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# PARALLELS BETWEEN CONTROL PDE'S AND SYSTEMS OF ODE'S

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## ABSTRACT

System theorists understand that the same mathematical objects which determine controllability for nonlinear control systems of ordinary differential equations also determine hypoellipticity for linear partial differential equations. Moreover, almost any study of o.d.e. systems begins with linear systems. It is remarkable that Hörmander's paper on hypoellipticity of second order linear p.d.e.'s starts with equations due to Kolmogorov, which we show are analogous to the linear o.d.e.'s. Eigenvalue placement by state feedback for a controllable linear system can be paralleled for a Kolmogorov equation if an appropriate type of feedback is introduced. Results concerning transformations of nonlinear systems to linear systems are similar to results for transforming a linear p.d.e. to a Kolmogorov equation.

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# I INTRODUCTION

We consider second order linear partial differential equations of the forms.

$$(1) \qquad -\frac{\partial u}{\partial t} + \sum_{j,k=1}^{n} A_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} B_j(x) \frac{\partial u}{\partial xj} + C(x)u = f(x,t)$$

$$(2) \qquad -\frac{\partial^2 u}{\partial t^2} + \sum_{j,k=1}^{n} A_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} B_j(x) \frac{\partial u}{\partial x_j} + C(x)u = f(x,t),$$

where the coefficients are $\mathscr{C}^\infty$ in some open set in $\mathbb{R}^n$ containing the origin. Here we assume the matrix $(A_{jk}(x))$ is symmetric, is positive semidefinite, and has constant rank $m$ . If we consider $f$ as the input to the p.d.e and $u$ as its output, then we can replace $f$ by $f - C(x)u$, so we consider $C(x) \equiv 0$.

Of particular interest to us is the spatial partial differential equation

$$(3) \qquad \sum_{j,k=1}^{n} A_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} B_j(x) \frac{\partial u}{\partial x_j} = f(x),$$

with the same assumptions on $(A_{jk}(x))$ as above. Hörmander [1] writes such an equation as

$$(4) \qquad \sum_{j=1}^{m} X_j^2 u + X_0^2 u = f(x),$$

where $X_0, X_1, \ldots, X_m$ are $\mathscr{C}^\infty$ vector fields on $\mathbb{R}^n$, and $X_1, X_2, \ldots, X_m$

are assumed to be linearly independent. Equations (3) and (4) are said to be <u>hypoelliptic</u> if $f$ being $\mathscr{C}^\infty$ implies that $u$ is $\mathscr{C}^\infty$.

Results concerning hypoellipticity of (4) can be compared with corresponding results concerning controllability of the nonlinear system

$$\dot{x} = X_0(x) + \sum_{j=1}^{m} X_j(x)v_j, \tag{5}$$

where $v_1, v_2, \ldots, v_m$ are the controls. The linear version of (5) (in which the usual roles of $A$ and $B$ are reversed to fit the standard upcoming p.d.e. notation) is

$$\dot{x} = Bx + Av, \tag{6}$$

where $A$ and $B$ are appropriate matrices.

Our purpose in this paper is to introduce a particular linear partial differential equation (called a Kolmogorov equation) that relates to the p.d.e. (3) as the linear system (6) relates to the o.d.e (5). For this particular p.d.e. we shall introduce an appropriate feedback that allows "eigenvalue placement" if the equation is hypoelliptic. We study the effect of this feedback on the spatial Fourier transform of the solution $u$. We also mention the problem of transforming (by state coordinate changes and feedback) the linear p.d.e. (3) to a Kolmogorow equation as one would transform the nonlinear system (5) to a controllable linear system (6).

We remark that the results of Hörmander on hypoellipticity have been applied in systems theory by Elliott [2], [3].

## II. P.D.E.'s AND FEEDBACK

An equation of special interest is

$$\sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j,k=1}^{n} b_{jk} x_j \frac{\partial u}{\partial x_k} = f, \tag{7}$$

where $A = (a_{jk})$ and $B = (b_{jk})$ are constant matrices, $A$ is symmetric, positive semidefinite of rank m, and the matrix $(A, BA, B^2A, \ldots, B^{n-1}A)$ has rank $n$ (this last condition implies hypoellipticity). Such a p.d.e. is called a Kolmogorov equation. In fact if $-\frac{\partial u}{\partial t}$ is added as in (1), Kolmogorov [4] indicates that certain probability density functions satisfy such equations.

By using the input $f$ we can introduce an appropriate type of feedback for the equation (7). This feedback should leave invariant the principal symbol of (7) and the hypoellipticity condition. Writing (7) in Hörmander's vector field notation yields

$$\sum_{j=1}^{m} \overline{X}_j^2 u + \overline{X}_0 u = f, \tag{8}$$

where $\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_m$ are linearly independent and constant coefficient and $\overline{X}_0$ is a linear vector field. Our linear feedback takes the form

$$f = \sum_{j=1}^{m} k_j x \overline{X}_j u, \tag{9}$$

where each $k_j$ is a $1 \times n$ matrix of constants. We can feed back a sum of terms involving a linear combination of $x$ variables times a vector field $\overline{X}_j$ applied to the solution $u$. Each $\overline{X}_j$ must be a vector field whose square appears in (8).

Example: Consider the p.d.e. on $\mathbb{R}^2$

$$\frac{\partial^2 u}{\partial x_2^2} + x_2 \frac{\partial u}{\partial x_1} = f. \tag{10}$$

This is of the form (7) with

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{11}$$

and of the form (8) with $m=1$ and

$$\begin{aligned} \overline{X}_1 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\partial}{\partial x_2} \\ \overline{X}_0 &= \begin{bmatrix} x_2 \\ 0 \end{bmatrix} = x_2 \frac{\partial}{\partial x_1}. \end{aligned} \tag{12}$$

If we let

$$f = (\alpha_0 x_1 + \alpha_1 x_2) \frac{\partial}{\partial x_2} + \overline{f}$$

where $\alpha_0$ and $\alpha_1$ are constants, then (10) becomes

$$\frac{\partial^2 u}{\partial x_2^2} + x_2 \frac{\partial u}{\partial x_1} - (\alpha_0 x_1 + \alpha_1 x_2) \frac{\partial}{\partial x_2} = \overline{f}. \tag{13}$$

Moreover, $A$ remains unchanged and $B$ moves to

(14) $$\begin{bmatrix} 1 & 0 \\ -\alpha_0 & -\alpha_1 \end{bmatrix}.$$

Neither the principal symbol or hypoellipticity of (10) is changed by the linear feedback. We can choose $\alpha_0$ and $\alpha_1$ to yield any desired characteristic polynomial for B.

We want to consider the effect of feedback of the form (9) on the p.d.e. (7). Following the argument in [1] we take Fourier transforms with $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ being the transform variables. Then (with $\hat{u}$ being the Fourier transform of u) we find

(15) $$-A(\xi,\xi)\ \hat{u} - \sum_{j,k=1}^{n} b_{jk}\xi_k \frac{\partial \hat{u}}{\partial \xi_j} = \hat{f}.$$

This is a first order linear p.d.e. which can be solved by the method of characteristics for a given noncharacteristic set of initial conditions. The <u>characteristic curves</u> are determined by (B' denotes B transpose)

(16) $$\frac{d\xi}{d\tau} = B'\xi$$

and $\hat{u}$ must satisfy

(17) $$\frac{d\hat{u}}{d\tau} = A(\xi,\xi) - \hat{f}.$$

Here $\tau$ denotes the parameter along the characteristic curves.

Hence the linear feedback in which the B matrix is altered simply changes the characteristic curves used in solving for $\hat{u}$. Equation (17) is actually unaltered.

Given the Kolmogorov equation (7) with matrices $A$ and $B$, we can form the controllable linear control system (6)

$$\dot{x} = Bx + Av.$$

We define the Kolmogorov indices $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_m$ of (7) to be the Kronecker indices $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_m$ of this linear system. Canonical forms for (7) which parallel canonical forms for linear systems can also be derived.

We state our main result without proof.

Theorem Assume that the matrices $A = (a_{jk})$ and $(B = b_{jk})$ from the p.d.e. (7)

$$\sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j,k=1}^{n} b_{jk} x_j \frac{\partial u}{\partial x_k} = f$$

satisfy rank $[A, BA, \ldots, B^{n-1}A] = n$. Then the eigenvalues of the $B$ matrix can be arbitrarily placed (with complex eigenvalues occuring in conjugate pairs) by linear feedback.

If rank $A = m$ is $n$, then equation (7) is elliptic. Then linear feedback can be used to eliminate all first order terms. This compares to $\dot{x} = Bx + Av$ with a control for each state.

Our Kolmogorov equation (7) has constant coefficient second order part and linear varying first order part. However, our original partial differential equation (3) has variable coefficients. An interesting problem is to derive necessary and sufficient conditions to transform (in a designated sense) the general hypoelliptic equation (3) to a

**Kolmogorov equation (7). This should compare to results for moving a nonlinear control system of o.d.e's to a controllable linear system** [5], [6], [7], [8], [9],[10].

Of course, partial differential equations of order higher than two can be considered. Also the implementations of the results of this paper by finite difference and finite element schemes when only point sensors and actuators are involved is an open problem.

## III CONCLUSION

We have drawn parallels between control theory for linear o.d.e's and the Kolmogorov p.d.e.'s. An appropriate type of linear feedback for p.d.e.'s was introduced. A transformation theory for more general partial differential equations is presently being researched.

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