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A SYSTEM OF THREE-DIMENSIONAL COMPLEX VARIABLES

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Abstract. This note reports some results of a new theory of multidimensional complex variables including, in particular, analytic functions of a three-dimensional (3-D) complex variable. Three-dimensional complex numbers are defined, including vector properties and rules of multiplication. The necessary conditions for a function of a 3-D variable to be analytic are given and shown to be analogous to the 2-D Cauchy-Riemann equations. A simple example also demonstrates the analogy between the newly defined 3-D complex velocity and 3-D complex potential and the corresponding ordinary complex velocity and complex potential in two dimensions.

1. Introduction. Early in the nineteenth century, mathematicians began a search for a "three-dimensional complex number and its algebra" that would be a generalization of the ordinary "two-dimensional" complex number [**1**, p. 90]. In 1843, William R. Hamilton introduced quaternions (see [**1**]), an important four-dimensional generalization of complex numbers and variables. *Hypercomplex analysis* has developed mainly as a further generalization of quaternions and, as such, is often referred to as *Clifford analysis*. The recent papers [**2**], [**3**], [**4**], [**5**] supply many references, including early work by Fueter (e.g., [**6**]). These algebras that generalize quaternions are noncommutative.

S. Bergman [**7**] has introduced a method based on E. T. Whittaker's [**8**, p. 390] general integral solution to Laplace's equation that provides a certain generalization of analytic functions of one complex variable. However, the present state has been summarized as follows by E. T. Copson [**9**, p. 207]: "The theory of harmonic functions in two dimensions can be made to depend on the theory of analytic functions of a complex variable, $x + iy$. There is nothing corresponding to the theory of functions of a complex variable $x + iy$ in three dimensions. The nearest approach is given by Whittaker's general solution ... of Laplace's equation."

The elements of the 3-D theory (a commutative algebra) to be described here are direct generalizations of corresponding elements of the classical 2-D theory. Therefore a direct comparison with 2-D is helpful for this description.

2. Basics in Two Dimensions for Comparison. A most important property of analytic functions of an ordinary complex variable is that from them are obtained vector functions g that are both *solenoidal* and *irrotational*. As a result, the components of g are harmonic functions.

Let $\mathbf{R}$ denote the set of all real numbers and $\mathbf{C}_2$ denote the set of all ordinary complex numbers. The complex variable $z = x + iy$ in $\mathbf{C}_2$ may be written also as $z = (x, y) = (1,0)x + (0,1)y$, which may be interpreted as a vector in $\mathbf{R}^2$ with real components x, y and with basis vectors $(1,0) = 1$ and $(0,1) = i$, whose *rules of multiplication* are: $1^2 = 1$, $1i = i1 = i$, $i^2 = -1$. However, the unit $(1,0) = 1$ as a factor is commonly omitted. If now $g = \phi_1 + i\phi_2$, in $\mathbf{C}_2$, is defined to be the vector (complex function) whose complex conjugate is an *analytic function* $\bar{g} = f(z) = \phi_1 - i\phi_2$, then the *conditions of analyticity* for $\bar{g} = f(z)$ are the Cauchy-Riemann equations: $\operatorname{div} g = \phi_{1x} + \phi_{2y} = 0$ and $\operatorname{curl} g = \phi_{2x} - \phi_{1y} = 0$.

(In two dimensions the result of the curl operation is defined as a scalar.) Therefore, g is solenoidal and irrotational (S and I).

Any 2-D S and I vector may be represented by a complex variable having the same form as g. For example, in 2-D ideal flow with velocity components v_1 and v_2 and with velocity potential ϕ and stream function ψ, the velocity vector $v = v_1 + iv_2$ and the vector $g = \phi_1 + i\phi_2$ (where $\phi_1 = \phi$ and $\phi_2 = -\psi$) are called, respectively, the complex velocity and the complex potential. Both v and g are S and I vectors, and their respective complex conjugates $\bar{v} = w(z) = v_1 - iv_2$ and $\bar{g} = f(z) = \phi_1 - i\phi_2$ may be represented by analytic functions for which $w = df/dz$.

3. Definitions and Results in Three Dimensions.

DEFINITION 1. *Let $\mathbf{C}_3$ denote the set of all "three-dimensional (3-D) numbers" of the form $Z = 1x + \delta y + \epsilon z$, in which* (i) *$Z$ may be interpreted as a vector with basis vectors $1, \delta, \epsilon$ and with components x, y, z in $\mathbf{C}_2$; and* (ii) *the rules of multiplication are as follows (or other equivalent forms of them):*

$$1^2 = 1, \qquad 1\delta = \delta 1 = \delta, \qquad 1\epsilon = \epsilon 1 = \epsilon,$$
$$\delta^2 = -\tfrac{1}{2}(1 + i\epsilon), \qquad \epsilon^2 = -\tfrac{1}{2}(1 - i\epsilon), \qquad \delta\epsilon = \epsilon\delta = -\tfrac{1}{2}(i\delta).$$

The 3-D unit 1 as a factor may be omitted (as the factor 1 in 2-D is omitted), with Z written generally as

$$Z = x + \delta y + \epsilon z \quad = Z_R + iZ_I,$$

where $Z_R = x_R + \delta y_R + \epsilon z_R$ and $Z_I = x_I + \delta y_I + \epsilon z_I$, with $x_R, y_R, z_R, x_I, y_I, z_I$ real.

DEFINITION 2. *Let $\mathbf{C}'_3$ be a subset of $\mathbf{C}_3$ such that, for every element $Z = x + \delta y + \epsilon z$ in $\mathbf{C}'_3$, the components x, y, z are real.*

Then, for Z in $\mathbf{C}_3$, Z_R and Z_I are in $\mathbf{C}'_3$, and the basis vectors 1, δ, and ϵ are in $\mathbf{C}'_3$. If Z is an *independent variable*, for which values can be prescribed, then one can set $Z_I = 0$, so that x, y and z are real and Z is in $\mathbf{C}'_3$.

The algebraic properties of these numbers in $\mathbf{C}_3$ are developed and discussed in papers by the author to be published. The multiplicative inverse, Z^{-1}, is of special significance. It can be found by setting $Z^{-1} = a_1 + \delta a_2 + \epsilon a_3$, where $a_k \in \mathbf{C}_2$, and by requiring $ZZ^{-1} = 1$. It is found that there are certain nonzero values of Z for which Z^{-1} is not defined, with results including the following:

THEOREM 3. *For $Z = x + \delta y + \epsilon z$ in $\mathbf{C}'_3$, the domain of definition of Z^{-1} includes all of the $\mathbf{R}^3$ space of (x, y, z) except the origin and any of the six rays in the plane $x = 0$ where $\vartheta = \tan^{-1}(z/y) = (\ell - 1)\pi/3$ for $\ell = 1, 2, \ldots, 6$.*

REMARK 4. The algebra of $\mathbf{C}_3$ is a *linear algebra of order 3* over the field of ordinary complex numbers, $\mathbf{C}_2$. Further, $\mathbf{C}_3$ is a *commutative ring with unity*, and not a field, since, for some nonzero elements Z, the inverse Z^{-1} is not defined.

Further discussion of Z^{-1} is beyond the scope of this note, but is included elsewhere.

DEFINITION 5. *For every $Z = x + \delta y + \epsilon z$ in $\mathbf{C}_3$, denote as the **bijugate** of Z the element of $\mathbf{C}_3$ given by $\overline{Z} = \tfrac{1}{2}x - \delta y - \epsilon z$.*

(The bijugate can be defined more generally.) The 3-D bijugate is in some ways analogous to the 2-D conjugate. The similar role in regard to analytic functions will be demonstrated here.

As an analogy to the variables z and g in $\mathbf{C}_2$ described in the previous section, consider the two variables in $\mathbf{C}_3$: $Z = x + \delta y + \epsilon z$ and $G = \phi_1 + \delta\phi_2 + \epsilon\phi_3$, which are also vectors in $\mathbf{C}_2^3$. Now let G be defined to be the vector (3-D complex function) whose *bijugate* is an *analytic function* $\overline{G} = F(Z) = \frac{1}{2}\phi_1 - \delta\phi_2 - \epsilon\phi_3$. The concepts of function, limit, derivative, and analytic function can be extended, with some care, to the set $\mathbf{C}_3$. Then, in analogy to the Cauchy-Riemann conditions in two dimensions, the following necessary conditions for the differentiability, and hence analyticity, of $F(Z)$ are found:

THEOREM 6. *For Z in some domain $\mathbf{D}_3 \subseteq \mathbf{C}_3$, and G in $\mathbf{C}_3$ with components ϕ_k in $\mathbf{C}_2$ such that $\overline{G} = F(Z)$, the necessary conditions for analyticity of $F(Z)$ are:*

$$\begin{aligned} \operatorname{div} G &= \phi_{1x} + \phi_{2y} + \phi_{3z} = 0, \\ \operatorname{curl} G &= 1(\phi_{3y} - \phi_{2z}) + \delta(\phi_{1z} - \phi_{3x}) + \epsilon(\phi_{2x} - \phi_{1y}) = 0, \end{aligned}$$

along with $\quad \phi_{1y} - i(\phi_{2z} + \phi_{3y}) = 0 \quad$ and $\quad \phi_{1z} - i(\phi_{2y} - \phi_{3z}) = 0.$

Since all the components of the curl must vanish, G is an S and I vector in three dimensions. Further, if we write $\phi_k = \phi_{kR} + i\phi_{kI}$ and $G = G_R + iG_I$, with the components ϕ_{kR} of G_R and components ϕ_{kI} of G_I real, then G_R and G_I are also S and I vectors (with the final two equations in Theorem 6 serving to connect the components ϕ_{kR} of G_R to the components ϕ_{kI} of G_I). In Theorem 6, x, y, and z are independent variables defined generally to be complex, but as independent variables, may be taken to be real (i.e., $Z \in \mathbf{C}_3'$).

COROLLARY 7. *If $W = v_1 + \delta v_2 + \epsilon v_3$, in $\mathbf{C}_3$, is defined to be the vector whose bijugate is the analytic function that is the derivative of $F(Z)$: $\overline{W} = V(Z) = dF/dZ = \frac{1}{2}v_1 - \delta v_2 - \epsilon v_3$, then W is also an* S and I *vector and*

$$\begin{aligned} v_1 &= \phi_{1x} = -(\phi_{2y} + \phi_{3z}), \\ v_2 &= \phi_{1y} = \phi_{2x} = i(\phi_{2z} + \phi_{3y}), \\ v_3 &= \phi_{1z} = \phi_{3x} = i(\phi_{2y} - \phi_{3z}), \\ &\ \phi_{3y} = \phi_{2z}, \end{aligned}$$

EXAMPLE 8. For Z in $\mathbf{C}_3'$ the product $Z^2 = Z\,Z$, with use of the rules of multiplication from Definition 1, is $Z^2 = x^2 - \frac{1}{2}(y^2 + z^2) + \delta(2xy) + \epsilon(2xz) - i\delta(yz) - i\epsilon\frac{1}{2}(y^2 - z^2)$. Then for $F(Z) = Z^2$, the results are $\phi_{1R} = 2x^2 - (y^2 + z^2)$, $\phi_{2R} = -2xy$, $\phi_{3R} = -2xz$, $\phi_{1I} = 0$, $\phi_{2I} = yz$, $\phi_{3I} = \frac{1}{2}(y^2 - z^2)$, which are readily seen to satisfy Theorem 6. The two S and I vectors G_R and G_I, with respective Cartesian components ϕ_{kR} and ϕ_{kI}, are thus generated by $F(Z) = Z^2$.

The (harmonic) components of either G_R or G_I can be related to a 3-D velocity potential and general 3-D stream functions, and either G_R or G_I can be taken to be a "3-D complex potential," with the corresponding "3-D complex velocity" then being either W_R or W_I.

A primary result here is that this theoretical structure can be used to generate S and I vectors and harmonic functions in three dimensions, as can the Whittaker-Bergman method, but without integration here, as in ordinary analytic-function theory for two dimensions.

Details, proofs, and further results are in [**10**].

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16. Abstract

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