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# Spectral dilation of $L(B,H)$ -valued measures and its application to stationary dilation for Banach space valued processes\*

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## Abstract

Let  $B$  be a Banach space and  $H$  and  $K$  two Hilbert spaces. The spectral dilation of  $L(B,H)$ -valued measures is studied and it is shown that the recent results of Makagon and Salehi (1986) and Rosenberg (1982) on the dilation of  $L(K,H)$ -valued measures can be extended to hold for the general Banach space setting of  $L(B,H)$ -valued measures. These  $L(B,H)$ -valued measures are closely connected to the Banach space valued processes. This connection is recalled and as application of spectral dilation of  $L(B,H)$ -valued measures the well known stationary dilation results for scalar valued processes is extended to the case of Banach space valued processes.

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**1. INTRODUCTION.** A simple dilation theorem for measures asserts that a Hilbert space  $H$  can be imbedded in a larger Hilbert space  $K$  so that a given  $H$ -valued measure  $\xi(\cdot)$  can be constructed by the projection of a simpler  $K$ -valued measure  $\zeta(\cdot)$ , more precisely,

$$\xi(\Delta) = P\zeta(\Delta), \text{ for all } \Delta \in \Sigma \quad (1.1)$$

where  $\Sigma$  is a fixed  $\sigma$ -algebra and  $P$  is the orthogonal projection on  $K$  onto  $H$ . In this case  $\zeta(\cdot)$  is called a dilation for  $\xi(\cdot)$ .

Niemi [16] proved the following dilation theorem: Every countably additive  $H$ -valued measure  $\xi(\cdot)$  on the Borel subsets of a locally compact Hausdorff space  $\Omega$  has a countably additive orthogonally-scattered dilation  $\zeta(\cdot)$  of the form (1.1). Chatterji [3] noted that the existence of Niemi's dilation is an algebraic property and it remains true for any bounded additive vector valued function on any  $\sigma$ -algebra  $\Sigma$ .

A popular dilation theorem for an  $H$ -valued process  $x_t$  asserts that one can express  $x_t$  as

$$x_t = Py_t, \quad t \in R, \quad (1.2)$$

where  $y_t$  is a stationary  $K$ -valued process and  $P$  is the orthogonal projection on  $K \supseteq H$  onto  $H$ . This kind of dilation theorems started by Abreu in [1] as early as 1970, where he proved any harmonizable process in the sense of Cramer [2] has a stationary dilation of the form (1.2). In [14] Miamee and Salehi characterized all the processes which have stationary dilations and showed that this class is exactly the same as the class of harmonizable processes in the sense of Rozanov [22]. Several other authors have established some useful dilation theorems [12], [15], [16], [18], and [19].

Of course dilation results of the form (1.1) are closely tied with those of the form (1.2). This is because of the well known fact that every  $H$ -valued stationary process  $x_t$  has a spectral representation of the form

$$x_t = \int e^{-i\omega t} \zeta(ds), \text{ for all } t \in R, \quad (1.3)$$

for some orthogonally scattered  $H$ -valued measure  $\zeta(\cdot)$ .

In (1982) Rosenberg [21] introduced the following dilation problem: Let  $H_1$  and  $H$  be two Hilbert spaces, does any  $L(H_1, H)$ -valued measure  $T(\cdot)$  on  $\Sigma$  has a spectral dilation of the form

$$T(\Delta) = PE(\Delta)S, \text{ for all } \Delta \in \Sigma, \quad (1.4)$$

where  $E(\cdot)$  is a spectral measure on a Hilbert space  $K \supseteq H$ ,  $S$  is a linear operator in  $L(H_1, K)$  and  $P$  is the orthogonal projection on  $K$  onto  $H$ . (We have stated this in slightly different way to make it consistent with the other dilation problems (1.1) and (1.2)). Among other important results Rosenberg proved an Equivalence Theorem which states the equivalence of this problem with the existence of a 2-majorant (to be defined in Section 2) and with some other kinds of dilation problems. (Note that Rosenberg's dilation theorem is a generalization of Niemi's dilation (1.1). In fact, if we replace the Hilbert space  $H_1$  in Rosenberg's dilation, by the complex numbers  $\mathcal{C}$  then we arrive at Niemi's dilation since  $L(\mathcal{C}, H)$  can be identified with  $H$ . Rosenberg was also able to show that if  $H_1$  is a finite dimensional Hilbert space, say  $\mathcal{C}^n$ , then his dilation problem has a positive answer.

Makagon and Salehi in their recent work [10] studied Rosenberg's question extensively. They first gave an example showing that Rosenberg's dilation question is not valid in general, and then proved several useful theorems concerning this problem including some results on the positive direction.

The main purpose of this note is to take one step further and replace the Hilbert space  $H_1$  by a Banach space  $B$ , to see whether an  $L(B, H)$ -valued measure  $T(\cdot)$  has a dilation of the form

$$T(\Delta) = PE(\Delta)S, \text{ for all } \Delta \in \Sigma, \quad (1.5)$$

where  $E(\cdot)$  is again a spectral measure on some Hilbert space  $K \supseteq H$ ,  $S \in L(B, K)$  and  $P$  is the orthogonal projection on  $K$  onto  $H$ . We will show that most of the dilation results of Rosenberg [21] and Makagon and Salehi [10] can be extended to our Banach space setting.

The study of these  $L(B,H)$ -valued measures are essential in the study of Banach space valued stochastic processes (for a discussion on these see Section 5) which have been already introduced and studied by several authors (see for example [4] and [13] ). In fact, this has been our motivation of studying these kind of measures and their dilations here.

After setting up the notation and preliminary facts in Section 2, we prove the main part of Rosenberg's Equivalence Theorem for the Banach space case in Section 3. In Section 4 we extend main dilation theorems of [10] to our case. Finally in Section 5 after recalling the definition of Banach space valued stochastic processes and revealing their close tie with  $L(B,H)$ -valued measures we use the results of Section 4 to obtain some stationary dilation results for these stochastic processes, thereby extending the well known dilation theorems for simple scalar valued stochastic processes to the Banach space valued stochastic processes.

**2. PRELIMINARIES.** In this section we set up the notation and preliminary facts which are frequently used in the sequel. It will be understood that:

**2.1 Notations.**

- (a)  $\Sigma$  is an algebra, of subsets of a set  $\Omega$ ,
- (b)  $W$  denotes a normed vector space over the complex numbers  $\mathcal{C}$  with norm being denoted by  $|\cdot|$ ,
- (c)  $B$  stands for a Banach space over  $\mathcal{C}$ , with norm  $|\cdot|$ ,
- (d)  $H$  and  $K$  denote Hilbert spaces over  $\mathcal{C}$  with inner product  $(\cdot, \cdot)$  and norm  $|\cdot|$ ,
- (e)  $W'$  and  $W^*$  denote the dual and adjoint space of  $W$  respectively, and the action of any functional  $f$  on  $W$  at some  $x \in W$  is denoted by  $f(x) = \langle x, f \rangle$ ,
- (f) For any two normed linear spaces  $W_1$  and  $W_2$ ,  $L(W_1, W_2)$  denotes the space of all continuous linear operators from  $W_1$  into  $W_2$  with the norm  $|\cdot|$ .

**2.2 REMARK.** The distinction between the adjoint space  $W^*$  and the dual space  $W'$  is important here:  $W'$  is the space of all continuous linear functions on  $W$  while  $W^*$  is the space of all continuous conjugate linear functionals on  $W$ . i.e.

$$W^* = \{\overline{f(\cdot)}: f(\cdot) \in W'\}.$$

For any  $A \in L(W_1, W_2)$  one can define the adjoint  $A^*$  and the transpose  $A'$  of  $A$  as follows

$$A^*: W_2^* \rightarrow W_1^*; A^*(f) = f \circ A; \text{ for all } f \in W_2^*,$$

$$A': W_2' \rightarrow W_1'; A'(f) = f \circ A; \text{ for all } f \in W_2'.$$

Thus, again  $A'$  and  $A^*$  are in general different. However  $|A^*| = |A'| = |A|$ .

**2.3 DEFINITION.** An operator  $M$  in  $L(B, B^*)$  is called hermitian if for all  $x, y \in B$ ,  $\langle x, My \rangle = \overline{\langle y, Mx \rangle}$ , and it is called nonnegative (in symbols  $\succeq 0$ ) if it is hermitian and for all  $x \in B$ ,  $\langle x, Mx \rangle \geq 0$ . The set of all nonnegative operators from  $B$  into  $B^*$  is denoted by  $L^+(B, B^*)$ .

**2.4 DEFINITION.** Let  $m$  be a finitely additive (f.a.)  $W$ -valued measure defined on  $\Sigma$ . The semi-variation  $\|m\|$  of  $m$  is defined to be

$$\|m\| = \sup \left\{ \left| \sum_{\Delta \in \pi} m(\Delta) t_\Delta \right| : \pi \in \Pi(\Omega), t_\Delta \in \mathcal{C}, |t_\Delta| \leq 1 \right\}$$

where  $\Pi(\Omega)$  denotes the set of all  $\Sigma$ -partitions of  $\Omega$ . Note that in general  $\|m\| \in [0, \infty]$ , in case  $\|m\| < \infty$ ; the measure  $m$  is said to have finite semi-variation. The space of all  $W$ -valued f.a. measures on  $\Sigma$  with finite semi-variation is denoted by  $M(W)$ . Clearly  $\|\cdot\|$  provides a norm on  $M(W)$ . In fact, if  $W$  is a Banach space then  $(M(W), \|\cdot\|)$  becomes a Banach space as well (see [5], p. 53).

We denote by  $S(W)$  the set of all  $\Sigma$ -simple,  $W$ -valued functions of the form  $f = \sum_{i=1}^n 1_{\Delta_i} x_i$ ,  $\Delta_i \in \Sigma$ ,  $x_i \in W$ ,  $i=1, \dots, n$ . For  $m \in M(W')$  and  $f = \sum_{i=1}^n 1_{\Delta_i} x_i \in S(W)$  define (cf. [5], § 7)

$$\int_{\Omega} \langle f, dm \rangle = \sum_{i=1}^n \langle x_i, m(\Delta_i) \rangle.$$

To any  $f \in S(W)$  besides the familiar sup-norm, namely

$$|f|_{sup} = \sup\{|f(w)| : w \in \Omega\}$$

we associate another norm defined by

$$|f|_{\infty} = \sup\{|\int \langle f, dm \rangle| : m \in M(W^*) : \|m\| \leq 1\}. \quad (2.5)$$

The following lemma (see [10, Lemma 2.4 & Remark 2.7] ) shows that  $| \cdot |_{\infty}$  is actually a norm on  $S(W)$  and gives some of its further properties.

**2.6 LEMMA.** With the notation just described

- (a)  $| \cdot |_{\infty}$  is a norm on  $S(W)$ ,
- (b) The mapping  $m \rightarrow \int \langle \cdot, dm \rangle$  is a linear isometry from  $(M(W^*), \| \cdot \|)$  onto  $(S(W), | \cdot |_{\infty})$ ,
- (c) For every  $f \in S(W)$ , we have  $|f|_{sup} \leq |f|_{\infty}$ ,
- (d) When  $W$  is finite dimensional these two norms  $| \cdot |_{sup}$  and  $| \cdot |_{\infty}$  on  $S(W)$  are equivalent but not in general.

**2.7 LEMMA.** Let  $T$  be an  $L(B,H)$ -valued f.a. measure on  $\Sigma$ . Then

(a)

$$\begin{aligned} \|T(\cdot)\| &= \sup\{\|T(\cdot)x\|, x \in B, |x| \leq 1\} = \sup\{\|T(\cdot)^*y\|, y \in H, |y| \leq 1\} \\ &\leq 4 \sup\{|T(\Delta)| : \Delta \in \Sigma\}, \end{aligned}$$

where  $\|T(\cdot)\|$ ,  $\|T(\cdot)x\|$  and  $\|T(\cdot)^*y\|$  denote the semi-variation of the measures  $T(\cdot)$ ,  $T(\cdot)x$  and  $T(\cdot)^*x$  respectively.

(b) If moreover  $\Sigma$  is a  $\sigma$ -algebra and  $T$  is weakly countably additive (w.c.a.) then

$$\sup\{|T(\Delta)| : \Delta \in \Sigma\} < \infty$$

*Proof.* (a) follows from the fact that for any operator  $T \in L(B,H)$

$$\begin{aligned} |T| &= \sup\{|Tx| : x \in B, |x| \leq 1\} \\ &= \sup\{|(Tx, y)| : x \in B, y \in H, |x| \leq 1, |y| \leq 1\}. \end{aligned}$$

(b) For each  $x \in B$  and  $y \in H$  consider the complex-valued measure  $(T(\cdot)x, y)$ . It is well known that

$$\sup\{|(T(\Delta)x, y)| : \Delta \in \Sigma\} = C(x, y) < \infty.$$

Now for each fixed  $x$  apply uniform bounded principle to the set  $\{(T(\Delta)x, \cdot) : \Delta \in \Sigma\}$  of functionals on  $H$  and conclude that

$$\sup\{|T(\Delta)x| : \Delta \in \Sigma\} = C(x) < \infty, \text{ for all } x \in B.$$

Now using this, one can apply the uniform boundedness principle again, this time to the set  $\{T(\Delta) : \Delta \in \Sigma\}$  of operators in  $L(B, H)$  to get  $\sup\{\|T(\Delta)\|\} < \infty$ .

Let  $T$  be a f.a.  $L(W_1, W_2)$ -valued measure on  $\Sigma$ . For each  $f = \sum_{i=1}^n 1_{\Delta_i} x_i$  in  $S(W_1)$  one can define (cf. [5], § 7).

$$\int f dT = \sum_{i=1}^n T(\Delta_i) x_i.$$

This integral provides a linear operator  $\Phi_T: S(W_1) \rightarrow W_2$  defined by  $\Phi_T(f) = \int f dT$ .

**2.9 LEMMA.** [10]. Let  $T$  be a f.a.  $L(W_1, W_2)$ -valued measure on  $\Sigma$ . Then  $\Phi_T: (S(W_1), \|\cdot\|_\infty) \rightarrow W_2$  is continuous if  $T$  has a finite semi-variation  $\|T\|$ . If this is the case we further have  $\|\Phi_T\| = \|T\|$ .

**2.10 REMARK.** As we know the set  $M(L(B, B^*))$  of all  $L(B, B^*)$ -valued f.a. measures with finite semi-variation equipped with the semi-variation norm is a Banach space. On the other hand  $L(B, B^*)$  is isometric to the conjugate space  $(B \otimes B)^*$  of the tensor product  $B \otimes B$  equipped with the projective norm ([25], p. 190). So  $M(L(B, B^*))$  can be identified with  $M((B \otimes B)^*)$ . But by Lemma 2.6,  $M((B \otimes B)^*)$  is isomorphic to  $(S(B \otimes B), \|\cdot\|_\infty)^*$ . Hence  $M(L(B, B^*))$  is isomorphic to  $(S(B \otimes B), \|\cdot\|_\infty)^*$  and for any  $T \in M(L(B, B^*))$  and its isomorph of  $\psi$  in  $(S(B \otimes B), \|\cdot\|_\infty)^*$  we have

$$\langle \phi, \psi \rangle = \int \langle \phi, dT \rangle, \text{ for all } \phi \in S(B \otimes B)$$

**2.11 DEFINITION.** (a) A f.a.  $L(K, K)$ -valued measure  $E$  defined on an algebra  $\Sigma$  is called a f.a. spectral measure in  $K$  if (i) for every  $\Delta \in \Sigma$ ;  $E(\Delta)$  is an orthogonal projection

in

$K$ , (ii)  $E(\Delta_1)E(\Delta_2) = 0$ , for every pair  $\Delta_1$  and  $\Delta_2$  of disjoint sets in  $\Sigma$ . If in addition  $\Sigma$  is a  $\sigma$ -algebra and  $E$  is w.c.a. then  $E$  is called a c.a. spectral measure or simply a spectral measure.

(b) We say that a f.a. (w.c.a.) measure  $T: \Sigma \rightarrow L(B, H)$  has a f.a. spectral dilation (spectral dilation) if there exist a Hilbert space  $K \supseteq H$ , a f.a. spectral dilation (spectral dilation)  $E(\cdot)$  in  $K$ , and an operator  $S \in L(B, K)$  such that

$$T(\Delta) = PE(\Delta)S, \text{ for all } \Delta \in \Sigma,$$

where  $P$  is the projection of  $K$  onto  $H$ .

(c) We say that a f.a. (w.c.a.) measure  $T$  defined on the algebra ( $\sigma$ -algebra)  $\Sigma$  with values in  $L(B, H)$  has a f.a. (w.c.a.) 2-majorant  $M$  if  $M$  is f.a. (w.c.a.)  $L^+(B, B^*)$ -valued measure on  $\Sigma$  such that

$$\left| \sum_{i=1}^n |T(\Delta_i)x_i|^2 \leq \sum_{i=1}^n \langle x_i, M(\Delta_i)x_i \rangle, \text{ for all } x_i \in B, \Delta_i \in \Sigma.$$

If  $\Sigma$  is a  $\sigma$ -algebra and  $\xi$  in a c.a.  $H$ -valued measure on  $\Sigma$  then the integral  $\int f d\xi$  is well defined for all bounded measurable complex-valued functions  $f$  (cf. [6], IV. 10 or [5] § 7). Now if  $T$  is a w.c.a.  $L(B, H)$ -valued measure on  $\Sigma$  then it is known that it is strongly c.a. ([6], IV 10.2). So for each  $x \in B$ , the  $H$ -valued measure  $T(\cdot)x$  is c.a. and hence  $\int f d(Tx)$  is defined. One can then define  $\int f dT$  by

$$\left( \int f dT \right) x = \int f d(Tx), \quad x \in B. \tag{2.12}$$

For this integral one can easily check that

$$\left| \int f dT \right| \leq \|T\| \|f\|_{\infty}, \text{ for all bounded functions } f, \tag{2.13}$$

$$U \left( \int f dT \right) V = \int f d(UTV), \text{ for all bounded functions } f, \tag{2.14}$$

$$\left( \int f d(Tx), y \right) = \int f d(Tx, y), \text{ for all } x \in B, y \in H. \tag{2.15}$$



Finally we will need the following lemmas in the proof of Theorem 4.4.

**2.16 LEMMA.** [21] Let  $\nu(\cdot)$  be a f.a. nonnegative real-valued measure on a  $\sigma$ -algebra  $\Sigma$ .

Define for each  $\Delta \in \Sigma$

$$\mu(\Delta) = \inf \left\{ \sum_{i=1}^{\infty} \nu(\Delta^i); \Delta^i \in \Sigma, \{\Delta^i\} \text{ is a countable partition for } \Delta \right\}.$$

Then  $\mu(\Delta)$  is a c.a. nonnegative real-valued measure on  $\Sigma$  such that

$$0 \leq \mu(\Delta) \leq \nu(\Delta) \leq \nu(\Omega) < \infty.$$

Next Lemma is the extension of Jordan-Von Neuman Theorem for the Banach space setting (see p. 124 in [27] and Lemma A.2 in [21]).

**2.17 LEMMA.** Jordan-Von Neuman. Let  $B$  be a Banach space over  $\mathcal{C}$  and let  $R(\cdot)$  be a nonnegative real-valued function defined on  $B$  such that

- (i)  $R(\cdot)^{1/2}$  is a seminorm,
- (ii)  $R(\cdot)^{1/2}$  satisfies the parallelogram law i.e.

$$R(x+y) + R(x-y) = 2R(x) + 2R(y), \text{ for all } x, y \in B,$$

- (iii) there exists  $K > 0$  such that  $R(x) \leq K|x|^2$ , for all  $x \in B$ .

Then (a)  $R(\cdot)$  can be recovered from a unique bounded nonnegative hermitian sesquilinear functional  $T(\cdot, \cdot)$  on  $B \times B$  to  $\mathcal{C}$ . i.e.

$$R(x) = T(x, x) \text{ and } |T(x, y)| \leq K|x| \cdot |y|. \quad (2.18)$$

(b) There exists a unique bounded nonnegative hermitian linear operator  $A$  on  $B$  to  $B^*$  such that  $|A| \leq K$  and  $T(x, y) = \langle y, Ax \rangle$ , for all  $x, y \in B$ .

*Proof.* (a) For each  $x, y \in B$  define  $T(x, y)$  by

$$T(x, y) = \frac{1}{4}(R(x+y) - R(x-y)) - \frac{i}{4}(R(ix+y) - R(ix-y)).$$

Then follow the proof of Jordan-Von Neuman Theorem [27, p. 124] to see that  $T(\cdot, \cdot)$  is a

nonnegative hermitian sesquilinear function on  $B \times B$  such that  $R(x) = T(x, x)$  for all  $x \in B$ . Then using Schwarz inequality for  $T$  we get  $|T(x, y)| \leq R(x)^{1/2}R(y)^{1/2} \leq K|x||y|$ , for all  $x, y \in B$ .

(b) For each  $x$  in  $B$  define  $g_x(\cdot)$  on  $B$  to  $\mathcal{O}$  by  $g_x(y) = T(x, y)$ , for all  $y \in B$ , clearly  $g_x$  is conjugate linear. Since

$$|g_x(y)| = |T(x, y)| \leq K|x||y| \quad (\text{by 2.18}),$$

we see that  $g_x$  is also bounded. So  $g_x \in B^*$ . In fact

$$|g_x| \leq K|x|, \text{ for all } x \in B. \quad (2.19)$$

Define  $A: B \rightarrow B^*$  by  $Ax = g_x$ , then one can easily see that  $A$  is linear.  $A$  is also bounded. In fact by (2.19) we have  $|A| \leq K$ . Since we can write

$$\langle y, Ax \rangle = \langle y, g_x \rangle = g_x(y) = T(x, y), \text{ for all } x, y \in B,$$

the proof is complete.

Having proved Jordan-Von Neuman Lemma for our Banach space setting the proof of the following lemma is exactly similar to its Hilbert space version in [21] and hence omitted.

**2.20 LEMMA .** Let  $B$  be a Banach space and let  $M(\cdot)$  be a f.a.  $L^+(B, B^*)$ -valued measure on  $\Sigma$ . Let for each  $x \in B$ ,  $\nu_x(\cdot) = \langle x, M(\cdot)x \rangle$  (which is clearly a f.a. nonnegative real-valued measure on  $\Sigma$ ) and let for each  $x \in B$ ,  $\mu_x(\cdot)$  be the nonnegative real-valued measure on  $\Sigma$  corresponding to  $\nu_x(\cdot)$  as in Lemma 2.16. Then

(a) for each  $\Delta \in \Sigma$  there is a unique operator  $F(\Delta)$  in  $L^+(B, B^*)$  such that

$$\langle x, F(\Delta)x \rangle = \mu_x(\Delta), \text{ for each } x \in B.$$

(b) the set function  $F(\cdot)$  is w.c.a. on  $\Sigma$  and

$$0 \lesssim F(\Delta) \lesssim M(\Delta) \lesssim N(\Omega), \text{ for each } \Delta \in \Sigma.$$

**3. THE EQUIVALENCE THEOREM.** In this section we extend a very useful result of

Rosenburg [21] namely his Equivalence Theorem for  $L(H_1, H)$ -valued measures to the case of  $L(B, H)$ -valued measures. Our interest in  $L(B, H)$ -valued measures is because of their essential role in the integral representation for Banach space valued stochastic processes (For some more precise comments see the beginning paragraph of Section 5). The proof of our Equivalence Theorem 3.2 goes along the same lines as the proof of Rosenburg's Equivalence Theorem.

We start by proving the following Lemma.

**3.1 LEMMA .** Let  $T$  be any  $L(B, H)$ -valued measure and  $M$  be any  $L(B, B^*)$ -valued measure on  $\Sigma$ . Consider the kernel  $K$  on  $\Sigma \times \Sigma$  defined by

$$K(\Delta, \Delta') = M(\Delta \cap \Delta') - T(\Delta')^* T(\Delta), \quad \Delta, \Delta' \in \Sigma.$$

Then, (a) for any  $\Delta_1, \dots, \Delta_n \in \Sigma$  and any  $x_1, \dots, x_n \in B$  we have

$$\sum_{i,j=1}^n \langle x_i, K(\Delta_i, \Delta_j) x_j \rangle = \sum_{i,j=1}^n \langle x_i, M(\Delta_i \cap \Delta_j) x_j \rangle - \left| \sum_{i=1}^n T(\Delta_i) x_i \right|^2.$$

(b)  $M$  is a 2-majorant for  $T$  iff the kernel  $K(\cdot, \cdot)$  is a positive definite kernel on  $\Sigma \times \Sigma$  (in the sense of Definition 2.5 in [12]).

*Proof.* (a) is clear. (b) Suppose  $M$  is a 2-majorant for  $T$  then by Lemma 2.6 in [12] it suffices to show that the scalar-valued kernel  $k$  on  $(\Sigma \times B) \times (\Sigma \times B)$  defined by  $k[(\Delta, x), (\Delta', x')] = \langle x', K(\Delta, \Delta') x \rangle$  is a positive definite kernel. For any  $c_1, \dots, c_n$  in  $\mathcal{C}$ , and any  $\alpha_1 = (\Delta_1, y_1), \dots, \alpha_n = (\Delta_n, y_n)$  in  $\Sigma \times B$  we have

$$\begin{aligned} \sum_{i,j=1}^n \bar{c}_j k(\alpha_i, \alpha_j) c_i &= \sum_{i,j=1}^n \bar{c}_j c_i \langle y_j, K(\Delta_i, \Delta_j) y_i \rangle \\ &= \sum_{i,j=1}^n \langle c_j y_j, K(\Delta_i, \Delta_j) c_i y_i \rangle = \sum_{i,j=1}^n \langle x_j, K(\Delta_i, \Delta_j) x_i \rangle, \end{aligned}$$

where  $x_i = c_i y_i$ ,  $i = 1, \dots, n$ . Thus by (a)

$$\sum_{i,j=1}^n \bar{c}_j k(\alpha_i, \alpha_j) c_i = \sum_{i,j=1}^n \langle x_i, M(\Delta_i \cap \Delta_j) x_j \rangle - \left| \sum_{i=1}^n T(\Delta_i) x_i \right|^2,$$

and this is nonnegative (since  $M$  is a 2-majorant for  $T$ ). Now to check the conjugate symmetry property of  $k$  on  $(\Sigma \times B) \times (\Sigma \times B)$  we note that for any  $\alpha = (\Delta, x) \in \Sigma \times B$  and  $\alpha' = (\Delta', x') \in \Sigma \times B$  we have

$$\begin{aligned} k(\alpha, \alpha') &= \langle x', K(\Delta, \Delta')x \rangle = \langle x', M(\Delta \cap \Delta')x \rangle - \langle x', T(\Delta')^* T(\Delta)x \rangle \\ &= \overline{\langle x, M(\Delta \cap \Delta')x' \rangle} - (T(\Delta')x', T(\Delta)x), \end{aligned}$$

this is because  $M(\Delta \cap \Delta')$  is nonnegative and hence hermitian (see Definition 2.11 (c) and Definition 2.3). We can further write

$$\begin{aligned} k(\alpha, \alpha') &= \overline{\langle x, M(\Delta' \cap \Delta)x' \rangle} - \overline{(T(\Delta)x, T(\Delta')x')} \\ &= \overline{\langle x, M(\Delta' \cap \Delta)x' \rangle} - \overline{\langle x, T(\Delta)^* T(\Delta')x' \rangle} = \overline{k(\alpha', \alpha)}. \end{aligned}$$

This completes the proof of one way. Proof of the other way is similar.

**3.2 EQUIVALENCE THEOREM.** (a) A f.a.  $L(B, H)$ -valued measure  $T$  on an algebra  $\Sigma$  has a f.a. spectral dilation iff it has a f.a. 2-majorant.

(b) A w.c.a.  $L(B, H)$ -valued measure  $T$  on a  $\sigma$ -algebra  $\Sigma$  has a spectral dilation iff it has a w.c.a. 2-majorant.

*Proof.* We will just prove part (b) since the proof of (a) is essentially the same. Suppose  $T$  has a w.f.a. spectral dilation i.e. suppose there exists a Hilbert space  $K \supseteq H$  and a spectral measure  $E(\cdot)$  in  $K$  such that

$$T(\Delta) = PE(\Delta)S; \text{ for all } \Delta \in \Sigma,$$

where  $P$  is the projection of  $K$  onto  $H$  and  $S$  is an operator in  $L(B, K)$ . Then for every collection  $x_1, \dots, x_n$  in  $B$  and every collection  $\Delta_1, \dots, \Delta_n$  in  $\Sigma$  we can write

$$\left| \sum_{i=1}^n T(\Delta_i)x_i \right|^2 = \left| P \left( \sum_{i=1}^n E(\Delta_i)Sx_i \right) \right|^2 \leq \sum_{i=1}^n \langle x_i, S^*E(\Delta_i)Sx_i \rangle$$

One can then easily check that  $M(\cdot) = S^*E(\cdot)S$  serves as a w.c.a. 2-majorant for  $T$ . To see the other way suppose  $T$  has a w.c.a. 2-majorant  $M$ . Define the kernel  $K(\cdot, \cdot)$  as in Lemma

3.1. By this Lemma  $K(\cdot, \cdot)$  is a positive definite kernel on  $\Sigma \times \Sigma$  in the sense of Definition 2.5 [12]. Thus one can apply The Kernel Theorem (see Theorem 2.10 in [12]) to conclude that there exists a Hilbert space  $H_1$  and an  $L(B, H_1)$ -valued function  $R(\cdot)$  on  $\Sigma$  such that

$$K(\Delta, \Delta') = R(\Delta')^* R(\Delta).$$

Now take  $K = H \oplus H_1$  and define  $\tilde{T}: B \rightarrow K$  by  $\tilde{T}(\Delta) = T(\Delta) + R(\Delta)$ . (Here we have identified  $H$  with  $H \otimes \{0\}$  in  $K$ ). Then for any  $x, x' \in B$ ,

$$\begin{aligned} \langle x, \tilde{T}(\Delta)^* \tilde{T}(\Delta') x' \rangle &= (\tilde{T}(\Delta) x, \tilde{T}(\Delta') x')_K = (T(\Delta) x, T(\Delta') x')_H + (R(\Delta) x, R(\Delta') x')_{H_1} \\ &= \langle x, T(\Delta)^* T(\Delta') x' \rangle + \langle x, R(\Delta)^* R(\Delta') x' \rangle = \langle x, [T(\Delta)^* T(\Delta') + K(\Delta', \Delta)] x' \rangle. \end{aligned}$$

This means  $\tilde{T}(\Delta)^* \tilde{T}(\Delta') = M(\Delta \cap \Delta')$ . Now, since  $M$  is w.c.a. one can adjust the proof of Lemma 8.6 in [11] to conclude that  $\tilde{T}$  is strongly countably additive. Now clearly we have

$$T(\Delta) = P\tilde{T}(\Delta), \text{ for each } \Delta \in \Sigma. \quad (3.3)$$

Let  $E(\Delta)$  be the projection on  $K$  onto the subspace spanned by  $\{\tilde{T}(\Delta') x: \Delta' \in \Sigma, \Delta' \subseteq \Delta, x \in B\}$ . Then one can show that  $E(\cdot)$  is a spectral measure in  $K$  and

$$\tilde{T}(\Delta) = E(\Delta) \tilde{T}(\Omega), \text{ for all } \Delta \in \Sigma.$$

Hence  $T(\Delta) = P\tilde{T}(\Delta) = PE(\Delta) \tilde{T}(\Omega)$  i.e.  $T(\Delta) = PE(\Delta) S$ , with  $S = \tilde{T}(\Omega)$ . (Here, to complete the proofs, the ideas of section 5 in [11] are needed. For more details see pages 443-444 there).

**4. SPECTRAL DILATION.** In this section we study the existence of dilations for  $L(B, H)$ -valued measures. When  $B$  is simply the complex number  $\mathcal{C}$ , one is just in the case of  $H$ -valued measures (Note that  $L(\mathcal{C}, H) \cong H$ ), and for this case it is shown by Niemi [18] that: A f.a.  $H$ -valued measured  $T$  with finite semi-variation has a f.a. spectral dilation iff there exists some constant  $C$  such that for every collection  $f_k$  of scalar valued simple functions

$$\sum_{k=1}^n \left| \int f_k dT \right|^2 \leq C \left| \sum_{k=1}^n |f_k|^2 \right|_{sup} \quad (4.1)$$

On the other hand fortunately Grothendieck's inequality guarantees the validity of (4.1). Thus each  $L(C, H) (\cong H)$ -valued measure with finite semi-variation has a spectral dilation. Rosenberg [21] among other important results showed that when  $\mathcal{C}$  is replaced by any finite dimensional Hilbert, space say  $\mathcal{C}^n$  still the problem of dilation has a positive answer, namely any  $L(\mathcal{C}^n, H)$ -valued measure  $T$  has a spectral dilation. When this space is replaced by any Hilbert space  $H_1$  the problem is extensively studied by Mahagan and Salehi [10] and they were able to show that in general an  $L(H_1, H)$ -valued measure does not have to have a spectral dilation. They then continued to study conditions which guarantee the existence of a dilation and in particular generalized Niemi's result mentioned above in the following way:

**4.2 THEOREM.** (a) Let  $T$  be a f.a.  $L(H_1, H)$ -valued measure on an algebra  $\Sigma$ . Then  $T$  has a f.a. spectral dilation iff there exists a constant  $C$  such that for every collection of simple functions  $f_k = \sum_{i=1}^n 1_{\Delta_i} x_i^k$  in  $S(H_1)$

$$\sum_{k=1}^n \left| \int f_k dT \right|^2 \leq C \sup \left\{ \sum_{k=1}^n \sum_{i=1}^N (F(\Delta_i) x_i^k, x_i^k) : F \in M(L(K, K)), \|F\| \leq 1 \right\} \quad (4.3)$$

(b) If  $T$  is a w.c.a. measure on a  $\sigma$ -algebra  $\Sigma$ , then (4.3) is a necessary and sufficient condition for  $T$  to have a spectral dilation.

Here in this section we first extend the above criteria for dilatability of  $L(\mathcal{C}, H)$ -valued and  $L(H_1, H)$ -valued measures to the case of  $L(H_1, H)$ -valued measures, where  $B$  is any Banach space (Theorem 4.4) and then use this to get a sufficient condition for dilatability.

**4.4 THEOREM.** (a) Let  $T$  be a f.a.  $L(B, H)$ -valued measure defined on an algebra  $\Sigma$ . Then  $T$  has a f.a. dilation iff there exists a constant  $C$  such that for all collection of simple functions  $f_k = \sum_{i=1}^N 1_{\Delta_i} x_i^k$  in  $S(B)$

$$\sum_{k=1}^n \left| \sum_{i=1}^N T(\Delta_i) x_i^k \right|^2 \leq C \sup \left\{ \left| \sum_{k=1}^n \sum_{i=1}^N \langle x_i^k, F(\Delta_i) x_i^k \rangle \right| : F \in M(L(W, W^*)), \right. \\ \left. \|F\| \leq 1 \right\} \quad (4.5)$$

(b) If  $T$  is a w.c.a. measure on a  $\sigma$ -algebra  $\Sigma$ . Then (4.5) is necessary and sufficient for  $T$  to have a spectral dilation.

*Proof.* For any two simple functions  $f$  and  $g$  in  $S(B)$  we define a new simple function  $f \circ g$  in  $S(B \otimes B)$  as follows: suppose  $f = \sum_{i=1}^N 1_{\Delta_i} x_i$  and  $g = \sum_{i=1}^N 1_{\Delta_i} y_i$  we let  $f \circ g = \sum_{i=1}^N 1_{\Delta_i} x_i \otimes y_i$ . Since by Remark 2.10 we know that  $(B \otimes B)^* = L(B, B^*)$  we have

$$\|f \circ g\|_{\infty} = \sup \left\{ \left| \sum_{i=1}^N \langle x_i, F(\Delta_i) y_i \rangle \right| : F \in M(L(B, B^*)), \|F\| \leq 1 \right\} \quad (4.6)$$

Using this condition (4.5) can be reformulated as

$$\sum_{k=1}^n \left| \int f_k dT \right|^2 \leq C \left| \sum_{k=1}^n f_k \circ f_k \right|_{\infty}, \text{ for all } f_1, \dots, f_n \in S(B). \quad (4.7)$$

Now if  $T$  has a f.a. dilation then by the Equivalence Theorem 3.2 the measure  $T$  must have a 2-majorant  $M$ . Thus (4.5) holds with  $C = \|M\| < \infty$  (Note that any 2-majorant is necessarily of finite semi-variation). This completes the proof of one way of (a). For the other way we assume that (4.5) or equivalently (4.7) holds and will prove that  $T$  has a f.a. 2-majorant which in view of The Equivalence Theorem completes the proof.

Consider the set  $U$  defined by

$$U = \left\{ \sum_{k=1}^n f_k \circ f_k : f_k \text{'s in } S(B) \text{ with } \sum_{k=1}^n \left| \int f_k dT \right|^2 = 1, n \in N \right\}.$$

from (4.7) it follows that  $\|\phi\|_{\infty} \geq \frac{1}{C}$ , for every  $\phi \in U$ . Since  $U$  is a convex set in  $S(B \otimes B)$  it follows from Hahn-Banach Theorem ([24], Theorem 3.4) and Lemma 2.10 that there exists some real number  $\gamma$  and a f.a. set function  $G \in M(L(B, B^*))$  with  $\|G\| = 1$  such that for every  $\phi, \psi \in S(B \otimes B^*)$  with  $\phi \in U$  and  $\|\psi\|_{\infty} \leq \frac{1}{C}$  we have

$$\operatorname{Re} \langle \phi, G \rangle \geq \gamma \geq \operatorname{Re} \langle \psi, G \rangle,$$

where  $\langle \phi, G \rangle = \int \langle \phi, dG \rangle$ . Since for  $\alpha = \frac{\overline{\langle \psi, G \rangle}}{|\langle \psi, G \rangle|}$ ,  $|\langle \psi, G \rangle| = \operatorname{Re} \langle \alpha \psi, G \rangle$  and  $|\psi|_\infty = |\alpha \psi|_\infty$  we have

$$\gamma \geq \sup \{ \operatorname{Re} \langle \psi, G \rangle : |\psi|_\infty \leq \frac{1}{C} \} = \sup \{ |\langle \psi, G \rangle| : |\psi|_\infty \leq \frac{1}{C} \} = \frac{1}{C} \|G\| = \frac{1}{C}.$$

Thus we have  $\operatorname{Re} \langle \phi, G \rangle \geq \frac{1}{C}$ , for all  $\phi \in U$ . Now put  $M(\Delta) = \frac{G(\Delta) + G(\Delta)^* J}{2}$ ,  $\Delta \in \Sigma$  where  $J$  is the standard linear isomorphism from  $B$  into  $B''$  defined by

$$\langle x^*, Jx \rangle = \overline{\langle x, x^* \rangle}, \text{ for all } x \in B, x^* \in B'. \quad (4.8)$$

Note that  $G(\Delta): B \rightarrow B'$  and hence  $G(\Delta)^*: B'' \rightarrow B'$ . Thus both  $G(\Delta)$  and  $G(\Delta)^* J$  belong to  $L(B, B')$  hence  $M(\Delta) \in L(B, B')$ , for each  $\Delta \in \Sigma$ . Thus  $M$  is an  $L(B, B')$ -valued measure. Furthermore  $M \in M(L(B, B'))$ , in fact, one can easily check that  $\|M\| \leq 1$ .  $M$  is also hermitian valued (see Definition 2.3), because for any  $x$  and  $y$  in  $B$  we have

$$\begin{aligned} \langle y, M(\Delta)x \rangle &= \frac{\langle y, G(\Delta)x \rangle + \langle y, G(\Delta)^* Jx \rangle}{2} = \frac{\langle y, G(\Delta)x \rangle + \langle G(\Delta)y, Jx \rangle}{2} \\ \text{by (4.8)} &= \frac{\langle y, G(\Delta)x \rangle + \overline{\langle x, G(\Delta)y \rangle}}{2} = \frac{\langle y, G(\Delta)x \rangle + \overline{\langle x, G(\Delta)y \rangle}}{2}. \end{aligned}$$

Thus

$$\langle y, M(\Delta)x \rangle = \frac{\overline{\langle y, G(\Delta)x \rangle + \langle x, G(\Delta)y \rangle}}{2} = \overline{\langle x, M(\Delta)y \rangle}. \quad (4.9)$$

Now for any collection of  $f_1, \dots, f_n$  in  $S(B)$  with  $\sum_{k=1}^n \int |f_k dT|^2 > 0$  using (4.9) we can write

$$\int \left\langle \left[ \sum_{k=1}^n \int |f_k dT|^2 \right]^{-1} \sum_{k=1}^n f_k \circ f_k, dM \right\rangle = \operatorname{Re} \int \left\langle \left[ \sum_{k=1}^n \int |f_k dT|^2 \right]^{-1} \sum_{k=1}^n f_k \circ f_k, dG \right\rangle \geq \frac{1}{C},$$

which implies that



$$\sum_{k=1}^n |\int f_k dT| \leq C \int < \sum_{k=1}^n f_k \circ f_k, dM >. \quad (4.10)$$

To prove that  $M$  is actually a 2-majorant for  $T$  we have to show that (4.9) remains valid in the other case namely when  $\sum_{k=1}^n |\int f_k dT|^2 = 0$ . This can be proved exactly as the proof of this fact in Theorem 4.3 in [10] and hence omitted.

(b) Since  $T$  is w.c.a. and hence of course f.a. by part (a) there exist a f.a. 2-majorant  $M$  such that

$$|\sum_{i=1}^n T(\Delta_i)x_i|^2 \leq \sum_{i=1}^n <x_i, M(\Delta_i)x_i>, \text{ for all } x_i \in B, \Delta_i \in \Sigma.$$

From Lemma 2.20 it follows that there exists a w.c.a. measure  $F$  on  $\Sigma$  such that for all  $x \in B, \Delta \in \Sigma$

$$<x, F(\Delta)x> = \inf\{\sum_{i=1}^{\infty} <x, M(\Delta^i)x> : \{\Delta^i\} \text{ is a countable partition for } \Delta\}.$$

Let  $\Delta_1, \dots, \Delta_n$  be some fixed disjoint elements of  $\Sigma$  and  $x_1, \dots, x_n$  some fixed elements in  $B$ . Given any  $\delta > 0$  let  $\{\Delta_j^i : i=1,2,\dots\}$  be countable partition of  $\Delta_j$  such that

$$\sum_{i=1}^{\infty} <x_j, F(\Delta_j^i)x_j> \leq <x_j, M(\Delta_j)x_j> + \frac{\delta}{n}.$$

Since  $T$  is w.c.a., and hence strongly countably additive ([6] IV. 10.1),  $T(\cdot)x$  is countably additive and we can write

$$|\sum_{j=1}^n T(\Delta_j)x_j|^2 = \lim_{k \rightarrow \infty} |\sum_{j=1}^n T(\bigcup_{i=1}^k \Delta_j^i)x_j|^2 \leq \lim_{k \rightarrow \infty} \sum_{j=1}^n <x_j, F(\bigcup_{i=1}^k \Delta_j^i)x_j>.$$

So

$$|\sum_{j=1}^n T(\Delta_j)x_j|^2 = \sum_{j=1}^n \lim_{k \rightarrow \infty} \sum_{i=1}^k <x_j, F(\Delta_j^i)x_j> \leq \sum_{j=1}^n <x_j, M(\Delta_j)x_j> + \delta.$$

Since  $\delta$  is arbitrary, this implies that  $M$  is a w.c.a. 2-majorant for  $T$ . Thus by The Equivalence Theroem 3.2,  $T$  has a spectral dilation.

The ideas of the proof of last theorem are similar to the proof of the proof of Theorem 4.2, as given in [10], which goes along the same lines as the proof of Pietsch's factorization theorem in [9]. However, the details of the proof, as one can see from the proof presented here, are quite different.

Using the main result of this section, namely Theorem 4.4, we can provide some sufficient conditions for a  $L(B,H)$ -valued measure  $T$  to have spectral dilation. The following theorem extends a result in [10] to the Banach space setting. The proof in [10] is quite Hilbertian and depends on taking an orthonormal basis in some Hilbert space which is now replaced by the Banach space  $B$ , which does not have such a nice basis.

We start with the following definition (cf. [26], [10] ).

**4.11 DEFINITION.** For any f.a.  $L(B,H)$ -valued measure  $T$  defined on an algebra  $\Sigma$  we define  $|||T||| \in [0, \infty]$  by  $|||T||| = \sup\{|\sum_{\Delta \in \pi} T(\Delta)x_{\Delta}| : \pi \in \Pi(\Omega), x_{\Delta} \in B, |x_{\Delta}| \leq 1\}$  or equivalently by  $|||T||| = \sup\{|\Phi_T(f)| : f \in S(B), |f|_{\text{sup}} \leq 1\}$ . Thus

$$|||T||| = \text{norm of the operator } \Phi_T: (S(B), | \cdot |_{\text{sup}}) \rightarrow H.$$

We will need the following lemma given in [10] (cf. also [26]).

**4.13 LEMMA .** Let  $H$  be a Hilbert space. There exists a constant  $C$  such that for every  $n, m \in \mathbb{N}; t_1, \dots, t_m \in \mathcal{C}^n; x_1, \dots, x_m \in H$ ,

$$\sum_{i,j=1}^m (t_i, t_j)(x_i, x_j) \leq C \int_{S_n} \left| \sum_{j=1}^m \overline{\text{sgn}(s, t_j)} |t_j| |x_j|^2 \mu_n(ds) \right|$$

where  $S_n$  is the unit sphere of the  $n$ -dimensional Euclidean space  $\mathcal{C}^n$ , and  $\mu_n$  is the normalized rotationally invariant measure on  $S_n$ . For any complex number  $z$ ,  $\text{sgn } z$  is equal to  $\frac{z}{|z|}$  if  $z \neq 0$  and 0 if  $z = 0$ .

**4.14 THEOREM .** Let  $T$  be a f.a. (w.c.a.)  $L(B,H)$ -valued measure on an algebra (a  $\sigma$ -algebra)  $\Sigma$ . If  $|||T||| < \infty$  or equivalently  $\Phi_T$  as an operator from  $(S(B), | \cdot |_{\text{sup}})$  into  $H$

is bounded then  $T$  is dilatible.

*Proof.* Let  $\Delta_1, \dots, \Delta_N$  be any disjoint nonempty sets in  $\Sigma$ ,  $x_i^k, k = 1, \dots, n; i = 1, \dots, N$  be any vectors in  $B$  we will prove that (4.5) holds. Take a normal basis  $e_1, \dots, e_r$  for  $\overline{\text{sp}}\{x_i^k; i = 1, \dots, N; k = 1, \dots, n\}$ . Then writing  $x_i^k = \sum_{p=1}^r a_{i,p}^k e_p$ , we have

$$\sum_{k=1}^n \left| \sum_{i=1}^N T(\Delta_i) x_i^k \right|^2 = \sum_{k=1}^n \left| \sum_{i=1}^N T(\Delta_i) \sum_{p=1}^r a_{i,p}^k e_p \right|^2 = \sum_{i,j=1}^N \sum_{p,q=1}^r (T(\Delta_i) e_p, T(\Delta_j) e_q) (t_{i,p}, t_{j,q}),$$

where  $t_{i,p} = (a_{i,p}^1, a_{i,p}^2, \dots, a_{i,p}^n) \in \mathcal{C}^n$ . Now apply Lemma 4.13 to get

$$\sum_{k=1}^n \left| \sum_{i=1}^N T(\Delta_i) x_i^k \right|^2 \leq C \int_{S_n} \left| \sum_{i=1}^N \sum_{p=1}^r \overline{\text{sgn}(s, t_{i,p})} |t_{i,p}| T(\Delta_i) e_p \right|^2 \mu_n(ds).$$

So

$$\sum_{k=1}^n \left| \sum_{i=1}^N T(\Delta_i) x_i^k \right|^2 \leq C \int_{S_n} \left| \sum_{i=1}^N T(\Delta_i) \left( \sum_{p=1}^r \overline{\text{sgn}(s, t_{i,p})} |t_{i,p}| e_p \right) \right|^2 \mu_n(ds).$$

Let

$$y_i(s) = \sum_{p=1}^r |t_{i,p}| \overline{\text{sgn}(s, t_{i,p})} e_p, \quad i = 1, \dots, N.$$

Then

$$|y_i(s)|^2 \leq \left( \sum_{p=1}^r |t_{i,p}| \right)^2 \leq r \left( \sum_{p=1}^r |t_{i,p}|^2 \right) = r \sum_{p=1}^r \sum_{k=1}^n |a_{i,p}^k|^2. \quad (4.15)$$

Now using Hahn-Banach theorem there exist linear functional  $e_p^* \in B^*$  such that  $e_p^*(e_q) = \delta_{pq}$ .

Then  $a_{i,p}^k = \langle x_i^k, e_p^* \rangle$  and (4.15) can be rewritten as

$$|y_i(s)|^2 \leq r \sum_{p=1}^r \sum_{k=1}^n |\langle x_i^k, e_p^* \rangle|^2 \leq r \sum_{k=1}^n \langle x_i^k, \sum_{p=1}^r \overline{\langle x_i^k, e_p^* \rangle} e_p^* \rangle. \quad (4.16)$$

Let's introduce the linear operator  $S: B \rightarrow B^*$  by

$$Sx = r \sum_{p=1}^r \overline{\langle x, e_p^* \rangle} e_p^*, \quad \text{for all } x \in B.$$

Then (4.16) becomes  $|y_i(s)|^2 \leq \sum_{k=1}^n \langle x_i, Sx_i^k \rangle$ . Thus we have

$$\sum_{k=1}^n \left| \sum_{i=1}^N T(\Delta_i) x_i^k \right|^2 \leq C |||T||| \sup \left\{ \sum_{k=1}^n \langle x_i^k, Sx_i^k \rangle : i = 1, \dots, N \right\}.$$

Let  $i_0$  be the integer between 1 and  $N$  which gives the sup on the right hand side of (4.17). Then

$$\sum_{k=1}^n \left| \sum_{i=1}^N T(\Delta_i) x_i^k \right|^2 \leq C |||T||| \sum_{k=1}^n \langle x_{i_0}^k, Sx_{i_0}^k \rangle.$$

This yields

$$\sum_{k=1}^n \left| \sum_{i=1}^N T(\Delta_i) x_i^k \right|^2 \leq C |||T||| \sup \left\{ \sum_{i=1}^N \sum_{k=1}^n \langle x_i^k, F(\Delta_i) x_i^k \rangle : F \in M(L(B, B^*)) \right\},$$

because  $\sum_{k=1}^n \langle x_{i_0}^k, Sx_{i_0}^k \rangle = \sum_{i=1}^N \sum_{k=1}^n \langle x_i^k, F_0(\Delta_i) x_i^k \rangle$ , for an  $L(B, B^*)$ -valued measure  $F_0$  with the property

$$F_0(\Delta_{i_0}) = S \quad \text{and} \quad F_0(\Delta_i) = 0, \quad \text{for all } i \neq i_0.$$

So we have proved (4.5).

Note that as shown in [10] this condition  $|||T||| < \infty$  is not necessary for  $T$  to have a dilation even in Hilbert space case.

**5. DILATION OF BANACH SPACE VALUED HARMONIZABLE PROCESSES.** In this section we will first review the definition of Banach space valued random variables and processes and set up the necessary definition and preliminary facts about Banach space valued  $V$ -bounded, harmonizable and stationary processes. Then we will apply Theorem 4.4 to study the stationary dilations of banach space valued harmonizable processes thereby extending the well known dilation theorems for scalar valued and Hilbert space valued processes ([1], [14], [10], [16], [18], [19]) to the Banach space valued processes.

Let  $B$  be a Banach space and  $X$  be a  $B$ -valued function defined on a probability space  $(\Omega, \Sigma, P)$ . We say  $X$  is a Banach space valued random variable if  $\langle X, x^* \rangle$  is measurable for

each  $x^* \in B^*$ . A Banach space valued random variable  $X$  is called of second order if

$$\int |\langle X, x^* \rangle|^2 dP < \infty \text{ for each } x^* \in B^* \text{ i.e. } \langle X, x^* \rangle \in L^2(\Omega, \Sigma, P), \text{ for all } x^* \in B^*.$$

Thus for each second order Banach space valued random variable  $X$  we can define an operator  $\tilde{X}: B^* \rightarrow L^2(\Omega, \Sigma, P)$  by  $\tilde{X}x^* = \langle X, x^* \rangle$ . Thus every second order Banach space valued random variable  $X$  gives an operator, namely  $\tilde{X}$  from  $B^*$  into  $H = L^2(\Omega, \Sigma, P)$ . Hence one can think of a second order Banach space valued random variable as an operator in  $L(B, H)$ , where  $B$  is a Banach space and  $H$  is a Hilbert space.

With this background by a Banach space valued stochastic process  $X_t, t \in R$  we mean a function  $X_t: R \rightarrow L(B, H)$  where  $H$  is a Hilbert space and  $B$  is a Banach space, and from now on we will call it simply an  $L(B, H)$ -valued process.

Let's now recall the definition of different type of usual scalar valued second order stochastic processes (which are also interpreted as  $H$ -valued processes [23]).

**5.1 DEFINITION.** An  $H$ -valued process  $x_t: R \rightarrow H$  is called

(a) Stationary (see [23]) if its correlation function  $\gamma(t, s) = \langle x_t, x_s \rangle$  is a function of only  $t - s$ , i.e.

$$\gamma(t, s) = \gamma(t - s, 0), \text{ for all } t, s \in R.$$

(b) harmonizable (see [22]) if its correlation function is of the form

$$\gamma(t, s) = \int \int e^{-i(tv - su)} \nu(dv, du)$$

where  $\nu$  is a positive definite bimeasure such that

$$\sup \left\{ \sum_{i,j=1}^n \alpha_i \bar{\alpha}_j \nu(\Delta_i, \Delta_j) : \alpha_i \in \mathcal{C}, |\alpha_i| \leq 1, \Delta_i \text{ are disjoint in } B \right\} < \infty.$$

(c)  $V$ -bounded (see [2]), if it is continuous and there exists a constant  $C$  such that for every  $\phi \in L^1(R, \mathcal{C})$  we have

$$|\int x_t \phi(t) dt| \leq C \|\hat{\phi}\|_{sup},$$

where the integral is in sense of Bachner [8] , and  $L^1(R, \mathcal{O})$  is the space of complex-valued Lebesgue integrable functions on  $R$  , and  $\hat{\phi}$  is the Fourier transform of  $\phi$  .

The following result is well known. It has been extended to the  $L(H_1, H)$ -valued processes ([10] Theorem 6.12 ) and we will extend it in a similar fashion to our Banach space setting.

**5.2 THEOREM .** ([16], [14]). For, any  $H$ -valued process  $x_t, t \in R$  the following are equivalent

- (i)  $x_t$  is harmonizable.
- (ii)  $x_t$  is  $V$ -bounded
- (iii) there exists an  $H$ -valued measure  $\xi$  on the Borel subsets of  $R$  such that

$$x_t = \int e^{-it} \xi(ds), \text{ for all } t \in R.$$

- (iv) there exists a Hilbert space  $K \supseteq H$  and a  $K$ -valued stationary process  $y_t$  such that

$$x_t = Py_t; \text{ for all } t \in R,$$

where  $P$  is the projection on  $K$  onto  $H$  . (For another version of harmonizability which is stronger see [1], [2] ).

Definition 5.1 and comments of the beginning paragraph of this section suggest the following definition (see also [7], [4], [13] ).

**5.3 DEFINITION.** An  $L(B, H)$ -valued process  $X_t$  is said to be

- (a) stationary if for each  $x \in B$  the  $H$ -valued process  $X_t x$  is stationary,
- (b) harmonizable if for each  $x \in B$  the  $H$ -valued process  $X_t x$  is harmonizable
- (c)  $V$ -bounded if for each  $x \in B$  the  $H$ -valued process  $X_t x$  is  $V$ -bounded.

The proof of the following theorem is similar to that of Theorem 6.12 in [10] for the Hilbert space setting and hence omitted.

**5.4 THEOREM.** Let  $X_t$  be an  $L(B,H)$ -valued process and let the operator  $\Gamma(t,s) = X_s^* X_t: B \rightarrow B^*$  be its correlation function. The following are equivalent

- (i)  $X_t$  is harmonizable
- (ii)  $X_t$  is  $V$ -bounded
- (iii) There exists a w.c.a.  $L(B,H)$ -valued measure  $Z$  such that

$$X_t = \int e^{-it\omega} Z(d\omega), \text{ for all } t \in \mathbb{R}.$$

**5.5. REMARK.** Last result partially extends the Theorem 5.2 to our case of Banach space valued stochastic processes. However, part (iv) of Theorem 5.2 which is probably the most interesting part is missing here in Theorem 5.4. It turns out that these definitions of harmonizability and  $V$ -boundedness are not strong enough to assure the existence of a stationary dilation. (For a counter example see Theorem 6.12 (B) in [10]). This problem was expected because the problem of having a stationary dilation for harmonizable  $L(B,H)$ -valued processes has a close tie with that of spectral dilation for  $L(B,H)$ -valued measures which was the subject of our study in Sections 3 & 4. As we recall from Theorem 4.14, for instance, in order to assure the existence of a spectral dilation there, we had to add an extra assumption (relative to the usual case of  $H$ -valued measures) namely  $\|T\| < \infty$ . This means that to get the existence of stationary dilation, namely a part (iv), we must impose similar extra assumptions on the other parts.

Here is the main result of this section

**5.6 THEOREM.** Let  $X_t$  be an  $L(B,H)$ -valued process and let  $\Gamma(t,s) = X_s^* X_t: B \rightarrow B^*$  be its correlation function.

- (a) The following are equivalent (i)  $X_t$  is harmonizable and the spectral measure  $Z$  in a harmonic representation of  $X_t$  (see Theorem 5.4 (iii)),

$$X_t = \int e^{-it\omega} Z(d\omega)$$

has the property  $|||Z||| < \infty$ ,

(ii)  $X_t$  is  $V$ -bounded and there exists a constant  $C$  such that

$$\left| \int X_t \phi(t) dt \right| \leq C \|\hat{\phi}\|_{\infty, p}; \text{ for all } \phi \in L^1(R, B),$$

where  $L^1(R, B)$  stands for the space of all  $B$ -valued Bochner integrable functions w.r.t. the Lebesgue measure,  $\hat{\phi}(s) = \int e^{-is} \phi(t) dt$  is the Fourier transform of  $\phi$  in  $L^1(R, B)$ , and all integrals are in Bochner sense.

(b) If the  $L(B, H)$ -valued process  $X_t$  satisfies (i) or (ii) then there exists a Hilbert space  $K \supseteq H$  and an  $L(B, K)$ -valued stationary process  $Y_t$  such that

$$X_t = PY_t, \text{ for all } t \in R,$$

where  $P$  is the projection on  $K$  onto  $H$ .

*Proof.* The proof of (i)  $\Leftrightarrow$  (ii) is similar to that of Theorem 6.17 in [10] and just needs some modifications. To see (i)  $\Rightarrow$  (ii), suppose  $X_t = \int e^{-it} Z(ds)$  with  $|||Z||| < \infty$ . It follows from Theorem 4.14 that the  $L(B, H)$ -valued w.c.a. measure  $Z$  has a spectral dilation. That is to say, there exists a Hilbert space  $K \supseteq H$  and  $S \in L(B, K)$  such that

$$Z(\Delta) = PE(\Delta)S, \text{ for all Borel subsets, } \Delta \text{ of } R,$$

where  $P$  is the projection on  $K$  onto  $H$ . Let  $Y_t = \int e^{-it} E(ds)S$ ,  $t \in R$ . Then one can see that  $Y_t$  is stationary (see [4], [13]) and by (2.14)  $X_t = PY_t$  for all  $t \in R$ .

**5.7 REMARK.** One can easily construct an  $L(B, H)$ -valued process  $X_t$  which has a stationary dilation but does not satisfy condition ((i) or equivalently (ii)) of last theorem. It is interesting to characterize the class of  $L(B, H)$ -valued or even  $L(H_1, H)$ -valued processes which have stationary dilations.



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