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DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
COLLEGE OF ENGINEERING AND TECHNOLOGY
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA 23508

GUIDANCE AND CONTROL STRATEGIES FOR AEROSPACE VEHICLES

By

Joseph L. Hibey, Principal Investigator

and

D. S. Naidu, Associate Research Professor

Progress Report
For the period July 1 to December 31, 1987

Prepared for the
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23665

Under
NASA Grant NAG-1-736
Dr. D. B. Price, Technical Monitor
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PROGRESS REPORT

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Summary of Research Work

A simplified method of matched asymptotic expansions has been developed where the common part in composite solution is generated as a polynomial in stretched variable instead of actually evaluating the same from outer solution (see items (i) and (ii) in the enclosed list of publications). This methodology has been applied to the solution of the exact equations for three dimensional atmospheric entry problem. A composite solution is formed in terms of an outer solution, an inner solution, and a common solution. The outer solution is obtained from gravitationally dominant region, whereas the aerodynamically dominant region contributes to the inner solution. The common solution accounts for the overlap between the outer and inner regions. In comparison to the previous works, the present simplified methodology yields explicit analytical expressions for various components of the composite solution without resorting to any type of transcendental equations to be solved only by numerical methods. (See item (iii) in the enclosed list of publications).

In the next stage, we address the optimal control problem arising in the noncoplanar orbital transfer employing aeroassist technology. The maneuver involves the transfer from high Earth orbit to low Earth orbit with a prescribed plane change and at the same time minimization of the time integral of the heating rate of the spacecraft. With a suitable performance index, we formulate the optimal control problem. Using Pontryagin minimum principle, the state and costate equations are obtained, leading to the nonlinear two-point boundary-value problem. This problem is solved numerically by using multiple shooting method (see item (iv) in the list of publications). On similar lines, the optimal control problem for coplanar orbital transfer is also being investigated using multiple shooting method.

During the same period, other related research works have been carried out and are briefly mentioned below.

1. An important work in the same period is the final preparation of the forthcoming book entitled, "SINGULAR PERTURBATION METHODOLOGY IN CONTROL SYSTEMS, authored by Dr. D. S. Naidu, and being published under IEE Control Engineering Series, by Peter Peregrinus Limited, Stevenage Herts, England. This book is scheduled to appear in February 1988.(See item (v) in the enclosed list of publications).

2. As an outgrowth of earlier work on singular perturbations and time scales in discrete control systems, it has been found that to a zeroth order approximation, these two approaches yield identical results. (See item (vi) in the enclosed list of publications).

3. Other work is concerned with the preparation of a NASA Technical Publication (see item (vii) in the list of publications).

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List of Publications

(i) D. S. Naidu and D. B. Price, "On the method of matched asymptotic expansions", SIAM Annual Meeting and 35th Anniversary, Denver, CO, October 12-15, 1987.

*(ii) D. S. Naidu and D. B. Price, "On the method of matched asymptotic expansions", accepted for publication in Journal of Guidance, Control and Dynamics, 1988.

*(iii) D. S. Naidu, "There-dimensional atmospheric entry problem using method of matched asymptotic expansions", Accepted for presentation at 1988 American Control Conference, Atlanta, GA, June 14-17, 1988.

*(iv) D. S. Naidu, "Optimal control of aeroassisted noncoplanar orbital transfer vehicles", Draft paper

(v) D. S. Naidu, "Singular Perturbation Methodology in Control Systems", IEE Control Engineering Series, Peter Peregrinus Ltd., Stevenage Herts, England, 1988. (in press)

(vi) D. S. Naidu and D. B. Price, "On singular perturbation and time scale approaches in discrete control systems", accepted for publication in Journal of Guidance, Control and Dynamics, 1988 (in press).

(vii) D. S. Naidu and D. B. Price, "Singular perturbations and time scales in digital flight control systems", NASA Technical Publication, Spacecraft Control Branch, Langley Research Center, Hampton (in preparation).

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ON THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

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INTRODUCTION

Singular perturbation problems, where suppression of a small parameter affects the order of the problems, have been solved by a wide variety of techniques.[1-5] Two of these techniques, singular perturbation method (SPM)[1,5] and the method of matched asymptotic expansions (MAE)[2,3] have been independently developed to a reasonable level of satisfaction. Essentially, the SPM consists of expressing the total solution in terms of an *outer* solution, an *inner* solution, and an *intermediate* solution. On the other hand, in the method of MAE, a *composite* solution is constructed as the outer solution, the inner solution and a *common* solution.

In this Engineering Note, a critical examination of the method of matched asymptotic expansions reveals that the various terms of the common solution of MAE can be generated as polynomials in stretched variable without actually solving for them from the outer solution as is done presently. This also shows that the common solution of the method of MAE and the intermediate solution of the SPM are the same and hence that these methods give identical results for a certain class of problems. An illustrative

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example is given.

METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

The method of matched asymptotic expansions has been extensively used in fluid mechanics.[2] In this method, a composite solution is expressed as an outer solution, plus an inner solution, and minus a common solution.

We describe briefly the method of MAE as applicable to initial value problems. Consider

$$\frac{dx}{dt} = f(x, z, \varepsilon, t) \tag{1a}$$

$$\varepsilon\frac{dz}{dt} = g(x, z, \varepsilon, t) \tag{1b}$$

where x, and z are n-and m-dimensional state vectors respectively and ε is a small positive parameter responsible for singular perturbation. We begin by representing the solutions in the form of a series in powers of ε as

$$x(t,\varepsilon) = \sum_{i=0}^{\infty} x^{(i)}(t)\varepsilon^{i}; \qquad z(t,\varepsilon) = \sum_{i=0}^{\infty} z^{(i)}(t)\varepsilon^{i} \tag{2}$$

and determine the various terms $x^{(i)}(t)$ and $z^{(i)}(t)$ by means of formal substitution of Eq. (2) in Eq. (1) and comparison of coefficients of equal powers of ε. Then the following set of recursive equations are obtained. For zeroth order approximation,

$$\frac{dx^{(0)}(t)}{dt} = f^{0}\ [x^{(0)}(t),\ z^{(0)}(t),\ 0,\ t] \tag{3a}$$

$$0 = g^{0}\ [x^{(0)}(t),\ z^{(0)}(t),\ 0,\ t] \tag{3b}$$

and for first order approximation, we have

$$\frac{dx^{(1)}(t)}{dt} = f^1 \, [x^{(1)}(t),\ z^{(1)}(t),\ x^{(0)}(t),\ z^{(0)}(t),\ t] \tag{4a}$$

$$\frac{dz^{(0)}(t)}{dt} = g^1 \, [x^{(1)}(t),\ z^{(1)}(t),\ x^{(0)}(t),\ z^{(0)}(t),\ t] \tag{4b}$$

where the notation f^0, and f^1 is used to indicate all the terms on the right hand side. Since the series of Eq. (2) corresponds to the solution outside the boundary layer, it is called an *outer* series.

The solution of Eq. (3) is obtained by using $x \quad (t=0) = x(0)$; and in general $z^{(0)}(t=0) \neq z(0)$. On the other hand, the solution of Eq. (4) poses a problem, since the initial condition $x^{(1)}(t=0)$ is not yet known. Once $x^{(1)}(t)$ is solved for, $z^{(1)}(t)$ is automatically known from Eq. (4b). In order to relate the outer series of Eq. (2) to the solution of (1) in the boundary layer, we use a *stretching* transformation

$$\tau = t/\varepsilon \tag{5}$$

Then using Eq. (5) in Eq. (1), the stretched or *inner* problem becomes

$$\frac{d\bar{x}(\tau)}{d\tau} = f[x(\tau),\ z(\tau),\ \varepsilon,\ \varepsilon\tau] \tag{6a}$$

$$\frac{d\bar{z}(\tau)}{d\tau} = g[x(\tau),\ z(\tau),\ \varepsilon,\ \varepsilon\tau] \tag{6b}$$

This has inner series expansions of the form

$$\bar{x}(\tau,\varepsilon) = \sum_{i=0}^{\infty} \bar{x}^{(i)}(\tau)\varepsilon^{i}; \qquad \bar{z}(\tau,\varepsilon) = \sum_{i=0}^{\infty} \bar{z}^{(i)}(\tau)\varepsilon^{i} \tag{7}$$

Substitution of Eq. (7) in Eq. (6) and comparison of coefficients result in for zeroth order approximation as

$$\frac{d\bar{x}^{(0)}(\tau)}{d\tau} = 0 \tag{8a}$$

$$\frac{d\bar{z}^{(0)}(\tau)}{d\tau} = \bar{g}^{0}[\bar{x}^{(0)}(\tau), \bar{z}^{(0)}(\tau)] \tag{8b}$$

and similarly we get equations for first order approximation. The inner problem of Eq. (6) has initial conditions as

$$\bar{x}^{(0)}(\tau=0) = x(t=0); \quad \bar{z}^{(0)}(t=0) = z(t=0) \tag{9a}$$

$$\bar{x}^{(i)}(\tau=0) = 0; \quad \bar{z}^{(i)}(\tau=0) = 0; \quad i > 0 \tag{9b}$$

Still, we have not resolved the problem of determining the initial value $x^{(1)}(t=0)$ of the outer problem of Eq. (4). This is done by using a matching principle of the method of MAE.[2,3] Thus the matching principle is stated as

$$\text{inner expansion of outer solution, } (x^{o})^{i} = \text{outer expansion of inner solution, } (x^{i})^{o} \tag{10}$$

To any order approximation, the composite solution x_c is given by

$$x_c = x^{o} + x^{i} - (x^{o})^{i} = x^{o} + x^{i} - (x^{i})^{o} \tag{11}$$

where x^{o}, and x^{i} are the outer and inner solutions respectively to any order of approximation and $(x^{o})^{i} = (x^{i})^{o}$ is also called the *common* solution. Similar expressions can be given for z also.

AN EXAMINATION OF COMMON SOLUTION

In this section, we will show that the common solution defined as the inner expansion of the outer solution is simply formulated as a polynomial in the stretched variable. The steps involved in obtaining the common solution are (i) express the outer solution in the inner variable τ, (ii) expand it around ε = 0, and (iii) rearrange the resulting solution in powers of ε. Thus, consider the outer solution as

$$x^{o}(t) = x^{(0)}(t) + \varepsilon x^{(1)}(t) + \ldots\ldots. \tag{12}$$

We express this outer solution in the inner variable $\tau = t/\varepsilon$ as

$$x^{o}(\varepsilon\tau) = x^{(0)}(\varepsilon\tau) + \varepsilon x^{(1)}(\varepsilon\tau) + \ldots.. \tag{13}$$

Expanding Eq. (14) around ε = 0, we get

$$\begin{aligned}(x^{o})^{i} = & \left[x^{(0)}(\varepsilon\tau)\Big|_{\varepsilon=0} + \varepsilon\frac{\partial x^{(0)}(\varepsilon\tau)}{\partial\varepsilon}\Big|_{\varepsilon=0} + \ldots.\right] + \\ & \varepsilon\left[x^{(1)}(\varepsilon\tau)\Big|_{\varepsilon=0} + \varepsilon\frac{\partial x^{(1)}(\varepsilon\tau)}{\partial\varepsilon}\Big|_{\varepsilon=0} + \ldots.\right]\end{aligned} \tag{14}$$

Now evaluation of function x $(\varepsilon\tau)$ at ε = 0 in τ- plane is the same as its evaluation at t = 0 in t-plane, and the partial derivative of function $x^{(i)}(\varepsilon\tau)$, with respect to ε in τ-plane is the same as its partial derivative w.r.t. t multiplied by τ in t-plane. Thus,

$$\begin{aligned}(x^{o})^{i} = & \left[x^{(0)}(t=0) + \varepsilon\tau\frac{\partial x^{(0)}(t)}{\partial t}\Big|_{t=0} + \ldots\right] + \\ & \varepsilon\left[x^{(1)}(t=0) + \varepsilon\tau\frac{\partial x^{(1)}(t)}{\partial t}\Big|_{t=0} + \ldots\right] \\ = & \; x^{(0)}(0) + \varepsilon\left[x^{(1)}(t=0) + \tau\dot{x}^{(0)}(0)\right] + \ldots\ldots\end{aligned}$$

$$(x^{o})^{i} = \tilde{x}^{(0)}(\tau) + \varepsilon\tilde{x}^{(1)}(\tau) + \ldots\ldots \tag{15}$$

where, $\tilde{x}^{(0)}(\tau) = x^{(0)}(0)$; $\tilde{x}^{(1)}(\tau) = x^{(1)}(0) + \tau\dot{x}^{(0)}(0)$, and the dot over x denotes differentiation of x w. r. t. t . Similar expression can be obtained for the function z. Let us note that the intermediate solution of SPM is obtained by (i) expanding the outer solution around $t = 0$, (ii) expressing it in the inner variable τ, and (iii) rearranging the resulting solution in powers of ε.[1,5] Then, the common solution of Eq. (16) of the method of MAE is found to be the same as the intermediate solution of the SPM. Thus, the outer and inner solutions being the same in the SPM and the method of MAE, we clearly see that these two methods give identical results. Essentially, this equivalence means that the expansion of the outer solution around $t = 0$ and transformation into τ-plane is the same as transformation of the outer solution into τ-plane first and then expansion around $\varepsilon = 0$. The main advantage of the present formulation of the common solution is that its various terms can be very easily *generated* as polynomials in τ and hence one need not have explicit outer solution to arrive at the common solution.

In this way, we suggest an improved method of MAE, where the outer and inner solutions are obtained as before and the common solution is generated simply as a polynomial in the stretched variable τ, instead of evaluating it from the explicit outer solutions as is done usually.[2]

EXAMPLE

Consider a simple second order system so that we can get explicit expressions for the solutions.

$$\frac{dx}{dt} = z \qquad x(t=0) = a \tag{16a}$$

$$\varepsilon\frac{dz}{dt} = -x - z \qquad z(t=0) = b \tag{16b}$$

Applying the method of MAE described in Section 2, we summarize the results as follows. The outer solutions corresponding to Eqs. (3) and (4) are

$$\left.\begin{aligned} x^{(0)}(t) &= ae^{-t}; \quad z^{(0)}(t) = -ae^{-t} \\ x^{(1)}(t) &= [x^{(1)}(0) - at]e^{-t} \\ z^{(1)}(t) &= [-x^{(1)}(0) + at - a]e^{-t} \end{aligned}\right\} \tag{17}$$

The inner solutions corresponding to Eqs. (8) and (9) are

$$\left.\begin{aligned} \bar{x}^{(0)}(\tau) &= a; \quad \bar{z}^{(0)}(\tau) = -a + (a + b)e^{-\tau} \\ \bar{x}^{(1)}(\tau) &= (a + b) - a\tau - (a + b)e^{-\tau} \\ \bar{z}^{(1)}(\tau) &= -(2a + b) + a\tau + [2a + b + (a + b)\tau]e^{-\tau} \end{aligned}\right\} \tag{18}$$

Considering the two-term expansions only, the common solution (CS) for x is obtained as

$$(CS)_x = (x^i)^o = (x^o)^i \tag{19}$$

From Eq. (18), we obtain $(x^i)^o$, the outer expansion of the inner solution by first expressing the inner solution in the outer variable $t = \varepsilon\tau$ and then expanding it around $\varepsilon = 0$. Thus

$$(x^i)^o = a(1 - t) + \varepsilon(a + b) \tag{20}$$

Next, from Eq. (17), we obtain $(x^o)^i$, the inner expansion of the outer solution as

$$(x^o)^i = a(1 - t) + \varepsilon x^{(1)}(0) \tag{21}$$

Alternatively, in the present approach, we formulate $(x^o)^i$ as

$$(x^o)^i = x^{(0)}(t=0) + \varepsilon[x^{(1)}(t=0) + \tau \dot{x}^{(0)}(t=0)]$$

$$= a(1 - t) + \varepsilon x^{(1)}(0) \tag{22}$$

Equating Eqs. (20) and (21), we get the value of undetermined coefficient $x^{(1)}(0) = (a + b)$. Similarly for z, we have

$$(CS)_z = (z^i)^o = (z^i)^o \tag{23}$$

From Eq. (18), we obtain $(z^i)^o$, the outer expansion of the inner solution as

$$(z^i)^o = -a(1 - t) + \varepsilon[-(2a + b)] \tag{24}$$

Next, we obtain $(z^o)^i$, the inner expansion of the outer solution as

$$(z^o)^i = -a(1 - t) + \varepsilon z^{(1)}(0) \tag{25}$$

Alternatively, in the improved method, we formulate $(z^o)^i$ as

$$(z^o)^i = z^{(0)}(t=0) + \varepsilon[z^{(1)}(t=0) + \tau \dot{z}^{(0)}(t=0)]$$

$$= -a(1 - t) + \varepsilon z^{(1)}(0) \tag{26}$$

Using Eqs. (23)-(25), we get the value of the undetermined coefficient $z^{(1)}(0)$ as

$$z^{(1)}(0) = -(2a + b) \tag{27}$$

The composite solution corresponding to Eq. (12) is

$$x_c(t,\varepsilon) = ae^{-t} + \varepsilon[(a + b)(e^{-t} - e^{-t/\varepsilon}) - ate^{-t}] \tag{28a}$$

$$z_c(t,\varepsilon) = -ae^{-t} + (a + b)(1 + t)e^{-t/\varepsilon} +$$

$$\epsilon[(2a + b)(e^{-t/\epsilon} - e^{-t}) + ate^{-t}] \qquad (28b)$$

CONCLUSION

In this paper, a critical examination of the method of matched asymptotic expansion have revealed that the terms of the common solution could be generated as polynomials in stretched variable without actually solving for them as it is done presently. We have also seen that the common solution of the method of matched asymptotic expansion is the same as the intermediate solution of the singular perturbation methods and hence these two methods give identical results. Two examples have been given for illustration.

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THREE-DIMENSIONAL ATMOSPHERIC ENTRY PROBLEM USING METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

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Abstract: The analysis of a three-dimensional atmospheric entry problem using the method of matched asymptotic expansions is considered. A composite solution is formed in terms of an outer solution, an inner solution and a common solution. The outer solution is obtained from gravitationally dominant region, whereas the aerodynamically dominant region contributes to the inner solution. The common solution accounts for the overlap between the outer and inner regions. In comparison to the previous works, the present simplified methodology yields explicit analytical expressions for various components of the composite solution without resorting to any type of transcendental equations to be solved only by numerical methods.

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Nomenclature

C_i : constants of integration for outer solution
$\bar{C}_i$: constants of integration for inner solution
C_D : drag coefficient
C_L : lift coefficient
D : drag force
g : gravitational acceleration
g_s : gravitational acceleration at surface level
h : nondimensional altitude
I : inclination of the plane of the osculating orbit
L : lift force
m : vehicle mass
r : distance from vehicle center of gravity to planet center
r_s : distance from vehicle center of gravity to surface level
S : aerodynamic reference area
t : time
V : velocity
v : nondimensional velocity
α : angle between the line of the ascending node and the position vector
β : inverse atmospheric scale height
γ : flight path angle
ψ : heading angle
σ : bank angle
θ : down range angle or longitude
ϕ : cross range angle or latitude
ρ : density
μ : gravitational constant of Earth
Ω : longitude of the ascending node

1. Introduction

In space transportation system, the concept of aeroassisted orbital transfer opens new mission opportunities, especially with regard to the initiation of a permanant space station. The atmospheric entry problem is of paramount importance for aeroassited orbital transfer vehicles (AOTV).

The atmospheric entry problem involves, in general, the solution of nonlinear differential equations by resorting to numerical integration. Analytical solutions of a simplified entry problem are important from the point of view of serving as a basis for investigating more complicated cases and providing a general understanding of the structure of solutions. Analytical solutions also provide a better foundation for the solution of guidance problems. With this in view, attempts have been made to obtain approximate analytical solutions for the entry problem using asymptotic methods such as the method of matched asymptotic expansions, singular perturbation method, and multiple scale method [1-11]. Most of these solutions were obtained either for the two-dimensional case, or under restrictive assumptions. For instance, in the three-dimensional atmospheric entry problem [9], the use of directly matched asymptotic expansions leads to a set of transcendental equations which can only be solved by resorting to numerical methods.

In this paper, we address a three-dimensional atmospheric entry problem, to be analyzed by the method of matched asymptotic expansions (MAE). The solution is expressed in three parts; outer, inner, and common solutions. The outer solution is valid in the region where gravity is predominant. On the other hand, the aerodynamically predominant region gives an inner solution. Since these two regions are bound to overlap, a matching process is required to identify the common solution. Thus, a composite solution, valid in the entire region, is constructed as the sum of the outer solution and inner solution from which we need to subtract the common solution. The matching principle, in other words, ties the constants of integration associated with the outer and inner solutions with given auxiliary conditions. Compared to the earlier work [9], the present method has the following features: (i) Analytical expressions have been obtained explicitly for the outer, inner and common solutions without facing a set of transcendental equations which can only be solved by numerical methods. (ii) The composite solution satisfies the given auxiliary conditions asymptotically. (iii) The common solution can be generated as a polynomial in the stretched variable without actually solving for it from the inner limit of the outer solution or the outer limit of the inner solution.

2. Equations of Motion

Consider a vehicle with constant point mass m, moving about a nonrotating spherical planet. The atmosphere surrounding the planet is assumed to be at rest, and the central gravitational field obeys the usual inverse square law. The equations of motion for three dimensional flight of the lifting vehicle are given by (Fig. 1) [7,9,10,12],

$$\frac{dr}{dt} = V\sin\gamma \tag{1a}$$

$$\frac{d\theta}{dt} = \frac{V\cos\gamma\cos\psi}{r\cos\phi} \tag{1b}$$

$$\frac{d\phi}{dt} = \frac{V\cos\gamma\sin\psi}{r} \tag{1c}$$

$$\frac{dV}{dt} = -\frac{D}{m} - g\sin\gamma \tag{1d}$$

$$V\frac{d\gamma}{dt} = \frac{L\cos\sigma}{m} - \left(g-V^2/r\right)\cos\gamma \tag{1e}$$

$$V\frac{d\psi}{dt} = \frac{L\sin\sigma}{m\cos\gamma} - \frac{V^2}{r}\cos\gamma\cos\psi\tan\phi \tag{1f}$$

where it is assumed that the aerodynamic drag and lift are

$$D = \frac{1}{2}\rho S C_D V^2; \qquad L = \frac{1}{2}\rho S C_L V^2 \tag{2}$$

the gravitational field is

$$g = \frac{g_s r_s^2}{r^2} = \mu/r^2; \qquad \mu = g_s r_s^2 \tag{3}$$

and the atmosphere is given by

$$\rho = \rho_s \exp\left[-\beta(r-r_s)\right] \tag{4}$$

For any particular flight program, the control functions C_L, C_D, and σ are given functions of time and the solution of (1) requires prescribing six initial conditions.

It is convenient to eliminate time t in (1). Then, we get

$$\frac{d\theta}{dr} = \frac{\cos\psi}{r\cos\phi\tan\gamma} \tag{5a}$$

$$\frac{d\phi}{dr} = \frac{\sin\psi}{r\tan\gamma} \tag{5b}$$

$$\frac{dV^2}{dr} = -\frac{\rho S C_D V^2}{m\sin\gamma} - 2g \tag{5c}$$

$$\frac{d\gamma}{dr} = \frac{\rho S C_L\cos\sigma}{2m\sin\gamma} - \left(g/V^2 - 1/r\right)\cot\gamma \tag{5d}$$

$$\frac{d\psi}{dr} = \frac{\rho S C_L\sin\sigma}{2m\sin\gamma\cos\gamma} - \frac{\cos\psi\tan\phi}{r\tan\gamma} \tag{5e}$$

Solution of the set of five first order nonlinear differential equations (5) requires integration by numerical methods. The aim of the present paper is to obtain approximate analytical solutions to (5) using some simplifications in the method of MAE.

3. Method of Matched Asymptotic Expansions(MAE)

In applying the method of MAE to the three-dimensional entry problem, we consider separately the flight in an outer region near the vacuum, where the gravity force dominates, and an inner region near the planetary surface where the aerodynamic force is predominant. There is bound to be an overlap or common region where both outer and inner solutions are approximately of equal strength. A matching principle is invoked to obtain the common part. An approximate solution called the composite solution valid over the entire region, is constructed from the outer, inner and common solutions. We see that the various components of the composite solution are separated depending on the altitude, and hence it is appropriate to choose the altitude as an independent variable for obtaining the solution of (5).

Let us define the following dimensionless quantities,

$$\left.\begin{array}{lll} h = (r-r_s)/r_s; & v = V^2/g_s r_s; & \varepsilon = 1/\beta r_s \\ B = \rho_s S C_D/2m\beta; & \lambda = C_L/C_D & \end{array}\right\} \tag{6}$$

Here the constant βr_s is large, i.e., for Earth's atmosphere βr_s = 900, and hence the parameter ε is a small quantity.

From (3)-(6), we get

$$\frac{d\theta}{dh} = \frac{\cos\psi\cot\gamma}{(1+h)\cos\phi} \tag{7a}$$

$$\frac{d\phi}{dh} = \frac{\sin\psi\cot\gamma}{(1+h)} \tag{7b}$$

$$\frac{dv}{dh} = -\frac{2Bv\exp(-h/\varepsilon)}{\varepsilon\sin\gamma} - \frac{2}{(1+h)^2} \tag{7c}$$

$$\frac{d\gamma}{dh} = \frac{B\lambda\cos\sigma\exp(-h/\varepsilon)}{\varepsilon\sin\gamma} + \left[\frac{1}{(1+h)} - \frac{1}{v(1+h)^2}\right]\cot\gamma \tag{7d}$$

$$\frac{d\psi}{dh} = \frac{B\lambda\sin\sigma\exp(-h/\varepsilon)}{\varepsilon\sin\gamma\cos\gamma} - \frac{\cos\psi\tan\phi\cot\gamma}{(1+h)} \tag{7e}$$

Although (7) is ready for analysis by the method of MAE, it is more convenient to replace the set of variables (θ,ϕ,ψ) with a new set of variables (α,Ω,I) which are related as (Fig.2) [9,10],

$$\left.\begin{aligned} \cos I &= \cos\phi\cos\psi \\ \sin\phi &= \sin I\sin\alpha \\ \cos\alpha &= \cos\phi\cos(\theta-\Omega) \end{aligned}\right\} \tag{8}$$

where I is the inclination of the plane of the osculating orbit, Ω is the longitude of the ascending node, and α is the angle between the line of the ascending node and the position vector.

Using (8) in (7), we get

$$\frac{d\alpha}{dh} = -\frac{\sin\alpha}{\tan I}\left\{\frac{B\lambda\sin\sigma\exp(-h/\varepsilon)}{\varepsilon\sin\gamma\cos\gamma}\right\} + \frac{\cot\gamma}{(1+h)} \tag{9a}$$

$$\frac{d\Omega}{dh} = \frac{\sin\alpha}{\sin I}\left\{\frac{B\lambda\sin\sigma\exp(-h/\varepsilon)}{\varepsilon\sin\gamma\cos\gamma}\right\} \tag{9b}$$

$$\frac{dI}{dh} = \cos\alpha\left\{\frac{B\lambda\sin\sigma\exp(-h/\varepsilon)}{\varepsilon\sin\gamma\cos\gamma}\right\} \tag{9c}$$

$$\frac{dv}{dh} = -\frac{2Bv\exp(-h/\varepsilon)}{\varepsilon\sin\gamma} - \frac{2}{(1+h)^2} \tag{9d}$$

$$\frac{d\gamma}{dh} = \frac{B\lambda\cos\sigma\exp(-h/\varepsilon)}{\varepsilon\sin\gamma} + \left[\frac{1}{(1+h)} - \frac{1}{v(1+h)^2}\right]\cot\gamma \tag{9e}$$

with initial conditions, α , Ω , I , v , and γ .

3.1 Outer (Keplerian) Region

The outer expansions describe the solution in the region near vacuum where the gravitational force is predominant. These are assumed as

$$\left.\begin{aligned} \alpha &= \alpha_o(h) + \varepsilon\alpha_1(h) + \ldots\ldots. \\ \Omega &= \Omega_o(h) + \varepsilon\Omega_1(h) + \ldots\ldots. \\ I &= I_o(h) + \varepsilon I_1(h) + \ldots\ldots. \\ v &= v_o(h) + \varepsilon v_1(h) + \ldots\ldots. \\ \gamma &= \gamma_o(h) + \varepsilon\gamma_1(h) + \ldots\ldots. \end{aligned}\right\} \tag{10}$$

By substituting the outer expansions (10) into the original set of equations (9), and equating coefficients of ε^o on either side, the set of equations for zeroth-order approximation is

$$\frac{d\alpha_o}{dh} = \frac{\cot\gamma_o}{(1+h)} \tag{11a}$$

$$\frac{d\Omega_o}{dh} = 0 \tag{11b}$$

$$\frac{dI_o}{dh} = 0 \tag{11c}$$

$$\frac{dv_o}{dh} = -\frac{2}{(1+h)^2} \tag{11d}$$

$$\frac{d\gamma_o}{dh} = \left[\frac{1}{(1+h)} - \frac{1}{v_o(1+h)^2}\right]\cot\gamma_o \tag{11e}$$

Let us note that the zeroth-order equations (11) are alternatively obtained by letting the small parameter ε tend to zero in (9). The effect of making ε $(=1/\beta r_s)$ zero is that the atmospheric density ρ $\{= \rho_s \exp(-h/\varepsilon)\}$ becomes zero and the resulting equations (11) describe the region near the vacuum.

Solving (11),

$$v_o = 2\left[C_1 + \frac{1}{(1+h)}\right] \tag{12a}$$

$$\cos\gamma_o = \frac{C_2}{(1+h)\sqrt{v_o}} \tag{12b}$$

$$\cos(\alpha_o - C_3) = \frac{C_2^2/(1+h)-1}{\sqrt{1+2C_1C_2^2}} \tag{12c}$$

$$\Omega_o = C_4 \tag{12d}$$

$$I_o = C_5 \tag{12e}$$

where C_i are the constants of integration to be determined. The first and higher order solutions are all equal to zero because at high altitude, in the limit, the atmospheric density is zero and the motion is Keplerian.

3.2 Inner (Aerodynamic) Region

The inner expansions are introduced to study the limiting condition of the solution near the planetary surface where the aerodynamic force is predominant. These are obtained by first using a stretching transformation

$$\bar{h} = h/\varepsilon \tag{13}$$

in (9) and then taking the limit $\varepsilon \to 0$. This corresponds to the region near $h = 0$, i.e., planetary surface. Thus the stretched system becomes

$$\frac{d\bar{\alpha}}{d\bar{h}} = -\frac{\sin\bar{\alpha}}{\tan\bar{I}}\left\{\frac{B\lambda\sin\sigma\exp(-\bar{h})}{\sin\bar{\gamma}\cos\bar{\gamma}}\right\} + \frac{\varepsilon\cot\bar{\gamma}}{(1+\varepsilon\bar{h})} \tag{14a}$$

$$\frac{d\bar{\Omega}}{d\bar{h}} = \frac{\sin\bar{\alpha}}{\sin\bar{I}}\left\{\frac{B\lambda\sin\sigma\exp(-\bar{h})}{\sin\bar{\gamma}\cos\bar{\gamma}}\right\} \tag{14b}$$

$$\frac{d\bar{I}}{d\bar{h}} = \cos\bar{\alpha}\left\{\frac{B\lambda\sin\sigma\exp(-\bar{h})}{\sin\bar{\gamma}\cos\bar{\gamma}}\right\} \tag{14c}$$

$$\frac{d\bar{v}}{d\bar{h}} = -\frac{2B\bar{v}\exp(-\bar{h})}{\sin\bar{\gamma}} - \frac{2\varepsilon}{(1+\varepsilon\bar{h})^2} \tag{14d}$$

$$\frac{d\bar{\gamma}}{d\bar{h}} = \frac{B\lambda\cos\sigma\exp(-\bar{h})}{\sin\bar{\gamma}} + \varepsilon\left\{\frac{1}{(1+\varepsilon\bar{h})} - \frac{1}{\bar{v}(1+\varepsilon\bar{h})^2}\right\}\cot\bar{\gamma} \tag{14e}$$

Let us note that λ and σ, assumed to be external control inputs do not undergo transformation. Let the inner solution be expressed as

$$\left.\begin{aligned}
\bar{\alpha} &= \bar{\alpha}_o(\bar{h}) + \varepsilon\bar{\alpha}_1(\bar{h}) + \ldots\ldots. \\
\bar{\Omega} &= \bar{\Omega}_o(\bar{h}) + \varepsilon\bar{\Omega}_1(\bar{h}) + \ldots\ldots. \\
\bar{I} &= \bar{I}_o(\bar{h}) + \varepsilon\bar{I}_1(\bar{h}) + \ldots\ldots. \\
\bar{v} &= \bar{v}_o(\bar{h}) + \varepsilon\bar{v}_1(\bar{h}) + \ldots\ldots. \\
\bar{\gamma} &= \bar{\gamma}_o(\bar{h}) + \varepsilon\bar{\gamma}_1(\bar{h}) + \ldots\ldots.
\end{aligned}\right\} \tag{15}$$

As before, substitution of (15) into (13) and collection of coefficients of powers of ε^o on either side gives the zeroth-order approximation as

$$\frac{d\bar{\alpha}_o}{d\bar{h}} = -\frac{\sin\bar{\alpha}_o}{\tan\bar{I}_o}\left\{\frac{B\lambda\sin\sigma\exp(-\bar{h})}{\sin\bar{\gamma}_o\cos\bar{\gamma}_o}\right\} \tag{16a}$$

$$\frac{d\bar{\Omega}_o}{d\bar{h}} = \frac{\sin\bar{\alpha}_o}{\sin\bar{I}_o}\left\{\frac{B\lambda\sin\sigma\exp(-\bar{h})}{\sin\bar{\gamma}_o\cos\bar{\gamma}_o}\right\} \tag{16b}$$

$$\frac{d\bar{I}_o}{d\bar{h}} = \cos\bar{\alpha}_o\left\{\frac{B\lambda\sin\sigma\exp(-\bar{h})}{\sin\bar{\gamma}_o\cos\bar{\gamma}_o}\right\} \tag{16c}$$

$$\frac{d\bar{v}_o}{d\bar{h}} = -\frac{2B\bar{v}_o\exp(-\bar{h})}{\sin\bar{\gamma}_o} \tag{16d}$$

$$\frac{d\bar{\gamma}_o}{d\bar{h}} = \frac{B\lambda\cos\sigma\exp(-\bar{h})}{\sin\bar{\gamma}_o} \tag{16e}$$

Solving (16), we get

$$\bar{v}_o = \bar{C}_1\exp\left(-2\bar{\gamma}_o/\lambda\cos\sigma\right) \tag{17a}$$

$$\cos\bar{\gamma}_o = B\lambda\cos\sigma\exp(-\bar{h}) + \bar{C}_2 \tag{17b}$$

$$\sin\bar{\alpha}_o\sin\bar{I}_o = \sin\bar{C}_3 \tag{17c}$$

$$\cos\bar{\alpha}_o = \cos\bar{C}_3\cos\left(\bar{C}_4-\bar{\Omega}_o\right) \tag{17d}$$

$$\cos\bar{I}_o = \cos\bar{C}_3\cos\left\{\tan\sigma\log\left[\tan(\bar{\gamma}_o/2+\pi/4)\right]+\bar{C}_5\right\} \tag{17e}$$

where $\bar{C}_i$ are the constants of integration. Here, the sequence of solutions is $\bar{\gamma}_o$ from (17b), $\bar{v}_o$ from (17a), $\bar{I}_o$ from (17e), $\bar{\alpha}_o$ from (17c), and finally $\bar{\Omega}_o$ from (17d). We now have the zeroth-order outer solutions (12) with C_i as the constants of integration and the zeroth-order inner solutions (17), where $\bar{C}_i$ are the integration constants. These constants are determined by a matching principle.

4. Matching Principle[13-15]

Matching is based on the notion that the outer solution valid in the Keplerian region and the inner solution valid near the planet surface, must both be valid in some overlap region. Thus matching is accomplished by extending the outer solution into the inner region by transforming the outer variable h to that of the inner variable $\bar{h}$ (=h/ϵ) and taking the limit as $\epsilon \to 0$. This is called the inner limit of the outer solution or expansion. Similarly, the outer limit of the inner solution or expansion is obtained by extending the inner solution into the outer region by transforming the inner variable $\bar{h}$ to that of the outer variable h (= $\epsilon\bar{h}$) and taking the limit as $\epsilon \to 0$. By equating the inner limit of outer expansion with the outer limit of inner expansion, we can determine the constants of integration and hence the common solution. A composite solution is formed as the sum of outer and inner solutions from which the common solution is subtracted.

In the earlier work [9], the matching principle yielded a relation for the constants $\bar{C}_i$ in terms of the constants C_i. Then the composite solution is expected to satisfy the given initial conditions. This procedure led to the formulation of a set of transcendental equations which can only be solved by resorting to numerical methods.

In the present method, we simplify the procedure by asking the outer solution to satisfy the given initial conditions and the matching principle gives the relation between the constants of integration [4]. Still, the composite solution satisfies the given initial conditions asymptotically. We note that in the simplified procedure, we are not faced with any kind of transcendental equations and explicit solutions are obtained for the composite solution. Moreover, we also get the common solution very easily by formulating or generating the various terms of the inner limit of the outer expansion as a polynomial in the stretched variable as [14,15],

$$\left.\begin{aligned}
v_o^m &= v_o(h=0) + \epsilon[v_1(h=0) + \bar{h}\dot{v}_o(h=0)] + \ldots\ldots \\
\gamma_o^m &= \gamma_o(h=0) + \epsilon[\gamma_1(h=0) + \bar{h}\dot{\gamma}_o(h=0)] + \ldots\ldots \\
\alpha_o^m &= \alpha_o(h=0) + \epsilon[\alpha_1(h=0) + \bar{h}\dot{\alpha}_o(h=0)] + \ldots\ldots \\
\Omega_o^m &= \Omega_o(h=0) + \epsilon[\Omega_1(h=0) + \bar{h}\dot{\Omega}_o(h=0)] + \ldots\ldots \\
I_o^m &= I_o(h=0) + \epsilon[I_1(h=0) + \bar{h}\dot{I}_o(h=0)] + \ldots\ldots
\end{aligned}\right\} \quad (18)$$

Here the dot denotes differentiation with respect to the independent variable h. We note that this is also called the intermediate solution in singular perturbation methods [14,15].

We now force the outer solution (12) to satisfy the given initial conditions, v_i, γ_i α_i, Ω_i, and I_i corresponding to $h = h_i$. This gives us

$$C_1 = v_i/2 - 1/(1+h_i) \quad (19a)$$

$$C_2 = \cos\gamma_i (1+h_i)\sqrt{v_i} \tag{19b}$$

$$C_3 = \alpha_i - \cos^{-1}\left\{\frac{\cos^2\gamma_i (1+h_i) v_i - 1}{\sqrt{1+\left[v_i(1+h_i)-2\right]\left[\cos^2\gamma_i (1+h_i) v_i\right]}}\right\} \tag{19c}$$

$$C_4 = \Omega_i \tag{19d}$$

$$C_5 = I_i \tag{19e}$$

Thus, we have the relation between the constants of outer solution explicitly in terms of the given initial conditions. In applying the matching principle, we first find the zeroth-order inner limit of the outer expansion (12), as

$$\left.\begin{aligned}
v_0^m &= 2(C_1+1) \\
\cos\gamma_0^m &= C_2 \Big/ \sqrt{2(C_1+1)} \\
\cos(\alpha_0^m - C_3) &= (C_2^2-1) \Big/ \sqrt{1+2C_1C_2^2} \\
\Omega_0^m &= C_4 \\
I_0^m &= C_5
\end{aligned}\right\} \tag{20}$$

The outer limit of inner expansion (17) is

$$\bar{v}_0^m = \bar{C}_1 \exp\left(-2\bar{\gamma}_0^m / \lambda\cos\sigma\right) \tag{21a}$$

$$\cos\bar{\gamma}_0^m = \bar{C}_2 \tag{21b}$$

$$\sin\bar{\alpha}_0^m \sin\bar{I}_0^m = \sin\bar{C}_3 \tag{21c}$$

$$\cos\bar{\alpha}_0^m = \cos\bar{C}_3 \cos(\bar{C}_4 - \bar{\Omega}_0^m) \tag{21d}$$

$$\cos\bar{I}_0^m = \cos\bar{C}_3 \cos\left\{\tan\sigma \log\left[\tan(\bar{\gamma}_0^m/2+\pi/4)\right] + \bar{C}_5\right\} \tag{21e}$$

Matching (21) with (20), we get constants $\bar{C}_i$ in terms of the constants C_i as

$$\bar{C}_1 = 2(C_1+1)\exp\left\{\frac{2}{\lambda\cos\sigma}\cos^{-1}\left[C_2/\sqrt{2(C_1+1)}\right]\right\} \tag{22a}$$

$$\bar{C}_2 = C_2/\sqrt{2(C_1+1)} \tag{22b}$$

$$\sin\bar{C}_3 = \sin\left[C_3+\cos^{-1}\left\{(C_2^2-1)\Big/\sqrt{1+2C_1C_2^2}\right\}\right]\sin C_5 \tag{22c}$$

$$\bar{C}_4 = C_4 + \cos^{-1}\left[\cos\left(C_3+\cos^{-1}\left\{(C_2^2-1)\Big/\sqrt{1+2C_1C_2^2}\right\}\right)\Big/\cos\bar{C}_3\right] \quad \ldots. \tag{22d}$$

$$\bar{C}_5 = \cos^{-1}\left(\cos C_5/\cos\bar{C}_3\right) - \tan\sigma\log\left[\tan\left\{\pi/4+\cos^{-1}\left[C_2/\sqrt{8(C_1+1)}\right]\right\}\right] \quad \ldots. \tag{22e}$$

We note that $\bar{C}_3$ in (22d) and (22e) are in turn related with the constants C_i via (22c).

5. Composite Solution

The composite solution or expansion is obtained as the sum of outer solution (12) and inner solution (17) from which the common solution (20) or (21) is subtracted. Thus

$$\left.\begin{aligned}
v_c &= v_o + \bar{v}_o - v_o^m(\text{ or } \bar{v}_o^m\ ) \\
\gamma_c &= \gamma_o + \bar{\gamma}_o - \gamma_o^m(\text{ or } \bar{\gamma}_o^m\ ) \\
\alpha_c &= \alpha_o + \bar{\alpha}_o - \alpha_o^m(\text{ or } \bar{\alpha}_o^m\ ) \\
\Omega_c &= \Omega_o + \bar{\Omega}_o - \Omega_o^m(\text{ or } \bar{\Omega}_o^m\ ) \\
I_c &= I_o + \bar{I}_o - I_o^m(\text{ or } \bar{I}_o^m\ )
\end{aligned}\right\} \quad \ldots.. \tag{23}$$

In terms of explicit expressions, we have

$$v_c = 2/(1+h) + \bar{C}_1 \exp\left(-2\bar{\gamma}_o/\lambda\cos\sigma\right) - 2.0 \tag{24a}$$

$$\gamma_c = \cos^{-1}\left\{\frac{C_2}{(1+h)\sqrt{2C_1+2/(1+h)}}\right\} + \cos^{-1}\left\{B\lambda\cos\sigma\exp(-h/\varepsilon)+\bar{C}_2\right\}$$
$$- \cos^{-1}\left\{C_2/\sqrt{2(C_1+1)}\right\} \tag{24b}$$

$$\Omega_c = \bar{C}_4 - \cos^{-1}\left[\cos\bar{\alpha}_o/\cos\bar{C}_3\right] \tag{24c}$$

$$\cos I_c = \cos\bar{C}_3 \cos\left[\tan\sigma\log\left(\tan(\bar{\gamma}_o/2+\pi/4)\right)+\bar{C}_5\right] \tag{24d}$$

$$\alpha_c = \cos^{-1}\left\{\frac{C_2^2/(1+h)-1}{\sqrt{1+2C_1C_2^2}}\right\} + \sin^{-1}\left\{\frac{\sin\bar{C}_3}{\sin\bar{I}_o}\right\} - \cos^{-1}\left\{\frac{C_2^2-1}{\sqrt{1+2C_1C_2^2}}\right\}$$

.... (24e)

The above composite solution is expressed in terms of h, constants $\bar{C}_i$, and C_i and the states from the inner solution.. The $\bar{C}_i$s are obtained from (22) and the C_is are obtained directly from the initial conditions via (19). The states of inner solution are obtained as explicit functions of the $\bar{C}_i$ and $\bar{h}$ via (17).

We need to check whether the composite solution (24) asymptotically satisfies the given initial conditions. Consider (24a) along with (17), (19), and (22). We have

$$v_c(h=h_i) = \frac{2}{(1+h_i)} + 2(C_1+1)\exp\left\{\frac{2}{\lambda\cos\sigma}\cos^{-1}\left[C_2/\sqrt{2(C_1+1)}\right]\right\}\times$$
$$\exp\left[-\frac{2}{\lambda\cos\sigma}\cos^{-1}\left\{B\cos\sigma\exp(-h_i/\varepsilon)+C_2/\sqrt{2(C_1+1)}\right\}\right] - 2.0$$

.... (25a)

As $\varepsilon \to 0$,

$$v_c(h=h_i) = \frac{2}{(1+h_i)} + 2C_1 = v_i \tag{25b}$$

Similarly, we can show that γ_c, I_c, α_c, and Ω_c satisfy their corresponding initial conditions asymptotically.

6. Conclusions

In this paper, we have addressed the solution of a three-dimensional atmospheric entry problem via a simplified method of matched asymptotic expansions. The solution has been expressed in three parts. An outer solution has been obtained in the gravitationally dominant region and an inner solution has been formed for the aerodynamically stronger region. A common solution has been formed as the outer (or inner) limit of the inner (or outer) solution. Finally, a composite solution has been constructed as the sum of the outer and inner solutions from which the common solution has been subtracted.

The special features of the present method are (i) The composite solution has been obtained in a simplified manner in the sense that analytical expressions have been obtained explicitly for the various components, outer, inner and commmon solutions, without resorting to any kind of transcendental equations which can only be solved by numerical methods. (ii) At the same time, the composite solution has satisfied the given initial conditions asymptotically. (iii) The common solution has been obtained in an easier manner by formulating or generating the inner limit of the outer solution in terms of a polynomial in the stretched variable. The numerical experimentation for the present method is under progress.

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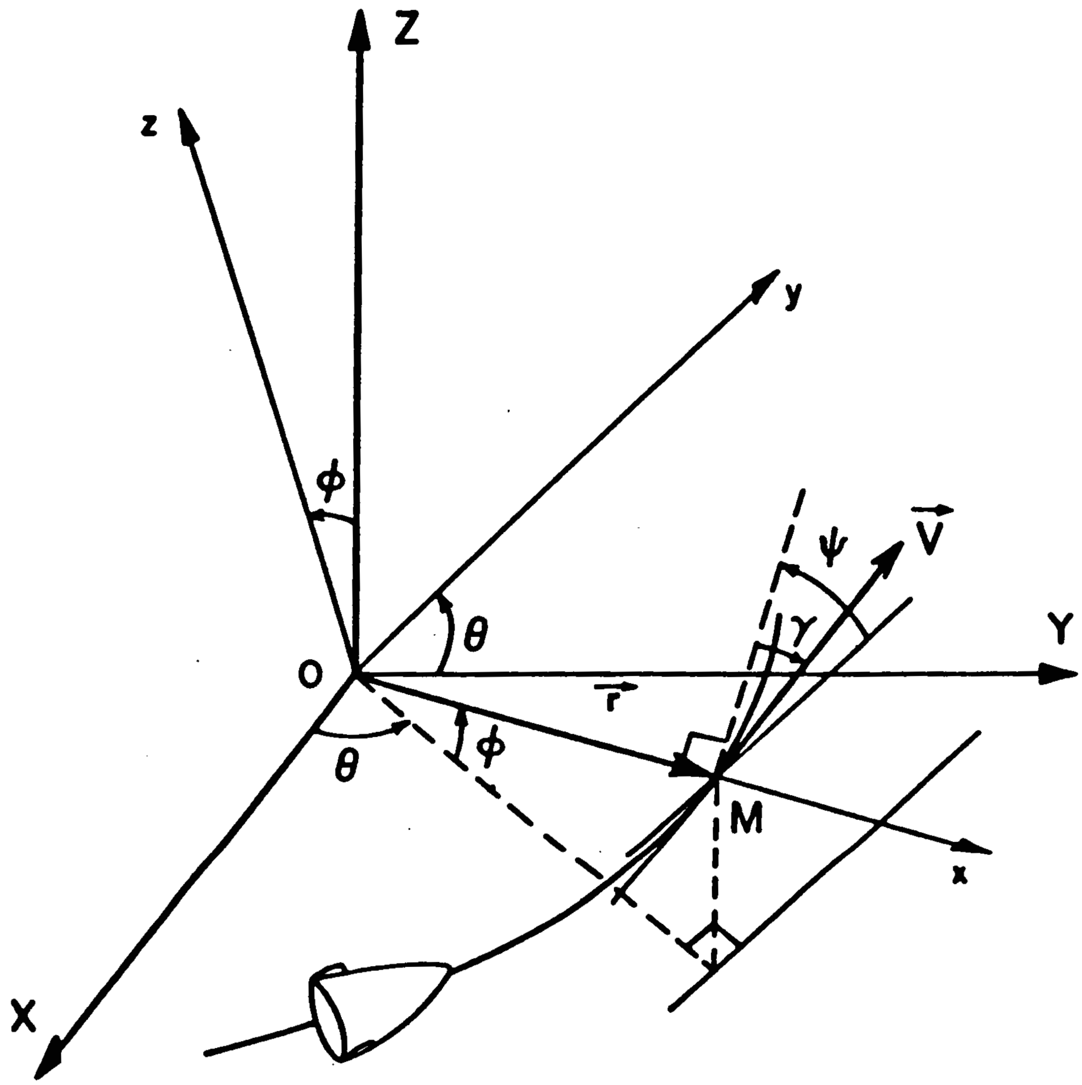

Fig. 1. Coordinate System

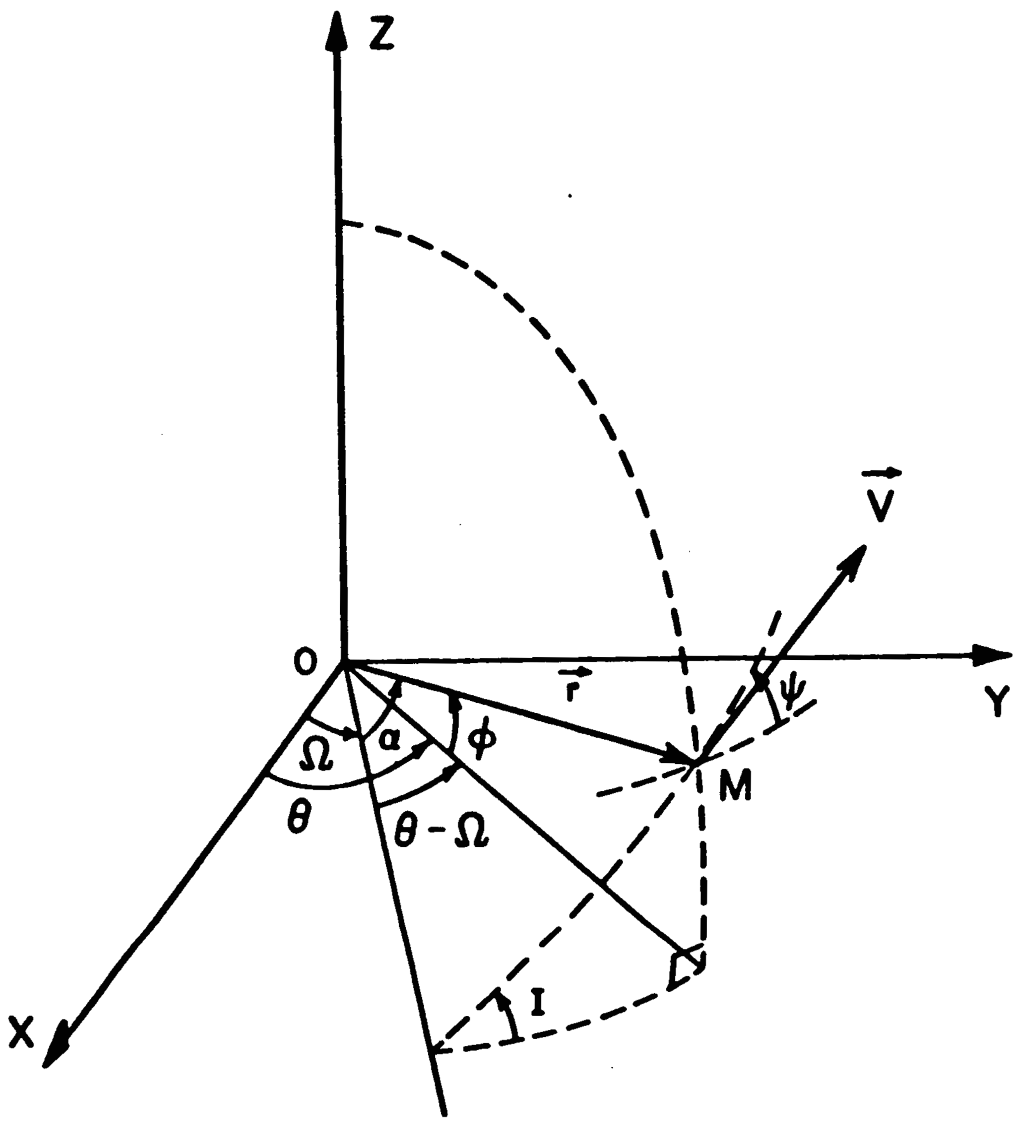

Fig. 2. Osculating Plane and Orbital Elements

OPTIMAL CONTROL OF AEROASSISTED NONCOPLANAR ORBITAL TRANSFER VEHICLES
(Rough draft only)

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Abstract: The optimal control problem arising in noncoplanar orbital transfer employing aeroassist technology is addressed. The maneuver involves the transfer from high Earth orbit to low Earth orbit with a presribed plane change. The performance index is chosen to minimize the time integral of the heating rate of the spacecraft. Using Pontryagin minimum principle, the state and costate differential equations are derived, leading to a nonlinear two-point boundary value problem. This problem is solved by using multiple shooting method.

Nomenclature

C_D : drag coefficient
C_{DO} : zero lift drag coefficient
C_L : lift coefficient
D : drag force
g : gravitational acceleration
g_s : gravitational acceleration at surface level
K : induced drag factor
L : lift force
m : vehicle mass
r : distance from vehicle center of gravity to planet center
r_s : distance from vehicle center of gravity to surface level
S : aerodynamic reference area
t : time
V : velocity
β : inverse atmospheric scale height
γ : flight path angle
ψ : heading angle
σ : bank angle
θ : down range angle or longitude
ϕ : cross range angle or latitude
ρ : density
μ : gravitational constant of Earth

1. Introduction

In space transportation system, the concept of aeroassisted orbital transfer opens new mission opportunities, especially with regard to the initiation of a permanant space station [1]. In a synergetic maneuver for aeroassited orbital transfer vehicles (AOTV's), the basic idea is to employ a hybrid combination of propulsive maneuvers in space and aerodynamic maneuvers in sensible atmosphere. Within the atmosphere, the trajectory control is achieved by means of lift and bank angle modulations [2-7]. In a typical maneuver, we start with a tangential propulsive burn, having a characterstic velocity ΔV_d for deorbitting from the high Earth orbit and entering into an elliptical transfer orbit. At point E the spacecraft enters the sensible atmosphere. The vehicle undergoes reduction in velocity due to atmospheric drag and in addition, the necessary plane change is performed. At point B, the spacecraft leaves the atmosphere augmented by a propulsive burn imparting ΔV_b for boosting. Once again, the transfer orbit is elliptical with a corresponding apogee. Finally, the maneuver ends with a reorbit burn having characteristic velocity ΔV_b to make the vehicle enter into the low Earth orbit. Thus, the maneuver consists of three impulses ΔV_d for deorbit, ΔV_b for boost, and ΔV_r for reorbit.

In this paper, we address the optimal control problem arising in noncoplanar orbital transfer employing aeroassist technology. The maneuver involves the transfer from high Earth orbit (HEO) to low Earth orbit (LEO) with a presribed plane change and at the same time minimization of the time integral of the heating rate of the spacecraft. With a suitable performance index, we formulateoptimal control problem. Using Pontryagin minimum principle, the state and costate differential equations are derived, leading to a nonlinear two-point boundary value problem (TPBVP). This problem is solved by using multiple shooting method [8-10].

2. Equations of Motion

Consider a vehicle with constant point mass m, moving about a nonrotating spherical planet. The atmosphere surrounding the planet is assumed to be at rest, and the central gravitational field obeys the usual inverse square law. The equations of motion for three dimensional flight of the lifting vehicle are given by (Fig. 1),

$$\frac{dr}{dt} = V\sin\gamma \tag{1a}$$

$$\frac{dV}{dt} = -AC_D V^2 \exp(-h/\beta) - (\mu/r^2)\sin\gamma \tag{1b}$$

$$\frac{d\gamma}{dt} = AC_L V\cos\sigma\exp(-h/\beta) + [V/r - \mu/r^2 V]\cos\gamma \tag{1c}$$

$$\frac{d\phi}{dt} = (V/r)\cos\gamma\sin\psi \tag{1d}$$

$$\frac{d\psi}{dt} = AC_L V\sin\sigma\exp(-h/\beta)/\cos\gamma - (V/r)\cos\gamma\cos\psi\tan\phi \tag{1e}$$

where $A = 0.5S\rho_s/m$, $h = r-r_s$, $\rho = \rho_s\exp(-h/\beta)$ and

$C_D = C_{DO} + KC_L^2$ for a drag polar.

3. Performance Index and Hamiltonian.

For optimal control problem regarding heating, it is required to choose the performance index to minimize the time integral of the heating rate at a particular point of the spacecraft, say, the stagnation point. Thus, the performance index is given by

$$J = K_h \int_0^T \sqrt{\sigma}\; V^{3.08} dt \tag{2}$$

where K_h is a proportional constant.

The first step in the optimization procedure using Pontryagin principle is to formulate Hamiltonian as

$$\begin{aligned} H = {} & K_h \sqrt{\sigma}\; V^{3.08} + \lambda_r V \sin\gamma \\ & + \lambda_v \left\{ - AC_D V^2 \exp(-h\beta) - (\mu/r^2)\sin\gamma \right\} \\ & + \lambda_\gamma \left\{ AC_L V \cos\sigma \exp(-h\beta) + [V/r - \mu/r^2 V]\cos\gamma \right\} \\ & + \lambda_\phi \left\{ (V/r)\cos\gamma \sin\psi \right\} \\ & + \lambda_\psi \left\{ AC_L V \sin\sigma \exp(-h\beta)/\cos\gamma - (V/r)\cos\gamma\cos\psi\tan\phi \right\} \end{aligned} \tag{3}$$

where λ's are the costates corresponding to the five states, x's.

4. Optimal Controls

The optimal control equations are given by

$$\frac{\partial H}{\partial C_L} = 0; \qquad \frac{\partial H}{\partial \sigma} = 0 \tag{4}$$

for lift and bank angle leading to

$$C_L = - \omega/2K\lambda_v V; \qquad \tan\sigma = \lambda_\psi/\lambda_\gamma \cos\gamma \tag{5}$$

where

$$\omega = \sqrt{\lambda_\gamma^2 + (\lambda_\psi/\cos\gamma)^2} \tag{6}$$

5. Costate Equations

The costate (auxiliary) variables λ's are given by

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial x} \tag{7}$$

leading to the corresponding five differential equations as

$$\begin{aligned}
\frac{d\lambda_r}{dt} = {} & (\beta/2)K_h\sqrt{\sigma_s}\,V^{3.08}\exp\{-\beta(r-r_s)/2\} \\
& - \lambda_V\left\{\beta AC_D V^2\exp\{-\beta(r-r_s)\} + (2\mu/r^3)\sin\gamma\right\} \\
& + \lambda_\gamma\left\{\beta AC_L V\cos\sigma\exp\{-\beta(r-r_s)\} + (V/r^2 - 2\mu/r^3V)\cos\gamma\right\} \\
& + \lambda_\phi\left\{(V/r^2)\cos\gamma\sin\psi\right\} + \lambda_\psi\left\{\beta AC_L V(\sin\sigma/\cos\gamma)\exp\{-\beta(r-r_s)\}\right. \\
& \left. - (V/r^2)\cos\gamma\cos\psi\tan\phi\right\}
\end{aligned} \tag{8a}$$

$$\begin{aligned}
\frac{d\lambda_V}{dt} = {} & -3.08K_h V^{2.08}\sqrt{\rho_s}\,\exp\{-\beta(r-r_s)/2\} \\
& - \lambda_r\sin\gamma + \lambda_V 2AC_D V\exp\{-\beta(r-r_s)\} \\
& - \lambda_\gamma\left\{AC_L\cos\sigma\exp\{-\beta(r-r_s)\} + [1/r + \mu/r^2V^2]\cos\gamma\right\} \\
& - \lambda_\phi\left\{(1/r)\cos\gamma\sin\psi\right\} \\
& - \lambda_\psi\left\{AC_L(\sin\sigma/\cos\gamma)\exp\{-\beta(r-r_s)\} - (1/r)\cos\gamma\cos\psi\tan\phi\right\}
\end{aligned} \tag{8b}$$

$$\frac{d\lambda_\gamma}{dt} = -\lambda_r V\cos\gamma + \lambda_V(\mu/r^2)\cos\gamma$$

$$+ \lambda_\gamma\left\{(V/r - \mu/r^2V)\sin\gamma\right\} + \lambda_\phi(V/r)\sin\gamma\sin\psi$$

$$- \lambda_\psi\left\{AC_L V(\tan\gamma/\cos\gamma)\sin\sigma\exp\{-\beta(r-r_s)\}\right.$$

$$\left. + (V/r)\sin\gamma\cos\psi\tan\phi\right\} \quad (8c)$$

$$\frac{d\lambda_\phi}{dt} = \lambda_\psi(V/r)\cos\gamma\cos\psi/(\cos\phi)^2 \quad (8d)$$

$$\frac{d\lambda_\psi}{dt} = - \lambda_\phi(V/r)\cos\gamma\cos\psi - \lambda_\psi(V/r)\cos\gamma\sin\psi\tan\phi \quad (8d)$$

6. Numerical Data

The various numerical values used for simulation purposes are given below [3].

C_{DO} = 0.01; K = 1.11; m/S = 275.0 kg/m^2
ρ_s = 1.225 kg/m^3; μ = 3.986×10^{14} m^3/sec^2
β = 1/6900 m^{-1}; r_E = 6378.0 KM; K_h = 10.0

A complete plane change maneuver hs the specifications of the initial and final orbit as boundary conditions. The initial and final boundary conditions for solving the state and costate equations (1) and (8) are given by

Inital:	Final:	
h(t=0) = 80 KM	h(t=T) = 80 KM	
V(t=0) = 7.95 KM/sec	V(t=T) = 5.0 KM/sec	
γ(t=0) = -1.25 deg	γ(t=T) = 2 deg	(9)
ϕ(t=0) = 0 deg	ϕ(t=T) = 3.83 deg	
ψ(t=0) = 0 deg	ψ(t=T) = 15 deg	

7. Multiple Shooting Method

The determination of optimal controls (5) requires the solution of tenth order, nonlinear, two-point boundary value problem (TPBVP) consisting of state equations (1) and costate equations (8) and the associated boundary conditions (9). This can only be done by numerical methods. The multiple shooting method is one of the powerful methods for solving nonlinear TPBVP's. The corresponding OPTSOL code was developed by DFVLR establishment at Oberpfaffenhofen, West Germany.

In solving any boudary value problem with the given initial and final conditions, we asume additional initial data and integrate forward so that the solution satisfies the given final condition as well. This is also called a simple shooting method.

Here, the convergence of the solution is highly sensitive to the assumed intial data. It is found that the error due to inaccurate intial data can be made orbitrarily small by performing the integration over sufficiently smaller subdivided panels within the given interval and thereby leading to multiple shooting method. Thus, the multiple shooting method is a simultaneous application of simple shooting method at several points within the interval of integration. Here, the trajectory may be restarted at intermediate points using new guesses. Jacobian matrices are formed for each segment. The resulting iteration scheme, based on reducing all discontinuities at internal grid points to zero, leads to a system of linear algebraic equations.

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