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Further Developments in Exact State Reconstruction in Deterministic Digital Control Systems

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I. INTRODUCTION

Books on modern digital control systems usually address the problem of controlling a continuous-time plant driven by a zero-order-hold with a sampled output as shown in Figure 1 (for example, see Reference 1, p. 126). A common solution to this problem is to reconstruct the state of the system at the sampling instant using a state observer and then feed back the reconstructed state [1, p. 195]. However, the state observer has two undesirable characteristics. First of all, it is a dynamical system in itself and, hence, adds additional states and eigenvalues to the system, which can affect system stability. Second, as a consequence, the reconstructed state is normally an approximation to the true state and is usually not a good one early in the state reconstruction process. Recently, Polites developed a new approach to state reconstruction which has neither of these problems [2]. Subsequently, he extended this work and developed what he called the ideal state reconstructor [3]. It was so named because: if the plant parameters are known exactly, its output will exactly equal the true state of the plant, not just approximate it. Besides that, it adds no new states or eigenvalues to the system. Nor does it affect the plant equation for the system in any way; it affects the measurement equation only. It is characterized by the fact that measurements prefiltered by a multi-input/multi-output moving-average (MA) process [4] are utilized in the state reconstruction process. Now, in this paper, a more-general version of the ideal state reconstructor is presented. It allows standard instantaneous measurements to supplement the MA-prefiltered ones in the state reconstruction process. Useful in the development of this more-general ideal state reconstructor are some results to date for continuous-time plants driven by a zero-order-hold with sampled outputs. These are reviewed in Section II, prior to the development of the more-general ideal state reconstructor presented in Section III. Section IV presents an example which illustrates the procedure for choosing the parameters in it. Section V contains the conclusions and recommendations.

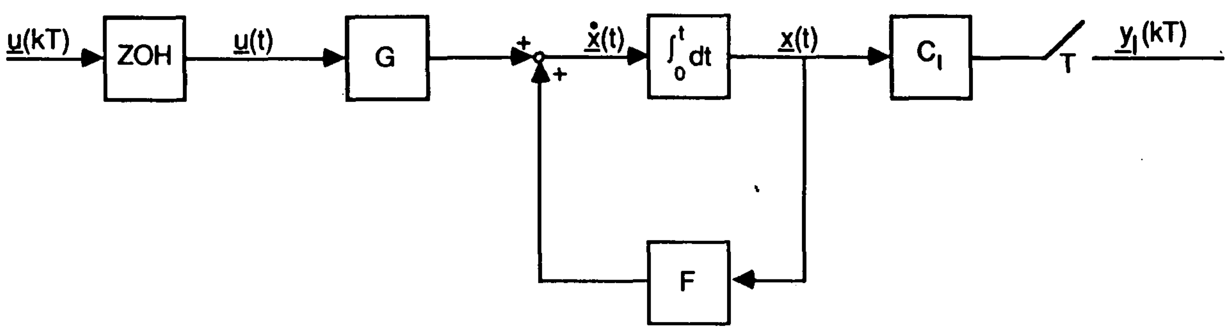

Figure 1. Continuous-time plant driven by a zero-order-hold with instantaneous measurements.

II. PRELIMINARY

For the plant in Figure 1, $\underline{x}(t)$ is an nx1 state vector, $\underline{u}(kT)$ is an rx1 control input vector, $\underline{y}_I(kT)$ is an mx1 output or measurement vector, F is an nxn system matrix, G is an nxr control matrix, and C_I is an mxn output matrix. Since $\underline{y}_I(k) = C_I\underline{x}(k)$, where k is the usual shorthand notation for time kT, $\underline{y}_I(k)$ represents an instantaneous measure of the system at the sampling instant kT. Hence, the plant in Figure 1 can be regarded as having instantaneous measurements for outputs. It is well known that this system can be modeled at the sampling instants by the discrete state equations [1, p. 126]

$$\underline{x}(k+1) = A\underline{x}(k) + B\underline{u}(k) \tag{1}$$

$$\underline{y}_I(k) = C_I\,\underline{x}(k) \quad , \tag{2}$$

where

$$\phi(t) = \mathcal{L}^{-1}\,[(sI-F)^{-1}] \quad , \tag{3}$$

$$A = \phi(T) \quad , \tag{4}$$

and

$$B = \left[\int_0^T \phi(\lambda)\,d\lambda\right] G \quad . \tag{5}$$

$\phi(t)$ is the nxn state transition matrix. A and B are the nxn system matrix and the nxr control matrix, respectively, for the discrete state equations (1) and (2).

A and B can be determined analytically using equations (3) through (5). An alternative approach, which is also quite suitable for numerical computation, is as follows [5]: $\phi(t)$ and $\int_0^t \phi(\lambda)d\lambda$ can be expressed in the form of matrix exponential series as

$$\phi(t) = \sum_{i=0}^{\infty} \frac{F^i\, t^i}{i!} \tag{6}$$

and

$$\int_0^t \phi(\lambda)d\lambda = \sum_{i=0}^{\infty} \frac{F^i t^{i-1}}{(i+1)!} \quad , \tag{7}$$

respectively. From equations (6) and (7),

$$\phi(t) = I + F\left[\int_0^t \phi(\lambda)d\lambda\right] \quad , \tag{8}$$

where I is an nxn identity matrix. Hence, $\int_0^T \phi(\lambda)d\lambda$ can be determined using equation (7) with t = T and this result substituted into equation (8) to get $\phi(T)$. With these results, A and B can be found using equations (4) and (5).

Now consider the plant in Figure 2, which is a generalization of the one in Figure 1. In addition to the instantaneous measurement vector $\underline{y}_I(kT)$, the plant in Figure 2 has the measurement $\underline{y}_F'(kT)$ generated as follows. First, the continuous-time output $\underline{z}(t)$ is sampled every T/N seconds. Every N samples are multiplied by the weighting matrices H_j, j = 0,1,...,N-1, and then summed to generate the output $\underline{y}_F(kT)$, every T seconds. Functionally, this is equivalent to passing the discrete measurements generated every T/N seconds through a multi-input/multi-output MA process with coefficient matrices H_j, j = 0,1,...,N-1 [4]. The output of the MA prefilter is sampled every T seconds to generate $\underline{y}_F(kT)$. Then, $\underline{y}_F(kT)$ has subtracted from it $E_{-}\underline{u}[(k-1)T]$, where E_{-} is a constant matrix, to produce the modified MA-prefiltered measurement vector $\underline{y}_F'(kT)$. Finally, $\underline{y}_F'(kT)$ is catenated with $\underline{y}_I(kT)$ to form the total measurement vector $\underline{y}_T(kT)$. In Figure 2, C_F is a pxn output matrix and $\underline{z}(t)$ is a px1 vector. The weighting matrices H_j, j = 0,1,...,N-1 are each qxp. Hence, $\underline{y}_F(kT)$ and $\underline{y}_F'(kT)$ are qx1 vectors. Since $\underline{y}_I(kT)$ is an mx1 vector, it follows that $\underline{y}_T(kT)$ is an (m+q)x1 vector. Since $\underline{u}[(k-1)T]$ is an rx1 delayed input vector, E_{-} is a qxr matrix.

Previously, Polites [6] showed that when

$$E_{-} = H\beta \quad , \tag{9}$$

where H is a qx(Np) matrix given by

$$H = [H_0 \mid H_1 \mid \ldots \mid H_{N-1}] \tag{10}$$

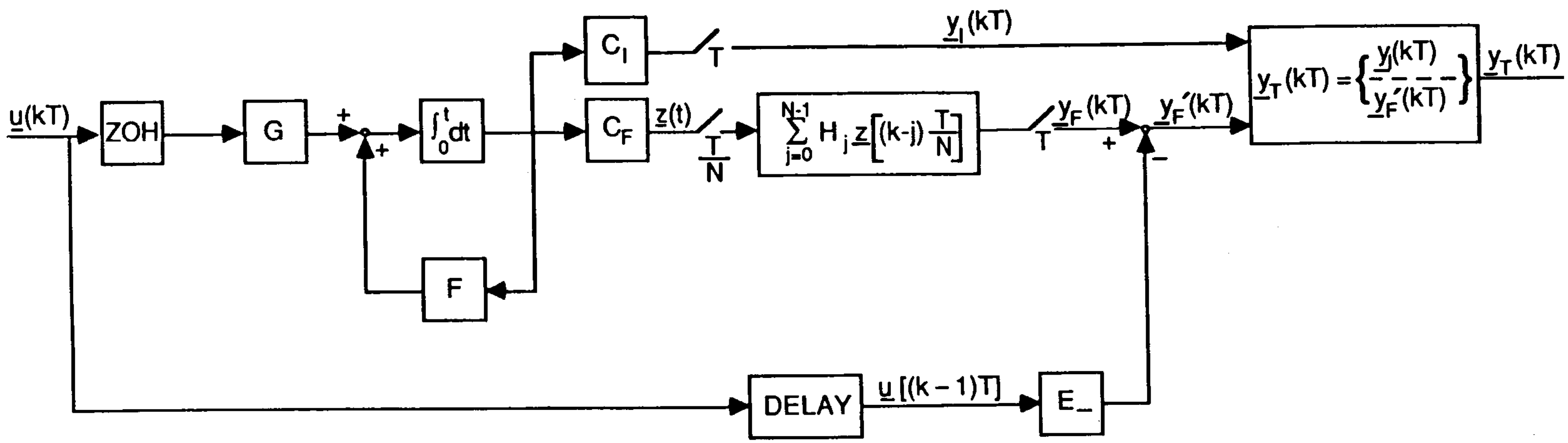

Figure 2. Continuous-time plant driven by a zero-order-hold with instantaneous and modified MA-prefiltered measurements.

and β is the (Np)xr matrix

$$\beta = \begin{bmatrix} C_F \left[\int_0^{0} \phi(\lambda)d\lambda\right] G \\ C_F \left[\int_0^{-(T/N)} \phi(\lambda)d\lambda\right] G \\ \vdots \\ C_F \left[\int_0^{-(N-1)(T/N)} \phi(\lambda)d\lambda\right] G \end{bmatrix} , \qquad (11)$$

the discrete state equations for the plant in Figure 2 become

$$\underline{x}(k+1) = A\underline{x}(k) + B\underline{u}(k) \qquad (12)$$

$$\underline{y}_T(k) = \begin{bmatrix} \underline{y}_I(k) \\ \hline \underline{y}_F'(k) \end{bmatrix} = \begin{bmatrix} C_I \\ \hline D_- \end{bmatrix} \underline{x}(k) = C_T\, \underline{x}(k) \quad , \qquad (13)$$

where D_- is a qxn matrix given by

$$D_- = H\alpha \qquad (14)$$

and α is the (Np)xn matrix

$$\alpha = \begin{bmatrix} C_F\, \phi(0) \\ C_F\, \phi(-\frac{T}{N}) \\ \vdots \\ C_F\, \phi\left[-(N-1)\,\frac{T}{N}\right] \end{bmatrix} . \qquad (15)$$

From equation (13),

$$C_T = \begin{bmatrix} C_I \\ --- \\ D_- \end{bmatrix} , \tag{16}$$

where C_T is an (m+q)xn matrix.

E_- and D_- can be evaluated analytically using equations (3), (9) through (11), (14), and (15). An alternative approach, which can be either analytical or numerical, is as follows. Let $t = -j(T/N)$, where $j = 0,1,\ldots,N-1$, and use equation (7) to determine $\int_0^{-j(T/N)} \phi(\lambda)d\lambda$, $j = 0,1,\ldots,N-1$. Substitute these results into equation (8) to get $\phi[-j(T/N)]$, $j = 0,1,\ldots,N-1$. At this point, E_- and D_- can be found using equations (9) through (11), (14), and (15).

III. THE MORE-GENERAL IDEAL STATE RECONSTRUCTOR

A general block diagram of the plant and the more-general ideal state reconstructor is shown in Figure 3. Notice the similarity between Figures 2 and 3. By virtue of this, if E_- is given by equation (9), then equations (12) to (15) define the discrete state equations for the system in Figure 3 up to the output $\underline{y}_T(k)$. Proceeding further, $\underline{y}_T'(k)$ is related to $\underline{y}_T(k)$ by the expression

$$\underline{y}_T'(k) = (C_T^T C_T)^{-1} C_T^T \underline{y}_T(k) \quad . \tag{17}$$

However, for equation (17) to be meaningful, $(C_T^T C_T)^{-1}$ must exist, and this occurs only when $(C_T^T C_T)$ is nonsingular. Recall that C_T is an (m+q)xn matrix. If $(m+q) \geqslant n$ and C_T has maximal rank (i.e. rank n), then $(C_T^T C_T)$ is positive definite and therefore nonsingular [7]. Hence, equation (17) requires that $(m+q) \geqslant n$ and rank $(C_T) = n$ for it and the more-general ideal state reconstructor to be meaningful. Assuming this is the case, it follows from equations (12), (13), and (17) that the discrete state equations for the system in Figure 3 are

$$\underline{x}(k+1) = A\underline{x}(k) + B\underline{u}(k) \tag{18}$$

$$\underline{y}_T'(k) = \underline{x}(k) \quad . \tag{19}$$

Hence, the output of the more-general ideal state reconstructor, $\underline{y}_T'(kT)$, exactly equals the true state of the plant, $\underline{x}(kT)$. Consequently, if one is given the plant in Figure 1 and modifies it to conform to Figure 3, he can exactly reconstruct the state of the plant without adding any new states, eigenvalues, or

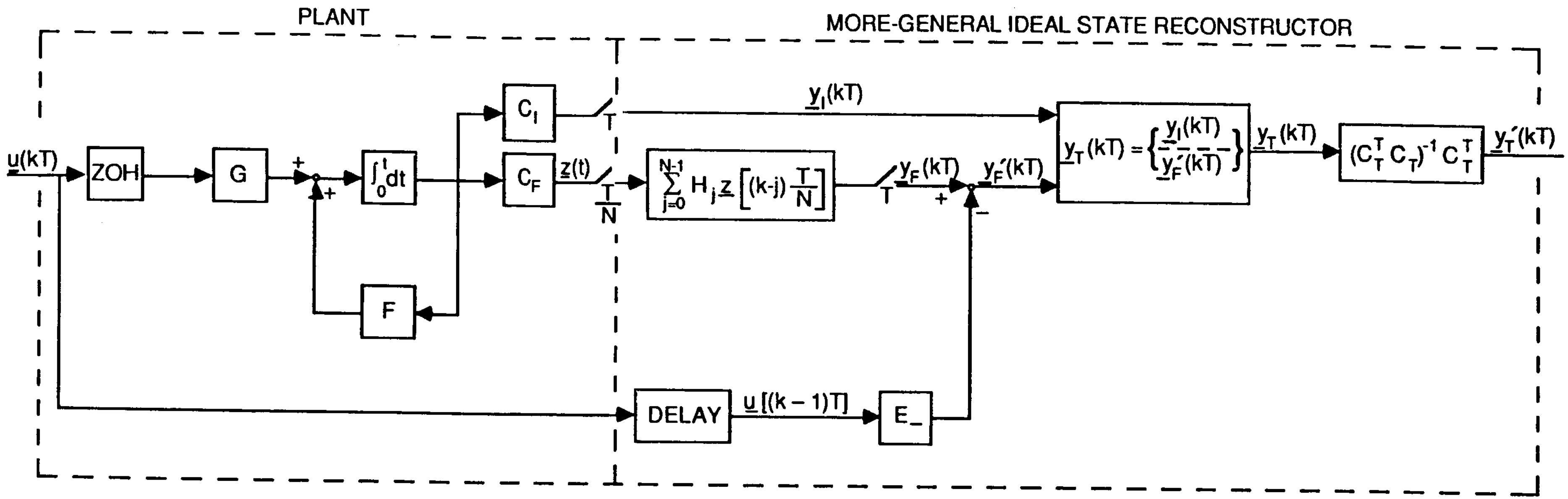

Figure 3. General block diagram of the plant and the more-general ideal state reconstructor.

dynamics to it, since the plant equation (18) for the system in Figure 3 is identical to the plant equation (1) for the plant in Figure 1. In Figure 3, exact state reconstruction is achieved when E_- is given by equation (9) and C_T is given by equation (16) where D_- is given by equation (14). Of course, C_T must satisfy the requirements previously imposed on it.

One of these is that the dimensions of C_T, namely (m+q)xn, satisfy the relationship $(m+q) \geq n$. This can also be written as $q \geq (n-m)$.Hence, the number of rows, q, in the weighting matrices H_j, j = 0,1,...,N-1 must equal or exceed the number of states, n, in the plant state vector, $\underline{x}(t)$ or $\underline{x}(kT)$, minus the number of elements, m, in the instantaneous measurement vector $\underline{y}_I(kT)$. Since q can be chosen arbitrarily, this requirement can be readily satisfied. The other requirement on C_T is that it have rank n. One approach to satisfying this requirement is as follows. Recall that C_T is defined by equation (16) where D_- is defined by equation (14). Assuming C_I is given, then one can choose D_- so that C_T has rank n and then find H to give the desired D_-. One solution to the problem of finding H to give the desired D_- is to let

$$H = D_- (\alpha^T \alpha)^{-1} \alpha^T \quad . \tag{20}$$

This follows from equation (14). However, like before, this requires that $(\alpha^T\alpha)$ be nonsingular. Recall that α is an (Np)xn matrix. If $(Np) \geq n$, or equivalently $N \geq n/p$, and α has maximal rank (i.e., rank n), then $(\alpha^T\alpha)$ is nonsingular. The first requirement can be easily satisfied because the number of weighting matrices N, where the weighting matrices are H_j, j = 0,1,...,N-1, can be arbitrarily chosen so that $N \geq n/p$. Recall that n is the number of states in the plant state vector, $\underline{x}(t)$, and p is the number of elements in the output vector $\underline{z}(t)$.

In summary, the procedure to achieve exact state reconstruction with the more-general ideal state reconstructor is as follows. Given the plant in Figure 1, modify it to conform to Figure 3. Choose the number of rows, q, in the weighting matrices H_j, j = 0,1,...,N-1 so that $q \geq (n-m)$ where n is the number of states in the plant state vector, $\underline{x}(t)$ or $\underline{x}(kT)$, and m is the number of elements in the instantaneous measurement vector $\underline{y}_I(kT)$. Choose the number of weighting matrices, N, so that $N \geq n/p$ where p is the number of elements in the output vector $\underline{z}(t)$. Assuming the (Np)xn matrix α, defined by equation (15), has maximal rank (i.e., rank n), let H be given by equation (20) where D_- is chosen so that the (m+q)xn matrix C_T, defined by equation (16), has maximal rank (i.e., rank n). The weighting matrices H_j, j = 0,1,...,N-1 are found by partitioning H as in equation (10). Finally, let E_- in Figure 3 be given by equation (9) where β is defined by equation (11). The discrete state equations for the system in Figure 3 are now given by equations (18) and (19). Hence, the output of the more-general ideal state reconstructor, $\underline{y}_T'(kT)$, exactly equals the true state of the system, $\underline{x}(kT)$.

In the event there are no instantaneous measurements, then $\underline{y}_I(kT)$ is a null vector, C_I is a null matrix, and the more-general ideal state reconstructor in Figure 3 degenerates to the one presented in Reference 3. In this case, the methods described in Reference 3 for choosing the parameters in the ideal state reconstructor to achieve exact state reconstruction, as well as the one presented here, are applicable.

IV. AN EXAMPLE

Consider the double integrator plant driven by a zero-order-hold as shown in Figure 4. Manipulating the plant in Figure 4 into the state variable format of Figure 1 yields

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \tag{21}$$

$$G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad , \tag{22}$$

and

$$C_I = [1 \quad 0] \quad . \tag{23}$$

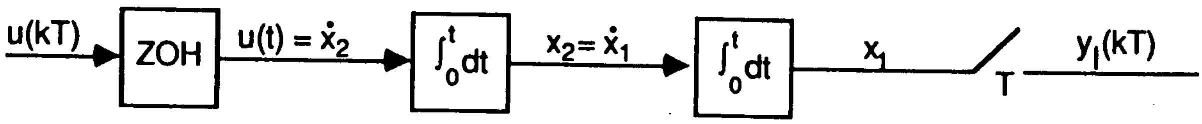

Figure 4. Plant for the example.

Since F, G, and C_I are nxn, nxr, and mxn matrices, respectively, it follows from equations (21) to (23) that $n=2$, $r=1$, and $m=1$. Using equations (21) and (22) and the formulas presented in Section II,

$$\phi(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad , \tag{24}$$

$$\int_0^t \phi(\lambda)d\lambda = \begin{bmatrix} t & \frac{t^2}{2} \\ 0 & t \end{bmatrix} \quad , \tag{25}$$

$$A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} ,$$

and

$$B = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} .$$

To exactly reconstruct the state of the plant in Figure 1 with the more-general ideal state reconstructor, modify it to conform to Figure 5. Comparing Figures 3 and 5, it is apparent that

$$C_F = [1 \quad 0] . \tag{26}$$

Since C_F is a pxn output matrix, it follows that $p = 1$. The requirement $q \geqslant (n\text{-}m)$ can be satisfied by letting $q = 1$. The requirement $N \geqslant n/p$ can be satisfied by letting $N = 4$. Now α and β can be evaluated using equations (11), (15), (22), (24), (25), and (26), and are found to be

$$\alpha = \begin{bmatrix} 1 & 0 \\ 1 & -\frac{T}{4} \\ 1 & -\frac{2T}{4} \\ 1 & -\frac{3T}{4} \end{bmatrix} \tag{27}$$

and

$$\beta = \begin{bmatrix} 0 \\ \frac{T^2}{32} \\ \frac{4T^2}{32} \\ \frac{9T^2}{32} \end{bmatrix} , \tag{28}$$

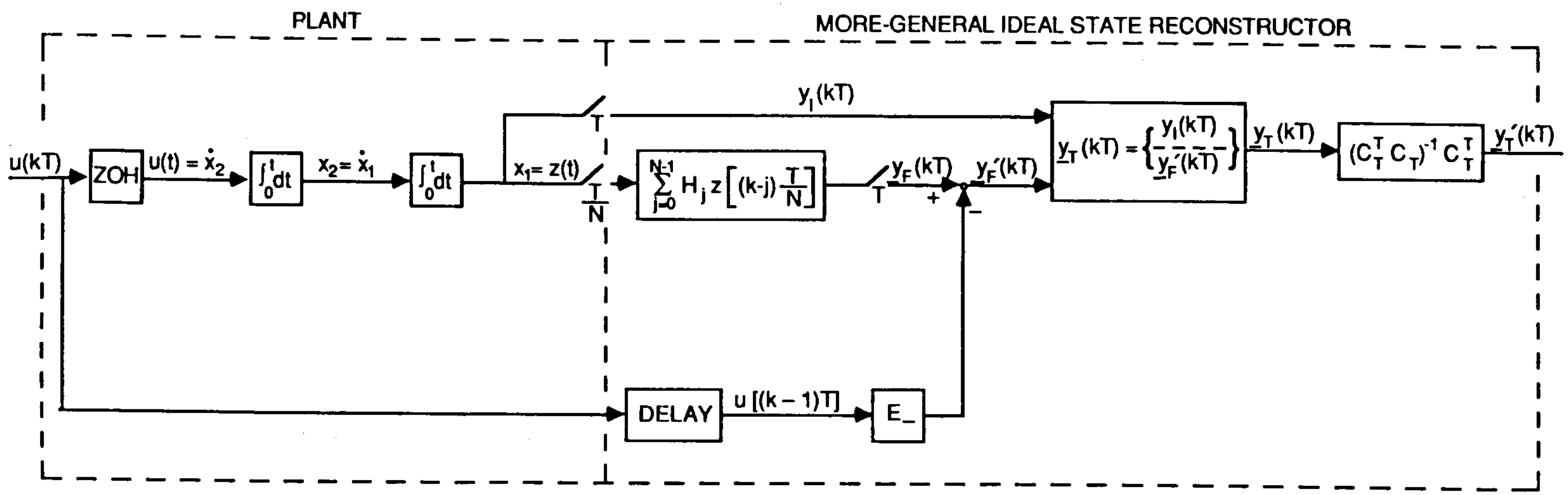

Figure 5. Plant and the more-general ideal state reconstructor for the example.

respectively. In equation (27), eliminating any two rows forms a 2x2 matrix with nonzero determinant, assuming of course that $T > 0$. Hence, rank $(\alpha) = 2 = n$ and so $(\alpha^T\alpha)$ is nonsingular. Consequently, $(\alpha^T\alpha)^{-1}\alpha^T$ exists and is found to be

$$(\alpha^T\alpha)^{-1}\alpha^T = \begin{bmatrix} \left(\frac{7}{10}\right) & \left(\frac{4}{10}\right) & \left(\frac{1}{10}\right) & \left(-\frac{2}{10}\right) \\ \left(\frac{6}{5T}\right) & \left(\frac{2}{5T}\right) & \left(-\frac{2}{5T}\right) & \left(-\frac{6}{5T}\right) \end{bmatrix} , \tag{29}$$

using equation (27). Since $(\alpha^T\alpha)^{-1}\alpha^T$ exists, H can be given by equation (20) where D_- needs to be chosen so C_T has maximal rank. Recall that C_T is an $(m+q)$xn matrix. In this example, $m = q = 1$, $n = 2$, and so C_T is a 2x2 matrix. Since C_T is defined by equation (16) where C_I is the 1x2 matrix in equation (23) for this example, D_- must be a 1x2 matrix. If D_- is chosen to be

$$D_- = [0 \quad 1] , \tag{30}$$

then

$$C_T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and rank $(C_T) = 2 = n$ since $\det(C_T) \neq 0$. Hence, $(C_T{}^TC_T)$ is positive definite and therefore nonsingular. Consequently, $(C_T{}^TC_T)^{-1}C_T{}^T$ exists and is found to be

$$(C_T{}^TC_T)^{-1}C_T{}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

From equations (10), (20), (29), and (30),

$$H = [H_0 \mid H_1 \mid H_2 \mid H_3] = \left[\left(\frac{6}{5T}\right) \mid \left(\frac{2}{5T}\right) \mid \left(-\frac{2}{5T}\right) \mid \left(-\frac{6}{5T}\right) \right] , \tag{31}$$

which reveals the weighting matrices H_j, j = 0,1,2,3. From equations (9), (28), and (31),

$$E_- = -3T/8 \quad .$$

The more-general ideal state reconstructor is now completely defined for this example.

V. CONCLUSIONS AND RECOMMENDATIONS

This paper has presented a more-general version of the ideal state reconstructor for deterministic digital control systems previously developed by Polites [3]. It is called an ideal state reconstructor because, unlike the popular state observer, it adds no new states, eigenvalues, or dynamics to the system, and, consequently, will not alter the stability of the system. In fact, adding the ideal state reconstructor to the system will not affect its plant equation in any way. It affects the measurement equation only. Also, if the plant parameters are known exactly, the reconstructed state will exactly equal the true state of the system, not just approximate it. The ideal state reconstructor is characterized by the fact that measurements prefiltered by a multi-input/multi-output moving-average (MA) process [4] are utilized in the state reconstruction process. It is called a more-general version because, unlike the original version [3], it allows for instantaneous measurements to supplement the MA-prefiltered ones in the state reconstruction process. The disadvantages of either version are that measurements must be made and calculations must be performed more frequently than with the state observer. Fortunately, this is not the problem it was 20 years ago, considering the speed of today's digital computers.

If the research in this paper is extended, two approaches are recommended. One is to explore advanced methods for choosing the parameters in the ideal state reconstructor. For example, the weighting matrices might be selected so the MA prefilter acts as a multi-input/multi-output low-pass filter for the case where measurement noise is present. The other approach is to investigate the robustness of the ideal state reconstructor and see how it compares with the state observer's. Specifically, the following questions should be addressed. What effect do modeling errors in the plant have on the ideal state reconstructor, and how does this compare with the state observer? What effect do plant process and measurement noise have on the ideal state reconstructor and how does this compare with the state observer, or even the Kalman filter? How can the robustness of the ideal state reconstructor be improved? Increasing the number of weighting matrices, N, may be one possibility. Catenating the ideal state reconstructor with a state observer, or a Kalman filter, may be another. This might produce a composite estimator which is better than either the ideal state reconstructor, the state observer, or the Kalman filter alone.

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15. SUPPLEMENTARY NOTES

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16. ABSTRACT

This paper presents a more-general version of the ideal state reconstructor for deterministic digital control systems previously developed by Polites. In the original version, measurements prefiltered by a multi-input/multi-output moving-average (MA) process were utilized in the state reconstruction process. In this version, the MA-prefiltered measurements can be supplemented by standard instantaneous measurements. The ideal state reconstructor is so named because: if the plant parameters are known exactly, its output will exactly equal the true state of the plant, not just approximate it. Furthermore, it adds no additional states or eigenvalues to the system. Nor does it affect the plant equation for the system in any way; it affects the measurement equation only. An example is presented which illustrates the procedure for choosing the parameters in it.

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