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DYNAMICS FORMULATIONS FOR THE REAL-TIME SIMULATION OF CONSTRAINED MOTION

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ABSTRACT

The Space Shuttle program has relied heavily on simulation throughout all phases of development and operation. Real-time, man-in-the-loop simulation has served the NASA manned-space-flight program by providing the means to evaluate systems design and integrated systems performance in a simulated flight environment as well as provide a means to train flight crews.

New challenges are presented by the development and operation of a permanently manned Space Station. The assembly of the Space Station, the transferral of payloads and the use of the Space Station manipulator to berth the Orbiter are operations critical to the success of the Space Station. All these operations are examples of constrained motion among the bodies associated with the Orbiter and Space Station system.

Current real-time simulations of the Orbiter and Space Station do not have the capability to model the dynamics of constrained motion. Determining an efficient, yet general, method for constrained motion in multibody systems is one essential key to high-fidelity of these critical operations.

This report described the state-of-the-art of formulating the governing dynamical equations of motion for constrained systems. The uses of the two basic problems in multibody dynamics are discussed. The most efficient formulations of the equations of motion are addressed from the point of view of completeness. The issues surrounding incorporating the constraints into the equations of motion are presented. Finally, an overview of the strengths and weaknesses of the current methods is given, along with some recommendations for further research.

INTRODUCTION

The planned scenarios involving the Shuttle Orbiter, the Space Station and their respective remote manipulators are examples of complex dynamical systems subjected to kinematic constraints, see Figure 1.

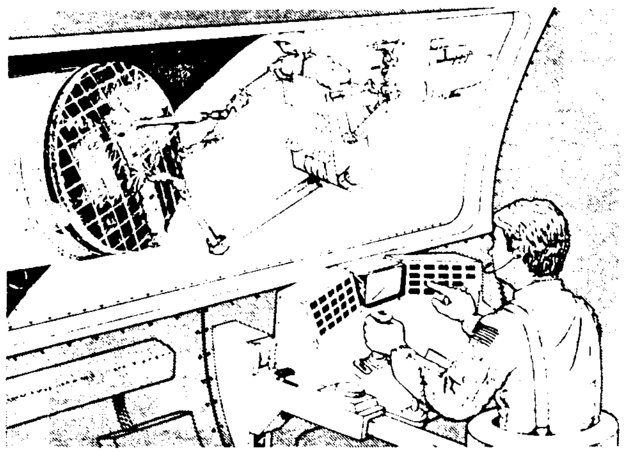

Figure 1 - Constrained motion between TRS and satellite

In order to develop realistic models of these activities, some of the bodies may be considered to be rigid and others flexible. An analysis of the flexibility of the bodies requires a complete knowledge of the system dynamics so that quantitative expressions for the joint forces and moments may be obtained. Having the capacity to formulate completely the equations of motion with the appropriate constraint equations and solve them is essential to any simulation of these systems.

NASA has relied heavily on simulation throughout the Space Shuttle program's development and operation. Real-time, man-in-the-loop simulation has provided the means to evaluate systems design and integrated systems performance in a simulated flight environment and to train flight crews.

Current real-time simulations of the Orbiter and Space Station do not have the capability to model the dynamics of constrained motion. Determining an efficient, yet general, method for simulating constrained motion in multibody systems would enhance the fidelity between the man-in-the-loop simulation and critical on-orbit operations.

This report addresses the efficiency and completeness of current formulations of the dynamics of constrained systems. Efficient formulations are necessary since the goal is to run the simulation in real-time. The formulation should also be complete since flexible bodies may be included.

The balance of this report is divided into three parts with the following part providing a discussion of the equations of motion. The next part discusses incorporating the constraint equations. Concluding remarks are given in the last part.

EQUATIONS OF MOTION

There are two basic problems in multibody dynamics. The first is the problem of determining the motion of a system from a set of applied forces; it is referred to as the forward dynamics problem. The second is the problem of determining a set of forces required to produce a prescribed motion in a system; it is referred to as the inverse dynamics problem.

A solution of the inverse dynamics is essential in the dynamic control of systems and many efficient formulations have been developed to implement real-time dynamic control. A solution of the forward dynamics is essential in the simulation of dynamic systems. Usually the forward dynamics are found by first formulating the inverse dynamics and then solving for the motion. Recently, several methods for improving the efficiency of the forward dynamics solution have been developed.

The three main approaches towards deriving the dynamic equations of motion for multibody systems with rigid elements have been the Newton-Euler, the Lagrange and the Kane methods. In all these methods efficiency is derived from the structure of the computation. First the linear and angular velocities and accelerations are computed recursively from the reference body to the end of the chain, then the forces and torques are computed recursively from the end to the reference body.

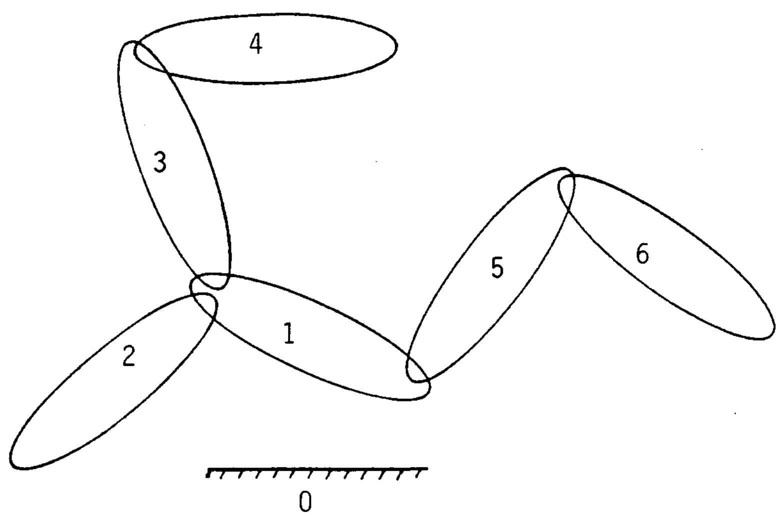

Figure 2 - A general chain system

Consider a multibody system such as shown in Figure 2. An accounting system for the system connectivity may be developed by arbitrarily selecting one of the bodies as a reference body and calling it B_1. Next, number the other bodies of the system arbitrarily. This numbering of the bodies can be used to describe the chain structure or topology through the "body connection array" as follows: Let $L(k)$, $k = 1, \ldots, N$ be the array of the adjoining lower body of B_k. The lower body is defined as the body adjoining B_k that is closest to the reference body [1].

The angular velocity of a typical body B_k in the inertial frame, B_o, is readily obtained by the recursion formula

$$\underline{\omega}^k = \underline{\omega}^{L(k)} + {}^{L(k)}\underline{\omega}^k \tag{1}$$

where $\underline{\omega}^k$ and $\underline{\omega}^{L(k)}$ are the angular velocities of B_k and $B_{L(k)}$ relative to the inertial frame and ${}^{L(k)}\underline{\omega}^k$ is the angular velocity of B_k relative to $B_{L(k)}$.

The angular acceleration of B_k in the inertial frame may be obtained by differentiating equation (1) in the inertial frame, leading to the recursion formula

$$\underline{\alpha}_k = \underline{\alpha}^{L(k)} + {}^{L(k)}\underline{\alpha}^k + \underline{\omega}^{L(k)} \times {}^{L(k)}\underline{\omega}^k \tag{2}$$

where $\underline{\alpha}^k$ and $\underline{\alpha}^{L(k)}$ are the angular accelerations of B^k and $B_{L(k)}$ relative to the inertial frame and ${}^{L(k)}\underline{\alpha}^k$ is the angular acceleration of B_k relative to $B_{L(k)}$.

Recursion formula can also be developed for the position, velocity and acceleration of a point on any link [1-4].

A chain system of N rigid bodies will, in general, have 6N or fewer degrees of freedom. This comes as a result of the fact that each rigid body can have up to 6 degrees of freedom. Kinematic constraints between members of the system reduce the number of degrees of freedom through contact between kinematic pairs. Usually, the analysis is simplified by introducing the kinematic constraints due to joints connecting the bodies of the system at an early stage. The systems's configuration is then said to be completely determined by n generalized coordinates, where n is the number of degrees of freedom of the system. However, it may occur that a contact between bodies may be made or broken as the motion of the system progresses. Figure 3 shows a case where two more bodies of the system in Figure 2 are in contact. In order to distinguish between the two types of constraints the former will be called a permanent constraint and the latter will be called a temporary constraint [5].

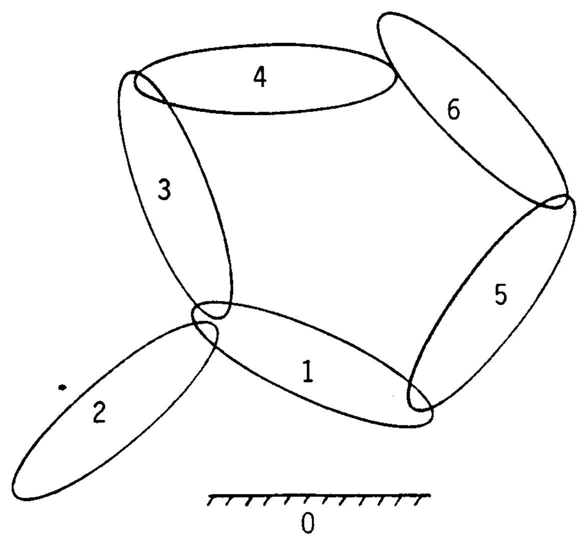

Figure 3 - Temporary constraint between bodies 4 and 6

Many procedures have been developed for both the inverse and forward dynamics of systems with permanent constraints using recursive relationships for efficiency. References [2, 3, 6] represent recently reported research efforts in rigid body inverse dynamics using recursive formulations of the Newton-Euler, Kane and Lagrange methods, respectively. Recursive formulations of the forward dynamics of rigid body systems with permanent constraints are significantly fewer in number and more intricate in their derivation. References [7-9] represent the currently reported procedures of this type.

Armstrong [7] hypothesized the existence of a linear recursive relationship between the motion of a forces applied to a body and the motion of and forces applied to its neighbors. He defines a set of recursion coefficients for each body and shows how the coefficients for a body may be calculated in terms of those of one of its neighbors. The coefficients are then used to calculate the acceleration of each body. The computational complexity of this method is $O(n)$, but the coefficient of n is quite large. The method is applicable to robots with spherical joints, but a modification is included in Appendix II [7] for revolute joints. This modification increases the computational requirement significantly, although the methods remains $O(n)$.

Walker and Orin [8] describe four methods of solving the forward dynamics problem for systems with revolute or prismatic joints. Three of these methods are based efficient techniques for solving the inverse dynamics problem. The computational complexity of each of these methods is $O(n^3)$, however the coefficients of the higher powers of n are relatively small. The fourth method discussed is based upon a recursive technique for constructing the moment of inertia matrix. The concept of composite center of mass and moment of inertia matrix of a partial set of bodies at the end of the chain is used in this recursion. The fourth method has a computational complexity of $O(n^2)$, however the coefficient of the n^2 term is large.

The basic approach of Walker and Orin's $O(n^3)$ algorithm can be summarized by following Featherstone [9]. The equations of motion can be expressed in the form

$$\underline{\tau} = A(\underline{q})\ \underline{\ddot{q}} + \underline{c}(\underline{q}, \underline{\dot{q}}, \underline{x}) \tag{3}$$

where $\underline{\tau}$ is the vector of generalized forces acting at the joints; $\underline{q}$, $\underline{\dot{q}}$, $\underline{\ddot{q}}$ are the generalized coordinate vector and its time derivatives; A is the generalized inertia matrix and $\underline{c}$ is the vector of Coriolis, centrifugal and external forces $\underline{x}$.

An algorithm for solving inverse dynamics can be thought of as an implementation of a vector function in the form

$$\underline{\tau} = \underline{d}(\underline{q}, \underline{\dot{q}}, \underline{\ddot{q}}, \underline{x}) \tag{4}$$

which states $\underline{\tau}$ is obtained from the generalized coordinates, the generalized speeds, the generalized acceleration and the external force acting on the system. It is seen that (cf (3))

$$\underline{c}(\underline{q}, \underline{\dot{q}}, \underline{x}) = \underline{d}(\underline{q}, \underline{\dot{q}}, \underline{0}, \underline{x}) \tag{5}$$

the calculation of $\underline{c}$ is the special case of calculating the inverse dynamics when the generalized acceleration is zero.

The forward dynamics calculation is conveniently partitioned into three steps: the calculation of $\underline{c}$, the calculation of A, and the solution of the set of simultaneous equations

$$A\underline{\ddot{q}} = \underline{\tau} - \underline{c} \tag{6}$$

For $\underline{\ddot{q}}$. The computational complexity of the first step is $O(n)$, the second step is $O(n^2)$ and that of the third is $O(n^3)$. This approach is referred to as the composite rigid-body method.

Featherstone [9] describes an extension of Armstrong's method to system with revolute and prismatic joints and uses a spatial notation consisting of 6X1 spatial vectors, 6X6 spatial transformation matrices and 6X6 spatial inertias. He refers to this method as the articulated-body method.

The articulated-body method starts with the equation

$$\ddot{\underline{q}} = \underline{h}(\underline{q}, \underline{\tau} - \underline{c}) \tag{7}$$

which states that $\ddot{\underline{q}}$ is a function of $\underline{q}$ and $\underline{\tau} - \underline{c}$ without implying the need to calculate a generalized inertia matrix.

The calculation is again conveniently partitioned into three steps: the calculation of $\underline{c}$, the calculation of the inhomogeneous articulated-body inertias for each body starting with the end effector and proceeding sequentially to the reference body, and the generalized accelerations are then calculated for each body starting at the reference body and proceeding sequentially to the end effector. The computational complexity of each step is $O(n)$.

The computational complexity of Armstrong's method is greater than either of the most efficient of Walker and Orin's $O(n^3)$ methods or Featherstone's articulated-body method. The best Walker and Orin method is more efficient than the Featherstone for values of n less than twelve [9]. Either the Armstrong nor the Walker and Orin nor the Featherstone methods provide for the calculation of the constraint forces of the joint. In this sense these algorithms are not complete. The next part describes some methods of incorporating the constraint equations and finding the constraint forces.

INCORPORATING CONSTRAINTS

References [5, 10-15] represent a partial list of recently reported research efforts on dynamics formulations for constrained motion. The equations of motion for a system of rigid bodies with open chain or tree structure (as in Figure 2) may be written as in equation (6). In systems where additional restrictions have been placed on the motion of the bodies, these constraints can usually be placed in the form

$$B\dot{\underline{q}} = \underline{g} \tag{8}$$

where B is an mxn matrix with m<n and $\underline{g}$ is n vector. Both B and $\underline{g}$ are functions of $\underline{q}$ and time t. Holonomic constraints can always be written in the form of equation (8). Simple or Pfaffian nonhomonomic constraints can also be written in the form of equation (8).

There are two fundamental approaches to solving the equations (6) and (8) for such systems [14]. One can introduce unknown generalized forces of constraint between the constrained bodies by the technique of undetermined multipliers and then solve the equations of motion simultaneously with the constraint equations to determine the generalized constraint forces as well as the kinematic variables. Alternatively, one can use the constraint equations to reduce the dimension of the equations of motion to be solved, eliminating the need to represent or solve for generalized constraint forces. The second approach has the advantage of providing, for solution, a set of equations of minimum dimension. The difficulty lies in its implementation.

In the special case of holonomic constraints, one might integrate equation (8) and solve explicitly for a subset m of the generalized coordinates in terms of the remaining $l \stackrel{\Delta}{=} n - m$. Substitution for these generalized coordinates in equation (6) permits reduction of the set of equations to be solved to l second order equations. In general, this reduction is not easily accomplished.

If equation (8) is nonintegrable (i.e., nonholonomic), then m of the generalized coordinates cannot be solved for in terms of the remaining l. One would like to solve for m of the elements of $\underline{\dot{q}}$, but the result cannot be substituted effectively into the second order equation (6). This obstacle can be avoided by converting equation (6) to first order form [10]. A large obstacle is the potential for the rank r of the matrix B in equation (8) to be less than the row dimension m. These state of affairs could occur in a system changing configuration, resulting in singularities in the numerical solution. Other methods which avoid all singularity problems exist and are described next.

The n generalized speeds, $\underline{\dot{q}}$, may be expressed in terms of l new generalized speeds, $\underline{u}$, by hypothesizing a relationship of the form

$$\underline{\dot{q}} = \beta \underline{u} + \underline{\delta} \tag{9}$$

where β is an nxl matrix and $\underline{\delta}$ is an n vector.

The solution to equation (8) for $\underline{\dot{q}}$ falls into one of two categories. If $\underline{g}$ does not lie in the column space of B (i.e., if $\underline{g}$ cannot be formed by a linear combination of the columns of B), the constraints are inconsistent, the problem is ill-posed and no solution can be found for $\underline{\dot{q}}$. If $\underline{g}$ does lie in the column space of B, at least one solution exists for $\underline{\dot{q}}$. Let Z be an nxl matrix with linearly independent columns that form a basis for the null space of B, that is,

$$BZ = 0 \tag{10}$$

and let Y be an nxm matrix whose columns complete a basis for the vector space R^n. The general soluation to equation (8) is

$$\underline{\dot{q}} = Z\underline{z} - Y\underline{y} \tag{11}$$

where $\underline{y}$ is a vector of m unique scalars such that

$$\underline{g} = BY\underline{y} \tag{12}$$

and $\underline{z}$ is a vector of l independent quantities describing the constrained system. The vector $\underline{u}$ of generalized speeds for the constrained system may be defined as

$$\underline{u} \triangleq \underline{z} \tag{13}$$

By substituting from equation (13) into (11), one obtains an equation of the form of equation (9) with

$$\beta = Z \tag{14}$$

and

$$\underline{\delta} = -Y\underline{y} \tag{15}$$

The preceding development demonstrates that it is always possible to find an equation of the form of equation (9) for a system subjected to constraints in the form of equation (8). However, it does not give an operational procedure for producing equations (14) and (15). The matrices Z and Y are not unique, and many approaches for constructing them can be envisioned. The problem is equivalent to that of constructing a generalized inverse of B, of which the Moore-Penrose generalized inverse is a particular example. Singular value decomposition [12, 14] and the zero eigenvalue theorem [11], are two methods of calculating this inverse. The remainder of this part will briefly outline two methods using Kane's equations of motion.

Kane's equations of motion [10] may be written in the remarkably simple form

$$\underline{f} + \underline{f}^* = 0 \tag{16}$$

where $\underline{f}$ is the n-vector of generalized active forces and $\underline{f}^*$ is the n vector of generalized inertia forces. The set of equations (6) may be efficiently derived from the set of equations (16) by relatively simple scalar products and recursive kinematic relationships [3, 4].

Kane's equations for constrained systems may be found by solving equation (8) for m of the $\underline{\dot{q}}$ in terms of the remain l, as described before. This leads to a reduced set of equations of motion.

$$\underline{k} + \underline{k}^* = 0 \tag{17}$$

where $\underline{k}$ and $\underline{k}^*$ are l-vectors of reduced generalized active and inertia forces. Equations (17) together with equations (8) then constitute a system of n equations for the n-vector $\underline{\dot{q}}$.

For complex multibody systems the solution of equations (8) for m of the $\underline{\dot{q}}$ may not be convenient or even possible. Also, the generalized constraint forces have been eliminated from the analysis. Although this reduces the dimension of the problem, it also means the analysis is incomplete.

To develop a method which includes the generalized constraint equations and retains the advantageous features of the previous method, let the constraining forces and moments associated with the constraint equations (8) be represented by a series of forces $\underline{f}'_i$ applied at points P_i, i = 1, ...,N', together with a series of couples with moments $\underline{m}'_j$ applied to bodies B_j, j = 1, ...,M'. Assuming that $m \leq 3\ (N' + M')$, the following analysis is applicable [15].

Let the P_i and B_j have specified motions given by $\underline{V}_{P_i}(t)$ and $\underline{\omega}_j(t)$. Let the scalar components of these vectors, relative to a set of basis vector fixed in the reference body, be collected into arrays $\underline{V}$ and $\underline{\omega}$ whose elements may be identified with the elements of $\underline{g}$ in equation (8). That is,

$$\left.\begin{array}{l} V_i \\ \omega_i \end{array}\right\} = g_i = \sum_{j=1}^{m} b_{ij} \dot{q}_j \tag{18}$$

for $i = 1,\ldots,3N'$ for the V_i and $i = 3N' + 1, \ldots, 3(N'+M')$ for the ω_i.

Next, let the components of $\underline{F}'_i$ and $\underline{M}'_j$ in the directions of the components of $\underline{V}_{Pi}$ and $\underline{\omega}_j$ be designated as ϕ_i and μ_j. Then the power, P, produced by the constraining forces and moments is

$$P = \sum_{i=1}^{3N'} \phi_i V_i \quad \sum_{j=1}^{3M'} \mu_j \omega_j \tag{19}$$

Let $\underline{\lambda}$ be an m-vector whose elements are ϕ_i and μ_j. P may now be written as

$$P = \underline{\lambda}^T \underline{g} = \underline{\lambda}^T B \underline{\dot{q}} \tag{20}$$

where the superscript T designates the transpose.

Since neither $\underline{f}'_i$ nor $\underline{m}'_j$ are functions of $\underline{q}$, the generalized forces associated with these constraining forces and moments are

$$\underline{f}' = \partial P / \partial \underline{\dot{q}} = B^T \underline{\lambda} \tag{21}$$

where $\underline{f}'$ is the generalized force vector.

In view of the foregoing analysis, let the generalized active forces be separated into two parts consisting of: those developed from constraint forces and moments and those developed from applied forces and moments. Let these be designated $\underline{f}'$ and $\underline{f}$. Then Kane's equations are

$$\underline{f}' + \underline{f} + \underline{f}^* = \underline{0} \tag{22}$$

Substitute for $\underline{f}'$ in equation (22) using equation (21)

$$B^T \underline{\lambda} + \underline{f} + \underline{f}^* = \underline{0} \tag{23}$$

Taken together with the constraint equations (8) they constitute a set of n+m equations for the n elements of $\underline{\dot{q}}$ and the m elements of $\underline{\lambda}$.

Suppose that an nxl matrix has been found such its columns form a basis for the null space of B. Then by premultiplying equation (23) by β^T we obtain

$$\beta^T \underline{f} + \beta^T \underline{f}^* = 0 \tag{24}$$

since

$$B\beta = \beta^T B^T = 0 \tag{25}$$

Equation (24) together with equations (8) then represent a set of n equations for the n elements of $\underline{\dot{q}}$. Once $\underline{\dot{q}}$ and $\underline{q}$ are determined, $\underline{\lambda}$ may be determined by back substitution into equation (23).

The method may be viewed as a generalization of the psuedo-coordinate method of Kane [10]. The analysis is developed with the system unrestrained. However, a difference is that the constraint forces and moments are formally introduced through a product of undetermined multipliers with coefficients of the constraint equations.

The concept is simple, but the implications extend beyond the task of finding constraint forces and moment components. For example, when compared with Lagrange's equations with undetermined multipliers, the procedures are similar. However, with Kane's equations the generalized speeds may be employed as the fundamental kinematic variables. Also, holonomic as well as simple nonholonomic constraints may be accommodated directly. Finally, the constraint force and moment components may be either evaluated or eliminated from the analysis. In the latter case the resulting equations are equivalent to the standard form of Kane's equations for constrained systems.

To illustrate these concepts consider Figure 4 which depicts the generalized forces in the n-dimensional space of the generalized speeds. The force triangle is a representation of equation (22). In this context, the generalized forces $\underline{k}$ and $\underline{k}^*$, used in Kane's equations for constrained systems, are seen to be projections of $\underline{f}$ and $\underline{f}^*$ onto the directions β^T, the generalized inverse of B.

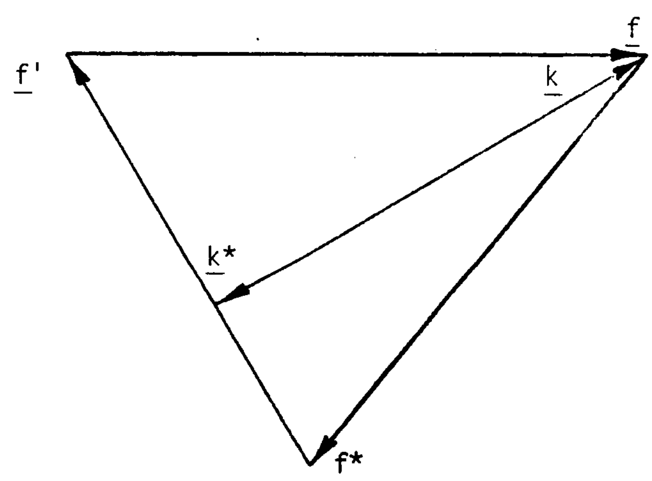

Figure 4 - A geometric interpretation of the generalized forces

CONCLUSIONS

Very efficient procedures have been developed for solving the inverse dynamics problem. Representative solution method using recursive formulatins of the Newton-Euler, Kane and Lagrange methods are described. A generalization of Kane's method [15] has been shown to be capable of efficiently and completely accommodating various constraints applied to a system whereas the others have not. This is not to say that this cannot be done. More research is certainly appropriate to develop parallel techniques using the Newton-Euler and Lagrange methods.

The forward dynamics problem has been more elusive. Although very efficient procedures have been developed for the special cases of permanent constraints of the revolute, prismatic and spherical types, a corresponding generalization has yet to be developed. Indeed, none of the forward dynamic algorithms available can efficiently computer the constraint forces and moments for even permanent constraints. Research is to determine efficient and complete methods of generalizing the solution to the forward dynamics problem, if this is possible.

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