# Improvement of Modal Scaling Factors Using Mass Additive Technique

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This paper presents a general investigation into the improvement of modal scaling factors of an experimental modal model using mass additive technique. Data base required by the proposed method consists of an experimental modal model (a set of complex eigenvalues and eigenvectors) of the original structure and a corresponding set of complex eigenvalues of the mass-added structure. Three analytical methods, i.e., first order and second order perturbation methods, and local eigenvalue modification technique, are proposed to predict the improved modal scaling factors. Difficulties encountered in scaling closely spaced modes are discussed. Methods to compute the necessary rotational modal vectors at the mass additive points are also proposed to increase the accuracy of the analytical prediction.

## INTRODUCTION

For most applications of experimental modal database, the identified modal vectors are expected to be normalized according to $\{y_r\}^T[U]\{y_r\} = 1$. But in practice, this relation is rarely satisfied and it becomes $\{y_r\}^T[U]\{y_r\} = 1/\alpha_r^2$, where $\alpha_r$ is an unknown scaler of the $r^{th}$ modal vector.

Modal scaling errors, characterized by $\alpha_r$, are disastrous for certain applications of experimental modal model, such as substructure synthesis, structural modification and adjustment finite element model[1-3]. Wei[4] analyzed the sources of modal scaling errors which are summarized as follows :

1. Local and global calibration errors.

2. Digital signal processing and FFT leakage errors.

3. Improper orientation of the force or response transducer at the driving point.

4. Low signal to noise ratio in the driving point measurement data.

Another possible cause of modal scaling errors is non-linearities of structures. A typical nonlinear example given by Lallement[5] showed that for a beam with nonlinear stiffness, lower natural frequencies and their corresponding mode shapes of a non-linear beam are very close to those of the same beam without the prescribed non-linear characteristics. However, $\alpha_r$ could vary from 0.4 to 1.3 .

From the above discussions, it can be concluded that modal scaling errors are almost inevitable in the applications of the prevailing modal testing techniques on practical structures. It is intended in this paper to determine the unknown scaler $\alpha_r$ by a supplementary testing technique, i.e., mass additive technique. Some of the work in correcting modal scaling errors using mass additive technique[6] has been done in the past using modal modeling technique. In this

paper, three analytical methods: first order and second order perturbation methods, and local eigenvalue modification technique are presented to predict the unknown modal scaling factors.

## FIRST ORDER PERTURBATION APPROACH

The equation of motion for the general case of nonproportional damping is :

$$M \ddot{y}(t) + C \dot{y}(t) + K y(t) = f(t) \tag{1}$$

and its associated eigenvalue problem can be written as :

$$\lambda_r \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} \psi_r \\ \lambda_r \psi_r \end{Bmatrix} = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \psi_r \\ \lambda_r \psi_r \end{Bmatrix} \tag{2}$$

or in the condensed form :

$$\lambda_r [U]\{y_r\} = [V]\{y_r\} \tag{3}$$

where :

$$\{y_r\} = \begin{Bmatrix} \psi_r \\ \lambda_r \psi_r \end{Bmatrix}, \quad \{y_r\} \in C^{2N}$$

$$[U] = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}, \quad [U] \in R^{2N}, \quad [U] = [U]^T$$

$$[V] = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \quad [V] \in R^{2N}, \quad [V] = [V]^T$$

Defining a perturbated system :

$$\hat{\lambda}_r [U + \varepsilon \underline{U}]\{\hat{\psi}_r\} = [V + \varepsilon \underline{V}]\{\hat{\psi}_r\} \tag{4}$$

Where $\varepsilon$ is the perturbation parameter, and the elements of matrices $\underline{V}$ and $\underline{U}$ are of the same order as those of matrices [U] and [V] respectively.

The eigenvalue $\hat{\lambda}_r$ may be expanded as power series in $\varepsilon$ :

$$\hat{\lambda}_r = \lambda_r + \varepsilon \lambda_r^{(1)} + \varepsilon^2 \lambda_r^{(2)} + \cdots \tag{5}$$

In the case of small perturbations ( $\varepsilon \ll 1$ ), the higher order terms of $\varepsilon$ can be neglected, therefore Equation 5 can be written as :

$$\hat{\lambda}_r = \lambda_r + \varepsilon \lambda_r^{(1)} + \bigcirc(\varepsilon^2) \tag{6}$$

The expression of $\lambda_r^{(1)}$ is given in numerous lectures [7,8] :

$$\lambda_r^{(1)} = \{y_r\}^T [\underline{V} - \lambda_r \underline{U}]\{y_r\} \tag{7}$$

Equation 7 is based on the assumption :

$$\{y_r\}^T[\,U\,]\{y_r\} = 1. \tag{8}$$

Considering the modal scaling errors, Equation 7 can be rewritten as :

$$\lambda_r^{(1)} = \{y_r\}^T[\,\underline{V} - \lambda_r \underline{U}\,]\{y_r\}\alpha_r^2 \tag{9}$$

Where $\{y_r\}$ is the identified modal vector with modal scaling errors:

$$\{y_r\}^T[\,U\,]\{y_r\} = 1/\alpha_r^2 \tag{10}$$

If the structural perturbation consists of only the mass matrix, $\Delta M = \varepsilon \underline{M}$, Equation 9 can be written as :

$$\lambda_r^{(1)} = -\lambda_r^2\{\psi_r\}^T\underline{M}\{\psi_r\}\alpha_r^2 \tag{11}$$

In the present problem, the mass added structure is considered as the perturbated system and the perturbated eigenvalues $\hat{\lambda}_r$ and both eigenvalues and eigenvectors of the original system have been identified. Using small added masses, the unknown scaler $\alpha_r$ can be directly derived from the first order approximation of $\hat{\lambda}_r$ :

$$\Delta\lambda_r = \hat{\lambda}_r - \lambda_r = \varepsilon\lambda_r^{(1)} = -\lambda_r^2\{\psi_r\}^T\Delta M\{\psi_r\}\alpha_r^2 \tag{12}$$

Thus :

$$\alpha_r = \sqrt{\Delta\lambda_r/(-\lambda_r^2\{\psi_r\}^T\Delta M\{\psi_r\})} \tag{13}$$

## SECOND ORDER PERTURBATION APPROACH

It is important to note that the first order perturbation approach is applicable only in the case of small perturbations. However, from a practical point of view, it is desirable to hold $\Delta\lambda_r$ from 5% to 10% of $\lambda_r$. Therefore, the second order perturbation approach is introduced to improve the accuracy of the estimated modal scaling factors :

$$\Delta\lambda_r = \hat{\lambda}_r - \lambda_r = \varepsilon\lambda_r^{(1)} + \varepsilon^2\lambda_r^{(2)} + \bigcirc(\varepsilon^3) \tag{14}$$

The expression of $\lambda_r^{(2)}$ is :

$$\lambda_r^{(2)} = \sum_{j=1, j\neq r}^{2N} [-\lambda_r^2\{\psi_j\}^T[\underline{M}]\{\psi_r\}\alpha_r\alpha_j]^2/(\lambda_r - \lambda_j) - 2\lambda_r\{\psi_r\}^T[\underline{M}]\{\psi_r\}\lambda_r^{(1)}\alpha_r^2 \tag{15}$$

Obviously Equation 14 is nonlinear in $\varepsilon_r$. Therefore an iterative process is applied to calculate $\varepsilon_r$. This iterative process consists of three steps :

1) Determining initial estimation of $\alpha_r^{(0)}$ :

$$\alpha_r^{(0)} = \sqrt{(\hat{\lambda}_r - \lambda_r)/(-\lambda_r^2\{\psi_r\}^T\Delta M\{\psi_r\})} \tag{16}$$

2) Calculating $\varepsilon^2\lambda_r^{(2)}$ :

$$\varepsilon^2\lambda_r^{(2)} = \sum_{j=1, j\neq r}^{2N} [-\lambda_r^2\{\psi_j\}^T[\Delta M]\{\psi_r\}\alpha_r^{(0)}\alpha_j^{(0)}]^2/(\lambda_r - \lambda_j) - 2\lambda_r^3[\{\psi_r\}^T[\Delta M]\{\psi_r\}(\alpha_r^{(0)})^2]^2 \tag{17}$$

3) Recalculating $\alpha_r^{(1)}$ :

$$\alpha_r{}^{(1)} = \sqrt{(\hat{\lambda}_r - \lambda_r - \varepsilon^2 \lambda_r{}^{(2)})/(-\lambda_r^2 \{\psi_r\}^T \Delta M \{\psi_r\})} \tag{18}$$

Steps 2 and 3 are repeated until the convergence condition :

$$| \alpha_r{}^{(k)} - \alpha_r{}^{(k-1)} | < \sigma$$

is satisfied by the $k^{th}$ iteration, where $\sigma$ is a given positive quantity. In general, $\alpha_r{}^{(k)}$ converges to a stable value very quickly.

## LOCAL EIGENVALUE MODIFICATION

In most practical cases, the second order perturbation approach gives a reasonable estimation of modal scaling factors $\alpha_r$ . But if the structure under study possesses very closed eigenvalues ( $|(\lambda_r - \lambda_{r+1})/\lambda_r| < 1\%$ ), the perturbation method is no longer valid. Zhang and Lallement[9] proposed a method based on modal space representation which would be suitable in this delicate case. This method requires not only the eigenvalues of the mass-added structure but also the modified eigenvectors. In this section, a method which does not require the eigenvectors of the mass added structure is presented : a method based on the local eigenvalue modification technique.

Recall[10] that if the structure is perturbated only at a single degree of freedom, the eigenvalues of the perturbated structure are of the roots of the following equation :

$$1/m = \sum_{r=1}^{N} (\hat{\lambda} \lambda_r^2 \psi_{kr}^2)/(\lambda_r - \hat{\lambda}) \tag{19}$$

Where :

$\lambda_r$ and $\hat{\lambda}$ are respectively the eigenvalues of the initial structure and the perturbated structure.
$\psi_{kr}$ is the $k^{th}$ component of the $r^{th}$ modal vector of the initial structure.
m is the mass attached at the degree of freedom k.

Because of the modal scaling error, Equation 19 becomes :

$$1/m = \sum_{r=1}^{n} (\hat{\lambda} \lambda_r^2 \psi_{kr}^2 \alpha_r^2)/(\lambda_r - \hat{\lambda}) \tag{20}$$

Since the eigenvalues of the perturbated structure $\hat{\lambda}_j$ ,j $= 1,2,\cdots,N$, are known, substitution of these values into Equation 20 yields :

$$\begin{Bmatrix} 1/m \\ \cdot \\ \cdot \\ \cdot \\ 1/m \end{Bmatrix} = \begin{bmatrix} (\hat{\lambda}_1 \lambda_1^2 \psi_{k1}^2)/(\lambda_1 - \hat{\lambda}_1) & \cdot\cdot & (\hat{\lambda}_1 \lambda_N^2 \psi_{kN}^2)/(\lambda_N - \hat{\lambda}_1) \\ & \cdot & & \cdot \\ & \cdot & \cdot\cdot & \cdot \\ & \cdot & \cdot\cdot & \cdot \\ (\hat{\lambda}_N \lambda_1^2 \psi_{k1}^2)/(\lambda_1 - \hat{\lambda}_N) & \cdot\cdot & (\hat{\lambda}_N \lambda_N^2 \psi_{kN}^2)/(\lambda_N - \hat{\lambda}_N) \end{bmatrix} \begin{Bmatrix} \alpha_1^2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_N^2 \end{Bmatrix} \tag{21}$$

or in the condensed form :

$$\{1/m\} = [Q]\{\alpha^2\} \tag{22}$$

and then the unknown modal scaling factors { $\alpha^2$ } are determined by solving Equation 22 :

$$\{\alpha^2\} = [Q]^{-1}\{1/m\} \tag{23}$$

There are several comments needed to be made on the proposed local eigenvalue modification technique:

1. Equation 20 implies the normalization relations :

$$\{y_r\}^T[U]\{y_r\} = 1/\alpha_r^2 \ , \quad \{y_r\}^T[V]\{y_r\} = \lambda_r/\alpha_r^2$$

where :

$$[U] = \begin{bmatrix} -C & 0 \\ 0 & K \end{bmatrix}, \quad [V] = \begin{bmatrix} C & K \\ K & 0 \end{bmatrix}, \quad [V] \in R^{2N} \ , \quad [V] = [V]^T$$

2. The major numerical difficulty in dealing with closely spaced modes is due to the following facts :

   - The difference between two adjacent eigenvalues $\lambda_r - \lambda_j$ appears in the denominators as shown in Equation 17.

   - It is difficult to make the correspondence between eigenvalues of the original structure and those of the mass-added structure.

   From Equation 21, it is noted that (i) the term ( $\lambda_r - \lambda_j$ ) does not appear in this equation, and (ii) the interchange of two modified eigenvalues does not affect the solution $\{\alpha^2\}$. Therefore, this method is very effective in dealing with closely spaced modes.

3. The disadvantage of this method is that it is only applicable in the case of a single point and a single degree of freedom modification. In practice, this implies that the dynamics effect of the added mass must be dominant by only one of the six degrees of freedom at the mass attachement point. This theoretical restriction can cause real difficulty in choosing the proper location of the added mass.

To avoid this problem, one of the options is to replace the additive mass by a feedback device , i.e., a device which can provide a force proportional to the acceleration at the mass attachment point i ( $f_i = m \ddot{x}_i$ ). Another alternative is to modify the structure by adding a single spring instead of a mass. In case of adding a spring, Equation 20 becomes :

$$-1/k = \sum_{r=1}^{n} (\psi_{kr}^2 \, \alpha_r^2)/(\lambda_r - \hat{\lambda}) \tag{24}$$

where k is the spring constant.

## PRACTICAL CONSIDERATIONS: Effective Mass

Rotational degrees of freedom information at the mass attachment point(s) is also important in the application of the mass additive technique when the moment of inertia of the added mass(es) is not negligible. Various methods have been proposed to measure or predict the rotational degree of freedom frequency response functions or modal coefficients at a point on the structure[11,12], but all methods have their technical limitations.

In this section, the concept of effective mass is introduced to include the dynamics effect of the unmeasured rotational degree of freedom information. Define the effective mass associated with the $r^{th}$ mode $m_{e\!f\!r}$ as :

$$m_{e\!f\!r} = m_{ad} + \Delta m_r \tag{25}$$

where $m_{ad}$ is the added mass and $\Delta m_r$ represents the additional mass due to the effect of the rotational degrees of freedom of mode r.

The estimation of $\Delta m$ is based on the assumption that the kinetic energy due to rotation ( $J \dot{\Theta}^2$ ) can be replaced by

an equivalent translational kinetic energy ( $\Delta m \, \dot{z}^2$ ) along the z axis.

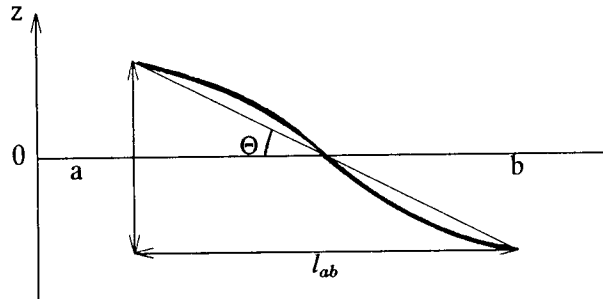For a one dimensional problem, total kinetic energy due to the added mass at point k can be written as :

$$E = m_{ad}\dot{z}_k^2 + J\dot{\Theta}_k^2$$

where J is the moment of inertia of the added mass.

The angle of rotation $\Theta_k$ as shown in Figure 1 can be approximately calculated :

$$\Theta_k \approx tangent\Theta_k = (z_a - z_b)/l_{ab} \tag{26}$$

where $l_{ab}$ is the distance between points a and b .



**Figure 1.** Estimation of Rotational DOF from Translational DOF

Equating $\Delta m\dot{z}_k^2$ to $J\dot{\Theta}_k^2$ , i.e., E = ( $m_{ad} + \Delta m$ )$\dot{z}_k^2$ , the scaler $\Delta m$ is determined :

$$\Delta m = J\dot{\Theta}_k^2/\dot{z}_k^2 = J(\dot{z}_a - \dot{z}_b)^2/(l_{ab}\dot{z}_k)^2 \tag{27}$$

Denoting $E_r$ as the total kinetic energy of the added mass associated with the $r^{th}$ mode :

$$E_r = \lambda_r^2(m_{ad} + \Delta m_r)\psi_{kr}^2$$

where :

$$\Delta m_r = J(\psi_{ar} - \psi_{br})^2/(l_{ab}\psi_{kr})^2 \tag{28}$$

Equation 28 shows that the effective mass $m_{ef} = m_{ad} + \Delta m_r$ is different for each mode.

## CASE STUDY

In order to demonstrate the effectiveness of the proposed methods, a case study was performed on a lightly damped steel T plate shown in Figure 2. The Polyreference time domain modal parameter estimation method[13] was used to identify the eigenvalues and the eigenvectors of the unmodified T plate (4 references and 24 measurement points). Eigenvalues of the modified structure with two additive masses,i.e, mass 1: 97 grams at point 3 and mass 2: 43 grams at point 22, were extracted from the frequency response functions measured at four reference and mass attachment points. The first mode shapes of the original T plate are plotted in Figure 3.
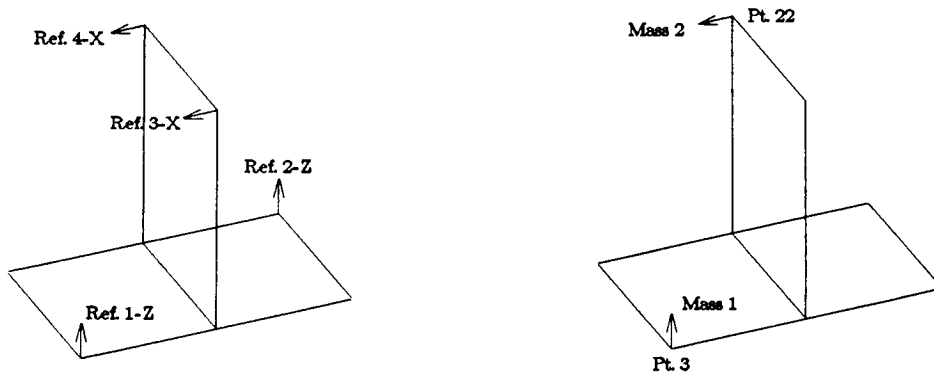
**Figure 2.** Test Structure: A Steel T Plate



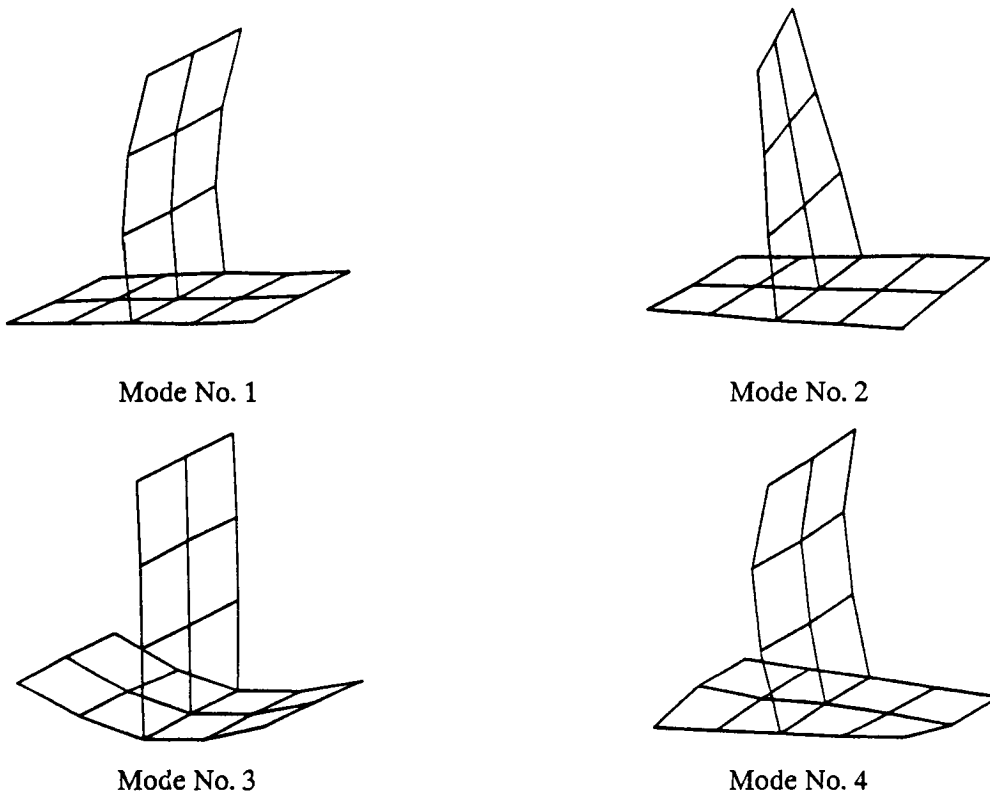Mode No. 1

Mode No. 2

Mode No. 3

Mode No. 4

**Figure 3.** First Four Mode Shapes

The first four damped natural frequencies and modal damping ratios of the original and the modified T plate are listed in Table 1. The identified complex modal vectors of the original T plate were normalized according to Equation 8. The modal scaling factors $\alpha_r$ calculated by using the first and second order perturbation methods are listed in Table 2.

In order to verify the calculated modal scaling factors listed in Table 2, a structural modification software (DYNOP developed by U.C. SDRL) is used to predict the eigenvalues of the modified T plate with a single mass added at point 3. Modified eigenvalues were derived for both the original and rescaled (using the modal scaling factors listed in Table 2) experimental modal model. The predicted eigenvalues and the measured data are listed in Table 3.

**TABLE 1.** Measured Complex Eigenvalues

| Measured Complex Eigenvalues | | | | |
|---|---|---|---|---|
| Mode No. | Original Structure | | Modified T Plate(2 masses) | |
| | f(Hz) | $\varsigma(\%)$ | f(Hz) | $\varsigma(\%)$ |
| 1 | 177.37 | .984 | 171.07 | 1.322 |
| 2 | 334.20 | .441 | 322.50 | .566 |
| 3 | 411.10 | .487 | 392.02 | .558 |
| 4 | 581.7 | .286 | 550.94 | .503 |

**TABLE 2.** Calculated Modal Scaling Factors

| Calculated Modal Scaling Factors, $\alpha_r$ | | | | |
|---|---|---|---|---|
| Mode No. | First Order Approach | | Second Order Approach | |
| | real | imaginary | real | imaginary |
| 1 | 0.95 | 0.05 | 0.96 | 0.06 |
| 2 | 1.05 | 0.02 | 1.09 | 0.01 |
| 3 | 0.98 | 0.04 | 1.01 | 0.03 |
| 4 | 0.76 | -0.002 | 0.68 | 0.002 |

**TABLE 3.** Complex Eigenvalues of the Modified T Plate

| Eigenvalues of the Modified T Plate(1 mass) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Mode No. | Measured Eigenvalues | | Predicted Eigenvalues Using Original $\{\psi_r\}$ | | | | Predicted Eigenvalues Using Rescaled $\{\psi_r\}$ | | | |
| | f(Hz) | $\varsigma(\%)$ | f(Hz) | Err(%) | $\varsigma(\%)$ | Err(%) | f(Hz) | Err(%) | $\varsigma(\%)$ | Err(%) |
| 1 | 172.8 | 1.289 | 173.1 | .16 | .894 | -30.6 | 172.8 | 0 | 1.166 | -9.5 |
| 2 | 331.8 | .521 | 333.2 | .41 | .469 | -10.0 | 333.4 | .48 | .468 | -10.0 |
| 3 | 392.5 | .594 | 389.7 | -.72 | .284 | -52.0 | 392.0 | -.13 | .483 | -18.7 |
| 4 | 556.5 | .529 | 550.5 | -1.06 | .437 | -17.3 | 563.9 | 1.34 | .305 | -42.3 |

From the results of this case study, it is noticed that:

1. Since all modal testings of the T plate were very carefully conducted, and the structure is perfectly linear, therefore, all but the last $\alpha_r$ are relatively small and within ±10% of 1.0 as shown in Table 2.

2. It was pointed out in Reference[14] that the analytical predictions of modified eigenvalues using modal modeling technique could be corrupted due to modal truncation error. It was also indicated that the last few predicted damped natural frequencies are usually greater than the true ones.

From Table 3, the 4th damped natural frequency derived from the original modal vectors is less than the measured data. On the contrary, the one from the rescaled modal vectors is greater. This indicates that the damped natural frequencies based on the rescaled modal vectors are more reliable.

## CONCLUSIONS

The three analytical methods proposed in this paper can be used to improve the modal scaling factors of the measured modal vectors from the structure with or without close-spaced modes. The main advantage of these methods is that it only requires complex eigenvalues of the mass-added structure as additional information, i.e., only a few additional measurements are needed to obtain the complex eigenvalues of the mass-added structure.

The concept of the effective mass is introduced to include the rotational effects of the additive mass(es) due to the lack of test data at the rotational degrees of freedom. Case studies based on this concept can be found in Reference [15].

The proposed methods also have the following limitations:

1. The perturbation technique is applicable only in the absence of closely spaced modes.

2. The local eigenvalue modification technique is limited to a single degree of freedom modification.

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