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CANONICAL COORDINATES FOR PARTIAL DIFFERENTIAL EQUATIONS

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I. Introduction

The emphasis of this paper is determining necessary and sufficient conditions under which the linear partial differential equation

$$(1)\qquad -\frac{\partial u}{\partial t}+\sum_{j,k=1}^{n} A_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k}+\sum_{j=1}^{n} B_j(x)\frac{\partial u}{\partial x_j}=f(x,t)$$

can be transformed to become either constant coefficient or of the Kolmogorov [1] type. Here we assume that the $A_{jk}(x)$ and $B_j(x)$ are C^∞ and consider C^∞ coordinate changes on $\mathbb{R}^n$.

Texts on partial differential equations develop the theory of canonical forms for second order linear partial differential equations in two variables (see Garabedian [2] and Courant and Hilbert [3]). Extensions to three or more variables are due to Cotton [4] and Fredricks [5]. These results involve finding C^∞ coordinates in which the principal part coefficients A_{jk} become constant. This is always possible in two dimensions with suitable necessary and sufficient conditions in three or more dimensions. Little attention is paid to the first order coefficients $B_j(x)$ after the new coordinates are introduced.

If there are C^∞ coordinates under which (1) becomes constant coefficient

$$(2)\qquad -\frac{\partial u}{\partial t}+\sum_{j,k=1}^{n} a_{jk}\frac{\partial^2 u}{\partial x_j \partial x_k}+\sum_{j=1}^{n} b_j\frac{\partial u}{\partial x_j}=f(x,t)\ ,$$

then Fourier transforms in the spatial variables yield

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$$(3)\qquad -\frac{\partial U}{\partial t} - \sum_{j,k=1}^{n} a_{jk}\,\xi_j\xi_k U - i\sum_{j=1}^{n} b_j\xi_j U = F(\xi,t)$$

an ordinary differential equation in U.

Similarly, if there are C^∞ coordinates in which (1) takes the form

$$(4)\qquad -\frac{\partial u}{\partial t} + \sum_{j,k=1}^{n} a_{jk}\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j,k=1}^{n} b_{jk}x_j\frac{\partial u}{\partial x_k} = f(x,t)\ ,$$

where a_{jk} and b_{jk} are constants, then Hormander [6] shows that spatial Fourier transforms lead to

$$(5)\qquad -\frac{\partial U}{\partial t} - \sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k U - \sum_{j,k=1}^{n} b_{jk}\xi_k\frac{\partial U}{\partial \xi_j} = F(\xi,t).$$

Under generic conditions, this first order linear partial differential equation can be solved by the method of characteristics (i.e. ordinary differential equations) and inverse Fourier transforms. The most important case involves a positive semidefinite matrix (a_{ij}) of constant rank m, and we shall call the corresponding equation (4) a Kolmogorov equation. Hormander writes the spatial partial differential operator in (4) as $\sum_{j=1}^{m} X_j^2 + X_0$, where X_j and X_0 are C^∞ vector fields on $\mathbb{R}^n$. In fact, if the $(A_{jk}(x))$ matrix in (1) is symmetric, constant rank m, and positive semidefinite, the spatial operator can be written as

$$(6)\qquad \sum_{j=1}^{m} X_j^2 + X_0.$$

For general partial differential operators of the form (6) Hormander proceeds to prove necessary and sufficient conditions for hypoellipticity (i.e. C^∞ right hand sides imply C^∞ solutions). We require that our Kolmogorov equations be hypoelliptic. Weber [7] constructed fundamental solutions for a class of equations related to those in (4).

In this light the problems considered in this paper are:

i) Given the partial differential operator (6) find necessary and sufficient conditions so that there exist nonsingular C^{∞} coordinate changes (local-near the origin) on $\mathbb{R}^n$ under which (6) becomes a constant coefficient partial differential operator. Standard differential geometry results (e.g. Spivak [8]) are employed and the results are of no surprise. The conditions are derived here for the sake of completeness.

ii) Given the partial differential operator (6) find necessary and sufficient conditions so that there exist nonsingular C^{∞} coordinate changes (local-near the origin) on $\mathbb{R}^n$ under which $X_1, X_2, \ldots, X_m$ in (6) are transformed to constant vector fields and X_0 becomes a linear vector field. This makes the partial differential operator (6) of <u>Kolmogorov type</u>, if the hypoellipticity conditions of Hormander are satisfied.

The spatial operator in (4) is a Kolmogorov operator if it is hypoelliptic.

We remark that both problems i) and ii) can be generalized to the partial differential operators $\sum_{j=1}^{m} X_j^2 - \sum_{j=1}^{p} Y_j^2 + X_0$, but we shall concentrate on the form (6). Moreover, we assume that $X_1, X_2, \ldots, X_m$ are linearly independent.

Our principle tools are taken from the field of systems and control. However, the purpose of this paper is not to draw a parallel between controllability of systems of nonlinear ordinary differential equations and hypoelliplicity of partial differential equations, as this has been well established in the literature and in conference presentations.

As we mentioned previously, problem i) is straightforward. Our work on problem ii) is analogous to the study of coordinate changes to transform a nonlinear control system on $\mathbb{R}^n$,

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(7) $$\dot{x} = f(x) + \sum_{j=1}^{m} u_j g_j(x)$$

to an n-dimensional controllable linear system

(8) $$\dot{\bar{x}} = F\bar{x} + Gu.$$

Here $\dot{x} = \frac{dx}{dt}$, $f, g_1, g_2, \ldots, g_m$ are C^{∞} vector fields on R^n, $f(0) = 0$, $u = (u_1, u_2, \ldots, u_m)$ consists of real-valued functions, F is an nxn constant matrix, and G is an nxm constant matrix. Also f,F, and u are obviously different objects in our control discussion than in our p.d.e. discussion.

We rely heavily on the results of Krener [9] and Respondek [10], and, in fact, our research essentially moves their results from the ordinary differential equation setting to the partial differential equation setting. Nonlinear control system (7) is replaced by equation (6) with X_j taking the place of g_j, $j = 1, 2, \ldots, m$, and X_0 taking the place of f. The linear system (8) is replaced by the Kolmogorov partial differential operator

(9) $$\sum_{j=2}^{m} \bar{X}_j^2 + \bar{X}_0^2,$$

where each $\bar{X}_j$ is a constant vector field and X_0 is linear. Here $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_m$ correspond to the m columns of G and $\bar{X}_0$ corresponds $F\bar{x}$ in (8). We want the span of the Lie brackets $\bar{X}_j, [\bar{X}_0, \bar{X}_j], \ldots, (ad^{n-1}\bar{X}_0, \bar{X}_j)$, $j=1,2,\ldots,m$ to be R^n, so we make the corresponding assumptions on the Lie brackets of vector fields in (6). As noted before we also suppose that $X_1, X_2, \ldots, X_m$ are linearly independent.

Section 2 of this paper contains basic definitions and consideration of those linear partial differential operators which can be made constant coefficient. In section 3, necessary and sufficient conditions are derived which classify those linear partial differential operators that can be moved to the Kolmogorov type.

II. Constant Coefficient Operators

We begin with a set of appropriate definitions.

If X and Y are C^{∞} vector fields on $\mathbb{R}^n$, then the Lie bracket of X and Y is

$$[X,Y] = \frac{\partial Y}{\partial x} X - \frac{\partial X}{\partial x} Y,$$

where $\frac{\partial Y}{\partial x}$ and $\frac{\partial X}{\partial x}$ are Jacobian matrices, x being the variable for $\mathbb{R}^n$. Successive Lie brackets such as $\left[X, [X,Y]\right]$, $\left[Y, [X,Y]\right]$, $\left[\left[X, [X,Y]\right],Y\right]$, etc. can be taken. A standard notation is

$$(ad^0 X,Y) = Y$$

$$(ad^1 X,Y) = [X,Y]$$

$$(ad^2 X,Y) = \left[X,[X,Y]\right]$$

$$\vdots$$

$$(ad^j X,Y) = [X,(ad^{j-1} X,Y)]$$

We let $\langle \cdot,\cdot \rangle$ denote the dual product of one forms and vector fields. Given a C^{∞} function h on R^n we define the Lie derivative of h with respect to the vector field X as

$$L_X h = \langle dh,f \rangle.$$

Successive Lie derivatives are

$$L_X^0 h = h$$

$$L_X^1 h = L_X h$$

$$L_X^2 h = L_X L_X h$$

$$\vdots$$

$$L_X^j h = L_X L_X^{j-1} h.$$

Moreover, the Lie derivative of the one form dh with respect to X is

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$$L_X(dh) = \left[\frac{\partial(dh)^*}{dx}X\right]^* + (dh)\frac{\partial X}{\partial x},$$

where * denotes transpose.

The three types of Lie derivatives satisfy the formula

$$L_X\langle dh,Y\rangle = \langle L_X(dh),Y\rangle + \langle dh,[X,Y]\rangle. \tag{10}$$

We motivate our study by the following example.

Example 2.1. Consider the partial differential operator

$$(1+x_1)^2\frac{\partial^2}{\partial x_1^2} + (1+x_1)\frac{\partial}{\partial x_1} + 2x_3\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \tag{11}$$

on $\mathbb{R}^3$. The local coordinate change (near the origin)

$$\begin{aligned} y_1 &= \ln(1+x_1) \\ y_2 &= x_2 - x_3^2 \\ y_3 &= x_3 \end{aligned} \tag{12}$$

moves (11) to the constant coefficient form

$$\frac{\partial^2}{\partial y_1^2} + \frac{\partial}{\partial y_3}. \tag{13}$$

This is discovered in the following way. First we write (11) as

$$(1+x_1)\frac{\partial}{\partial x_1}\left[(1+x_1)\frac{\partial}{\partial x_1}\right] + 2x_3\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \tag{14}$$

and set

$$X_1 = \begin{bmatrix} 1+x_1 \\ 0 \\ 0 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 0 \\ 2x_3 \\ 1 \end{bmatrix}. \tag{15}$$

Thus (11) becomes

$$X_1^2 + X_0. \tag{16}$$

Since X_1 and X_0 are linearly independent and the Lie bracket $[X_1, X_0] \equiv 0$, standard differential geometry results (see [8]) imply that the transformation (12) takes

$$X_1 \text{ to } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } X_0 \text{ to } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, the partial differential operator (11) is moved to the constant coefficient form (13).

We now prove our result concerning transformation to constant coefficient operators. Again, this result is trivial from the differential geometry viewpoint.

Theorem 2.1. Given the C^∞ partial differential operator on $\mathbb{R}^n$

$$\sum_{j=1}^{m} X_j^2 + X_0, \tag{17}$$

where $X_0, X_1, X_2, \ldots X_m$, are linearly independent for $m<n$ and $X_1, X_2, \ldots, X_m$ are linearly independent for $m=n$, there exists a non-singular (local) coordinate change on $\mathbb{R}^n$ under which $X_0, X_1, \ldots, X_m$ become constant vector fields if and only if

$$[X_r, X_s] = 0 \text{ for all } 0 \le r, s \le m. \tag{18}$$

Proof. If $m<n$ results from [8] indicate there are nonsingular coordinate taking

$$X_1 \text{ to } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \; X_2 \text{ to } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \ldots, X_m \text{ to } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow (n-m)^{th} \text{ place}$$

$$X_0 \text{ to } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow (n-m-1)^{th} \text{ place .}$$

if and only if (18) is satisfied.

If m = n, we can move

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$$X_1 \text{ to } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} X_2 \text{ to } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \ldots, X_m \text{ to } \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

if and only if $\left[X_r, X_s\right] \equiv 0$ for all $1 \leq r,s \leq m$. Setting

$$X_0 = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix},$$

we find the only possible nonzero column of

$$\left[X_j, X_0\right] \text{ is } \begin{bmatrix} \frac{\partial \gamma_1}{\partial x_j} \\ \frac{\partial \gamma_2}{\partial x_j} \\ \vdots \\ \frac{\partial \gamma_n}{\partial x_j} \end{bmatrix}$$

for $j = 1,2,\ldots,n$. Then X_0 is a constant vector field if and only if $\left[X_j, X_0\right] \equiv 0$, $j = 1,2,\ldots,n$. □

We now study our problem ii), the main consideration of this paper.

III. Kolmogorov Operators

We examine the partial differential operator (6) $\sum_{j=1}^{m} X_j^2 + X_0$ where $X_1, X_2, \ldots, X_m$ are linearly independent and X_0 vanishes at the origin in $\mathbb{R}^n$. We derive conditions under which C^∞ coordinate changes (local-near the origin) exist taking (6) to the Kolmogorov operator (9).

As stressed in the introduction, the main contribution of this paper is realizing that the results of Krener [9] and Respondek [10] in the nonlinear systems (o.d.e.) and control area can be applied to partial differential

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operators

(19) $$\sum_{j,k=1}^{n} A_{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} B_j(x) \frac{\partial}{\partial x_j} .$$

The linear controllable system

(20) $$\dot{\bar{x}} = F\bar{x} + Gu$$

is replaced by the hypoelliptic Kolmogorov partial differential operator (with a_{jk} and b_{jk} constant)

(21) $$\sum_{j,k=1}^{n} a_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j,k=1}^{n} b_{jk} x_j \frac{\partial}{\partial x_k} .$$

Of course we shall study operators (19) and (21) in our vector field notation. First we wish to examine parallels between system (20) and operator (21).

For the control system (20) the Kronecker indices and eigenvalues of the F matrices are invariants under coordinate changes. For the operator (21) we introduce Kolmogorov indices and note that these and the eigenvalues of the B matrix are invariants. Canonical forms which parallel controllable canonical forms for (20) will occur in our work.

If the matrix $A = (a_{jk})$ in (21) has rank m, let $\bar{X}_0, \bar{X}_1, \bar{X}_2, \ldots, \bar{X}_m$ be linearly independent vector fields so that (21) becomes $\sum_{j=1}^{m} \bar{X}_j^2 + \bar{X}_0$. We set B = Jacobian matrix of the vector field $\bar{X}_0$. If the partial differential operator in (21) is hypoelliptic (in this case $\bar{X}_j, B\bar{X}_j, \ldots, B^{n-1}\bar{X}_j$, j = 1,2,...,m, span $\mathbb{R}^n$) we introduce the following process:

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1) Write out the grid

$$\begin{array}{cccc}
\bar{X}_1 & \bar{X}_2 & \cdots & \bar{X}_m \\
B\bar{X}_1 & B\bar{X}_2 & \cdots & B\bar{X}_m \\
B^2\bar{X}_1 & B^2\bar{X}_2 & \cdots & B^2\bar{X}_m \\
\vdots & \vdots & & \vdots \\
B^{n-1}\bar{X}_1 & B^{n-1}\bar{X}_2 & \cdots & B^{n-1}\bar{X}_m
\end{array}$$

2) Start at $\bar{X}_1$ and move left to right across the first row, then start at $B\bar{X}_1$ and move across the second row, etc.

3) Throw out any vector field that is a linear combination of the preceding vector fields in the grid. Discard all vector fields in the column below this vector field.

4) Continue until n linearly independent vector fields are found and all others have been discarded.

5) Let ℓ_j = number of entries remaining in the j^{th} column of the grid, $j = 1,2,\ldots,m$.

6) Renumber $\bar{X}_1,\bar{X}_2,\ldots,\bar{X}_m$, if necessary, so that $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_m$.

Def. The integers $\ell_1,\ell_2,\ldots,\ell_m$ are called the Kolmogorov indices of the partial differential operator (21). If the vector field notation for (21) is

(9) $\sum_{j=1}^{m} \bar{X}_j^2 + \bar{X}_0$ we also have the integers $\ell_1,\ell_2,\ldots,\ell_m$ associated with (9).

By assumption $\ell_1 + \ell_2 + \ldots + \ell_m = n$.

We ask if the partial differential operator (6) $\sum_{j=1}^{m} X_j^2 + X_0$ is a coordinate change away from the Kolmogorov partial differential operator (9) $\sum_{j=1}^{m} \bar{X}_j^2 + \bar{X}_0$ having given indices $\ell_1,\ell_2,\ldots,\ell_m$. Before stating the general theorem we present an example.

Example 3.1. Consider the partial differential operator

$$(22)\qquad 4x_3^2 \frac{\partial^2}{\partial x_2^2} + 4x_3 \frac{\partial^2}{\partial x_2 \partial x_3} + \frac{\partial^2}{\partial x_3^2} + (2+x_3) \frac{\partial}{\partial x_3} + \left[x_2 - x_3^2 + 2(x_2-x_3^2)\ x_3\right] \frac{\partial}{\partial x_1}$$

on $\mathbb{R}^3$. We write this as

$$(23)\qquad \left[2x_3 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right]\left[2x_3 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right] + \left[x_2 - x_3^2 + 2(x_2-x_3^2)\ x_3\right] \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} .$$

Letting

$$X_1 = \begin{bmatrix} 0 \\ 2x_3 \\ 1 \end{bmatrix} , \quad X_0 = \begin{bmatrix} x_2-x_3^2+2(x_2-x_3^2)x_3 \\ x_3 \\ 0 \end{bmatrix}$$

we find (23) becomes

$$(24)\qquad X_1^2 + X_0 .$$

The local coordinate changes

$$\bar{x}_1 = x_1 - x_2^3 + 2x_2x_3^2 - x_3^4$$

$$\bar{x}_2 = x_2 - x_3^2 = L_{X_0}\bar{x}_1$$

$$\bar{x}_3 = x_3 = L_{X_0}\bar{x}_2$$

take (24) to

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$$(25)\qquad \bar{X}_1^2 + \bar{X}_0 ,$$

where $\bar{X}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\bar{X}_0 = \begin{bmatrix} \bar{x}_2 \\ \bar{x}_3 \\ 0 \end{bmatrix}$.

This yields the Kolmogorov partial differential operator

$$(26)\qquad \frac{\partial^2}{\partial \bar{x}_3^2} + \bar{x}_2 \frac{\partial}{\partial \bar{x}_1} + \bar{x}_3 \frac{\partial}{\partial \bar{x}_2} .$$

Since

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$\bar{X}_1$, $B\bar{X}_1$, $B^2\bar{X}_1$ span $\mathbb{R}^3$, and the single Kolmogorov index is $\ell_1 = 3 = n$.

In equation (24) we remark that

$$X_1, [X_0, X_1], (ad^2 X_0 . X_1) \text{ span } \mathbb{R}^3 \text{ (near } (0,0,0))$$

and

$$[(ad^r X_0, X_1), (ad^s X_0, X_1)] \equiv \text{ for } 0 \leq r,s \leq 3$$

We now state and prove our main result. The general proof will follow from analogous results from [9] and [10], but we shall present a proof in the case $m = 1$ for some sort of completeness.

<u>Theorem 3.1.</u> The linear partial differential operator (6) $\sum_{j=1}^{m} X_j^2 + X_0$, with $X_1, X_2, \ldots, X_m$ linearly independent on $\mathbb{R}^n$ and $X_0(0) = 0$, can be transformed by nonsingular coordinate changes (local-near the origin) to the Kolmogorov partial differential operator (9) $\sum_{j=1}^{m} \bar{X}_j^2 + \bar{X}_0$ having indices $\ell_1, \ell_2, \ldots, \ell_m$ if and only if

α) the set $\{X_1, [X_0, X_1], \ldots, (ad^{\ell_1 - 1} X_0, X_1), X_2, [X_0, X_2], \ldots, (ad^{\ell_2 - 1} X_0, X_2), \ldots, X_m, [X_0, X_m], \ldots, (ad^{\ell_m - 1} X_0, X_m)\}$ is linearly independent.

and β) the Lie brackets of every pair of vectors fields in

$$\{X_1, [X_0, X_1], \ldots, (ad^{\ell_1} X_0, X_1), X_2, [X_0, X_2], \ldots, (ad^{\ell_2} X_0, X_2), \ldots, X_m, [X_0, X_m], \ldots, (ad^{\ell_m} X_0, X_m)\}$$ is zero.

<u>Proof.</u> (For $m = 1$, $\ell_1 = n$, and the operator $X_1^2 + X_0$):

Since $X_1, [X_0, X_1], \ldots, (ad^{n-1} X_0, X_1)$ are linearly independent and have zero

Lie brackets, we have coordinates so that

$$X_1 \text{ becomes } \bar{X}_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$[X_0, X_1] \text{ becomes } [\bar{X}_0, \bar{X}_1] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$$

$$\vdots$$

$$(\mathrm{ad}^{n-1} X_0, X_1) \text{ becomes } (\mathrm{ad}^{n-1} \bar{X}_0, \bar{X}_1) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

If $\bar{x}_0 = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \delta_{n-1} \\ \delta_n \end{bmatrix}$, then $[\bar{X}_0, \bar{X}_1] = 0$ implies $\dfrac{\partial \delta_i}{\partial \bar{x}_n} = 0, i=1,2,\ldots,n-2,n,\ \dfrac{\partial \delta_{n-1}}{\partial \bar{x}_n} = 1$

$$(\mathrm{ad}^2 \bar{X}_0, \bar{X}_1) = 0 \text{ implies } \frac{\partial \delta_i}{\partial \bar{x}_{n-1}} = 0, i=1,2,\ldots,n-3,n-1,n,\ \frac{\partial \delta_{n-2}}{\partial \bar{x}_{n-1}} = 1$$

(28) $\quad\vdots$

$$(\mathrm{ad}^{n-1} \bar{X}_0, \bar{X}_1) = 0 \text{ implies } \frac{\partial \delta_i}{\partial \bar{x}_2} = 0,\ i = 2,3,\ldots,n,\ \frac{\partial \delta_1}{\partial \bar{x}_2} = 1.$$

Now $[(\mathrm{ad}^n \bar{X}_0, \bar{X}_1), \bar{X}_1] = 0$ yields $\dfrac{\partial^2 \delta_i}{\partial \bar{x}_j \partial \bar{x}_n} = 0,\ i,j = 1,2\ldots,n$

$$[(\mathrm{ad}^n \bar{X}_0, \bar{X}_1),\ [\bar{X}_0, \bar{X}_1]] = 0 \text{ yields } \frac{\partial^2 \delta_i}{\partial \bar{x}_j \partial \bar{x}_{n-1}} = 0,\ i,j = 1,2,\ldots,n.$$

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$$[(ad^n\bar{X}_0,\bar{X}_1),\ (ad^{n-1}\bar{X}_0,X_1)] = 0 \text{ yields } \frac{\partial^2 \delta_i}{\partial \bar{x}_j \partial \bar{x}_i} = 0, i,j=1,2,\ldots,n .$$

Hence $\bar{X}_0$ is a linear vector field as promised and $\bar{X}_1^{\,2} + \bar{X}_0$ is a Kolmogorov partial differential operator. □

In the above proof

$$(30) \qquad \bar{X}_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \text{ and } \bar{X}_0 = \begin{bmatrix} b_{11} & 1 & 0 & \ldots & 0 \\ b_{21} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \\ b_{(n-1)1} & 0 & 0 & \ldots & 1 \\ b_{n1} & 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_{n-1} \\ \bar{x}_n \end{bmatrix},$$

and $\bar{X}_1^{\,2} + \bar{X}_0$ is in a canonical form. If we set

$$z_1 = \bar{x}_1$$

$$z_2 = L_{\bar{X}_0}\bar{z}_1$$

$$\vdots$$

$$z_n = L_{\bar{X}_0}\bar{z}_{n-1} ,$$

we have a canonical form in which the matrix defined by the new $\bar{X}_0$ is in rational canonical form and $\bar{X}_1$ is as before. If m≥1, we get the analogue of controllable canonical form.

We have developed a theory giving necessary and sufficient conditions that a second order linear partial differential operator be a coordinate change away from a Kolmogorov operator.

Future research will be in two directions:

1) Expand the transformations used to include "appropriate types" of feedback. This research is presently underway, and first thoughts were to include results in this paper. However, the process of feedback, as applied in transformation theory, is not well

addressed in the partial differential equation literature, where coordinate changes are standard fare. Therefore, we decided that separate papers are appropriate.

2) Extend all results to the discrete setting. A Ph.D. student of the first author is currently working on this project.

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