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THE USE OF THE QR FACTORIZATION IN THE PARTIAL REALIZATION PROBLEM

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Abstract. The use of the QR factorization of the Hankel matrix in solving the partial realization problem is analyzed in this paper.

Straightforward use of the QR factorization results in a new realization scheme that possess all of the computational advantages of Rissanen's realization scheme. These latter properties are computational efficiency, recursiveness, use of limited computer memory, and the realization of a system triplet having a condensed structure. Moreover, this new scheme is robust when the order of the system corresponds to the rank of the Hankel matrix.

When this latter condition is violated, an "approximate" realization could be determined via the QR factorization. In this second scheme, the given Hankel matrix is approximated by a low-rank non-Hankel matrix. Furthermore, it is demonstrated that column pivoting might be incorporated in this second scheme.

The results presented in this paper are derived for a single-input/single-output system, but this does seem not to be a restriction.

Keywords: Partial realization, QR factorization, Hankel matrix, Markov parameters

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1. Introduction.

The so-called Hankel approach [1] to solve the *partial realization* problem [2] is studied in this paper. The cornerstone of this approach consists of the following two properties of the Hankel matrix, constructed from the Markov parameters: (1) the so-called shift invariance property and (2) the correspondence of the "numerical" rank of the Hankel matrix with the order of the system. Basically from these two properties "any" factorization of the Hankel matrix allows to solve the partial realization problem.

This observation was first made by J. Rissanen [3] in 1971. The algorithm described in [3] possesses a number of appealing properties: it is *recursive* and, computationally as well as storage related, *very efficient*. However, the original presented scheme was not numerically stable. This was observed by L.S. De Jong [4], who in the same paper proposed algorithmic modifications to make the scheme *numerically stable*.

The remaining drawback of this computational scheme was that it only used a very small amount of the available Markov parameters. Hence, making it only applicable when the Markov parameters are infinite-accurate. For the latter circumstances, which of course are very unrealistic, Rissanen's realization scheme will be indicated in this paper as an *elegant* solution, summarizing the above mentioned algorithmic properties. Additionally, this realization scheme directly resulted in a state space description in a canonical form [3], [4], what will be indicated as an *attractive* property of the realization scheme.

To make the Hankel approach applicable for practical realization problems, S.Y. Kung proposed to use the singular value decomposition (SVD) to factorize the Hankel matrix [1]. This allowed first to reliably determine the rank of the Hankel matrix and second to consider all available Markov parameters. Although the SVD might lead to a numerical robust solution of the partial realization problem, it is still, for large systems and/or marginally stable systems, very storage and computational inefficient. Also, it does not result in a special canonical form and therefore certainly does not possess the elegantness and attractiveness of Rissanen's numerical stable scheme.

An inherent drawback in the Hankel approach is that even (relatively) small errors on the Markov parameters cause the second basic property to be violated. This is due to the repetition of the individual Markov parameters and hence, so do their corresponding errors. An illustration of this phenomena will be given in this paper. Therefore, only factorizing

the Hankel matrix to solve the partial realization problem is not sufficient, no matter which factorization technique is used. This drawback has stimulated additional research which has been indicated as *reduced-order modelling*. Here one focuses on "approximating" the given Hankel matrix by a low-rank Hankel matrix [5], [6], and [7]. For certain practical applications [8] the latter constraint might be relaxed to approximating the given Hankel matrix by a low-rank non-Hankel matrix. Also here the key tool used in solving this type of realization problems is the SVD.

The incentive of this paper is to investigate the usefulness of the *QR factorization of the Hankel matrix* in solving partial realization problems. Two types of realization problems are investigated: (1) when the two fundamental properties of the Hankel matrix, mentioned above, hold, and (2) when the given Hankel matrix is approximated by a low-rank non-Hankel matrix.

The organization of this paper is as follows. After (re)stating the partial realization problem in section 2, the QR factorization is first analyzed for the type of partial realization problem, where the two basic properties are not violated. In this case, the Markov parameters are indicated to be "accurately" given. Section 4 then discusses the use of the QR factorization in solving the realization problem from a low-rank non-Hankel approximation of the given Hankel matrix. This is indicated as "approximate" reduced-order modelling. The presented two realization schemes in sections 3 and 4 are then evaluated experimentally via a number of numerical examples in section 5. Finally, section 6 present the concluding remarks.

The discussion throughout this paper is restricted to the single-input/single-output (SISO) case. But the results rather straightforwardly extend to the multi-input/multi-output (MIMO).

2. The partial realization problem.

In this section we outline the partial realization problem and indicate the two fundamental properties of the Hankel matrix, which form the cornerstone in solving this problem. This was also indicated in [1].

Let $h_1, h_2, \cdots$ be the impulse response of a linear time-invariant system:

$$x_{k+1} = Ax_k + bu_k \tag{1}$$

$$y_k = c^T x_k \tag{2}$$

where $x_k \in R^n$. Generally, the terms h_i are called the *Markov parameters.* Given the system description (1-2), they could be stated as follows:

$$h_i = c^T A^{i-1} b \tag{3}$$

The so-called *partial realization problem* is one of recovering the system triplet $\{A, b, c^T\}$ from the finite sequence $\{h_i\}$ for $i = 1, 2, \cdots M + N - 1$. Here the integers M, N have to be larger than n, however since n is not known a priori they are generally (assumed to be) taken much larger than n.

The Hankel matrix $H_q(k, \ell)$ that is used to solve this problem is constructed from from the Markov parameters as follows:

$$H_q(k,\ell) = \begin{pmatrix} h_q & h_{q+1} & \cdots & h_{q+\ell-1} \\ h_{q+1} & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ h_{q+k-1} & . & \cdots & h_{q+k+\ell-2} \end{pmatrix} \tag{4}$$

The two fundamental properties of this Hankel matrix are:

Property 2.1: The rank of the Hankel matrix $H_1(M, N)$, taken M and N "sufficiently large" as indicated above, is the order of the state space system given in (1-2).

This property can easily be understood by using the Cayley-Hamilton theorem [2] and factorizing $H_1(M, N)$ as:

$$H_1(M,N) = \begin{pmatrix} c^T \\ c^T A \\ \vdots \\ c^T A^{M-1} \end{pmatrix} \begin{pmatrix} b & \cdots & A^{N-1} b \end{pmatrix} = \mathcal{O}\mathcal{C} \tag{5}$$

where $\mathcal{O} \in R^{M \times n}$ and $\mathcal{C} \in R^{n \times N}$.

Property 2.2: Using a similar factorization for the Hankel matrix $H_2(M, N)$ the *shift-invariance* property of this Hankel matrix is:

$$H_2(M,N) = \mathcal{O} A \mathcal{C} \tag{6}$$

Remark 2.1: From (5-6) we directly observe that any factorization of the Hankel matrix as given in (5), allows to determine a state space realization. The advantage of using the SVD is twofold: first, the calculation of the system state matrix A, via (6), can reliably and easily be done once the SVD of the Hankel matrix is available and secondly, the order of the system n is also reliably revealed by a SVD.

However even for (relatively) small errors on the Markov parameters, the information about the order of the system, which is crucial in the solution of the partial realization problem, cannot be retrieved from the rank of the Hankel matrix. For these circumstances, which will be defined precisely in the sequel, properties (5-6) are no longer sufficient to solve the partial realization problem. Therefore, additional conditions are required [4], [5].

In the next section we start treating the realization problem where both fundamental properties hold.

3. Accurate Markov parameters.

3.1. The order of the system given. Let us first assume in this section that the Markov parameters are exactly known and that the order of the system (1-2) n is also given. Then, consider the QR factorization of the Hankel matrix $H_1(M, n+1)$:

$$H_1(M, n+1) = (Q_1 \mid Q_2) \begin{pmatrix} R \\ \hline 0 \end{pmatrix} = Q_1 R \tag{7}$$

Here $M \geq n$ and $Q_1 \in R^{M \times n}$ with $Q_1^T Q_1 = I_n$. Furthermore let us introduce the following notation:

$\underline{Q}_1 \in R^{(M-1)\times n}$ — denotes Q_1 without its last row

$\overline{Q}_1 \in R^{(M-1)\times n}$ — denotes Q_1 without its first row

$R \in R^{n\times(n+1)}$ — is an upper trapezoidal matrix given as $[r_1 \cdots r_{n+1}]$

The fact that R is upper trapezoidal and of rank n will be shown later on in corollary 3.3.

Using the above notation, the solution to the partial realization problem becomes:

$$c_c^T = (q_{11} \;\; q_{12} \cdots q_{1n}) \quad \text{denoting the first row of } Q_1 \tag{8}$$

$$b_c = r_1 \tag{9}$$

and A_c can be computed from either:

$$\underline{Q}_1 A_c = \overline{Q}_1 \tag{10}$$

or,

$$A_c[r_1 \cdots r_n] = [r_2 \cdots r_{n+1}] \tag{11}$$

The solution, given by equations (8, 9 and 10 or 11) is completely similar with the one base on the SVD [1]. However, now the ***algorithmic complexity is less.*** For example, when using Eq. (10) to compute A_c, Q_1 only has to be accumulated and this does not require iteration as in the SVD to compute the left singular vectors.

In the decomposition (7) $[r_1 \cdots r_n]$ can be considered as the ***controllability matrix*** of the realizing pair $\{A_c, b_c\}$. This then should imply that this pair is in controller-Hessenberg form. This assertion is proved in the following theorem. Let us first define the controller-Hessenberg form [9].

Definition 3.1: When the compound matrix $(B \mid A)$ of the system triplet $\{A, B, C\}$ is upper trapezoidal, the state space description $\{A, B, C\}$ is in *controller-Hessenberg* form.

Theorem 3.1: The system triplet given by Eqs. (8, 9 and 10 or 11), obtained via a QR factorization of the Hankel matrix given in (7) is in *controller-Hessenberg* form.

Proof: From definition 3.1 for SISO, the controller-Hessenberg form, requires A_c to be in Hessenberg form. This can directly be concluded form (11) since $[r_2 \cdots r_{n+1}]$ is Hessenberg and $[r_1 \cdots r_n]$ is upper triangular.
This in combination with the structure of the input-vector b_c given in (9) defines the controller-Hessenberg form.

From this theorem, we can now derive a constructive algorithm to solve the partial realization problem with the order of the system given ***without explicitly*** accumulating the used orthogonal transformations in the QR factorization of the Hankel matrix.

Algorithm 3.1:

1. Construction of A_c, given as $[a_1 \cdots a_n]$ from (11):

   $a_1 = r_2 r_{11}^{-1}$

for i = 2:n

$$a_i = (r_{i+1} - [a_1 \cdots a_{i-1}] \begin{pmatrix} r_{1i} \\ \vdots \\ r_{(i-1)i} \end{pmatrix}) r_{ii}^{-1} \tag{12}$$

end

2. Construction of c_c^T from:

$$c_c^T [r_1 \cdots r_n] = [h_1 \cdots h_n] \tag{13}$$

3. Construction of b_c given by (9).

Corollary 3.1: Using the RQ factorization of the Hankel matrix $H_1(n+1, M)$ for $M \geq n$, results in a similar way into the *observer-Hèssenberg* form, defined in [9].

Remark 3.1: Since Eq. (13) can straightforwardly be executed *recursively*, algorithm 3.1 is a recursive scheme to realize a system triplet $\{A_c, B_c, C_c^T\}$ once the Hankel matrix has been triangularized. And the latter operation can also be done recursively.

Remark 3.2: In [3], the observer-Hessenberg form was also derived without explicit notice. However, the developed implementation was numerically unstable [4] and furthermore the realization was obtained only based on a very restricted number of Markov parameters. This latter drawback may even for "very small" errors on the Markov parameters, result in "large errors" on the realization. This is indicated explicitly in the experimental evaluation study in section 5.1. These two restrictions do not apply to algorithm 3.1.

Remark 3.3: Algorithm 3.1 also applies to the MIMO case. For this case, we recognize two situations. First, the dimension of the input is a multiplicity of the state dimensions. Then, algorithm 1 straightforwardly holds, except now that the scalars r_{ii} and vectors a_i in (12) are respectively square and rectangular matrices. Secondly, when the above multiplicity does not hold, the algorithm remains similar, except for the computation of the last row of A_c. This requires the use of pseudo-inverses [10].

3.2. Detecting the order of the system. Generally, the order n of the state space model is not *a priori* known. Based on property 2.1 it can however be retrieved from

the Hankel matrix. Particularly, the QR decomposition of this matrix ***directly displays*** the order of the system. In order to show this, let us first state the following theorem.

Theorem 3.2: When the state space system (1-2) is of order n, then the $n+1, n+2, \cdots, N$ columns of $H_1(M,N)$ with $N > n$ are ***linearly dependent*** on the first n-columns.

Proof: Let the characteristic polynomial of the considered n^{th} order system (1-2) be given as:

$$\alpha(z) = z^n + d_1 z^{n-1} + \cdots + d_{n-1} z + d_n \tag{14}$$

Then according to the Cayley-Hamilton theorem [2], we can write the $(n+1)^{\text{th}}$ column of $H_1(M,N)$ as:

$$\begin{pmatrix} h_{n+1} \\ h_{n+2} \\ \vdots \\ h_{n+M} \end{pmatrix} = H_1(M,n)\underline{d} \tag{15}$$

where $\underline{d} = (d_n d_{n-1} \cdots d_1)^T \neq 0$. This completes the proof for the $(n+1)^{th}$ column vector of $H_1(M,N)$. The $(n+2)^{th}$ column of $H_1(M,N)$ can then be written as:

$$\begin{pmatrix} h_{n+2} \\ h_{n+3} \\ \\ \vdots \\ h_{n+M+1} \end{pmatrix} = H_1(M,n) \begin{pmatrix} 0 \\ d_n \\ d_{n-1} \\ \vdots \\ d_2 \end{pmatrix} + d_1 H_1(M,n)\underline{d} = H_1(M,N)\underline{d}^1 \tag{16}$$

with $\underline{d}^1 \neq 0$.

In a similar way the $n+3, \cdots, N$ columns can be expressed as a linear combination of the first n columns of $H_1(M,N)$, what completes the proof.

Corollary 3.2: Based on property 2.1 and theorem 3.2, the first n columns of $H_1(M,N)$ are ***linearly independent.***

Corollary 3.3: When the Markov parameters are exactly given, the QR factorization of $H_1(M,N)$ becomes:

$$H_1(M,N) = Q \left(\begin{array}{c|c} R_{11} & R_{12} \\ \hline 0 & 0 \end{array} \right) \begin{matrix} \} n \\ \} N-n \end{matrix} \tag{17}$$

with $Q \in R^{M \times N}$.

Hence, a QR factorization ***without column pivoting*** of a Hankel matrix of sufficiently large dimensions allows us to directly ***read-off*** the order of the system.

Generally, the conditions on the Markov parameters as imposed in corollary 3.3 are not met in practice. Therefore, we are interested for the size of the errors on the Markov parameters for which Corollary 3.3 holds. This is investigated in more detail in the next section.

3.3. Perturbation analysis. Let us assume that the Markov parameters h_i, defined in (3), are perturbed by a data acquisition error e_i, which satisfies:

$$|e_i| \leq \sigma \tag{18}$$

Hence, the Hankel matrix is now given as $H_1(M,N) + E$, where the perturbation matrix E is a Hankel matrix itself. In this section we additionally assume that these perturbations are *acute*. This latter notion is taken from [11] and is defined as follows.

Definition 3.2: The matrix $(A + E)$ is an ***acute perturbation*** of A if and only if:

$$rank(A) = rank(A + E) \tag{19}$$

This condition implies that $\|A^{\dagger}\|_2 \|E\|_2 < 1$ [10], where $A^{\dagger}$ denotes the pseudo-inverse of A [11] and $\|(\cdot)\|_2 = \sup_{\|x\|_2 = 1} \|(\cdot)x\|_2$.

Under the presence of the perturbations defined above, the QR factorization given in (17) becomes:

$$\left(\; H_1(M,n) + E_1 \mid H_{n+1}(M, N-n) + E_2 \; \right) = (Q + W) \left(\begin{array}{c|c} R_{11} + F_1 & R_{12} + F_2 \\ \hline 0 & R_{22} \end{array} \right) \tag{20}$$

This decomposition clarifies that theorem 3.2 no longer holds "exactly". For this case it is however more natural to refer to the ***linear dependency*** in least squares sense. Let us therefore define the following least squares problem associated to the decomposition (20).

$$\min_x \|(H_1(M,n) + E_1)x - (H_{n+1}(M,1) + E_2)\|_2 \tag{21}$$

Here we focus on the way the $(n+1)^{\text{th}}$ column of $(H_1(M,N) + E)$ is lying in the space spanned by its first n columns. This information is incorporated in the residual of (21).

Therefore, the task now is to examine the influence of the perturbations $\{E_1, E_2\}$ on this residual. The actual influence is of course very difficult to calculate. However based on [11], we can formulate an upperbound for the perturbation on this residual due to the errors $\{E_1, E_2\}$. This can be retrieved from the following lemma, taken from [11].

Lemma 3.1: Let the residual of the following least squares problem:

$$\min_x \|(A + E_1)x - (b + E_2)\|_2$$

be denoted by $\bar{r}$ and the actual residual of the unperturbed least squares problem $\min_x \|Ax - b)\|_2$ be denoted by r, then an upperbound for the error on r is:

$$\|\bar{r} - r\|_2 \leq \|E_2\|_2 + \frac{\|(A + E_1)^\dagger\|_2 \|E_1\|_2}{[1 + (\|(A + E_1)^\dagger\|_2 \|E_1\|_2)^2]^{\frac{1}{2}}} \tag{22}$$

The upperbound in (22) can for the errors under consideration be taken as:

$$\|\bar{r} - r\|_2 \leq \|E_2\|_2 + \|(A + E_1)^\dagger\|_2 \|E_1\|_2 \tag{23}$$

Assuming now that the original system was of order n, the application of lemma 3.1 yields the following upperbound for the residual r_{22} of (21):

$$|r_{22}| \leq (1 + \sqrt{n}\|(H_1(M, n) + E_1)^\dagger\|_2)\sigma\sqrt{M} \tag{24}$$

Therefore, by calculating this upperbound for different values of n, we just have to inspect the magnitude of the diagonal elements of the R-factor in (20), relative to these bounds, in order to detect the order of the system. This can be incorporated during the calculation of the QR factorization of the Hankel matrix, as is demonstrated by algorithm 3.2.

Remark 3.4: The diagonal elements r_{ii} of the R-factor in (20) can be expressed into the following product:

$$b_c(1) \cdot a_c(2, 1) \cdots a_c(i, i - 1)$$

what exactly corresponds to the controllability measure, defined for SISO in [12]. In this perspective, a small r_{ii} indicates the inclusion of a "weakly" controllable mode in the realization. The performed error analysis allows to precisely define the notion "weakly" here.

Algorithm 3.2:

$H_2^0 = H_1(M, N)$

$H_1^0 = \{\ \}$

for $i = 1 : N$

$$\left(\begin{array}{c|c} I_{i-1} & 0 \\ \hline 0 & Q_i \end{array}\right)\left(\begin{array}{c|c} H_1^{i-1} & \star \\ \hline 0 & H_2^{i-1} \end{array}\right) = \left(\begin{array}{c|c} H_1^i & \star \\ \hline 0 & H_2^i \end{array}\right) \tag{25}$$

(where I_i denotes the identity matrix of order i (with $I_0 = \{\ \}$)

$H_1^i \in R^{i \times i}$

$H_2^i \in R^{(M-i)\times(N-i)}$)

$$\text{if} \quad \|H_2^i(:,1)\|_2 \leq (1 + \sqrt{i}\|(H_1^i)^\dagger\|_2)\sigma\sqrt{M} \tag{26}$$

(where $H_2^i(:,1)$ denotes the first column of H_2^i)

then go to OUT:

end

OUT: $n = i$

Define the first $(i+1)$ columns of $[H_1^i \mid \star]$ in (25) as $[r_1 \cdots r_i | r_{i+1}]$ which is then used as an input for algorithm 3.1. to compute the triplet $\{A_c, b_c, c_c^T\}$

Remark 3.4: The quantity $\|(H_1^i)^\dagger\|_2$ in (26) can be computed by the inverse iteration method [13]. The iterations involved here are very efficient since H_1^i is upper triangular.

Remark 3.5: The calculation of $\|H_2^i(:,1)\|_2$ in (26), each step of the do-loop in algorithm 3.2, can be done very efficiently and recursively. This is because each two adjacent columns of the Hankel matrix only differ in their first and last element.

4. Approximate reduced-order modelling.

Because of the repetition of the Markov parameters and hence also their errors, the practical applicability in applying algorithm 3.2 straightforwardly might be limited.

The limitations are that the actual size of these errors on the Markov parameters have to be "relatively" very small in order to satisfy condition (19). Of course depending on the size of the dimensions of the taken Hankel matrix, the assumed upperbound σ in (18) might still vary. However, here we are facing another intrinsic limitation of the Hankel approach. Namely, that when the accuracy of the Markov parameters is "poor," the dimensions of the Hankel matrix should be "large" in order to get "good" error averaging.

This brief exposure demonstrates that the Hankel approach imbeds a number of major drawbacks when addressing realization problems where the given set of Markov parameters are contaminated by "significant" errors.

On the same time, these drawbacks have stimulated additional research in an area indicated nowadays as "reduced order modelling". The problem here is to approximate the given full-rank Hankel matrix by a low-rank matrix, which also has to be Hankel.

For a number of applications [8], the latter stringent requirement might be relaxed to only finding a low-rank approximation, which not necessarily possesses the Hankel structure. For this approximate reduced-order modelling problem, a realization scheme also based on the QR factorization of the given Hankel matrix, is summarized in algorithm 4.1. Here the order n of the reduced-order system is assumed to be given. Furthermore, it is demonstrated that column pivoting might be incorporated in the QR factorization.

Algorithm 4.1:

STEP 1: Compute a QR factorization ***with column pivoting***, indicated by the column permutation matrix Π, of the given Hankel matrix $H_1(M,N)$:

$$H_1(M,N)\Pi = (Q_1 \mid Q_2) \left(\begin{array}{c|c} R_{11} & R_{12} \\ \hline 0 & R_{22} \end{array} \right) \quad \Big\} n \tag{27}$$

From this decomposition, the Hankel matrix H_1 can be approximated by the matrix H_1^α, given as:

$$H_1^\alpha = Q_1[R_{11} \mid R_{12}]\Pi^T = Q_1[r_{c_1} \cdots r_{c_n} \mid r_{c_{n+1}} \cdots r_{c_N}] \tag{28}$$

with $\|H_1 - H_1^\alpha\|_2 = \|R_{22}\|_2$

STEP 2: From the decomposition given in (28), the state space triplet $\{A_c^\alpha, b_c^\alpha, (c_c^\alpha)^T\}$ is then realized by solving the following least squares problems:

$$\min_{A_c^\alpha} \|A_c^\alpha [r_{c_1} \cdots r_{c_{N-1}}] - [r_{c_2} \cdots r_{c_N}]\|_2 \tag{29}$$

$$\min_{c_c^\alpha} \|(c_c^\alpha)^T [r_{c_1} \cdots r_{c_{N-1}}] - [h_1 \cdots h_{N-1}]\|_2 \tag{30}$$

and b_c^α is given as:

$$b_c^\alpha = r_{c_1} \tag{31}$$

Equations (29-30) of algorithm 4.1 clearly demonstrate that the system triplet $\{A_c^\alpha, b_c^\alpha, c_c^\alpha\}$ is realized via the solution of a set of least squares problems. This is completely similar to the realization scheme based on the SVD [1]. On the same time, we observe from the same set of equations (29-30) that the condensed system structure of the realization is no longer preserved. Therefore, in comparison with algorithm 3.2, algorithm 4.1 is no longer *attractive*, however it still remains *elegant*. This elegantness is due to the fact that algorithm 4.1 also only needs the upper triangular factor of the QR factorization.

Furthermore, it is not necessary to incorporate column pivoting in algorithm 4.1. The necessity depends on the following trade-off. On the one hand, it makes the realization scheme computationally more *complex*, but on the other hand it allows the error in approximating the given Hankel matrix, as may be expressed by $\|R_{22}\|$, to be *minimized*. If, for example, the minimization of this error is crucial, it is recommended to use the column pivoting strategy described in [14] or [15].

In addition to the drawback of approximating the given Hankel matrix by a non-Hankel matrix (from which the system is realized), algorithm 4.1 also required the specification of the order of the reduced-order system. Since it was assumed that this information could not be retrieved from the second fundamental property of the Hankel approach, additional information is required.

5. Experimental evaluation.

In this section, algorithm 3.2 and 4.1 are evaluated experimentally. In this analysis, a third-order SISO system was taken as a test vehicle. The transfer function $\beta(z)$ of this plant is:

$$\beta(z) = \frac{0.005496z^2 + 0.020285z + 0.004672}{z^3 - 2.70066z^2 + 2.424258z - 0.72253} \tag{32}$$

and its impulse response is shown in Fig. 1.

The experimental study begins with an evaluation of the size of the errors e_i, for which algorithm 3.2 is still applicable. In judging the applicability we will evaluate the difference between the actual impulse response (calculated from (32)) and the one calculated from the realized state space triplet.

5.1. Determining the applicability of algorithm 3.2. In this part of the evaluation, we furthermore compare algorithm 3.2 with the realization technique based on the SVD [1] and the one by Rissanen [3].

In a *first* experiment no additional errors e_i on the Markov parameters, calculated from (32), are assumed. These Markov parameters were arranged in the Hankel matrix $H_1(200,6)$. From this Hankel matrix a state space triplet $\{A_{QR}, b_{QR}, c_{QR}^T\}$ was realized via algorithm 3.2 and using the SVD as described in [1] the triplet $\{A_{SVD}, b_{SVD}, c_{SVD}^T\}$ resulted. The impulse response calculated from both realizations coincides with the actual given one. This becomes clear from a comparison of the relative errors on the Markov parameters computed from each triplet. These relative errors are given as:

$$\frac{\|h_i^{\{A_{QR}, b_{QR}, c_{QR}^T\}} - h_i\|_2}{\|h_i\|_2} \tag{33}$$

and

$$\frac{\|h_i^{\{A_{SVD}, b_{SVD}, c_{SVD}^T\}} - h_i\|_2}{\|h_i\|_2} \tag{34}$$

for the triplet realized by the algorithm 3.2 and the one based on the SVD [1] respectively. Figure 2 graphically represents both these quantities and from this figure we clearly observe that the Markov parameters calculated from both realizations are of the same accuracy.

In a *second* experiment, additional errors e_i were considered. For the sequence $\{e_i\}$, defined in (18), a zero-mean white-noise sequence with standard deviation $\sigma_{e_i} = 3 \cdot 10^{-6}$ is taken. This resulted in very small errors on the Markov parameters relative to their original magnitude. The Hankel matrix $H_1(200,6)$ constructed from these perturbed Markov parameters is now used to realize two state space triplets based on the QR factorization

and the SVD. The relative errors (33) and (34) for this experimental condition are plotted in Fig. 3. From this figure, we observe that both relative errors are small and of the same order. However, we now observe from Fig. 3 that the results based on the SVD are more accurate. This discrepancy between the algorithm 3.2 and the method based on the SVD will increase when the size of the errors e_i increases. Whether this discrepancy remains very small for the region where algorithm 3.2 remains applicable is investigated more closely in the next experiment. Let us first evaluate the performance of the technique developed by J. Rissanen [3] for the perturbed Markov parameters of the second experiment. The computed impulse response from this realization is plotted in combination with the one given in Fig. 4. This clearly reveals the severe limitations of this realization scheme.

In the *third* experiment, we investigate the applicability of the order-detection mechanism, summarized by equation (26). For this purpose, three different values of σ_{e_i} were chosen, i.e. 10^{-7}, 10^{-6} and 10^{-5} respectively. The information used in detecting whether the system is of order **two** is graphically represented in Fig. 5a. From this figure we clearly observe (the original system being *third order*) that $\sigma_{e_i} = 10^{-5}$ determines the maximal size of the errors, for which (26) still detects the correct order. Of course the range of applicability of (26) could be somewhat extended for this application by decreasing the number of samples, however the size of the errors for which (26) detects the correct order, will remain *relatively very small.*

For $\sigma_{e_i} = 10^{-5}$, the calculation of the impulse response based on the calculated triplet via algorithm 3.2 remains very small. The relative error (33) is again of the same order as those depicted in Fig. 3. This third experiment clarifies that the size of the errors e_i for which algorithm 3.2 remains *applicable* have relatively to be *very small.*

In Fig. 5b, the information necessary for determining whether the system is of order **three** is graphically represented. Here, obviously no restrictions imply.

When the errors e_i increase in magnitude, by making σ_{e_i} larer than 10^{-5}, even the rank of the Hankel matrix becomes *ill-defined* via the SVD. This is because no longer a "clear gap" is present in the sequence of the singular values of the "perturbed" Hankel matrix. Therefore, this confirms that when the size of the errors e_i is outside the region of applicability of algorithm 3.2, also the scheme based on the SVD presented in [1] no

longer solves the partial realization problem straightforwardly. In this case, one focusses on the *reduced-order modelling* problem. The performance of algorithm 4.1 for this purpose is evaluated in the next section.

5.2. Approximate reduced-order modelling. In this experiment we considered errors e_i outside the region of applicability of algorithm 3.2, and took $\sigma_{e_i} = 10^{-3}$. The detection of the rank based on finding a gap in the singular values of the constructed Hankel matrix $H_1(100, 10)$ is *not possible*. Therefore, let us just postulate the order of the system. In this way, a system of order 3 and 9 were realized via algorithm 3.2. The computed impulse response from these realizations are respectively depicted in Fig. 6 and compared with the the given (noisy) impulse response. From Fig. 6 we clearly observe the bad performance of the third-order realized system and also the match of the ninth-order system is unsatisfactory. Another disadvantage of this last realization is although it is in its "condensed" controller Hessenberg form, its huge dimension (compared with the order of the actual system) will severely *penalize* the operation count of subsequent calculations based on that realization.

Using the same Hankel matrix $H_1(100, 10)$, we compared in, Fig. 7, the performance of algorithm 4.1 when no column pivoting was used with case where the column pivoting strategy described in [14] or [15] is used. In this same figure, we furthermore plotted the impulse response calculated from a realization obtained from a SVD of the given Hankel matrix. From this figure, we clearly observe the improvement and therefore also the necessity of using column pivoting in algorithm 4.1.

6. The conclusions.

In this paper, the use of the QR factorization is studied in solving the partial realization problem via the so-called Hankel approach [2].

Two different types of partial realization problems have been considered.

In the first type, the Markov parameters are indicated to be "accurately" given, meaning that the errors on these parameters do not destroy the correspondence between the numerical rank of the Hankel matrix and the order of the system.

For this type of realization problem, a new realization scheme has been proposed based on a straightforward QR factorization of the Hankel matrix. It has been clearly

demonstrated that this scheme outperforms the as "originally indicated computational attrative schemes" developed by J. Rissanen [3], and L.S. De Jong [4]. This is because while the new scheme considers all the given Markov parameters and is due to the use of orthogonal tranfomations numerically robust, it has also resulted in a realization which is in condensed system structure, i.e. the observer-Hessenberg or controller-Hessenberg form [9]. This allows the direct use of the vast amount of algorithms to solve in an *efficient and reliable manner* different system theoretical problems (see, for example, [9] for an overview).

In an experimental comparison, it has been demonstrated that for the type of realization problem under consideration, the new scheme has a numerical reliability which is similar to the realization scheme based on the SVD [1]. However, the computational complexity of this realization scheme as well as the retrieval of a condensed system structure make this new scheme much more attractive.

A second type of realization problems considered the errors on the Markov parameters to violate the correspondence between the numerical rank of the Hankel matrix and the order of the system. For this type of realization problems, an algorithmic scheme was presented which approximated the given Hankel matrix by a low-rank non-Hankel matrix also based on the QR factorization. This type of realization problem was referred to in this paper as "approximate reduced-order modelling" in order to distinguish from the treated "reduced-order modelling problems" [5], [6], and [7], where the Hankel structure is imposed on the low-rank approximation. It was demonstrated here that column pivoting might be incorporated in this scheme. In comparison with the solution presented for the previous type of realization problems, this second scheme (with column pivoting) possesses the same advantages over the SVD, except that now no condensed system structure is directly obtained.

The main conclusion of this research is that the QR factorization with or without column pivoting might replace the SVD as computational tool in solving partial realization problems. The challenging question now is whether it might also help conceptually to better understand difficult areas, such as approximating the given Hankel matrix by a low-rank matrix, which also has the Hankel structure, in the partial realization problem.

References

[1] S. Kung, *A new identification and model reduction algorithm via singular value decomposition*, Proc. 12th Asilomar Conf. Circuits, Syst., Comput., Pacific Grove, CA, pp. 705-714, Nov. 1979.

[2] T. Kailath, *Linear Systems*, Prentice Hall information and system sciences series, 1980.

[3] J. Rissanen, *Recursive identification of linear systems*, SIAM J. Control, Vol. 9, No. 3, pp. 420-430, Aug. 1971.

[4] L.S. de Jong, *Numerical aspects of recursive realization algorithms*, SIAM J. Control and Optimization, Vol 16, No. 4, pp. 646-659, July 1978.

[5] S. Kung, D.W. Sin, *Optimal Hankel-norm model reductions: multivariable systems*, IEEE Trans. AC-26, pp. 832-852, Aug. 1981.

[6] V.M. Adamjan, D.Z. Arov and M.G. Krein, *Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem*, Math. USSR Sbornik, Vol. 15, no. 1, pp. 31-73, 1971.

[7] K. Glover, *All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds*, Int. J. Control, Vol. 39, No.6, pp. 1115-1193, 1984.

[8] J.N. Juang and R.S. Pappa, *An eigensystem realization algorithm for modal parameter identification and model reduction*, J. Guidance, Control and Dynamics, Vol.8, No.5, pp. 620-627, 1985.

[9] P. Van Dooren, M.H. Verhaegen, *On the use of unitary state-space transformations*, in the Special Issue of Contemporay Mathematics on Linear Algebra and its Role in System theory, AMS, 1985.

[10] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc., 51, pp. 506-513, 1955.

[11] G.W. Stewart, *On the perturbations of pseudo-inverses, projections and linear least squares problems*, SIAM Review, Vol. 19, No.4, pp. 634-662, Oct. 1977.

[12] D.L. Boley, *Computing the Controllability/Observability Decomposition of a Linear Time-Invariant Dynamic System, A numerical approach*, PhD dissertation, Stanford University, Dept. Comp. Sc., 1981.

[13] G.H. Golub, C.F. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, 1983.

[14] L.V. Foster, *Rank and Null Space calculations using matrix decomposition without column interchanges*, to appear in Linear Algebra and its Applications.

[15] T. Chan, *Rank Revealing QR-factorizations*, Research Report YALEU/DCS/RR-398, Yale University, dept. Comp. Sc., June 1985.

Fig. 1: Impulse response of the third-order system with transfer function (32).

Fig. 2: The relative error on the Markov parameters calculated from a state space triplet realized from a QR and SVD decomposition assuming noise-free Markov parameters given.

Fig. 3: The relative error on the Markov parameters calculated from a state space triplet realized from a QR and SVD decomposition assuming perturbed Markov parameters ($\sigma_{e_i} = 3 \cdot 10^6$).

Fig. 4: Comparison of the impulse response realized via the technique developed by J. Rissanen with the given noisy impulse response ($\sigma_{e_i} = 3 \cdot 10^6$).

Fig. 5a: Evaluation of (26) in algorithm 3.2 in detecting whether the underlying system is of order two, for different error levels σ_{e_i}.

Fig. 5b: Evaluation of (26) in algorithm 3.2 in detecting whether the underlying system is of order three, for different error levels σ_{e_i}.

Fig. 6: The impulse response calculated from a state space model of orders 3 and 9 realized by algorithm 3.1.

Fig. 7: The performance of algorithm 4.1 with and without column pivoting.

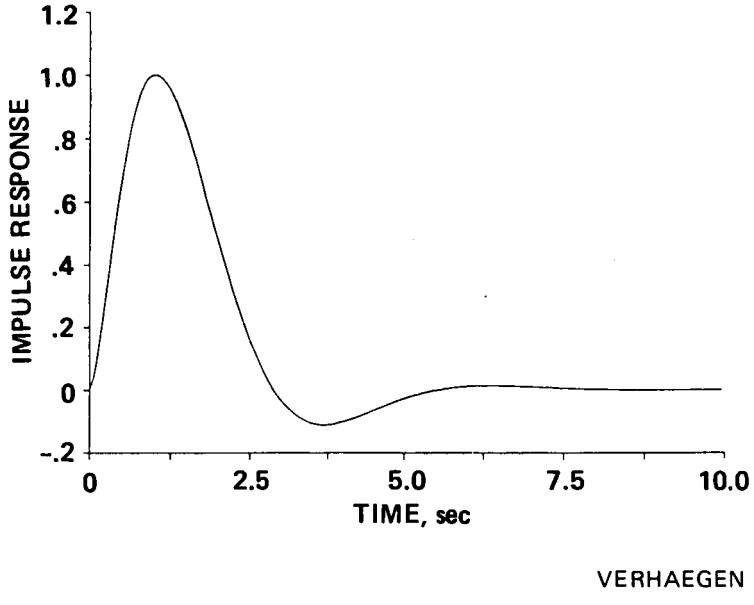
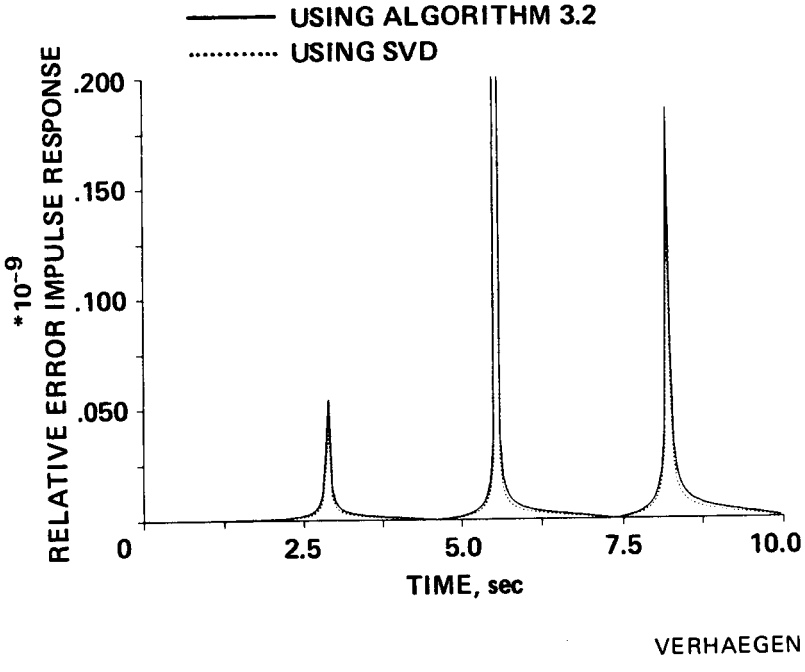

1.2
1.0
.8
.6
.4
.2
0
-.2
IMPULSE RESPONSE
0
2.5
5.0
7.5
10.0
TIME, sec
VERHAEGEN

Fig. 1


USING ALGORITHM 3.2
USING SVD
RELATIVE ERROR IMPULSE RESPONSE
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.100
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0
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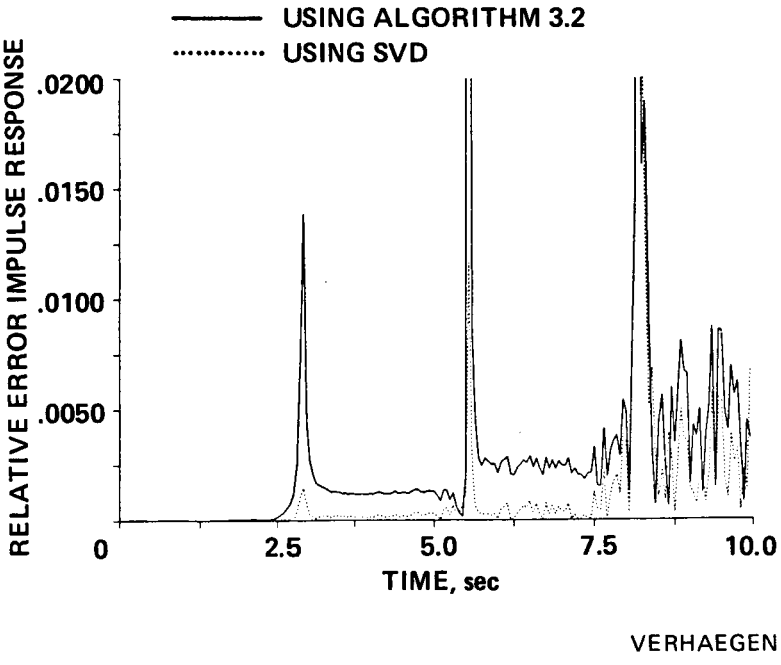
5.0
7.5
10.0
TIME, sec
VERHAEGEN

Fig. 2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USING SVD
RELATIVE ERROR IMPULSE RESPONSE
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7.5
10.0
TIME, sec
VERHAEGEN

Fig. 3

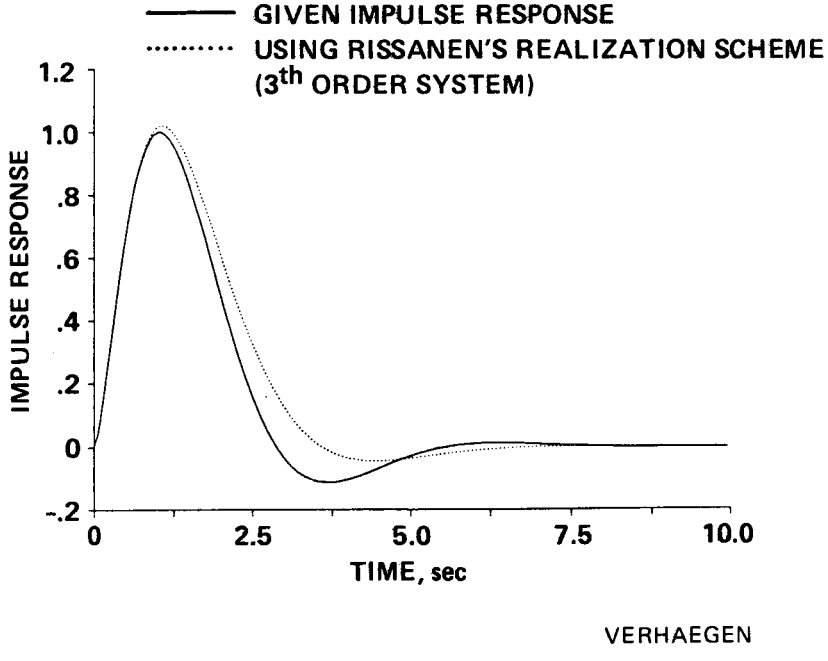
GIVEN IMPULSE RESPONSE
USING RISSANEN'S REALIZATION SCHEME
(3th ORDER SYSTEM)
IMPULSE RESPONSE
1.2
1.0
.8
.6
.4
.2
0
-.2
0
2.5
5.0
7.5
10.0
TIME, sec
VERHAEGEN

Fig. 4

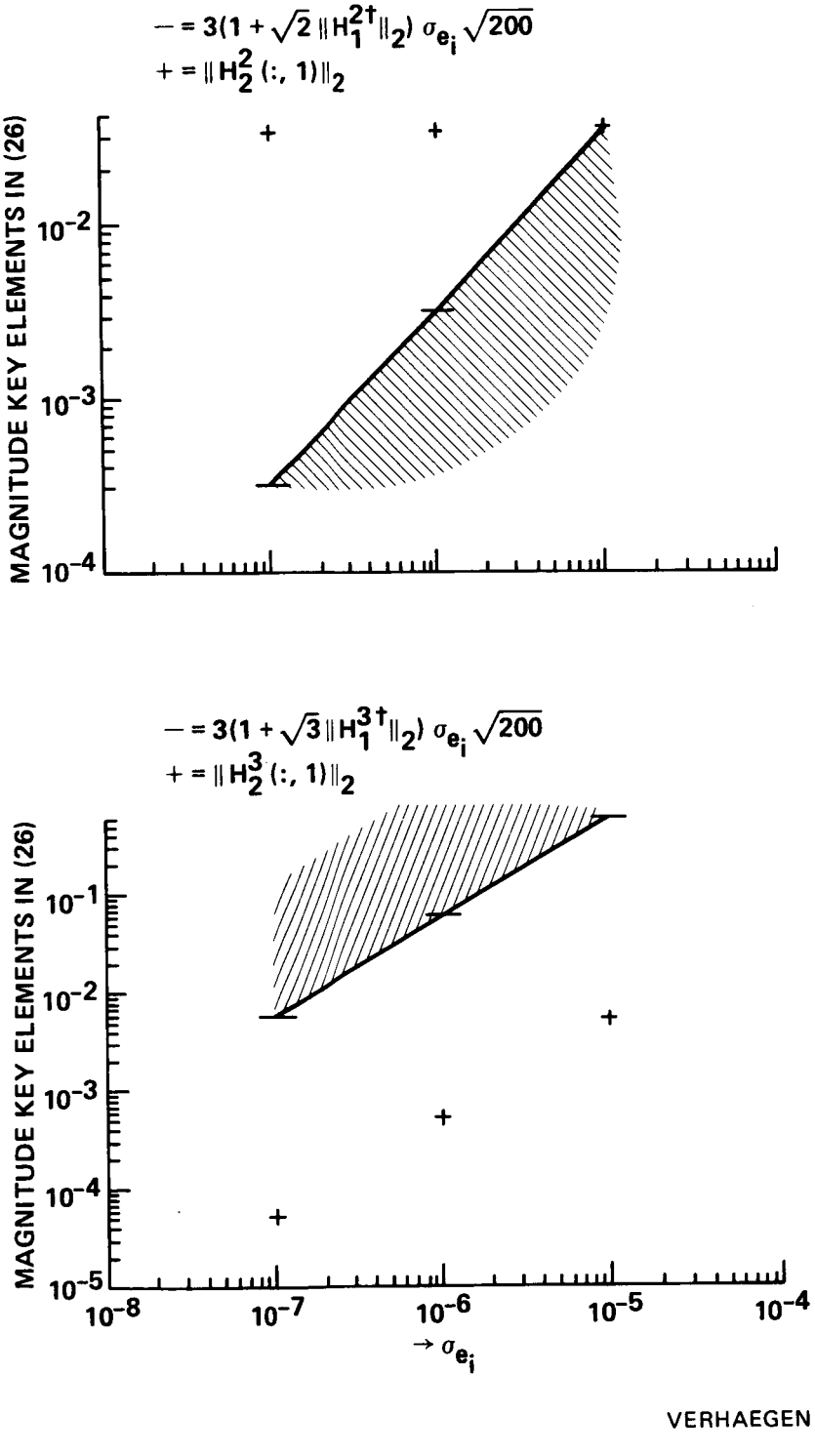
— = 3(1 + √2 ‖H₁^{2†}‖₂) σ_{e_i} √200
+ = ‖H₂²(:, 1)‖₂
MAGNITUDE KEY ELEMENTS IN (26)
10⁻²
10⁻³
10⁻⁴
— = 3(1 + √3 ‖H₁^{3†}‖₂) σ_{e_i} √200
+ = ‖H₂³(:, 1)‖₂
MAGNITUDE KEY ELEMENTS IN (26)
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
10⁻⁸
10⁻⁷
10⁻⁶
10⁻⁵
10⁻⁴
→ σ_{e_i}
VERHAEGEN

Fig. 5

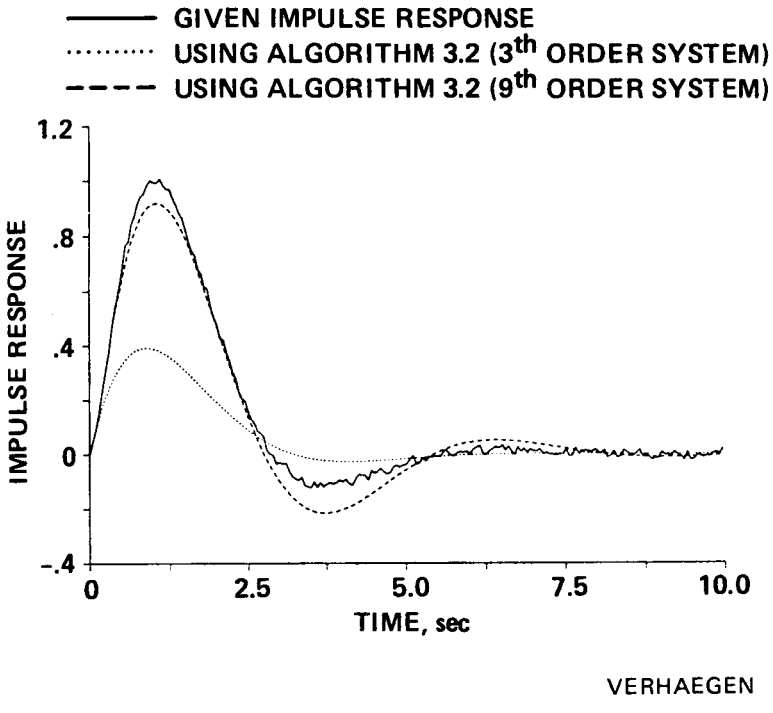
GIVEN IMPULSE RESPONSE
USING ALGORITHM 3.2 (3th ORDER SYSTEM)
USING ALGORITHM 3.2 (9th ORDER SYSTEM)
IMPULSE RESPONSE
1.2
.8
.4
0
-.4
0
2.5
5.0
7.5
10.0
TIME, sec
VERHAEGEN

Fig. 6

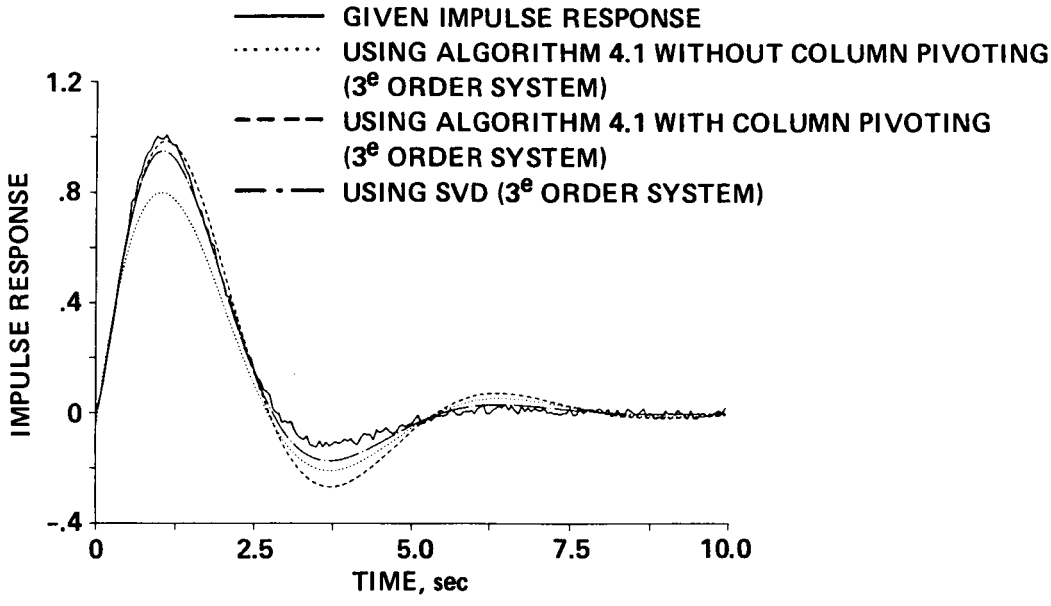
GIVEN IMPULSE RESPONSE
USING ALGORITHM 4.1 WITHOUT COLUMN PIVOTING
(3e ORDER SYSTEM)
USING ALGORITHM 4.1 WITH COLUMN PIVOTING
(3e ORDER SYSTEM)
USING SVD (3e ORDER SYSTEM)
IMPULSE RESPONSE
1.2
.8
.4
0
-.4
0
2.5
5.0
7.5
10.0
TIME, sec
VERHAEGEN

Fig. 7

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16. Abstract

The use of the QR factorization of the Hankel matrix in solving the partial realization problem is analyzed in this paper.

Straightforward use of the QR factorization results in a new realization scheme that possesses all of the computational advantages of Rissanen's realization scheme. These latter properties are computational efficiency, recursiveness, use of limited computer memory, and the realization of a system triplet having a condensed structure. Moreover, this new scheme is robust when the order of the system corresponds to the rank of the Hankel matrix.

When this latter condition is violated, an "approximate" realization could be determined via the QR factorization. In this second scheme, the given Hankel matrix is approximated by a low-rank non-Hankel matrix. Furthermore, it is demonstrated that column pivoting might be incorporated in this second scheme.

The results presented in this paper are derived for a single-input/single-output system, but this does seem not to be a restriction.

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