frac{\partial p}{\partial\xi} - g^{i2}\frac{\partial p}{\partial\eta} - g^{i3}\frac{\partial p}{\partial\zeta} \tag{4.17}$$

The normal-derivative component is equal to zero , in both of the surface directions of the boundary layer (i = 1 or 2) because of the zero value of the metric terms g^{31} , g^{23} . If , in addition, the contravariant metrics are replaced by the covariant ones, the following expressions are obtained

$$P^1 = \frac{g_{33}\, g_{12}}{g} \frac{\partial p}{\partial \eta} - \frac{g_{22}\, g_{33}}{g} \frac{\partial p}{\partial \xi} \qquad (4.18)$$

$$P^2 = \frac{g_{33}\, g_{12}}{g} \frac{\partial p}{\partial \xi} - \frac{g_{11}\, g_{33}}{g} \frac{\partial p}{\partial \eta} \qquad (4.19)$$

Finally, there are the curvature terms, F^i, of the left-hand side (LHS) of the momentum equations

$$F^i = \{{}^{\,i}_{j\,k}\}\, u^j u^k \qquad (4.20)$$

For i=1 or 2, the curvature terms of the ξ– or of the η–momentum equation are obtained:

$$F^1 = \{{}^{\,1}_{1\,1}\}U^2 + 2\{{}^{\,1}_{1\,2}\}UV + \{{}^{\,1}_{2\,2}\}V^2 \qquad (4.21a)$$

$$F^2 = \{{}^{\,2}_{1\,1}\}U^2 + 2\{{}^{\,2}_{1\,2}\}UV + \{{}^{\,2}_{2\,2}\}V^2 \qquad (4.21b)$$

In the equations (4.21a) and (4.21b) the terms including the velocity component W have been eliminated, as they are of the order of ε.

4.5 Normal Pressure Gradient

In the first-order boundary layer approximation, the η–momentum equation is commonly not considered , because a dimensional analysis shows that its terms are of the order of ε. However in the present case, though the viscous and the convection terms are of this order, the curvature term is $O(1) < F^3 < O(1/\delta)$ because this term includes the normal derivative of the metric coefficients

$$F^3 = \{^{\,3}_{1\,1}\}U^2 + 2\{^{\,3}_{1\,2}\}UV + \{^{\,3}_{2\,2}\}V^2 = -\frac{1}{2}g^{33}\left(\frac{\partial g_{11}}{\partial \zeta}U^2 + 2\frac{\partial g_{12}}{\partial \zeta}UV + \frac{\partial g_{22}}{\partial \zeta}V^2\right) \quad (4.22)$$

The corresponding pressure gradient is

$$P^3 = -g^{33}\frac{\partial p}{\partial \zeta} \quad (4.23)$$

So, in case of bodies of large curvature , a normal pressure gradient exists, given by:

$$\frac{\partial p}{\partial \zeta} = \frac{1}{2}\frac{\partial g_{11}}{\partial \zeta}U^2 + \frac{\partial g_{12}}{\partial \zeta}UV + \frac{1}{2}\frac{\partial g_{22}}{\partial \zeta}V^2 \quad (4.24)$$

4.6 Final Equations for Large Surface-Curvature

If the boundary-layer approximations of the various terms ,derived in this section, are substituted in the original Navier-Stokes equations of §3, the following higher-order boundary layer equations are found

For the continuity equation

$$\frac{\partial}{\partial t}\left(\frac{\rho}{J}\right) + \frac{\partial}{\partial \xi}\left(\rho\frac{U}{J}\right) + \frac{\partial}{\partial \eta}\left(\rho\frac{V}{J}\right) + \frac{\partial}{\partial \zeta}\left(\rho\frac{W}{J}\right) = 0 \quad (4.25)$$

For the momentum equations

1. ξ–momentum

$$\rho\frac{\partial U}{\partial t} + \rho U\frac{\partial U}{\partial \xi} + \rho V\frac{\partial U}{\partial \eta} + \rho W\frac{\partial U}{\partial \zeta} + K_{a1}\rho U^2 + K_{a2}\rho UV + K_{a3}\rho V^2 =$$

$$K_{a4}\frac{\partial p}{\partial \eta} - K_{a5}\frac{\partial p}{\partial \xi} + \frac{J}{Re}\frac{\partial}{\partial \zeta}\left(\frac{\mu}{J g_{33}}\left(\frac{\partial U}{\partial \zeta} + K_{a6}U + K_{a7}V\right)\right) + \frac{\mu}{Re}K_{a8}\frac{\partial U}{\partial \zeta} + \frac{\mu}{Re}K_{a9}\frac{\partial V}{\partial \zeta}$$

(4.26)

2. η–momentum

$$\rho\frac{\partial V}{\partial t}+\rho U\frac{\partial V}{\partial \xi}+\rho V\frac{\partial V}{\partial \eta}+\rho W\frac{\partial V}{\partial \zeta}+K_{b1}\rho U^2+K_{b2}\rho UV+K_{b3}\rho V^2=$$

$$K_{b4}\frac{\partial p}{\partial \xi}-K_{b5}\frac{\partial p}{\partial \eta}+\frac{J}{Re}\frac{\partial}{\partial \zeta}\left(\frac{\mu}{Jg_{33}}\left(\frac{\partial V}{\partial \zeta}+K_{b6}U+K_{b7}V\right)\right)+\frac{\mu}{Re}K_{b8}\frac{\partial U}{\partial \zeta}+\frac{\mu}{Re}K_{b9}\frac{\partial V}{\partial \zeta} \tag{4.27}$$

and for the energy equation

$$\frac{\partial}{\partial t}\left(\frac{e}{J}\right)+\frac{\partial}{\partial \xi}\left(\left(\frac{e+p}{J}\right)U\right)+\frac{\partial}{\partial \eta}\left(\left(\frac{e+p}{J}\right)V\right)+\frac{\partial}{\partial \zeta}\left(\left(\frac{e+p}{J}\right)W\right)=$$

$$\frac{1}{Re}\frac{\partial}{\partial \zeta}\left(\mu K_{e1}\frac{\partial U^2}{\partial \zeta}+\mu K_{e2}\frac{\partial (UV)}{\partial \zeta}+\mu K_{e3}\frac{\partial V^2}{\partial \zeta}\right)+\frac{1}{Re}\frac{\partial}{\partial \zeta}\left(\frac{\mu}{Jg_{33}\,Pr(\gamma-1)}\frac{\partial a^2}{\partial \zeta}\right) \tag{4.28}$$

These equations describe the most general case of a body highly curved in both the principal directions of the body axes. In this case, variation of the normal pressure gradient , equation (4.24), must be considered.

The curvature coefficients appearing in equations (4.27) and (4.28) are defined by

$$K_{a1}=\left\{{1 \atop 1\,1}\right\},\quad K_{a2}=2\left\{{1 \atop 1\,2}\right\},\quad K_{a3}=\left\{{1 \atop 2\,2}\right\},\quad K_{a4}=\frac{g_{33}g_{12}}{g},\quad K_{a5}=\frac{g_{22}g_{33}}{g}$$

$$K_{a6}=\left\{{1 \atop 3\,1}\right\},\quad K_{a7}=\left\{{1 \atop 2\,3}\right\},\quad K_{a8}=g^{33}\left\{{1 \atop 3\,1}\right\},\quad K_{a9}=g^{33}\left\{{1 \atop 2\,3}\right\}$$

$$K_{b1}=\left\{{2 \atop 1\,1}\right\},\quad K_{b2}=2\left\{{2 \atop 1\,2}\right\},\quad K_{b3}=\left\{{2 \atop 2\,2}\right\},\quad K_{b4}=\frac{g_{33}g_{12}}{g},\quad K_{b5}=\frac{g_{33}g_{11}}{g}$$

$$K_{b6}=\left\{{2 \atop 3\,1}\right\},\quad K_{b7}=\left\{{2 \atop 2\,3}\right\},\quad K_{b8}=g^{33}\left\{{2 \atop 3\,1}\right\},\quad K_{b9}=g^{33}\left\{{2 \atop 2\,3}\right\}$$

$$K_{e1}=\frac{g_{11}}{2g_{33}},\quad K_{e2}=\frac{g_{12}}{g_{33}},\quad K_{e3}=\frac{g_{22}}{2g_{33}}$$

The Christoffel symbols appearing in these coefficients are given in the appendix.

4.7 Final Equations for Moderate Surface Curvature

In the case of bodies of moderate curvature, the terms including the derivative of the metrics in the normal direction may be omitted. Then the viscous terms of the momentum equations are considerably simplified. Only the first viscous term of the RHS has to be kept. Furthermore, in this case the metrics do not depend on the ζ–derivative . Equations (4.27) and (4.28) become

$$\rho \frac{\partial U}{\partial t} + \rho U \frac{\partial U}{\partial \xi} + \rho V \frac{\partial U}{\partial \eta} + \rho W \frac{\partial U}{\partial \zeta} + K_{a1}\rho U^2 + K_{a2}\rho UV + K_{a3}\rho V^2 =$$

$$K_{a4}\frac{\partial p}{\partial \eta} - K_{a5}\frac{\partial p}{\partial \xi} + \frac{1}{g_{33}\mathrm{Re}}\frac{\partial}{\partial \zeta}(\mu \frac{\partial U}{\partial \zeta}) \qquad (4.29)$$

$$\rho \frac{\partial V}{\partial t} + \rho U \frac{\partial V}{\partial \xi} + \rho V \frac{\partial V}{\partial \eta} + \rho W \frac{\partial V}{\partial \zeta} + K_{b1}\rho U^2 + K_{b2}\rho UV + K_{b3}\rho V^2 =$$

$$K_{b4}\frac{\partial p}{\partial \xi} - K_{b5}\frac{\partial p}{\partial \eta} + \frac{1}{g_{33}\mathrm{Re}}\frac{\partial}{\partial \zeta}(\mu \frac{\partial V}{\partial \zeta}) \qquad (4.30)$$

In the first order of magnitude approximation it is sufficient to estimate the metrics and the Jacobian just on the surface (ref. 3), assuming that there is no significant variation across the boundary layer. The standard notation in this case is: $g_{ij} = a_{\alpha\beta}$, $(i,j,\alpha,\beta = 1,2)$, where the symbol "a" denotes surface quantities. Evidently, if this approximation is applied, the number of operations required for a numerical solution will be reduced considerably.

The K factors that appear in equations (4.29), (4.30) are given by

$$K_{a1} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial \xi} + \frac{1}{2} g^{12} (2 \frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}), \quad K_{a2} = g^{11} \frac{\partial g_{11}}{\partial \eta} + g^{12} \frac{\partial g_{22}}{\partial \xi}$$

$$K_{a3} = \frac{1}{2} g^{12} \frac{\partial g_{22}}{\partial \eta} + \frac{1}{2} g^{11} (2 \frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}), \quad K_{a4} = \frac{g_{33} g_{12}}{g}, \quad K_{a5} = \frac{g_{22} g_{33}}{g}$$

$$K_{b1} = \frac{1}{2} g^{12} \frac{\partial g_{11}}{\partial \xi} + \frac{1}{2} g^{22} (2 \frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}), \quad K_{b2} = g^{12} \frac{\partial g_{11}}{\partial \eta} + g^{22} \frac{\partial g_{22}}{\partial \xi}$$

$$K_{b3} = \frac{1}{2} g^{22} \frac{\partial g_{22}}{\partial \eta} + \frac{1}{2} g^{12} (2 \frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}), \quad K_{b4} = \frac{g_{33} g_{12}}{g}, \quad K_{b5} = \frac{g_{11} g_{33}}{g}$$

If a fully-orthogonal coordinate system is examined, these relations are considerably simplified, because in this case the metrics g_{12} and g^{12} become equal to zero. In addition, the covariant metric terms simply become the inverse of the covariant terms ($g^{ij} = g_{ij}$) and the Jacobian is simplified to: $g=g_{11}g_{22}g_{33}$. If the metric terms are substituted by the h_i-parameters(§2.1) and the physical-contravariant components are used

$$h_j = \sqrt{g_{jj}} , \qquad A(j) = A^j \sqrt{g_{jj}} = A^j h_j \qquad \text{(no sum on j)}$$

equations (4.29) and (4.30) simplify to

$$\rho \frac{\partial U}{\partial t} + \rho \frac{U}{h_1} \frac{\partial U}{\partial \xi} + \rho \frac{V}{h_2} \frac{\partial U}{\partial \eta} + \rho \frac{W}{h_3} \frac{\partial U}{\partial \zeta} + \rho U V K_1 - \rho V^2 K_2 = - \frac{1}{h_1} \frac{\partial p}{\partial \xi} + \frac{1}{h_3^2 Re} \frac{\partial}{\partial \zeta} (\mu \frac{\partial U}{\partial \zeta}) \tag{4.31}$$

$$\rho \frac{\partial V}{\partial t} + \rho \frac{U}{h_1} \frac{\partial V}{\partial \xi} + \rho \frac{V}{h_2} \frac{\partial V}{\partial \eta} + \rho \frac{W}{h_3} \frac{\partial V}{\partial \zeta} + \rho U V K_2 - \rho U^2 K_1 = - \frac{1}{h_2} \frac{\partial p}{\partial \eta} + \frac{1}{h_3^2 Re} \frac{\partial}{\partial \zeta} (\mu \frac{\partial V}{\partial \zeta}) \tag{4.32}$$

where K_1, K_2 denote the geodesic curvatures of the surface lines

$$K_1 = \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial \eta} \quad , \qquad K_2 = \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial \xi}$$

In these well-known equations, the basic contravariant symbols (U, V, W) have been used for denoting the physical velocity components.

In the appendix, the first-order boundary layer equations are given in the case where the normal to the surface is denoted by $x^2 = \eta$.

5. CONCLUSION

A set of higher-order boundary-layer equations have been derived for a nearly-general curvilinear coordinate system. In this system, the surface coordinates can be nonorthogonal, whereas the third axis is restricted to be normal to the surface. For the derivation of the equations, an order of magnitude analysis has been performed in the Navier-Stokes equations, written in a fully nonorthogonal coordinate system using the tensor notation. Assuming that the normal derivatives of the metrics are: $O(1) < g_{ij,3} < O(1/\delta)$, (i,j =1 or 2), a number of curvature-dependent terms appear in the viscous part of the equations. Thus, it will be possible to obtain results of higher accuracy in the case of bodies that have large curvature.

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APPENDIX

Metric Tensors

By application of the definition it is found that the elements of the covariant metric tensor are given by:

$$g_{11} = x_\xi^2 + y_\xi^2 + z_\xi^2 \;,\quad g_{12} = x_\xi x_\eta + y_\xi y_\eta + z_\xi z_\eta \;,\quad g_{13} = x_\xi x_\zeta + y_\xi y_\zeta + z_\xi z_\zeta$$
$$g_{22} = x_\eta^2 + y_\eta^2 + z_\eta^2 \;,\quad g_{23} = x_\eta x_\zeta + y_\eta y_\zeta + z_\eta z_\zeta$$
$$g_{33} = x_\zeta^2 + y_\zeta^2 + z_\zeta^2 \qquad \text{(a.1)}$$

The elements of the contravariant metric tensor can be estimated similarly by simply interchanging in equations (a.1) the Greek letters (ξ, η, ζ) by the Latin letters (x, y ,z). Alternatively they are given as the inverse elements of the covariant metric tensor (equation 2.13)

$$g^{11} = \frac{g_{22}\,g_{33} - g_{23}^2}{g} \;,\quad g^{12} = \frac{g_{23}\,g_{31} - g_{21}\,g_{33}}{g} \;,\quad g^{13} = \frac{g_{21}\,g_{32} - g_{22}\,g_{31}}{g}$$

$$g^{22} = \frac{g_{11}\,g_{33} - g_{13}^2}{g} \;,\quad g^{23} = \frac{g_{21}\,g_{31} - g_{11}\,g_{32}}{g}$$

$$g^{33} = \frac{g_{11}\,g_{22} - g_{12}^2}{g} \qquad \text{(a.2)}$$

Note that both of the metric tensors are symmetric.

Some of the elements of the metric tensors become equal to zero in the case of the partially orthogonal system assumed in this paper, because the following relations are valid between the base vectors

$$\overline{e_1} \cdot \overline{e_3} = x_\xi . x_\zeta + y_\xi . y_\zeta + z_\xi . z_\zeta = g_{13} = 0$$

$$\overline{e}_2 . \overline{e}_3 = x_\eta . x_\zeta + y_\eta . y_\zeta + z_\eta . z_\zeta = g_{23} = 0 \qquad \text{(a.3)}$$

Thus, the metric tensor and the Jacobian are simplified considerably

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix} \qquad \text{(a.4)}$$

$$\frac{1}{J} = \sqrt{g} = \sqrt{g_{22}\,(g_{11}\,g_{33} - g_{13}^2)} \qquad \text{(a.5)}$$

By considering the relation that connects the covariant and the contravariant metric tensors it is easy to show that the contravariant metric terms g^{13} and g^{23} are also equal to zero.

Christoffel Symbols

The Christoffel symbols are given here only for the particular semiorthogonal coordinate system examined in § 4

$$\left\{{1 \atop 1\,1}\right\} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial \xi} + \frac{1}{2} \left(g^{13}\, 2 \frac{\partial g_{13}}{\partial \xi} - g^{13} \frac{\partial g_{11}}{\partial \zeta}\right) \ , \quad \left\{{1 \atop 1\,2}\right\} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial \eta} + \frac{1}{2} g^{13} \frac{\partial g_{13}}{\partial \eta}$$

$$\left\{{1 \atop 1\,3}\right\} = \frac{1}{2} \left(g^{11} \frac{\partial g_{11}}{\partial \zeta} + g^{13} \frac{\partial g_{33}}{\partial \xi}\right) \ , \quad \left\{{1 \atop 2\,3}\right\} = \frac{1}{2} g^{11} \frac{\partial g_{13}}{\partial \eta} + \frac{1}{2} g^{13} \frac{\partial g_{33}}{\partial \eta}$$

$$\left\{{1 \atop 3\,3}\right\} = \frac{1}{2} g^{13} \frac{\partial g_{33}}{\partial \zeta} + \frac{1}{2} \left(g^{11}\, 2 \frac{\partial g_{13}}{\partial \zeta} - g^{11} \frac{\partial g_{33}}{\partial \xi}\right) \ , \quad \left\{{3 \atop 1\,2}\right\} = \frac{1}{2} g^{13} \frac{\partial g_{11}}{\partial \eta} + \frac{1}{2} g^{33} \frac{\partial g_{13}}{\partial \eta}$$

$$\left\{{3 \atop 1\,1}\right\} = \frac{1}{2} g^{13} \frac{\partial g_{11}}{\partial \xi} + \frac{1}{2} \left(g^{33}\, 2 \frac{\partial g_{13}}{\partial \xi} - g^{33} \frac{\partial g_{11}}{\partial \zeta}\right) \ , \quad \left\{{3 \atop 2\,3}\right\} = \frac{1}{2} g^{13} \frac{\partial g_{13}}{\partial \eta} + \frac{1}{2} g^{33} \frac{\partial g_{33}}{\partial \eta}$$

$$\left\{ {}^{\;3}_{3\,3} \right\} = \frac{1}{2} g^{33} \frac{\partial g_{33}}{\partial \zeta} + \frac{1}{2} \left(g^{13} 2 \frac{\partial g_{13}}{\partial \zeta} - g^{13} \frac{\partial g_{33}}{\partial \xi} \right) \;,\; \left\{ {}^{\;3}_{1\,3} \right\} = \frac{1}{2} g^{13} \frac{\partial g_{11}}{\partial \zeta} + \frac{1}{2} g^{33} \frac{\partial g_{33}}{\partial \xi}$$

$$\left\{ {}^{\;2}_{1\,1} \right\} = - \frac{1}{2} g^{22} \frac{\partial g_{11}}{\partial \eta} \quad , \quad \left\{ {}^{\;2}_{1\,2} \right\} = \frac{1}{2} g^{22} \frac{\partial g_{22}}{\partial \xi} \quad , \quad \left\{ {}^{\;2}_{1\,3} \right\} = - \frac{1}{2} g^{22} \frac{\partial g_{13}}{\partial \eta}$$

$$\left\{ {}^{\;2}_{2\,2} \right\} = \frac{1}{2} g^{22} \frac{\partial g_{22}}{\partial \eta} \quad , \quad \left\{ {}^{\;2}_{2\,3} \right\} = \frac{1}{2} g^{22} \frac{\partial g_{22}}{\partial \eta} \quad , \quad \left\{ {}^{\;2}_{3\,3} \right\} = - \frac{1}{2} g^{22} \frac{\partial g_{33}}{\partial \eta}$$

The Boundary Layer Equations in an Alternative System

If the normal direction is designated by $x^2 = \eta$ and the crossflow direction by $x^3 = \zeta$, the boundary layer equations in the case of the first order of approximation become

for the ξ–momentum

$$\rho \frac{\partial U}{\partial t} + \rho U \frac{\partial U}{\partial \xi} + \rho V \frac{\partial U}{\partial \eta} + \rho W \frac{\partial U}{\partial \zeta} + K_{a1} \rho U^2 + K_{a2} \rho U W + K_{a3} \rho W^2 =$$

$$K_{a4} \frac{\partial p}{\partial \zeta} - K_{a5} \frac{\partial p}{\partial \xi} + \frac{1}{g_{22} Re} \frac{\partial}{\partial \eta} \left(\mu \frac{\partial U}{\partial \eta} \right) \tag{a.5}$$

for the ζ–momentum

$$\rho \frac{\partial W}{\partial t} + \rho U \frac{\partial W}{\partial \xi} + \rho V \frac{\partial W}{\partial \eta} + \rho W \frac{\partial W}{\partial \zeta} + K_{b1} \rho U^2 + K_{b2} \rho U W + K_{b3} \rho W^2 =$$

$$K_{b4} \frac{\partial p}{\partial \xi} - K_{b5} \frac{\partial p}{\partial \zeta} + \frac{1}{g_{22} Re} \frac{\partial}{\partial \eta} \left(\mu \frac{\partial W}{\partial \eta} \right) \tag{a.6}$$

In this case the K values are given by

$$K_{a1} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial \xi} + \frac{1}{2} g^{13} \left(2 \frac{\partial g_{13}}{\partial \xi} - \frac{\partial g_{11}}{\partial \zeta} \right), \quad K_{a2} = g^{11} \frac{\partial g_{11}}{\partial \zeta} + g^{13} \frac{\partial g_{33}}{\partial \xi}$$

$$K_{a3} = \frac{1}{2} g^{13} \frac{\partial g_{33}}{\partial \zeta} + \frac{1}{2} g^{11} \left(2 \frac{\partial g_{13}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \xi} \right), \quad K_{a4} = \frac{g_{22} g_{13}}{g}, \quad K_{a5} = \frac{g_{22} g_{33}}{g}$$

$$K_{b1} = \frac{1}{2} g^{13} \frac{\partial g_{11}}{\partial \xi} + \frac{1}{2} g^{33} \left(2 \frac{\partial g_{13}}{\partial \xi} - \frac{\partial g_{11}}{\partial \zeta} \right), \quad K_{b2} = g^{13} \frac{\partial g_{11}}{\partial \zeta} + g^{33} \frac{\partial g_{33}}{\partial \xi}$$

$$K_{b3} = \frac{1}{2} g^{33} \frac{\partial g_{33}}{\partial \zeta} + \frac{1}{2} g^{13} \left(2 \frac{\partial g_{13}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \xi} \right), \quad K_{b4} = \frac{g_{22} g_{13}}{g}, \quad K_{b5} = \frac{g_{11} g_{22}}{g}$$

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16. Abstract

A set of higher-order boundary-layer equations are derived that are valid for three-dimensional compressible flows. The equations are written in a generalized curvilinear coordinate system, in which the surface coordinates are nonorthogonal; the third axis is restricted to be normal to the surface. Also, higher-order viscous terms which are retained depend on the surface curvature of the body. Thus, the equations are suitable for the calculation of the boundary layer about arbitrary vehicles. As a starting point, the Navier-Stokes equations are derived in a tensorian notation. Then by means of an order-of-magnitude analysis, the boundary-layer equations are developed. To provide an interface between the analytical partial-differentiation notation and the compact tensor notation, a brief review of the most essential theorems of the tensor analysis related to the equations of the fluid dynamics is given. Many useful quantities, such as the contravarient and the covariant metrics and the physical velocity components, are written in both notations.

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