NASA Technical Memorandum 100699

Effect of Atmospheric Turbulence on the Bit Error Probability of a Space to Ground Near Infrared Laser Communications Link Using Binary Pulse Position Modulation and an Avalanche Photodiode Detector

(NASA-TM-100699) EFFECT OF ATMOSPHERIC TURBULENCE ON THE BIT ERROR PROBABILITY OF A SPACE TO GROUND NEAR INFRARED LASER COMMUNICATIONS LINK USING BINARY PULSE POSITION MODULATION AND AN AVALANCHE N88-21189 Unclas G3/17 0136061

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INTRODUCTION

Over the last two decades, there has been substantial interest within NASA in the use of lasers for communication between orbiting spacecraft. In the environment of space, the inherent advantages of lasers (i.e., narrow beamwidth and high directivity) are fully exploited and the problem of weather dependent link outages vanishes. Prior to their operational deployment in space, however, it is generally agreed that a comprehensive set of experiments must be carried out to verify the performance and feasibility of these systems.

Currently, the most promising possibility for conducting such experiments centers on the NASA Advanced Communication Technology Satellite (ACTS). Experiments are being considered which involve a laser system on the ACTS and one or more ground stations; the ACTS-to-ground laser communication link would use a near infrared wavelength of about 0.8 micrometer and binary pulse position modulation, with an avalanche photodiode detector in the ground station receiver.

This report investigates the degree to which the communication link may be expected to be degraded by atmospheric turbulence, by calculating the expected increase in bit error probability due to turbulence, including the ameliorating effect of a finite aperture receiver.

The calculations indicate that, for representative values of the system parameters, the bit error probability for a realistic, turbulent atmosphere increases by only a few percent over the value for a hypothetical nonturbulent atmosphere, for a signal strength of less than two hundred incident photons per bit. With the same system parameter values, the percent increase in bit error probability becomes greater as the signal strength increases, until the increase reaches about thirty percent at a signal strength of four hundred photons per bit. For greater signal strengths the increase becomes even larger, but this is not important because the bit error probability itself decreases to less than one error in a billion.

If these results are translated into the increase in signal strength needed to compensate for the effects of turbulence, it turns out that, to maintain a bit error rate of one error in a million, the signal must be increased from the approximately 254 photons per bit needed for this error level for a quiet atmosphere to about 257 photons per bit for an atmosphere with an average amount of turbulence, a signal increase of about 0.05 db. To maintain an error rate of one in ten million, the corresponding figures are about 300 photons per bit for a quiet atmosphere and about 303 for an average turbulent atmosphere, a signal increase of about 0.04 db. For greater degrees of turbulence the required increase in signal strength will, of course, be greater. For example, if the degree of turbulence, as measured by the variance of the signal fluctuation, is greater than the average degree of turbulence by a factor of five, then the signal must be increased by 0.17 db to maintain an error rate of one in a million, or by 0.18 db to maintain an error rate of one in ten million. (All these figures are for the representative system parameter values referred to above.)

In the course of the analysis in this report, formulas are developed for the bit error probability as a function of signal strength for an avalanche photodiode detector and binary pulse position modulation. Formulas are also developed for the mean and variance of the bit error probability, which fluctuates as a result of the turbulence-induced fluctuations of the signal strength; because these formulas involve quadratures and are time consuming to calculate numerically, approximate formulas are developed which are easily calculated and are also sufficiently accurate for system feasibility studies.

MATHEMATICAL TREATMENT OF THE EFFECT OF TURBULENCE ON THE BIT ERROR PROBABILITY

Formulas for the Bit Error Probability Statistics

Fig. 1 below shows diagramatically the space to ground laser communication link being investigated.

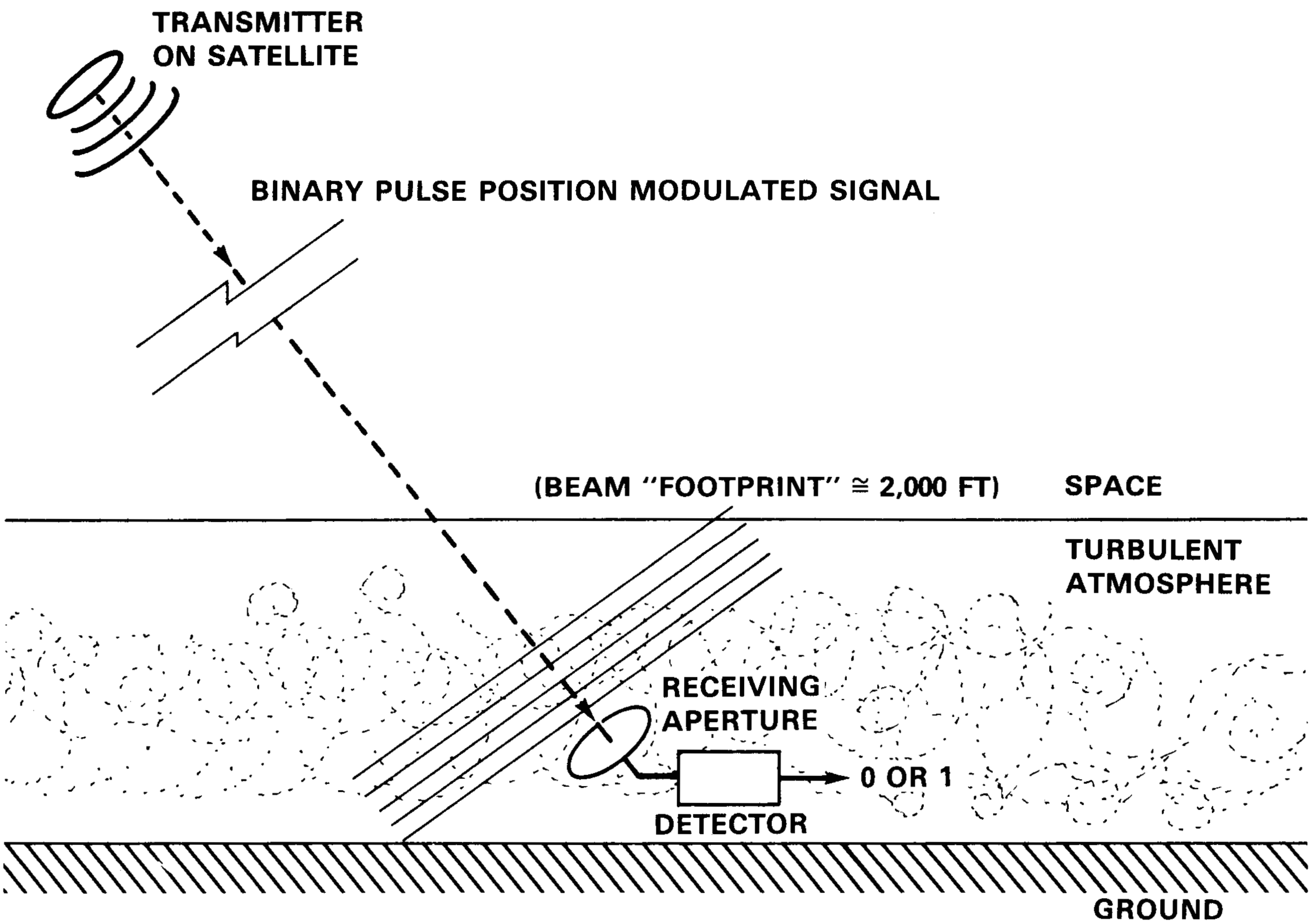

Figure 1. Space to Ground Laser Communication Link

The transmitter is assumed to be on the ACTS satellite, which is in geosynchronous orbit, but the analysis that follows holds for any situation of this type.

The signal consists of a series of pulses with binary pulse position modulation. In this modulation scheme the data to be sent are digitized, so that each pulse represents a binary digit, or bit (zero or one). For each bit to be sent a time interval, called a word, is allocated. Each word is divided into two equal subintervals, called slots. If the data pulse is sent in the first slot it represents a zero; if it is sent in the second slot it represents a one. If there were no background radiation and no noise in the receiver, there would be no problem in sensing which slot contains the pulse. Because of the presence of background radiation and receiver noise, plus the fact that the transmitter sends some signal even when it is turned off, each of the slots in a given word will contain, at the output of the detector, a non-zero power level. Because of random noise fluctuations there will be some appreciable probability, for a weak enough signal pulse, that the slot containing the signal pulse will contain less total energy than the slot not containing it. Since the detector decides whether a zero or a one has been sent by comparing the total energy content of the two slots (by means of an integrating circuit, a delay circuit and a comparator), it follows that the detector has some probability of making a wrong decision, which may become intolerably large as the signal becomes very weak. It is this "bit error probability," or, in other words, the bit error rate (the ratio of the number of errors to the total number of bits sent, for a large number of bits) that will be analyzed in this report, especially the effect of atmospheric turbulence on it.

For the ACTS to ground configuration the beam, which starts at the transmitter in geosynchronous orbit with a very small diameter, will be approximately 2,000 feet across when it reaches the earth's atmosphere. The receiving aperture (a telescope mirror) will thus intercept only a very small part of the beam; so small that the radiant intensity, neglecting turbulence effects, will be essentially constant across the aperture. Turbulence in the atmosphere will result in intensity fluctuations across the aperture. If the aperture is large compared to the distance over which the random fluctuations become uncorrelated, the collection and effective summation by the aperture of a number of independent random fluctuations will tend to smooth them out; i.e., the variance of the aperture averaged signal will be less than the variance of the signal at a point. This is in fact the case in the ACTS to ground configuration. For example, the receiving telescope at the Goddard Space Flight Center has a 48-inch diameter, whereas the turbulence fluctuations tend to become independent over a distance of a few centimeters. Because a large collecting aperture tends to reduce the size of the signal fluctuation, such an aperture averaged signal will result in a smaller increase in the bit error probability than would a signal collected by a very small aperture.

For any particular type of detector, the bit error probability (i.e., the relative frequency of errors) is functionally related to the signal strength. The signal strength will be defined here as the number of signal photons per bit incident on the detector; it will be denoted by s, and will be treated as a continuous variable, even though it would appear to be discrete. This is justifiable if the number of photons is regarded as the intensity of the radiation (which in the classical theory of radiation is a continuous quantity) divided by the energy of a photon. It is also mathematically convenient, because the log-normal probability distribution, which is generally used to describe the turbulence induced fluctuations, is a continuous distribution. The bit error probability E may thus be written as:

$$E = f(s;p) \tag{1}$$

where the quantity p represents a vector of values of the system parameters. If attention is focused on s and no confusion results, p will usually be omitted and f(s;p) will be written simply as f(s), or even E(s), if there is no danger of confusing the variable E with the function f. The quantities E and s are here regarded as ordinary variables; i.e., E is the bit error probability for a signal pulse of strength s.

If the signal strength is regarded as fluctuating randomly (due to atmospheric turbulence), s becomes a random variable with a probability distribution which will be denoted by P(s). The bit error probability also becomes a random variable, with a probability distribution which will be denoted by Q(E). The two random variables are still related by the function f, so that the probability distribution of E is determined if the probability distribution of s is known. The relation between the two distributions is (see, for example, Reference 1, Chapter V):

$$Q(E) = P(s) \cdot |ds/dE| \tag{2}$$

provided the function f is monotonic (so that the derivative is nowhere zero). In the present case the function f is monotonically decreasing (as will be shown below), so Equation (2) is valid. It is worth noting that the time scale of the turbulence fluctuations is measured in milliseconds, so that the turbulence fluctuations constitute only a very slow drift compared to the data rate, which in this case is measured in hundreds of megabits per second.

Since $s = f^{-1}(E)$, Equation (2) gives the probability distribution of E in terms of E, in principle. In practice, however, the quantities on the right side of the equation may not be explicitly representable in closed form. This happens to be the case in this particular problem, so that (2) is of little direct use; even though the distribution Q(E) contains complete information about the behavior of the random variable E, this information is not easily accessible. However, the statistical information about E required for this problem is mostly contained in the first two moments of the distribution (the mean and variance), and these may be represented explicitly in terms of the distribution of s by the following formulas (see any text on probability and statistics, such as Reference 1):

$$\overline{E} = \int_0^\infty E\, P(s)\, ds \tag{3}$$

$$\mathrm{Var}(E) = \int_0^\infty (E - \overline{E})^2\, P(s)\, ds \tag{4}$$

The bar over E indicates the mean, and Var denotes the variance.

The rest of this report will be devoted to developing explicit formulas for the functional relation between E and s and for the probability distribution of s, and to developing approximate formulas for the mean and variance of E which are easier to calculate than the ones given by the above expressions.

It is worth noting that the expression for the mean of E given by Equation (3) reduces (as it clearly must) to $E(\overline{s})$ as the variance of s goes to zero. To see this, one need only note that the integral of P(s) is unity (because P is a probability distribution) and that $P(\overline{s})$ therefore becomes infinite as the variance of s goes to zero. Thus the function P(s), for vanishing Var(s), is a Dirac delta function, so that:

$$\text{As } \mathrm{Var}(s) \rightarrow 0, \quad \overline{E} \rightarrow \int_0^\infty E\, \delta(s-\overline{s})\, ds = E(\overline{s}) \tag{5}$$

Before proceeding with the development of the formulas given by (3) and (4), alternative forms will be developed for the mean and variance of E as series in the moments of the distribution P, in the hope that approximate formulas can be obtained by taking only the first few terms of the series. It will turn out that this can be done; in fact, quite accurate formulas result from using only the first term of each series.

Expression of the Formulas as Series of Moments

If, in Equations (3) and (4) above, the quantities E and $(E-\overline{E})^2$ are expanded as Taylor series about $\overline{s}$, the series are integrated term by term and note is taken of the facts that

$$\int_0^\infty P(s)\,ds = 1 \qquad \text{and} \qquad \int_0^\infty (s-\overline{s})\,P(s)\,ds = 0$$

(since the integral of a probability distribution is unity and the first moment about the mean vanishes), the following series are obtained:

$$\overline{E} = E(\overline{s}) + \sum_{i=2}^{\infty} \frac{E^{(i)}(\overline{s})}{i!} M_i \tag{6}$$

$$\mathrm{Var}(E) = (E(\overline{s})-\overline{E})^2 + \sum_{i=2}^{\infty} \frac{D^{(i)}\,(E(\overline{s})-\overline{E})^2}{i!} M_i \tag{7}$$

where M_i is the i th moment of P(s) about $\overline{s}$:

$$M_i = \int_0^\infty (s-\overline{s})^i\,P(s)\,ds \tag{8}$$

and $E^{(i)}$, $D^{(i)}$ denote, respectively, the i th derivatives of E(s) and of $(E(s)-\overline{E})^2$, evaluated at $\overline{s}$.

These series may be somewhat simplified if it is assumed that P(s) is approximately a normal distribution; that this is in fact true for this case, since P(s) is a log-normal distribution with a small variance-to-mean ratio, will be shown later. With this assumption, P(s) may be written:

$$P(s) \cong \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma_s} \cdot e^{-\frac{(s-\overline{s})^2}{2\sigma_s^2}} \tag{9}$$

It is then clear that all the odd moments M_i vanish. (In fact this will clearly be true for any distribution which is symmetric about its mean.) Further, for the normal distribution, the even moments may be represented by the general formula (see Reference 2):

$$M_i = \frac{\sigma_s^i}{2^{i/2}} \times \frac{i!}{(i/2)!} \qquad \text{for } i = 2, 4, 6, \ldots \tag{10}$$

Thus, for example, $M_2 = \sigma_s^2$, $M_4 = 3\,\sigma_s^4$, etc. The series (6) and (7) then become:

$$\bar{E} \cong E(\bar{s}) + \sum_{i=2,4,6\ldots}^{\infty} \frac{E^{(i)}(\bar{s})}{2^{1/2}\,(i/2)!}\,\sigma_s^i \tag{11}$$

$$\mathrm{Var}(E) \cong (E(\bar{s})-\bar{E})^2 + \sum_{i=2,4,6\ldots}^{\infty} \frac{D^{(i)}\,(E(\bar{s})-\bar{E})^2}{2^{1/2}\,(i/2)!}\,\sigma_s^i \tag{12}$$

For now these series will be left as they are, since no further progress can be made until a specific form is obtained for the function E(s).

Before proceeding to obtain formulas for E(s), it will be shown why the signal fluctuations caused by atmospheric turbulence result in an increase in the bit error probability.

Why Atmospheric Turbulence Increases the Bit Error Probability

In deriving Equation (6) the function E(s) was expanded in an infinite Taylor series about $\bar{s}$. If instead, E(s) is expressed as a finite Taylor formula about $\bar{s}$, with a second degree error term, it may be written:

$$\bar{E} = E(\bar{s}) + 1/2 \int_0^{\infty} E^{(2)}(\bar{s} + \theta(s)\cdot(s-\bar{s}))\,(s-\bar{s})^2\,P(s)\,ds \tag{13}$$

where $E^{(2)}$ is the second derivative of E(s), and θ (s) is between zero and one for each value of s, but is otherwise completely unknown. From this formula it is clear that $\bar{E}$ is greater than $E(\bar{s})$ if the second derivative of E(s) is positive for all s; in other words:

> A sufficient condition for atmospheric turbulence to cause an increase in the bit error rate is that the second derivative of the function E(s) be everywhere positive. (14)

Intuitively, this can be understood by visualizing the curve of E vs. s, which (as will be shown below) is decreasing and highly nonlinear, with a positive second derivative. As the signal fluctuates about its mean, the value of E given by E(s) oscillates along the curve, to either side of the point ($\bar{s}$, $E(\bar{s})$); but because of the fact that the curve is decreasing with s, but more slowly as s increases, the increase in E as the point moves to the left is greater in magnitude than the decrease in E as the point moves to the right, so that the average value of E is biased upward from its value at $\bar{s}$.

Derivation of Formula for the Bit Error Probability

Fig. 2 shows a schematic diagram of the detector.

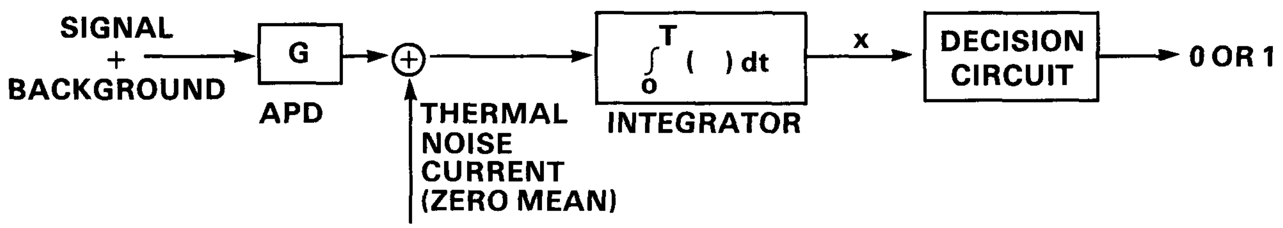

Figure 2. Schematic Diagram of Detector

A series of signal pulses with some distribution of intensities, which will be assumed to be a log-normal distribution arising from atmospheric turbulence, is incident on the avalanche photodiode (APD). (The intensity distribution is regarded as already having been partially smoothed by the collecting aperture; this effect will be described in the next section.) Since the signal has binary pulse position modulation (BPPM), each signal pulse occupies either the first or second slot of a two-slot word. Each pulse consists of the transmitted signal plus background noise; also, the slot not containing the signal pulse actually has some signal in it, due to the fact that the transmitter cannot be completely turned off during that time slot.

The incident photons in each pulse give rise to primary photoelectrons which are then randomly amplified by the avalanche photodiode; the resulting current, which includes bulk and surface leakage currents, is summed with the noise current from an external preamp and is integrated. The integrated outputs for the two slots are compared and a decision is made as to which of the two slots contains the signal pulse, by choosing the slot which gives rise to the greater integrated output.

The output of the integrator, denoted by x, may be written as follows, for a slot containing a signal pulse:

$$x = [(n_e + i_b T/q)G + i_s T/q] + (1/q) \int_0^T i_n(t)\,dt \tag{15}$$

The quantities in this formula are defined as follows:

n_e = number of primary photoelectrons for a slot containing a signal pulse plus background noise; n_e is a (dimensionless) discrete random variable with a Poisson distribution:

$$R(n_e) = \frac{e^{-\eta p}\,(\eta p)^{n_e}}{n_e!} \tag{16}$$

for $n_e = 0,1,2,3,\ldots$; the mean of the distribution is:

$$\bar{n}_e = \eta p \tag{17}$$

η = quantum efficiency of detector (dimensionless)

p = s + b = total number of photons per pulse (signal + background)

s = number of signal photons per pulse (dimensionless; log-normally distributed continuous random variable; fluctuates slowly, on a millisecond time scale, due to atmospheric turbulence)

b = number of background photons per pulse (dimensionless; assumed constant)

b = $r_b T$

r_b = rate of background photons (per second; assumed constant)

T = duration of pulse (seconds; T is the duration of a slot; each word has two slots in BPPM)

G = gain of avalanche photodiode (dimensionless; discrete random variable with a rather complicated distribution)

i_b = APD bulk leakage current (subject to gain; amperes; assumed constant)

i_s = APD surface leakage current (not subject to gain; amperes; assumed constant)

q = charge of electron (1.602×10^{-19} coulombs)

i_n = random noise current from preamp (amperes; continuous normally distributed random variable with zero mean)

The quantity x is a (dimensionless) random variable which measures the output of the integrator as an amount of charge in units of the charge of an electron. Because the random variables n_e and G are discrete, while the random variable i_n is continuous, the probability distribution of x is the convolution of discrete and continuous distributions. Adding to this the fact that the gain G of the avalanche photodiode has a rather complicated distribution, it would seem that further analysis would be difficult.

Fortunately, approximations have been developed for this problem which greatly simplify the analysis. It has been shown by Gagliardi and Prati (Ref. 3; see also References 4-6, on which Ref. 3 is based) that a random variable such as x, which represents the output, integrated over the pulse duration, of an avalanche photodiode with added thermal noise, may be replaced with quite good accuracy (under certain restrictions which are generally satisfied in practice) by a continuous random variable with a gaussian (normal) distribution. Gagliardi did not explicitly take account of the bulk and surface leakage currents of the photodiode in his analysis; Abshire (Ref. 7) has modified Gagliardi's formulas to take account of these effects. The result of these analyses allows replacement of the random variable x by the gaussian random variable y; the distribution of y may be written:

$$P_y(y) = \frac{1}{\sqrt{2\pi} \cdot \sigma_y} e^{-\frac{(y-\bar{y})^2}{2\sigma_y^2}} \tag{18}$$

where the mean $\bar{y}$ and the variance σ_y^2 of y are given by:

$$\bar{y} = [(s + r_b T)\eta + i_b T/q]\,\bar{G} + i_s T/q \tag{19}$$

$$\sigma_y^2 = \bar{G}^2 [\beta\bar{G} + (1-\beta)(2-1/\bar{G})]\,[(s + r_b T)\eta + i_b T/q] + i_s T/q + (2kT_R T)/(q^2 R) \tag{20}$$

The first two terms in the formula for the variance arise from the photodiode; the last term is the variance of the thermal noise. The quantities in these formulas are defined as follows:

- $\bar{G}$ = mean gain of the avalanche photodiode (a constant; dimensionless)
- β = effective ratio of the APD hole and electron ionization coefficients (dimensionless; approx. 0.02 or less for silicon devices)
- k = Boltzmann's constant (1.3806×10^{-23} joules/K)
- T_R = receiver noise temperature (K)
- R = detector load resistance (ohms)

Equations (18) - (20) give a good approximation for the probability distribution of the output of the integrator, for a time slot containing a signal pulse. The analysis is the same for a time slot which does not contain a signal pulse, except that the number of signal photons s is reduced by a factor M called the modulation extinction ratio; M is a measure of how well the transmitter succeeds in extinguishing the signal during a no-pulse (zero) time slot. For a no-pulse time slot, then, the output of the integrator may be represented by a random variable y_o, with a probability distribution given by Equations (18) - (20), except that the quantity s is replaced by s/M.

A random variable Y may now be defined which is the difference of the variables y and y_o:

$$Y = y - y_o \tag{21}$$

The probability distribution of the variable Y gives the probability that the output of the integrator for a time slot containing a signal pulse is greater than or less than the output for a time slot not containing a signal pulse by some amount. (In the absence of background radiation and detector noise, the variable Y would of course have zero probability of being negative.) The probability of a bit error is clearly just the probability that Y is negative; i.e., that a time slot with no signal pulse will produce an integrator output greater than a time slot containing a signal pulse:

$$\text{Probability of a bit error} = \int_{-\infty}^{0} P_Y(Y)\, dY \tag{22}$$

where $P_Y(Y)$ is the probability distribution of Y. (It might be noted that the bit error probability, calculated in this way, gives exactly the same result as would be obtained from the formula P(1/0) × P(0) + P(0/1) × P(1), where P(1) is the probability that a signal pulse is present in a given time slot, P(0) is the probability that no signal is present, P(1/0) is the probability that a signal is observed when none is actually present and P(0/1) is the probability that no signal is observed when one actually is present.) The variable Y, being the difference of two gaussian variables, is also a gaussian variable. Further, the two variables y and y_o are independent, because they are associated with different time slots. It follows that the mean and variance of Y are given by:

$$\bar{Y} = \bar{y} - \bar{y}_o \tag{23}$$

$$\sigma_Y^2 = \sigma_y^2 + \sigma_{y_o}^2 \tag{24}$$

Thus, the probability distribution of Y is given by the formula:

$$P_Y(Y) = \frac{1}{\sqrt{2\pi}\,(\sigma_y^2 + \sigma_{y_o}^2)^{1/2}}\; e^{\dfrac{-(Y - (\bar{y} - \bar{y}_o))^2}{2\,(\sigma_y^2 + \sigma_{y_o}^2)}} \tag{25}$$

If one puts into this formula the expressions for the means and variances and forms the bit error probability according to (22), a formula is obtained which can be put in the following form:

$$E = \frac{1}{\sqrt{2\pi}\,(c_2\, s + c_3)^{1/2}} \int_{-\infty}^{0} e^{\dfrac{-(Y - c_1\, s)^2}{2(c_2\, s + c_3)}}\, dY \tag{26}$$

where, as before, E is the bit error probability, s is the number of photons in a signal pulse, and the quantities c_1, c_2, c_3, are constants formed from the system parameters:

$$c_1 = (1 - 1/M)\,\eta\bar{G} \tag{27}$$

$$c_2 = \bar{G}^2\,[\beta\bar{G} + (1 - \beta)(2 - 1/\bar{G})]\,(1 + 1/M)\,\eta \tag{28}$$

$$c_3 = \bar{G}^2\,[\beta\bar{G} + (1 - \beta)(2 - 1/\bar{G})]\,[2r_b\, T\eta + 2i_b T/q] + 2i_s T/q + (4kT_R\, T)/(q^2 R) \tag{29}$$

This way of displaying the bit error probability E was chosen to show explicitly the dependence of E on the signal strength s, with the system parameters lumped into the three constants.

The formula given by (26) may be put into a simpler form by changing variables:

$$z = \frac{Y - c_1 s}{\sqrt{2}\ (c_2 s + c_3)^{1/2}}$$

and noting that

$$\frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} dz = 1/2;$$

one thus obtains:

$$E(s) = 1/2 \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\alpha(s)} e^{-z^2} dz \right] \tag{30}$$

where $$\alpha(s) = \frac{c_1 s}{\sqrt{2}\ (c_2 s + c_3)^{1/2}} \tag{31}$$

This formula exhibits the bit error probability in a very simple form, involving only the evaluation of an error function.

Some Properties of the Bit Error Probability

Monotonicity and First Two Derivatives. It is clear from Eqs. (30) and (31) that:

$\alpha(s)$ increases monotonically with increasing s (32)

E(s) decreases monotonically with increasing s (33)

As s varies from 0 to ∞, E decreases from 1/2 to 0 (34)

The first two derivatives of E(s) are easily computed (note that the definite integral must be differentiated with respect to its upper limit); the following formulas are obtained:

$$E^{(1)}(s) = -\frac{1}{\pi^{1/2}}\ e^{-\alpha^2(s)} \bullet \alpha^{(1)}(s) \tag{35}$$

$$E^{(2)}(s) = -\frac{1}{\pi^{1/2}}\ e^{-\alpha^2(s)} \bullet [\alpha^{(2)}(s) - 2\,\alpha(s) \bullet \alpha^{(1)^2}(s)] \tag{36}$$

where the superscripts in parentheses denote derivatives; the derivatives of $\alpha(s)$ are given by:

$$\alpha^{(1)}(s) = \frac{\sqrt{2}\; c_1\,(c_2\,s + 2\,c_3)}{4\,(c_2\,s + c_3)^{3/2}} \tag{37}$$

$$\alpha^{(2)}(s) = \frac{-\sqrt{2}\; c_1\,c_2\,(c_2\,s + 4\,c_3)}{8\,(c_2\,s + c_3)^{5/2}} \tag{38}$$

Note that $\beta << 1$, while M, G >> 1; it follows from Eqs. (27) - (29) that c_1, c_2 and c_3 are all positive. From (37) and (38) it then follows that $\alpha^{(1)}$ is positive and $\alpha^{(2)}$ is negative, and therefore from (35) and (36) that $E^{(1)}$ is negative and $E^{(2)}$ is positive:

$$E^{(1)}(s) < 0 \qquad \text{for all } s \geqslant 0 \tag{39}$$

$$E^{(2)}(s) > 0 \qquad \text{for all } s \geqslant 0 \tag{40}$$

The fact that the second derivative of E(s) is positive justifies the earlier assertion that atmospheric turbulence causes an increase in the bit error rate (see (14) above).

Dependence on System Parameters. If the expressions for the c's given by Eqs. (27)-(29) are put into the formula for α (s) given by (31), the resulting expression may be cast into the following form:

$$\alpha = \frac{\frac{1}{\sqrt{2}}\left(1 - \frac{1}{M}\right)}{\left\{\left[\beta\left(\bar{G} + \frac{1}{\bar{G}} - 2\right) + \left(2 - \frac{1}{\bar{G}}\right)\right]\left[\frac{1 + 1/M}{\eta s} + \frac{2 r_b\,T}{\eta s^2} + \frac{2 i_b\,T}{q \eta^2\,s^2}\right] + \frac{2 i_s\,T}{q\,\bar{G}^2\,\eta^2\,s^2} + \frac{4\,k\,T_R\,T}{q^2\,R\,\bar{G}^2\,\eta^2\,s^2}\right\}^{1/2}} \tag{41}$$

From this formula the behavior of α with respect to the various parameters is apparent. Aside from the clear monotonic increase of α with s, one sees that α increases monotonically with M, η and R, and decreases monotonically with β, r_b, T, i_b, i_s and T_R. If these observations are combined with the fact that E decreases monotonically with α, as is evident from Eq. (30), it follows that:

E increases monotonically with β, r_b, T, i_b, i_s and T_R; (42)

E decreases monotonically with M, η and R. (43)

With respect to the mean gain $\bar{G}$, the only parameter not yet considered, the behavior of E is more complex. To investigate the dependence of α, and thus E, on $\bar{G}$, Eq. (41) may be written in the following form:

$$\alpha = \frac{a}{\left\{b\,\bar{G} - \frac{c}{\bar{G}} + \frac{d}{\bar{G}^2}\right\}^{1/2}} \tag{44}$$

where: $a = (1/\sqrt{2})\,(1 - 1/M) > 0$, since $M >> 1$;

c = constant $\times (1-\beta) > 0$, since $\beta << 1$;

$b, d > 0$.

Also, it is clear from Equation (41) that $\alpha > 0$ for all $\bar{G}$ and the expression in curly brackets > 0 for all $\bar{G}$.

Equation (44) may be written in the alternate form:

$$\alpha = \frac{a\bar{G}}{\left\{ b\bar{G}^3 - c\bar{G} + d \right\}^{1/2}} \tag{45}$$

From (45) it follows that: For $\bar{G} = 0$, $\alpha = 0$ (46)

and from (44) that: As $\bar{G} \rightarrow \infty$, $\alpha \rightarrow 0$ (47)

The first derivative of α with respect to $\bar{G}$ may be written in the two alternate forms:

$$\frac{d\alpha}{d\bar{G}} = \frac{-\frac{a}{2}\left[b + \frac{c}{\bar{G}^2} - \frac{2d}{\bar{G}^3} \right]}{\left\{ b\bar{G} - \frac{c}{\bar{G}} + \frac{d}{\bar{G}^2} \right\}^{3/2}} \tag{48}$$

or

$$= \frac{-\frac{a}{2}\left[b\bar{G}^3 + c\bar{G} - 2d \right]}{\left\{ b\bar{G}^3 - c\bar{G} + d \right\}^{3/2}} \tag{49}$$

From (48) it is clear that:

As $\bar{G} \rightarrow \infty$, $d\alpha/d\bar{G} \rightarrow 0$ and $d\,\alpha/d\bar{G} < 0$ (50)

and from (49) it is clear that:

As $\bar{G} \rightarrow 0$, $d\alpha/d\bar{G} \rightarrow a/d^{1/2} > 0$ (51)

From (46), (47), (50) and (51), and the fact that α is always positive, it follows that the curve of α vs. $\bar{G}$ starts at zero at $\bar{G}=0$, with a positive slope, remains positive, and decreases to zero as $\bar{G}\rightarrow\infty$; but how many maxima does the curve have? This question may be answered by setting the derivative equal to zero and noting from (49) that this is equivalent to solving the equation:

$$f(G) = b\bar{G}^3 + c\bar{G} - 2d = 0 \tag{52}$$

This is a cubic equation, so it must have either one or three real roots.

Differentiation of f with respect to $\bar{G}$ yields:

$$df/d\bar{G} = 3b\bar{G}^2 + c \tag{53}$$

Note that this derivative is everywhere positive, and also note from (52) that $f(\bar{G}) < 0$ for $\bar{G} = 0$ and $f(\bar{G}) > 0$ for $\bar{G} \to \infty$; it follows that the cubic equation has only one real root. Thus α $(\bar{G})$ has only one maximum. From the fact that E decreases monotonically with α it then follows, from this last result and from (30), (46) and (47), that:

$$E \rightarrow 1/2 \text{ as } \bar{G} \rightarrow 0 \text{ or } \infty$$

and (54)

$$E(\bar{G}) \text{ has only one minimum.}$$

In other words, as $\bar{G}$ increases from 0 to ∞, E decreases from a value of 1/2, goes through a single minimum and increases back toward 1/2 (if all the other system parameters and the signal strength are held constant).

Statistics of the Signal Fluctuation Due to Turbulence, Including Finite Aperture Averaging

Approximately Log-Normal Distribution of the Aperture-Averaged Signal. In accordance with general practice it may be assumed that, at any point of the receiving aperture, the fluctuating signal has a log-normal distribution. This may be intuitively understood as resulting from the large number of turbulence-induced index of refraction fluctuations over the atmospheric path and the exponential form of the absorption over the path:

$$n_r = n_o \, e^{-\int_0^L a(l)\,dl} \tag{55}$$

where n_o is the number of signal photons per unit area at the top of the atmosphere (the laser beam may be assumed to be a plane wave, because of the great distance of the transmitter in geosynchronous orbit), n_r is the number of signal photons per unit area at some point on the receiver aperture (after absorption and scattering over the atmospheric path) and a is the total extinction coefficient at the point l along the path, where l varies from 0 at the receiver to L at the top of the atmosphere. Because the extinction coefficient a at each point of the path varies randomly due to turbulence, and these random variables are independent except for correlations over a short distance, the integral may be regarded as the sum of a large number of essentially independent random variables and thus as a random variable itself with a distribution which is approximately normal. The variable n_r is thus approximately a log-normally distributed random variable.

The effect of the finite receiving aperture is to sum a number of these (possibly correlated) log-normal variables, one for each small area element of the aperture. It is not clear, without further analysis, what the distribution of the sum will be. It turns out that the sum (i.e., the total number of photons incident on the aperture, which was called s in the preceding analysis, assuming no losses in the receiving optics) is generally approximately log-normally distributed (see Refs. 8 and 9). This is surprising, because the central limit theorem states that the distribution of the sum of independent random variables approaches a normal distribution as the number of variables becomes large, whatever the individual distributions may be. Yet experimental measurements have shown (see Ref. 9) that the distribution is log-normal, regardless of the size of the collecting aperture. The reason for this phenomenon is discussed in Ref. 9, where it is shown that, because of the skewness and large tail of the log-normal distribution, the convergence is slow, so that the log-normal nature of the distribution persists longer than would be expected, although of course the distribution will become normal for a large enough number of variables.

The distribution of the signal strength s (incident on the detector after aperture averaging) may therefore be written in the form:

$$P(s) = \frac{e^{-\frac{(\frac{1}{2}\ln(s/\bar{s}) + \sigma_l^2)^2}{2\sigma_l^2}}}{2\sqrt{2\pi}\, s\, \sigma_l} \tag{56}$$

where the quantity σ_l^2, called the "log-amplitude variance," is given by:

$$\sigma_l^2 = 1/4 \cdot \ln\left(1 + \sigma_s^{\,2}/\bar{s}^2\right) \tag{57}$$

and $\bar{s}$, σ_s^2 are the mean and variance, respectively, of the log-normal distribution of s. In the next subsection a formula will be given for the variance σ_s^2.

<u>Variance of the Aperture-Averaged Signal</u>. In Ref. 8 a formula is derived (Eq. 6-50, p. 6-24) for the variance of the signal intensity after averaging by a finite receiving aperture, assuming plane wave propagation. This formula may be adapted to the problem considered in this report by simple changes in units and in some of the quantities used; the result is the following formula:

$$\sigma_s^{\,2} \cong 13.1 \,\frac{\bar{s}^2 \sec^3\theta}{A^{7/6}} \int_{h_o}^{h_{max}} (h - h_o)^2 \, C_n^2(h)\, dh \tag{58}$$

where s is the number of photons incident on the detector per signal pulse (considered, as above, as a continuous random variable), $\bar{s}$ is the mean of s, θ is the zenith angle, A is the area of the receiving aperture in square meters, h is the altitude in meters, h_o is the altitude of the receiver in meters, h_{max} is the altitude (in meters) above which the structure constant C_n^2 is zero and C_n^2 is the index of refraction structure constant in meters $^{-2/3}$, assuming the Kolmogorov theory of turbulence.

The ratio of the standard deviation to the mean is particularly important for the current problem. From Eq. 58, and recalling that the standard deviation is the square root of the variance, this ratio is given by:

$$\frac{\sigma_s}{\bar{s}} \cong 3.62 \,\frac{\sec^{3/2}\theta}{A^{7/12}} \left[\int_{h_o}^{h_{max}} (h - h_o)^2 \, C_n^2(h)\, dh\right]^{1/2} \tag{59}$$

It is interesting to note that these expressions do not involve the wavelength. This is the case only for large apertures; for wavelengths of approximately one micrometer, Eq. (58) will be a good approximation provided the aperture diameter is greater than about ten inches (see Ref. 8, page 6-24). The proposed receiver at the Goddard Space Flight Center has a diameter of 48 inches; thus the formula should be quite accurate, since the proposed wavelength is approximately 0.8 micrometers.

Smallness of the Ratio of the Standard Deviation to the Mean. To evaluate the integral in Eq. (58) one needs to know the dependence of the index of refraction structure constant on altitude. The variation of the structure constant with altitude is, in general, extremely complex and it also varies in time. However, a very simple approximate model for the structure constant is given in Ref. 8 (Eq. 6-20, page 6-14) which is time invariant, but (according to Ref. 8) accurate enough to represent a median of the real world. Because of its simplicity, this model will be used in some of the following investigations, but there is always the option of using more detailed models. The model is defined as follows:

$$C_n^2(h) = \begin{cases} b/h & \text{below 20,000 meters above sea level} \\ 0 & \text{above 20,000 meters above sea level} \end{cases} \tag{60}$$

where $b = 1.5 \times 10^{-13}$; h is the altitude in meters above local ground (assumed to be less than 2500 meters above sea level) and $C_n^2(h)$ has the same units as given above. (Note that, although h is the altitude above local ground, the cutoff of 20,000 meters relates to elevation above sea level.)

The model of the structure constant may be used to obtain an approximate value for the ratio of the standard deviation to the mean, given by Eq. (59). For the receiver at the Goddard Space Flight Center one may use the approximate values: $\theta \cong 51$ degrees; $h_o \cong 10$ meters; $h_{max} \cong 20{,}000$ meters; A (for the 48″ telescope) $\cong 1.17$ square meters. The integral is easily evaluated to obtain:

$$\frac{\sigma_s}{\bar{s}} \cong 0.03 \tag{61}$$

In other words, the standard deviation of the signal strength fluctuation due to turbulence is only about three percent of the mean signal strength, for this configuration. The use of a more sophisticated model of the structure constant is not likely to qualitatively change this result, so it may be concluded that the fluctuation of the signal due to turbulence is quite small compared to the signal itself.

Normal Approximation to the Log-Normal Distribution. The smallness of the ratio of the standard deviation to the mean of the fluctuating signal leads to the result that the log-normal distribution given by Eq. (56) may be approximated by a normal distribution, with a degree of accuracy which depends on the ratio of the standard deviation to the mean.

To prove this result, one may start by making use of the series expansions:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \qquad \text{for } x^2 < 1; \tag{62}$$

$$\ln x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right] \qquad \text{for } x > 0. \tag{63}$$

Application of these series to the logarithms appearing in Equations (56) and (57) yields:

$$\ln(1+\gamma^2) = \gamma^2 - \frac{1}{2}\gamma^4 + \frac{1}{3}\gamma^6 - \frac{1}{4}\gamma^8 + \cdots \tag{64}$$

where $$\gamma = \sigma_s/\bar{s} \tag{65}$$

$$\ln(s/\bar{s}) = 2\left[\varepsilon + \frac{1}{3}\varepsilon^3 + \frac{1}{5}\varepsilon^5 + \cdots\right] \tag{66}$$

where $$\varepsilon = \frac{(s-\bar{s})/\bar{s}}{2+(s-\bar{s})/\bar{s}}$$

If one writes $(s - \bar{s}) = n\sigma_s$, where n (not an integer) measures the distance of s from the mean $\bar{s}$ in units of the standard deviation, then:

$$\varepsilon = \frac{n\gamma}{2+n\gamma} \tag{67}$$

In Eq. (56) there is an s that appears in the denominator; if this s is written in the form $s = \bar{s} + n\gamma\bar{s}$ and use is made of the series expansions above, the log-normal distribution given by Eq. (56) may be written in the form:

$$P(s) = \frac{e^{\dfrac{-\left\{\left[\left(\frac{n\gamma}{2+n\gamma}\right) + \frac{1}{3}\left(\frac{n\gamma}{2+n\gamma}\right)^3 + \cdots\right] + \frac{1}{4}\left[\gamma^2 - \frac{1}{2}\gamma^4 + \cdots\right]\right\}^2}{\frac{1}{2}\left[\gamma^2 - \frac{1}{2}\gamma^4 + \cdots\right]}}}{\sqrt{2\pi}\,(\bar{s} + n\gamma\bar{s})\left[\gamma^2 - \frac{1}{2}\gamma^4 + \cdots\right]^{1/2}} \tag{68}$$

If γ approaches zero in Eq. (68), a limiting form is obtained by dropping all terms after the first in each series expansion, dropping the entire second series in the numerator of the exponential (because it starts with γ^2) and dropping the term involving γ in the quantity $(\bar{s} + n\gamma\bar{s})$ in the denominator; the resulting limiting form is:

$$\underset{\gamma \to 0}{P(s)} \rightarrow \frac{1}{\sqrt{2\pi}\,\sigma_s}\, e^{\dfrac{-(s-\bar{s})^2}{2\sigma_s^2}} \tag{69}$$

which is a normal distribution. It is thus seen that, as the standard deviation becomes small compared to the mean, the log-normal distribution approaches the normal distribution with the same mean and variance. However, due to the presence of the quantity n in Eq. (68), it is clear that the convergence is not uniform. Because of the smallness of the ratio $\sigma_s/\bar{s}$ in this case, as shown in Eq. (61), it may be concluded from the form given by Eq. (68) that the log-normal distribution of s may be quite accurately approximated by the normal distribution (69).

Approximate Formulas for the Bit Error Probability Statistics, Obtained by Truncating the Series of Moments

The fact that the probability distribution of the aperture averaged signal is nearly normal, as shown above, allows the use of Eqs. (11) and (12), which were derived on the assumption that the distribution is normal, to develop approximate formulas for the mean and variance of the bit error probability E.

Only the first term of each summation (in Eqs. (11) and (12)) will be used; it will be shown below that the resulting formulas are in fact good approximations. Taking only the first term of the summation in Eq. (11) yields:

$$\bar{E} \cong E(\bar{s}) + \frac{1}{2} E^{(2)}(\bar{s}) \cdot \sigma_s^2 \tag{70}$$

If one now takes only the first term of the summation in Eq. (12), substitutes Eq. (70) into the resulting expression and simplifies, the result is:

$$\mathrm{Var}\,(E) \cong E^{(1)^2}(\bar{s}) \cdot \sigma_s^2 - \frac{1}{4} E^{(2)^2}(\bar{s}) \cdot \sigma_s^4$$

It can be shown that the second term of this expression is much smaller than the first; if it is assumed that it is (the numerical calculations described in the following section will justify this), the approximate formula for the variance may be written:

$$\mathrm{Var}\,(E) \cong E^{(1)^2}(\bar{s}) \cdot \sigma_s^2 \tag{71}$$

The derivatives of E which occur in Eqs. (70) and (71) are given by Eqs. (35) through (38), together with Eqs. (27) through (29) and Eq. (31).

In the next section, a computer program will be described which evaluates the accuracy of the formulas given by (70) and (71).

COMPUTER PROGRAM TO NUMERICALLY EVALUATE THE ACCURACY OF THE APPROXIMATE FORMULAS

Description of Program

The computer program, which is written in FORTRAN, evaluates the accuracy of the approximate formulas for the mean and variance of the bit error probability E by comparing the values computed from the approximate formulas against the values computed from the exact expressions given by Eqs. (3) and (4). Listings of the program are given in Appendix A.

The functions E(s) and P(s) occurring in Eqs. (3) and (4) are given by Eqs. (30) (along with Eqs. (31) and (27) through (29)) and (56) (along with (57)), respectively. Thus, the exact expressions use the log-normal probability distribution, whereas the approximate expressions given by Eqs. (70) and (71) use the normal approximation to the log-normal distribution. It should be noted that, although Eqs. (3) and (4) involve no approximations, the formula for E(s) given by Eq. (30) is only approximate, because it assumes the approximate gaussian representation of the photodiode statistics given by Eq. (18); thus even the "exact" values as calculated from (3) and (4) are not strictly exact. This should cause no problem, however, because the approximation given by (18) is quite accurate.

The computer program is designed to be used interactively, on a microcomputer. Upon starting the program, the user is informed by the program (on the terminal screen) of the current values of the system parameters (used in the last run). The user then has the option of changing any or all of these values, along with the value of the signal strength. The program then computes the exact and approximate values and prints them out, along with the values of the system parameters that were used in the calculation. A sample of the printout is given below in Fig. 3. In this calculation the signal strength was taken as 100 photons per bit, the structure constant profile given by Eq. (60) was used and the system parameters used are fairly representative values.

From the results shown in Fig. 3, it is seen that the bit error probability for a non-turbulent atmosphere is about 0.0026, that this value increases by about 1.8 percent when turbulence is considered, and that this increase, as calculated from the approximate formula, is only about 0.2 percent below the increase as calculated from the exact formula. For the variance of the bit error probability, the value calculated from the approximate formula is about 3.2 percent below the value calculated from the exact formula. The bottom three lines in Fig. 3 describe the one standard deviation band about the mean bit error probability; note that the bottom of the band (i.e., one s.d. below the mean value of E) lies below the value E(s). (It should be noted that, in the computer program, the signal strength s is denoted by the variable N, its mean by NBAR, the probability of a bit error by PBE, etc; these variables are clearly defined in the program listings.)

EVALUATION OF APPROXIMATE FORMULAS FOR THE MEAN AND VARIANCE OF THE BIT ERROR PROBABILITY

The signal strength is 100.0 photons/bit

The current values of the system parameters are:

1.	Modulation extinction ratio	=	100.0	
2.	Quantum efficiency	=	0.7	
3.	APD mean gain	=	250.0	
4.	APD ionization coeff. ratio	=	0.02	
5.	Background radiation rate	=	0.1E+10	photons/sec
6.	Data rate	=	0.5E+09	bits/sec
7.	APD bulk leakage current	=	0.1E−09	amperes
8.	APD surface leakage current	=	0.1E−07	amperes
9.	Receiver noise temperature	=	600.0	degrees K
10.	Detector load resistance	=	200.0	ohms

The current structure constant profile is 1/h.

The current values of the parameters associated with the signal strength fluctuation are:

1.	Receiver aperture area	=	1.167	sq. meters
2.	Zenith angle	=	51.0	degrees
3.	Height of receiver (above local ground)	=	30.0	feet
4.	Altitude of receiver site (above sea level)	=	143.0	feet

Formula for mean

****

Bit error probability at nbar	=	0.00257	
"Exact" increase in BEP	=	1.77880	percent
Approx. increase in BEP	=	1.77471	percent
Error of approx. formula for increase	=	−0.23	percent

Formula for variance

****

"Exact" variance of BEP	=	0.24115E-06	
Error of approx. formula for variance	=	−3.24808	percent

Band of one standard deviation about mean BEP

****

Mean BEP + 1 s.d.	=	0.0031091
Mean BEP	=	0.0026180
Mean BEP − 1 s.d.	=	0.0021269

Fig. 3. Sample Output from Computer Program

The computer program uses double precision to calculate the integrals; this is necessary to avoid loss of significance due to the sensitivity of some of the calculations. The integrals are computed by subroutine DSIMP, which is a double precision subroutine using Simpson's rule. The user specifies the accuracy required; the routine then successively doubles the number of subintervals until the computed value converges to the required accuracy. In the calculations an accuracy of one part in a million was generally required, with one exception: in computing the table of values of the error function for use by subroutine ERF (which computes the error function using either interpolation in the table or a convergent or asymptotic series, depending on the value of the argument), an accuracy of one part in ten to the ninth power was required.

Additional complications in the computation of the integrals in Eqs. (3) and (4) are the infinite limits and the narrow structure of the integrand, which latter is due to the small value of the ratio of the standard deviation to the mean of E as shown in a previous section of this report. To avoid skipping over the narrow peak in the integrand, it was found necessary to require DSIMP to start with a large number of subintervals; a value of one million subintervals was used, just to be safe. To handle the infinite limits, the integrals were computed with successively larger deviations from the mean of the P(s) distribution until convergence was achieved; step increases of five standard deviations were used. As might be expected from the preceding description, the calculations were quite time consuming; this was not a problem, however, because the calculations only had to be done once, to establish the accuracy of the approximate formulas.

From the results shown in Fig. 3 and from other calculations not shown here, it became clear that the approximate formulas are sufficiently accurate for system feasibility studies. In the next subsection the results of these calculations are shown as plots of the accuracy of the approximations versus signal strength. A plot is also included which shows the increase in the bit error probability due to turbulence, as a function of the signal strength (as calculated from the exact formula, using the same system parameter values given in Fig. 3).

Computed Results

Accuracy of the Approximate Formulas. The computer program described above was used to compute the accuracy of the approximate formulas for the mean and variance of the bit error probability E, for values of the system parameters shown in Fig. 3, which are realistic values. The results are shown in Figs. 4 and 5.

Fig. 4 shows the percent error of the increase in the bit error probability due to turbulence as computed from the approximate formula (70). Note that the quantity plotted is not $\overline{E}$, but rather $\overline{E} - E(\overline{s})$, since it is the increase that we are interested in, not the value of $\overline{E}$ itself. From Fig. 4 it is seen that the percent error is only a few tenths of one percent for a signal strength of less than 100 photons per bit; for greater signal strengths the error increases, reaching about 5.4 percent for a signal strength of 300 photons per bit. Beyond this point, although the error continues to increase, the bit error probability itself becomes so small (less than 10^{-7}) that the error is not important.

Fig. 5 shows the percent error of the variance of the bit error probability, as computed from the approximate formula (71). The percent error is less than three for a signal strength of under 100 photons per bit, increasing to about 32 percent at a signal strength of 300 photons per bit. Again, although the error continues to increase, the bit error probability itself becomes so small as to make the error irrelevant.

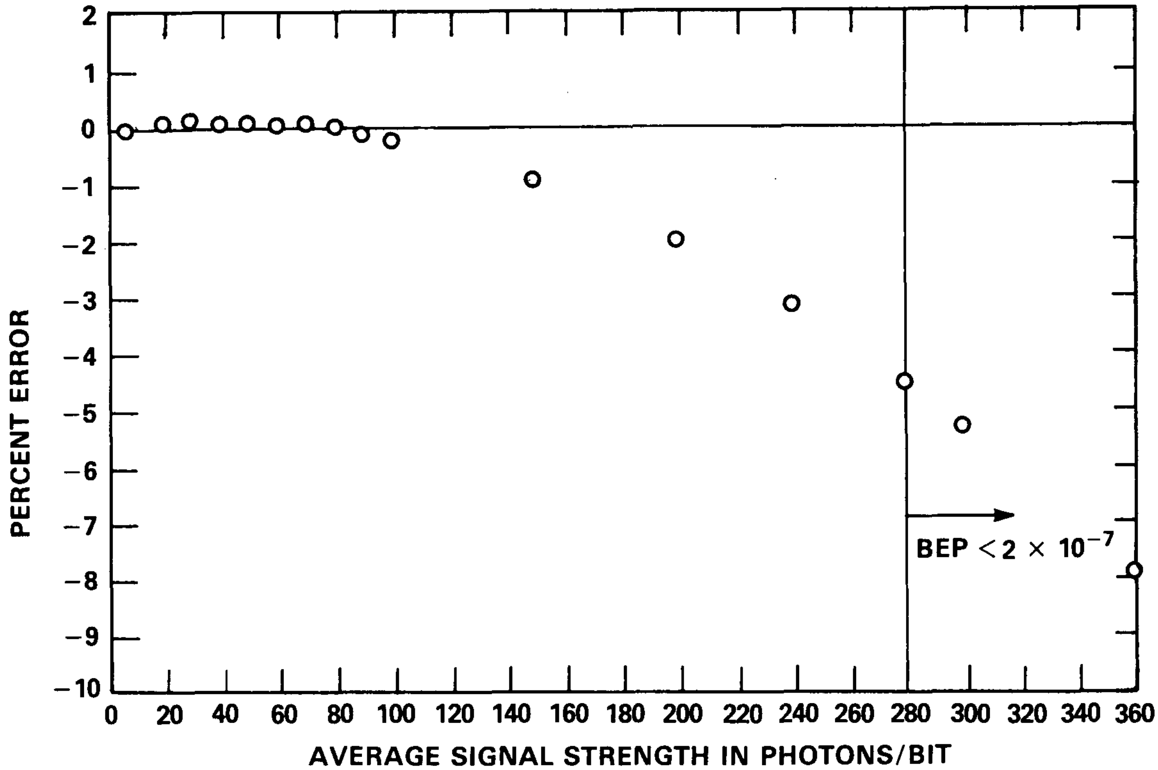

Figure 4. Accuracy of Approximate Formula for BEP Increase

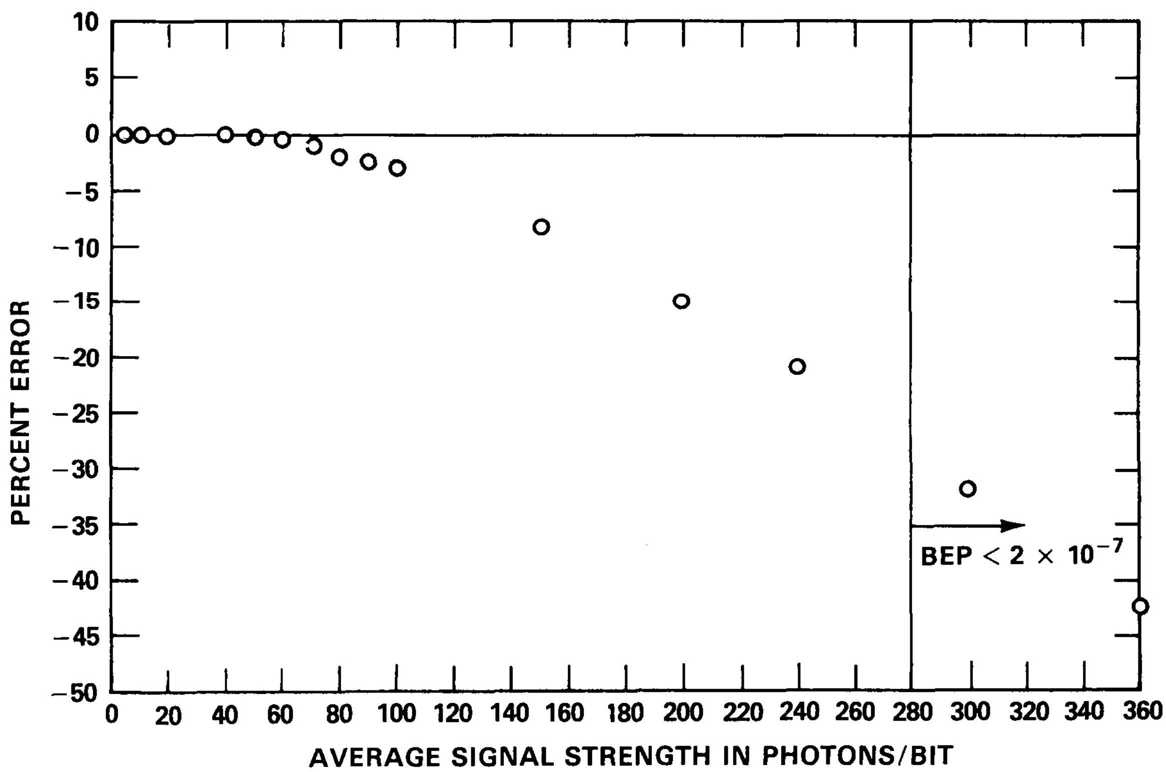

Figure 5. Accuracy of Approximate Formula for BEP Variance

Increase in Bit Error Probability as a Function of Signal Strength. Table 1 below shows the increase in the bit error probability due to turbulence (i.e., $\bar{E} - E(\bar{s})$), as computed from the exact formula (3), using Eq. (30) to compute E(s) and Eq. (56) for P(s); the bit error probability $E(\bar{s})$ is also shown. The values of the system parameters used are the same as were used above (see Fig. 3).

Table 1. Bit Error Probability (BEP) and Increase in BEP due to Turbulence as Functions of Signal Strength in Photons per Bit, for Representative Values of the System Parameters

Mean Signal Strength $\bar{s}$ in photons/bit	Bit Error Prob. $E(\bar{s})$	Percent Increase in Bit Error Prob.
1.	0.4754	0.00014
2.	0.4519	0.00058
5.	0.3873	0.0040
10.	0.2980	0.017
20.	0.1751	0.074
30.	0.1026	0.17
40.	0.6016 E−1	0.30
50.	0.3536 E−1	0.46
60.	0.2084 E−1	0.66
70.	0.1231 E−1	0.89
80.	0.7290 E−2	1.2
90.	0.4326 E−2	1.5
100.	0.2572 E−2	1.8
150.	0.1956 E−3	3.9
200.	0.1530 E−4	6.9
240.	0.2020 E−5	10.0
280.	0.2691 E−6	13.6
300.	0.9846 E−7	15.7
360.	0.4869 E−8	23.1
400.	0.6599 E−9	29.0

Increase in Signal Required to Compensate for the Increased Bit Error Probability. The information in Table 1 is plotted in Fig. 6 as the curves labeled "quiet atmosphere" and "turbulent atmosphere (average)." The other four curves in Fig. 6 are obtained as follows: division of Eq. (70) by $E(\bar{s})$ yields:

$$\frac{\bar{E} - E(\bar{s})}{E(\bar{s})} \cong \left(\frac{1}{2} \frac{E^{(2)}(\bar{s},p)}{E\ (\bar{s},p)} \right) \sigma_s^2$$

This equation shows that the percent increase in E is approximately proportional to the signal variance, for a given value of $\bar{s}$ (and given values of the system parameters). The last column in Table 1 shows the percent increase in E for the signal variance computed from Eq. (58), using the structure constant for an atmosphere of average turbulence given by Eq. (60). Because the percent increase in E is approximately proportional to the signal variance, as shown by the above equation, the values in the last column of Table 1 may be multiplied by any constant to obtain the increase in E for various degrees of turbulence, as measured by the signal variance. The top four curves in Fig. 6 were constructed by using degrees of turbulence of two, three, four and five times the average amount.

The increase in the signal needed to maintain the bit error probability at a given level when turbulence is taken into account may be read off from the curves in Fig. 6. For example, to maintain E at a level of one error in a million ($E = 10^{-6}$), the signal must be increased from about 254 photons per bit to about 257 photons per bit, or an increase of about 0.05 db,for an average degree of turbulence. For five times the average degree of turbulence, the corresponding signal increase is about 0.17 db. To maintain E at a level of one error in ten million ($E = 10^{-7}$), the corresponding numbers are about 0.04 db and 0.18 db, respectively. Thus it is seen that the signal increase required to compensate for the effect of turbulence is not large.

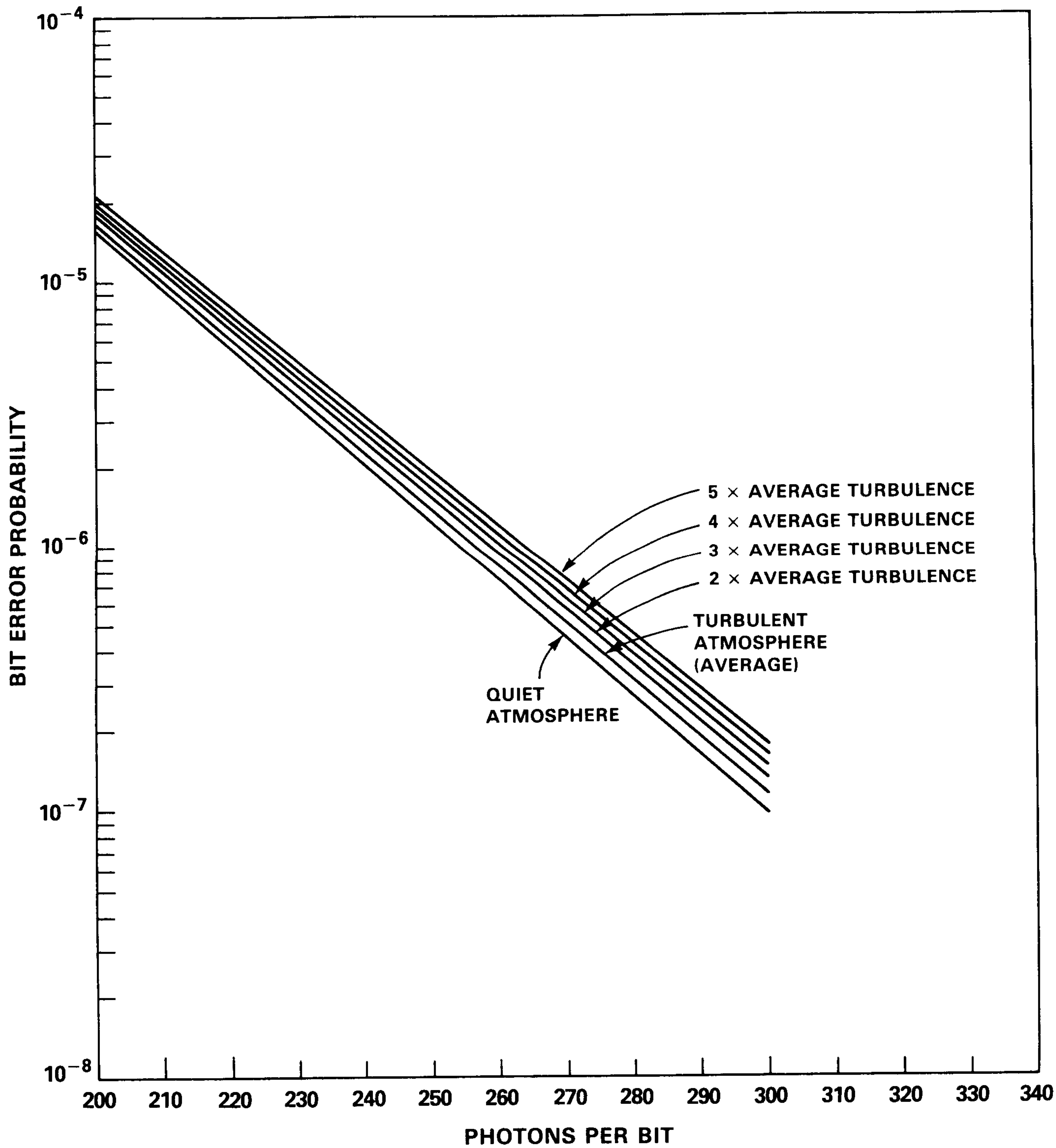

Figure 6. Increase in Bit Error Probability for Various Degrees of Turbulence (For Representative Values of System Parameters)

SUMMARY AND CONCLUSIONS

This report investigates the effect of atmospheric turbulence on the performance of a space to ground laser communications link operating in the near infrared, using binary pulse position modulation and an avalanche photodiode detector.

Specifically, the effect of turbulence on the bit error rate is analyzed; general formulas are presented for the mean and variance of the bit error rate, which is a random variable, being a function of the signal strength which fluctuates randomly due to turbulence. The general formulas are not of much practical use, however, because they involve integrals with infinite limits and integrands which contain very narrow peaks, so that care must be exercised in numerically integrating them.

Because of the limited usefulness of the general formulas, approximate formulas were developed which are easily computed and sufficiently accurate for system feasibility studies. In the process of developing these approximate formulas, a very simple formula was developed for the bit error rate as a function of the signal strength, expressed in photons per bit. This formula expresses the bit error rate in terms of a single error function, in which the upper limit is a function of the signal strength.

To assess the accuracy of the approximate formulas, a computer program (written in FORTRAN) was developed, which compares the mean and variance of the bit error rate as computed from the approximate formulas against the same quantities as computed from the general formulas by numerical integration. The results obtained with this program showed that the approximate formulas are very accurate for signal strengths of about one hundred photons per bit or less, and reasonably accurate (within about five percent for the mean and about thirty percent for the variance) for signal strengths of up to three hundred photons per bit. For greater signal strengths the accuracy deteriorates, but the bit error rate itself becomes so small (less than one in ten million) that the decreased accuracy, and in fact the increase in the bit error rate, becomes unimportant.

In the course of the calculations to assess the accuracy of the approximate formulas, the bir error rate and its increase due to turbulence were also calculated as functions of the signal strength. These results show that the bit error rate is an extremely rapidly decreasing function of signal strength, varying from near one half for signals of only a few photons per bit to less than one error in a billion for signals of four hundred photons per bit (for the representative system parameter values given in Fig. 3). Over the same range of signal strength, the increase in the bit error rate due to turbulence varies from less than 0.004 percent to about thirty percent.

Translation of these results into the increase in signal strength needed to compensate for the effect of turbulence shows that the signal must be increased by about 0.05 db to maintain a bit error rate of one error in a million, for an average degree of turbulence, and by about 0.17 db for five times the average degree of turbulence (as measured by the signal variance). For an error rate of one error in ten million the figures are practically the same. (These numbers apply for representative values of the system parameters.)

From the foregoing results it may be concluded that, for realistic values of the system parameters and for an average degree of turbulence, the increase in the bit error rate due to turbulence is rather mild, not exceeding thirty percent for signal strengths small enough to warrant even considering the effects of turbulence. The corresponding increase in signal strength required to maintain an error rate of one in a million to one in ten million is not more than one or two tenths of a db, even for as much as five times the average degree of turbulence.

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APPENDIX A

COMPUTER PROGRAM LISTINGS

```
      PROGRAM TAPROX
C
C     This routine tests the approximate formulas for the mean and variance of the bit error probability, for
C     the avalanche photodiode, by checking computed values from the approximate formulas against values
C     computed by numerically integrating the exact formulas.
C
C     The routine may be run any number of times, by responding ''yes'' to a prompt. The user may re-
C     specify the system parameters and/or the signal strength before each run.
C*******************************************************************************************
C
      BYTE            REPLY, NARRAY(10,6), BLANK
      INTEGER*2       ITMLST(10)
      REAL*4          M,MU,IB,IS, PARAM(10), NBAR, LOWLIM
      REAL*8          PBARIG, Q, QPREV, PBE, ALPHA, DPBEBR
      EXTERNAL        PBARIG, PVARIG
C
      COMMON             C1,C2,C3, NBAR, SIGSQN, DPBEBR
      COMMON/PRFIL/   HO, IPRTSW
C
      DATA            NARRAY/'M','M','G','B','R','B','I','I','T','R',
     *                       ' ','U',' ','E','B','I','B','S','R',' ',
     *                       ' ',' ',' ','T',' ','T',' ',' ','E',' ',
     *                       ' ',' ',' ','A',' ','R',' ',' ','C',' ',
     *                       ' ',' ',' ',' ',' ','A',' ',' ',' ',' ',
     *                       ' ',' ',' ',' ',' ','T',' ',' ',' ',' '/
C*******************************************************************************************
C
C     Mathematical and physical constants
C     ******************************
C
C     Pi
C     --
      PI = 3.1415926535
C
C     Boltzmann's constant
C     --------------------------
      BK = 1.3806E-23                          ! joules/degree K
C
C     Electron charge
C     -------------------
      E = 1.602E-19                            ! coulombs
C
C
C     Specify the values of the system parameters
C     ************************************
C
C     Modulation extinction ratio (dimensionless)
C     ---------------------------------
      M = 100.
      PARAM (1) = M
```

```
C
C          Quantum efficiency (dimensionless)
C          -----------------------
           MU = 0.7
           PARAM(2) = MU
C
C          APD MEAN gain (dimensionless)
C          ---------------------
           G = 250.
           PARAM(3) = G
C
C          APD ionization coefficient (dimensionless)
C          --------------------------------
           BETA = 0.02
           PARAM(4) = BETA
C
C          Incident background rate (per second)
C          ----------------------------
           RB = 1.E9
           PARAM(5) = RB
C
C          Data rate (bits per second; one bit per two-slot word)
C          -----------
           BITRAT = 500.E6
           TAU    = 0.5 * (1./BITRAT)
           PARAM(6) = BITRAT
C
C          APD bulk leakage current (amperes)
C          -------------------------------
           IB = 0.1E-9
           PARAM(7) = IB
C
C          APD surface leakage current (amperes)
C          ----------------------------------
           IS = 10.E-9
           PARAM(8) = IS
C
C          Receiver noise temperature degrees K)
C          --------------------------------
           TREC = 600.
           PARAM(9) = TREC
C
C          Detector load resistance (ohms)
C          ----------------------------
           R = 200.
           PARAM(10) = R
C
C
C          Prompt user to change any of these values, if desired
C          ********************************************
C
```

```
 100     TYPE 110, M,MU,G,BETA,RB,BITRAT,IB,IS,TREC,R
 110     FORMAT(///T2,'The current values of the system parameters are:'//
   *       T10,'1. Modulation extinction ratio     =  ',t43,f8.1/
   *       T10,'2. Quantum efficiency              =  ',t47,f7.4/
   *       T10,'3. APD mean gain                   =  ',t43,f8.1/
   *       T10,'4. APD ionization coeff. ratio     =  ',t47,f7.4/
   *       T10,'5. Background radiation rate       =  ',t46,e12.4,t61,'photons/sec'/
   *       T10,'6. Data rate                       =  ',t46,e12.4,t61,'bits/sec'/
   *       T10,'7. APD bulk leakage current        =  ',t46,e12.4,t61,'amperes'/
   *       T10,'8. APD surface leakage current     =  ',t46,e12.4,t61,'amperes'/
           T10,'9. Receiver noise temperature      =  ',t42,f10.2,t61,'degrees K'/
   *       T9,'10. Detector load resistance        =  ',t42,f10.2,t61,'ohms'//
   *       T2,'Do you want to change any of these values? (Y or N): ',$)
         ACCEPT 111, REPLY
 111     FORMAT (A1)
         IF (REPLY .EQ. 'N')  GO TO 300
C
         TYPE 121
 121     FORMAT(//T2,'Type the numbers for the items you wish to change,'/
   *               T2,'one at a time, each followed by a carriage return;'/
   *               T2,'follow the last item by a zero and a carriage return:')
         DO 1240   I=1,10
 122     ACCEPT 123, ITMLST(I)
 123     FORMAT(I3)
         IF (ITMLST(I) .EQ. 0)  GO TO 1241
1240     CONTINUE
C
1241     DO 1242  J=1,10
         JJ = J
         ITEM = ITMLST(JJ)
         IF (ITEM .EQ. 0)  GO TO 127
         TYPE 125, (NARRAY(ITEM,JJ),JJ=1,6), PARAM(ITEM)
 125     FORMAT(T10,'Change ',t17,6A1,t24,'=',t26,G12.4,T40,'to: ',$)
         ACCEPT 126, PARAM(ITEM)
 126     FORMAT(G12.4)
1242     CONTINUE
C
 127     M        =  PARAM(1)
         MU       =  PARAM(2)
         G        =  PARAM(3)
         BETA     =  PARAM(4)
         RB       =  PARAM(5)
         BITRAT   =  PARAM(6)
         IB       =  PARAM(7)
         IS       =  PARAM(8)
         TREC     =  PARAM(9)
         R        =  PARAM(10)
C
         TAU      =  0.5 * (1./BITRAT)
C
```

```
C         Calculate values of the C's
C         *********************
C
  300     C1 = (1.-1./M) * MU*G
C
          BRACKC  =  BETA*G + (1.-BETA)*(2.-1./G)
          FACT1   =  (G**2)*BRACKC
          C2      =  FACT1 * (1.+1./M) * MU
C
          FACT2   =  2.*RB*TAU*MU + 2.*IB*TAU/E
          TERM2   =  2.*IS*TAU/E
          TERM3   =  4.*BK*TREC*TAU/(R*E**2)
          C3      =  FACT1*FACT2 + TERM2 + TERM3
C
C
C         Ask user to input value of mean signal strength
C         *************************************
C
          TYPE 301
  301     FORMAT(///T2,'Type mean signal strength in photons/bit (real): ',$)
          ACCEPT 302, NBAR
  302     FORMAT(G12.4)
C
C
C         Calculate the variance of the signal strength
C         (sgfluc uses dsimp to evaluate the integral of
C         the structure constant over altitude)
C         ***********************************
C
          SIGMAN = SGFLUC (NBAR)
          SIGSQN  = SIGMAN**2
C
C         Compute PBEbar, the mean bit error probability
C         (use DSIMP; start with an upper limit of mean
C         plus 10 sigma and add 5- sigma's until integral
C         converges)
C         **************************************
C
          UPPLIM  = NBAR + 10.*SIGMAN
          LOWLIM = AMAX1 ((NBAR - 10.*SIGMAN), 0.)
          INCR = 10
          Q = 0.D0
C
  400     UPPLIM  = UPPLIM + 5.*SIGMAN
          LOWLIM = AMAX1 ((LOWLIM - 5.*SIGMAN), 0.)
          INCR = INCR + 5
          QPREV = Q
C
```

```
      ERRBD    = 1.E-6
      MAXDUB   = 14
  402 CALL DSIMP (DBLE(LOWLIM), DBLE(UPPLIM),
     *           ERRBD, 6, MAXDUB, MAXDUB+1, PBARIG,
     *           Q, INDERR, RELERR, NUMDUB)
C
C
C     Display intermediate values of the integral for the mean,
C     so the user can follow the calculation
C     *********************************************
      TYPE 403, LOWLIM,UPPLIM,Q,INDERR,ERRBD,RELERR,MAXDUB,NUMDUB
  403 FORMAT(//T2,'PBE mean:'//
     *        T14,'LOWLIM   =  ',T22,F8.2/
     *        T14,'UPPLIM   =  ',T22,F8.2/
     *        T14,'INTGRL   =  ',T24,D22,14//
     *        T14,'INDERR   =  ',T25,I2/
     *        T14,'ERRBD    =  ',T24,E10.2/
     *        T14,'RELERR   =  ',T24,E10.2/
     *        T14,'MAXDUB   =  ',T25,I2/
     *        T14,'NUMDUB   =  ',T25,I2//)
C
C
      IF (INDERR .EQ. 0)    GO TO 404
      IF (INDERR .EQ. 2)    MAXDUB = MAXDUB+1
      IF (MAXDUB .GT. 16)  GO TO 404
      GO TO 402
C
  404 IF (INCR .EQ. 15)  GO TO 400
      IF (INCR .GE. 30)  GO TO 405
      IF (DABS(Q-QPREV)/QPREV .GT. 1.D-4)  GO TO 400
  405 PBEBAR  = SNGL (Q)
      DPBEBR  = Q
C
C
C     Compute the bit error probability for NBAR
C     ***********************************
C
      PBENBR = SNGL (PBE(DBLE(NBAR)))
C
C
C     Compute PBEbar from the approximate formula
C     **************************************
C
      SALPHA = SNGL (ALPHA (DBLE(NBAR)))
C
      Q1  =  SQRT(2.) * C1
      Q2  =  C2*NBAR + 2.*C3
      Q3  =  (C2*NBAR + C3)**1.5
      ALPHA1 = (Q1*Q2)/(4.*Q3)
C
```

```
      Q2  =  Q2 + 2.*C3
      Q3  =  Q3 * (C2*NBAR + C3)
      ALPHA2 = -(Q1*C2*Q2)/(8.*Q3)
C
      Q4  =  ALPHA2 - 2.*SALPHA*ALPHA1**2
      Q5  =  EXP(-SALPHA**2)
      PBE2 = -(1.*SQRT(PI)) * Q5 * Q4
C
      APBEBR = PBENBR + 0.5*PBE2*SIGSQN
C
C     Compute the fractional error of the approximate formula
C     for the increase in bit error probability
C     ********************************************
C
      IF (PBEBAR .EQ. PBENBR)  ERRMN = 0.
      IF (PBEBAR .NE. PBENBR)  ERRMN = (APBEBR-PBEBAR)/(PBEBAR-PBENBR)
C
C
C     Display values of the parameters for this run
C     ***********************************
C
      TYPE 710,  NBAR,  M,MU,G,BETA,RB,BITRAT,IB,IS,TREC,R
710   FORMAT(
     *  T18,'EVALUATION OF APPROXIMATE FORMULAS FOR THE MEAN'/
     *  T18,'AND VARIANCE OF THE BIT ERROR PROBABILITY'/
     *  T18,' ********************************************************'//
     *  T2,'The signal strength is ',t25,g12.4,t39,'photons/bit'//
     *  T2,'The current values of the system parameters are:'//
     *  T10,'1.  Modulation extinction ratio     =  ',t43,f8.1/
     *  T10,'2.  Quantum efficiency              =  ',t47,f7.4/
     *  T10,'3.  APD mean gain                   =  ',t43,f8.1/
     *  T10,'4.  APD ionization coeff. ratio     =  ',t47,f7,4/
     *  T10,'5.  Background radiation rate       =  ',t46,e12.4,t61,'photons/sec'/
     *  T10,'6.  Data rate                       =  ',t46,e12.4,t61,'bits/sec'/
        T10,'7.  APD bulk leakage current        =  ',t46,e12.4,t61,'amperes'/
        T10,'8.  APD surface leakage current     =  ',t46,e12.4,t61,'amperes'/
        T10,'9.  Receiver noise temperature      =  ',t42,f10.2,t61,'degrees K'/
        T9,'10.  Detector load resistance        =  ',t42,f10.2,t61,'ohms')


      Display the fractional increase in BEP and the accuracy
      of the approximate formula for the fractional increase
      ********************************************

      TYPE 410, PBENBR,
                100.*(PBEBAR-PBENBR) / PBENBR,
                100.*(APBEBR-PBENBR) / PBENBR,
                100.*ERRMN
```

```
      410 FORMAT(/T2,'Formula for mean'/
          *             T2,'**************'//
          * T2,' Bit error probability at nbar           =  ',t49,g14.6/
          *  t2,'''Exact'' increase in BEP                 =  ',t48,g14.6,
          *                                                    t64,'percent'/
          *  t2,' Approx.  ''  ''  ''                      =  ',t48,g,14.6,
          *                                                    t64,'percent'/
          *  t2,' Error of approx. formula for increase    =  ',t48,g14.6,
C         *                                                    t64,'percent')

C         Compute PBEvar, the variance of the bit error probability
C         (use DSIMP; start with an upper limit of mean
C         plus 10 sigma and add 5- sigma's until integral
C         converges)
C         ********************************************
          UPPLIM   = NBAR + 10.*SIGMAN
          LOWLIM  = AMAX1 ((NBAR - 10.*SIGMAN), 0.)
          INCR = 10
          Q = 0.D0
C
      500 UPPLIM   = UPPLIM + 5.*SIGMAN
          LOWLIM  = AMAX1 ((LOWLIM - 5.*SIGMAN), 0.)
          INCR = INCR + 5
          QPREV = Q
C
          ERRBD     = 1.E-6
          MAXDUB  = 14
      502 CALL DSIMP (DBLE(LOWLIM), DBLE(UPPLIM),
          *              ERRBD, 6, MAXDUB, MAXDUB+1, PVARIG,
          *              Q, INDERR, RELERR, NUMDUB)
C
C         Display intermediate values of the integral
C         ********************************
C
          TYPE 503, LOWLIM,UPPLIM,Q,INDERR,ERRBD,RELERR,MAXDUB,NUMDUB
      503 FORMAT(//T2,'PBE variance:'//
          *          T14,'LOWLIM    =   ',T22,F8.2/
          *          T14,'UPPLIM    =   ',T22,F8.2/
          *          T14,'INTGRL    =   ',T24,D22.14//
          *          T14,'INDERR    =   ',T25,I2/
          *          T14,'ERRBD     =   ',T24,E10.2/
          *          T14,'RELERR    =   ',T24,E10.2/
          *          T14,'MAXDUB =   ',T25,I2/
          *          T14,'NUMDUB =   ',T25,I2//)
C
          IF (INDERR   .EQ. 0)  GO TO 504
          IF (INDERR   .EQ. 2)  MAXDUB = MAXDUB+1
          IF (MAXDUB  .GT. 16) GO TO 504
          GO TO 502
C
```

```
504     IF (INCR .EQ. 15)  GO TO 500
        IF (INCR .GE. 30)  GO TO 505
        IF (DABS(Q-QPREV)/QPREV .GT. 1.D-4)  GO TO 500
505     PBEVAR = SNGL(Q)

        Compute the variance of the bit error probability
        from the approximate formula
        ***************************************

        PBE1     = -(1./SQRT(PI)) * Q5 * ALPHA1
        APBEVR =    PBE1**2 * SIGSQN


        Compute the fractional error of the
        approximate formula for the increase
        *****************************

        ERRVR = (APBEVR-PBEVAR)/PBEVAR


        Display the accuracy of the approximate
        formula for the variance and the ratio
        of the standard deviation to the increase
        in the bit error probability
        ********************************

        TYPE 610, PBEVAR, 100.*ERRVR
610     FORMAT(//T2,'Formula for variance'/
   *              t2,'*****************'//
   *      t2,''Exact'' variance of bit error probability        = ',t52,g14.6/
   *      t2,'  Error of approx. formula for variance         = ',t52,g14.6,
   *                                                          t68,'percent')


        Print the results of this run, if desired
        ******************************

        TYPE 650
650     FORMAT(//
   *    T2,'Do you want to print the values
        from this run? (Y or N): ',$)
        ACCEPT 651, REPLY
651     FORMAT(A1)
        IF (REPLY .EQ. 'N')  GO TO 4999
C
        PRINT 710, NBAR,   M,MU,G,BETA,RB,BITRAT,IB,IS,TREC,R
C
        IPRTSW   = 0
        DUMMY  = SGFLUC(0.)
        IPRTSW   = 1
C
```

```
        PRINT 410, PBENBR,
     *              100.*(PBEBAR-PBENBR) / PBENBR,
     *              100.*(APBEBR-PBENBR) / PBENBR,
     *              100.*ERRMN
        PRINT 611, PBEVAR, 100.*ERRVR
 611    FORMAT(/T2,'Formula for variance'/
     *              t2,'******************'//
     *   t2,'"Exact" variance of bit error probability      =  ',t52,g14.6/
     *   t2,' Error of approx. formula for variance         =  ',t52,g14.6,
     *                                              t68,'percent')
C
        PRINT 612, AMIN1 (PBEBAR+SQRT(PBEVAR), 0.5),
     *              PBEBAR,
     *              AMAX1 (PBEBAR-SQRT(PBEVAR), 0.)
 612    FORMAT(/
     *   t2,'Band of one standard deviation about mean BEP'/
     *   t2,'*********************************************'//
     *   t2,'Mean BEP + 1 s.d. = ',t23,g14.6/
     *   t2,'Mean BEP          = ',t23,g14.6/
     *   t2,'Mean BEP - 1 s.d. = ',t23,g14.6/////////////)
C
C
C       Run another test?
C       *************
C
 4999   TYPE 5000
 5000   FORMAT(///T2,'Do you want to run another test, for different'/
     *              t2,values of the system parameters and/or the '/
     *              t2,'signal strength? (Y or N) :',$)
        ACCEPT 5001, REPLY
 5001   FORMAT(A1)
        IF (REPLY .EQ. 'Y')  GO TO 100
C
        STOP
        END
C
C
        REAL FUNCTION PBE*8 (N)
C       *************************
C
C    This routine computes the bit error probability as a function of
C    the signal strength N, for the APD detector.
C**************************************************************************************
C
        REAL*8         N, ERF, ALPHA
C
C**************************************************************************************
C
        PBE = 0.5D0 * (1.D0 - ERF(ALPHA(N)))
C
        RETURN
        END
```

```
      REAL FUNCTION ERF*8 (ALPHA)
C     ******************************
C
C  This routine computes the error function, for a given value alpha.
C
C***********************************************************************************************
C
      INTEGER  OPENSW
      REAL*8   ROOTPI,TERM,SUM,FACT1,FACT2,ALPHSQ,EXPN,DENOM,
     *         E(401), E1,E2,E3, ALPHA1,ALPHA2,ALPHA3,
     *         TERM1,TERM2,TERM3, ALPHA
      DATA     ROOTPI/1.772453850905516D0/, OPENSW/1/
C
C
C     Use different computational methods,
C     depending on the value of alpha
C     ******************************
C
      IF (ALPHA .LT. 1.D0)                           GO TO 1000
      IF (ALPHA .GE. 1.D0 .AND.  ALPHA .LE. 5.D0)    GO TO 2000
      IF (ALPHA .GT. 5.D0)                           GO TO 3000
C
C
C     ALPHA LESS THAN 1; use convergent series
C     (first eleven terms)
      *************************************
C
 1000 TERM    = ALPHA
      ALPHSQ  = ALPHA**2
      FACT1   = 1.D0
      FACT2   = 0.D0
      SUM     = TERM
C
      D0 1100  I=1,10
      FACT1=  FACT1 + 2.D0
      FACT2=  FACT2 + 1.D0
      TERM =  -(TERM*ALPHSQ*(FACT1-2.D0))/(FACT1*FACT2)
      SUM  =  SUM + TERM
 1100 CONTINUE
C
      ERF = SUM*2,D0/ROOTPI
      GO TO 5000
C
C
C     ALPHA between 1 and 5; use quadratic interpolation in a
C     table read in from a disk file (pre-computed by DSIMP)
C     ********************************************
C
 2000 IF (OPENSW .EQ. 0)  GO TO 2100
C
```

```
      OPEN (UNIT=1, NAME='VM:ERFTAB.DAT', TYPE='OLD',
     *       DISP='KEEP', FORM='FORMATTED'
      OPENSW = 0
      D0 2010 I=1,401
      READ (1,2011)  E(I)
 2010 CONTINUE
 2011 FORMAT(F21.14)
C
 2100  L = IDINT(100.D0*ALPHA) -100 +1
      E1 = E(L)
      E2 = E(L+1)
      E3 = E(L+2)
C
      ALPHA1 = DBLE(FLOAT(IDINT(100.D0*ALPHA))) * 0.01D0
      ALPHA2 = ALPHA1 + 0.01D0
      ALPHA3 = ALPHA1 + 0.02D0
C
      TERM1  =  (ALPHA2-ALPHA)*(ALPHA3-ALPHA)*E1
      TERM2  =  2.D0*(ALPHA-ALPHA1)*(ALPHA3-ALPHA)*E2
      TERM3  =  -(ALPHA-ALPHA1)*(ALPHA2-ALPHA)*E3
      SUM    =  TERM1 + TERM2 + TERM3
C
      ERF = 0.5D4*SUM
      GO TO 5000
C
C
C     ALPHA greater than 5; use an asymptotic series
C     (first eleven terms)
C     ***************************************
C
 3000 SUM    =  1.D0
      TERM   =  1.D0
      FACT1  =  -1.D0
      ALPHSQ =  ALPHA**2
      FACT2  =  2.D0 * ALPHSQ
C
      D0 3100 =  I=1,10
      FACT1  =  FACT1 + 2.D0
      TERM   =  -TERM*(FACT1/FACT2)
      SUM    =  SUM + TERM
 3100 CONTINUE
C
      EXPN   =  DEXP(-ALPHSQ)
      DENOM  =  ROOTPI * ALPHA
      ERF    =  1.D0 - SUM*(EXPN/DENOM)
C
C
 5000 RETURN
      END
```

```
      REAL FUNCTION ALPHA*8 (N)
C     ****************************
C
C  This function computes the value of alpha for a given value of
C  the signal strength N, for given values of the C's
C*****************************************************************************************
C
      REAL*4     NBAR
      REAL*8     N, ROOT, XNUM,DENOM, ALPHA, DPBEBR
C
      COMMON     C1,C2,C3, NBAR, SIGSQN, DPBEBR
C
C*****************************************************************************************
C
      DC1 = DBLE (C1)
      DC2 = DBLE(C2)
      DC3 = DBLE(C3)
C
      ROOT   =  DSQRT (DC2*N + DC3)
      XNUM   =  DC1 * N
      DENOM  =  DSQRT(2.D0) * ROOT
      ALPHA  =  XNUM/DENOM
C
      RETURN
      END




      SUBROUTINE DSIMP (XL,XU, ERRBD, BCTMIN, BCTMAX, RACTVT, FCT,
     *                 SF, ERR IND, ERROR, B COUNT)
C     ************************************************************************
C
C
C     PURPOSE
C     ////////
C
C     TO COMPUTE AN APPROXIMATION FOR THE DEFINITE INTEGRAL OF THE
C     FUNCTION FCT(X) (SUPPLIED BY THE USER) BETWEEN THE LIMITS
C     XL AND XU. THE INTEGRATION IS ACCOMPLISHED BY THE REPEATED
C     APPLICATION OF SIMPSON'S RULE, SUCCESSIVELY DOUBLING THE
C     NUMBER OF SUB-INTERVALS UNTIL THE DESIRED ACCURACY IS REACHED.
C     THE SUBROUTINE IS WRITTEN IN DOUBLE PRECISION, TO MINIMIZE
C     ROUND-OFF PROBLEMS.
C
C
```

```
C       DESCRIPTION OF ARGUMENTS
C       ///////////////////////
C
C       XL      -  LOWER BOUND OF THE INTERVAL OF INTEGRATION (REAL*8)
C       XU      -  UPPER BOUND OF THE INTERVAL OF INTEGRATION (REAL*8)
C
C       ERRBD   -  DESIRED RELATIVE ACCURACY OF THE COMPUTED VALUE
C                  OF THE INTEGRAL (REAL*4)
C                  (FOR EXAMPLE, IF THE INTEGRAL IS DESIRED TO BE
C                   ACCURATE TO ONE PART IN A MILLION, SET ERRBD=1.E-6.)
C
C       BCTMIN  -  THE MINIMUM NUMBER OF SUB-INTERVAL DOUBLINGS
C                  NECESSARY TO GET A FINE ENOUGH GRID TO HIT ALL
C                  THE STRUCTURE OF THE INTEGRAND FUNCTION
C
C       BCTMAX  -  THE MAXIMUM NUMBER OF SUB-INTERVAL DOUBLINGS
C                  (INTEGER, BETWEEN 3 AND 30, INCLUSIVE)
C
C       RACTVT  -  A SWITCH TO ACTIVATE THE TEST FOR LIMITATION OF
C                  ACCURACY DUE TO THE ACCUMULATION OF ROUND-OFF ERROR
C                  (INTEGER - CAN BE 0 OR ANY POSITIVE INTEGER)
C
C                  THE TEST FOR ACCUMULATION OF ROUND-OFF ERROR IS
C                  ACTIVATED WHEN THE NUMBER OF SUB-INTERVAL DOUBLINGS
C                  REACHES THE VALUE OF RACTVT.
C
C                  THE VALUE OF RACTVT SHOULD BE CHOSEN LARGE ENOUGH SO
C                  THAT THE APPROXIMATION GIVEN BY SIMPSON'S RULE FOR
C                  RACTVT+1 DOUBLINGS IS CLEARLY CLOSER TO THE TRUE
C                  VALUE OF THE INTEGRAL THAN THE APPROXIMATION FOR
C                  RACTVT DOUBLINGS. (THE PURPOSE HERE IS AVOID
C                  POSSIBLE SPURIOUS INDICATIONS OF ROUND-OFF ERROR
C                  ACCUMULATION DUE MERELY TO PECULIARITIES OF THE
C                  INTEGRAND.)
C
C                  IF IT IS NOT EASY TO JUDGE AN APPROPRIATE VALUE FOR
C                  RACTVT, THE USER MAY SET RACTVT = 0.
C                  (FOR MOST INTEGRANDS THIS SHOULD NOT CAUSE ANY
C                  SPURIOUS INDICATIONS.)
C
C                  IF THE USER DOES NOT WISH TO APPLY THE TEST FOR
C                  ROUND-OFF ERROR, HE MAY SET RACTVT TO ANY INTEGER
C                  GREATER THAN BCTMAX.
C
C       FCT     -  DUMMY NAME FOR THE DOUBLE-PRECISION FUNCTION SUB-
C                  PROGRAM USED TO COMPUTE THE INTEGRAND.
C
C                  THIS FUNCTION SUB-PROGRAM MUST BE SUPPLIED BY THE
C                  USER; ITS ACTUAL NAME MUST BE PASSED TO DSIMP BY
C                  THE CALLING PROGRAM.
C
```

```
C                     (NOTE THAT IN THE CALLING PROGRAM THE ACTUAL NAME
C                     MUST BE SPECIFIED AS THE NAME OF AN EXTERNAL ROUTINE
C                     BY AN ''EXTERNAL'' STATEMENT.)
C
C       SF         -  FINAL APPROXIMATION TO THE INTEGRAL RETURNED TO THE
C                     CALLING PROGRAM (REAL*8)
C
C       ERRIND     -  AN INDICATOR WHICH TELLS WHETHER THE ROUTINE DSIMP
C                     WAS ABLE TO COMPUTE THE INTEGRAL TO THE DESIRED
C                     ACCURACY (INTEGER) :
C
C                        ERRIND = 0  -  IT WAS POSSIBLE TO REACH THE RE-
C                                       QUIRED ACCURACY WITHIN BCTMAX
C                                       DOUBLINGS.
C
C                        ERRIND = 1  -  IT WAS IMPOSSIBLE TO REACH THE
C                                       REQUIRED ACCURACY BECAUSE OF THE
C                                       ACCUMULATION OF ROUNDING ERRORS.
C
C                        ERRIND = 2  -  THE REQUIRED ACCURACY COULD NOT
C                                       BE REACHED WITHIN BCTMAX DOUBLINGS
C                                       OF THE NUMBER OF SUB-INTERVALS.
C                                       BCTMAX SHOULD BE INCREASED.
C
C       ERROR      -  THE ESTIMATED MAGNITUDE OF THE RELATIVE ERROR OF SF
C                     (REAL*4).
C
C       BCOUNT     -  THE NUMBER OF SUB-INTERVAL DOUBLINGS CARRIED OUT
C                     DURING THE CALCULATION OF THE INTEGRAL.
C
C       SPECIFICATION STATEMENTS
C       ////////////////////////////////////////
C
        INTEGER  BCTMIN, BCTMAX, ERRIND, BCOUNT, M, S3CT, R3CT, K,
     *           M2, RACTVT
        REAL*4   ERRBD, ERROR
        REAL*8   XL, XU, FCT, SF, H, SUM, SN, RANGEN, S(3), RANGE(3),
     *           SEND, SEVEN, SODD
C
C
        SF = 0.D0                             ! Set integral to zero if the
        IF (XU .EQ. XL)  RETURN               ! lower and upper limits are
C                                               the same.
C
C
```

```
C         INITIALIZE
C         ////////////////
C
          BCOUNT = 0
          H  = XU-XL
          M  = 1
          SEND = FCT(XL) + FCT(XU)
          SEVEN = 0.
          SODD = 0.
          S3CT = 0
          R3CT = 0
C
C         COMPUTE SUCCESSIVE APPROXIMATIONS TO INTEGRAL
C         ///////////////////////////////////////////////////////////////////////////////
C
  100     BCOUNT = BCOUNT + 1
C
          H  =  H/2.D0
          M  =  2*M
C
          S EVEN = S EVEN + S ODD
C
          SODD = 0.
          M2 = M-1
          D0 101 K=1,M2,2
  101     SODD = SODD + FCT(XL + K*H)
          SUM = SEND + 2.D0*SEVEN + 4.D0*SODD
          SN = (H/3.D0) * SUM
          IF (BCOUNT .LT. BCTMIN)  GO TO 100
C
          GO TO 200
C
  110     IF (BCOUNT .LT. BCTMAX)  GO TO 100
C
          SF = SN
          ERRIND = 2
          ERROR = RANGEN/DABS(S(3))
          RETURN
C
C         TEST FOR CONVERGENCE - COMPUTE SUCCESSIVE RANGES
C         /////////////////////////////////////////////////////////////////////////////////////
C
  200     S3CT = S3CT + 1
C
          S(S3CT) = SN
C
          IF (S3CT .LT. 3)  GO TO 110
C
          RANGEN = DMAX1 (S(1),S(2),S(3)) - DMIN1 (S(1),S(2),S(3))
C
```

```
         IF (RANGEN/DABS(S(3)) .LT. ERRBD)  GO TO 250
         GO TO 300
C
  210    S(1)  = S(2)
         S(2)  = S(3)
         S3CT = 2
         GO TO 110
C
  250    SF = S(3)
         ERRIND = 0
         ERROR = RANGEN/DABS(S(3))
         RETURN
C
C
C        TEST GROUPS OF THREE SUCCESSIVE RANGES
C        FOR ROUND-OFF ERROR LIMITATION
C        //////////////////////////////////////////////////////////////////
C
  300    R3CT = R3CT +1
C
         RANGE(R3CT) = RANGEN
C
         IF (R3CT .LT. 3)  GO TO 210
C
         IF (RANGE(2) .GE. RANGE(1)  .AND. RANGE (3) .GE. RANGE(1)
     *                               .AND. BCOUNT .GE. RACTVT)  GO TO 350
C
         RANGE(1) = RANGE(2)
         RANGE(2) = RANGE(3)
         R3CT = 2
         GO TO 210
C
  350    SF = S(1)
         ERRIND = 1
         ERROR = RANGE(1)/DABS(S(1))
         RETURN
C
C
         END
```

```
      REAL FUNCTION PN*8 (N)
C     ************************
C
C   This routine computes the log-normal probability distribution of the
C   signal strength N, for given values of the mean nbar and variance
C   sigsqn.
C
C*********************************************************************************
C
      REAL*4    NBAR
      REAL*8    N,Q1,SIGSQL,SIGMAL,BRACK,EXPT,DENOM,PI,DPBEBR
C
      COMMON    C1,C2,C3, NBAR, SIGSQN, DPBEBR
C
      DATA      PI/3.141592653589793D0/
C
C*********************************************************************************
C
      IF (N .EQ. 0.D0)  GO TO 100
C
      DNBAR   = DBLE(NBAR)
      DSGSQN  = DBLE(SIGSQN)
C
C
C     Compute the log-amplitude variance
C     ******************************
C
      Q1 = 1.D0 + DSGSQN/DNBAR**2
      SIGSQL = DLOG(Q1)/4.D0
      SIGMAL = DSQRT(SIGSQL)
C
C
C     Compute the exponent
C     *****************
C
      BRACK  = 0.5D0*DLOG(N/DNBAR) + SIGSQL
      EXPT   = - BRACK**2/(2.D0/SIGSQL)
C
C
C     Compute the denominator
C     ********************
C
      DENOM = 2.D0*DSQRT(2.D0*PI)*N*SIGMAL
C
C
C     Compute the distribution
C     *******************
C
      PN = DEXP(EXPT)/DENOM
      RETURN
```

```
C
  100     PN = 0.D0
          RETURN
          END
C
C
C
          REAL FUNCTION PBARIG*8 (N)
C         *****************************
C      This routine computes the product of the functions PBE(N) and
C      PN(N), to form the integrand of the integral defining PBEbar;
C      this integrand is used by DSIMP and so must be in double
C      precision.
C*************************************************************************************
C
          REAL*8          N, PBE, PN
C
C*************************************************************************************
C
          PBARIG = PBE(N) * PN(N)
C
          RETURN
          END




          REAL FUNCTION PVARIG*8(N)
C         ****************************
C
C      This routine computes the product of the functions (pbe(n)-pbebar)**2
C      and pn(n), to form the integrand of the integral defining the variance
C      pbevar of the bit error probability; this integrand is used by dsimp
C      and so must be in double precision.
C*************************************************************************************
C
          REAL*4       NBAR
          REAL*8       N, PBE, PN, DPBEBR
C
          COMMON       C1,C2,C3, NBAR, SIGSQN, DPBEBR
C
C*************************************************************************************
C
          PVARIG = ((PBE(N)-DPBEBR)**2) * PN(N)
C
          RETURN
          END
```

```
      REAL FUNCTION SGFLUC*4 (NBAR)
C     *********************************
C
C     This routine uses DSIMP to calculate the integral of the index of
C     refraction structure constant over altitude, to get the variance
C     of the signal strength fluctuation.
C
C     The user may choose one of several models for the structure constant
C     altitude profile, as well as the values of several parameters.
C*************************************************************************************
C
      INTEGER*2     PROFIL, PROFL
C
      REAL*4        NBAR, INTGRL
      REAL*8        PRFL1, XIGRL
C
      EXTERNAL      PRFL1
C
      COMMON/PRFIL/  HO, IPRTSW
C
      DATA          PROFIL/1/, AREA/1.167/, THETA/51./, RCVALT/143./,
     *              HO/30./, IPRTSW/1/
C*************************************************************************************
C
      IF (IPRTSW .EQ. 1)  GO TO 99
C
C     PRINTOUT MODE:
C     Print out profile model name and parameter values
C     ****************************************
C
      IF (PROFIL .EQ. 1)  PRINT 10
   10 FORMAT(/T2,'The current structure constant profile is 1/h.')
C
      PRINT 20, AREA, THETA, H0, RCVALT
   20 FORMAT(/T2,'The current values of the parameters associated'/
     *   t2,'with the signal strength fluctuation are:'//
     *   t10,'1.  Receiver aperture area       =  ',t47,f10.3,t65,'sq. meters'/
     *   t10,'2.  Zenith angle                 =  ',t47,f10.3,t65,'degrees'/
     *   t10,'3.  Height of receiver           =  ',t47,f10.3,t65,'feet'/
     *   t10,'    (above local ground)'/
     *   t10,'4.  Altitude of receiver site    =  ',t47,f10.3,t65,'feet'/
     *   t10,'    (above sea level)'/
C
      SGFLUC = 0.
      RETURN
```

```
C         Choose the index of refraction structure constant
C         altitude profile and specify values of parameters
C         ***************************************
C
C
C         Structure constant profile
C         ------------------------------
C
   99     TYPE 100, PROFIL
  100     FORMAT(///T2,'This is subroutine SGFLUC - '//
     *       t2,/The following structure constant profiles are provided:'//
     *           t10,'1.   1/h   profile'/
     *           t10,'2.   ...   profile'//
     *       t2,'The current profile is,t25,I2,t27,';'/
     *       t2,'if you want to change this, type the'/
     *       t2,' number of the profile desired; if not, type zero:   ',$)
C
          ACCEPT 101, PROFL
  101     FORMAT(I2)
          IF (PROFL .NE. 0)  PROFIL = PROFL
C
C
C         Area of receiver aperture
C         ------------------------------
C
          TYPE 110, AREA
  110     FORMAT(///T2,'Current area of receiver aperture = ',t39,f10.4,
     *              t51,'sq. meters;'/
     *       t2,'if you want to change this,'/
     *       t2,'type the new value (real);'/
     *       t2,'if not, type 0. :                   ',$)
C
          ACCEPT 111, ARA
  111     FORMAT(G14.4)
          IF (ARA .NE. 0.)  AREA = ARA
C
C
C         Zenith angle of atmospheric path
C         ----------------------------------------
C
          TYPE 120, THETA
  120     FORMAT(///T2,'Current zenith angle = ',t25,f5.1,t32,'degrees;'/
     *       t2,'if you want to change this,'/
     *       t2,'type the new value (real);'/
     *       t2,'if not, type 0. :                   ',$)
C
          ACCEPT 121, THET
  121     FORMAT(G10.2)
          IF (THET .NE. 0.)  THETA = THET
```

```
C         Height of receiver (above local ground)
C         ------------------------------------------------
C
          TYPE 130, H0
  130     FORMAT(///T2,'Current height of receiver (above local ground = ',
     *              t55,f8.1,t65,'feet;'/
     *      t2,'if you want to change this,'/
     *      t2,'type the new value (real)''/
     *      t2,'if not, type 0.   :                ',$)
C
          ACCEPT 131, H00
  131     FORMAT(G14.2)
          IF (H00 .NE. 0.)  H0 = H00
C
C
C         Altitude (above sea level) of receiver site (not receiver)
C         ------------------------------------------------------------------
C
          TYPE 140, RCVALT
  140     FORMAT(///T2,'Current altitude (above sea level)'/
     *      t2,'of receiver site (not receiver) = ',t27,f8.1,t47,'feet;'/
     *      t2,'if you want to change this,'/
     *      t2,'type the new value (real);'/
     *      t2,'if not, type 0.   :                ',$)
C
          ACCEPT 141, RCVAL
  141     FORMAT(G14.2)
          IF (RCVAL .NE. 0.)  RCVALT = RCVAL
C
C
C         Call DSIMP to compute the integral of the profile
C         ****************************************
C
          MAXDUB = 14
C
  200     IF (PROFIL .EQ. 1)     CALL DSIMP  (DBLE(H0*0.3048),
     *                                        DBLE(20.E3-RCVALT*0.3048),
     *                                        1.E-6, 8, MAXDUB, MAXDUB+1,
     *                                        PRFL1,
     *                                        XIGRL, INDERR, RELERR, NUMDUB)
C
C         IF (PROFIL .EQ. 2)     CALL DSIMP  (.....)
C
          IF (INDERR .EQ. 0)  GO TO 250
          MAXDUB = MAXDUB + 1
          GO TO 200
```

```
C         Compute standard deviation of signal strength fluctuation
C         ********************************************
C
  250     INTGRL  =  SNGL (XIGRL)
          ROOTIG  =  SQRT (INTGRL)
          SECTHE  =  1./COS (THETA*(3.14159/180.))
          COEFF   =  3.62 * (SECTHE**1.5) ' AREA**(7./12.)
          FACTOR  =  COEFF * ROOTIG
C
          SGFLUC  =  FACTOR * NBAR
C
C
C         Display ratio of standard deviation to mean signal strength
C         *********************************************
C
          TYPE 300, FACTOR
  300     FORMAT(///
     *       T2,'Ratio of standard deviation to mean signal strength = ',
     *       t57,f8.4//)
C
          RETURN
          END
C
C
C
C
C
          REAL FUNCTION PRFL1*8 (H)
C         **************************
C
C     This routine computes the integrand of the altitude integral of the
C     index of refraction structure constant, for use by DSIMP
C***********************************************************************************
C
          REAL*4        H0
          REAL*8        H, B, H0MTRS
C
          COMMON/PRFIL/  H0, IPRTSW
C
          DATA          B/1.5D-13/
C
C***********************************************************************************
C
          HOMTRS  =  DBLE (H0*0.3048)
          PRFL1   =  ((H-HOMTRS)**2) * (B/H)
C
          RETURN
          END
```

```
      PROGRAM ERFTAB
C
C   This routine computes a table of values of the error function
C   and stores them in the file DY1:ERFTAB.DAT. The Table is computed
C   at intervals of 0.01, from 1.00 to 5.00, to nine significant digits,
C   for use by routine ERF in computing the error function for values
C   between 1.00 and 5.00; series are used for values of less than 1.00
C   or greater than 5.00.
C***************************************************************************
C
      INTEGER*2  EIND, BCOUNT
      REAL*8     ERF(401), X, ERFIGD, Q, ROOTPI
      EXTERNAL   ERFIGD
      DATA        ROOTPI/1.772453850905516D0/
C
C***************************************************************************
C
C     Compute the error function values by using DSIMP
C     (display them as they are computed)
C     *******************************************
C
      X = 0.99D0
C
      DO 100 I=1,401
      X = X + 0.01D0
      CALL DSIMP (0.D0, X, 1.E-9, 15,0, ERFIGD,
     *            Q, EIND, ERR, BCOUNT)
      ERF(I) = (2.D0/ROOTPI) * Q
      TYPE 101, X,ERF(I)
  100 CONTINUE
  101 FORMAT(/T15,'ERF(',T19,F10.6,T29,') = ',T34,F10.6)
C
C     Open a file in DY1 and write the ERF array to it
C     *****************************************
      OPEN (UNIT=1, NAME='DY1:ERFTAB.DAT',  TYPE='NEW',
     *      DISP='KEEP', FORM='FORMATTED',  RECORDSIZE=21)
C
      DO 200  I=1,401
      WRITE(1,201)  ERF(I)
  200 CONTINUE
  201 FORMAT(F21.14)
C
      CLOSE(UNIT=1,DISP='KEEP')
      STOP
      END
```

APPENDIX B

TABLE OF VALUES OF THE ERROR FUNCTION

Table of values of the error function, accurate to nine significant digits, precomputed by routine DSIMP, for use by routine ERF in computing the error function for values of the argument between 1.00 and 5.00, at intervals of 0.01 (series are used for values less than 1.00 or greater than 5.00).

The error function is defined by the formula:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2}\, dt$$

1.00	0.842700793	1.49	0.964897865	1.98	0.994892000
1.01	0.846810496	1.50	0.966105146	1.99	0.995111413
1.02	0.850838018	1.51	0.967276748	2.00	0.995322265
1.03	0.854784211	1.52	0.968413497	2.01	0.995524849
1.04	0.858649947	1.53	0.969516209	2.02	0.995719451
1.05	0.862436106	1.54	0.970585690	2.03	0.995906348
1.06	0.866143587	1.55	0.971622733	2.04	0.996085810
1.07	0.869773297	1.56	0.972628122	2.05	0.996258096
1.08	0.873326158	1.57	0.973602627	2.06	0.996423462
1.09	0.876803102	1.58	0.974547009	2.07	0.996582153
1.10	0.880205070	1.59	0.975462016	2.08	0.996734409
1.11	0.883533012	1.60	0.976348383	2.09	0.996880461
1.12	0.886787890	1.61	0.977206837	2.10	0.997020533
1.13	0.889970670	1.62	0.978038088	2.11	0.997154845
1.14	0.893082328	1.63	0.978842840	2.12	0.997283607
1.15	0.896123843	1.64	0.979621780	2.13	0.997407023
1.16	0,899096203	1.65	0.980375585	2.14	0.997525293
1.17	0.902000399	1.66	0.981104921	2.15	0.997638607
1.18	0.904837427	1.67	0.981810442	2.16	0.997747152
1.19	0.907608286	1.68	0.982492787	2.17	0.997851108
1.20	0.910313978	1.69	0.983152587	2.18	0.997950649
1.21	0.912955508	1.70	0.983790459	2.19	0.998045943
1.22	0.915533881	1.71	0.984407008	2.20	0.998137154
1.23	0.918050104	1.72	0.985002827	2.21	0.998224438
1.24	0.920505184	1.73	0.985578500	2.22	0.998307948
1.25	0.922900128	1.74	0.986134595	2.23	0.998387832
1.26	0.925235942	1.75	0.986671671	2.24	0.998464231
1.27	0.927513629	1.76	0.987190275	2.25	0.998537283
1.28	0.929734193	1.77	0.987690942	2.26	0.998607121
1.29	0.931898633	1.78	0.988174196	2.27	0.998673872
1.30	0.934007945	1.79	0.988640549	2.28	0.998737661
1.31	0.936063123	1.80	0.989090502	2.29	0.998798606
1.32	0.938065155	1.81	0.989524545	2.30	0.998856823
1.33	0.940015026	1.82	0.989943156	2.31	0.998912423
1.34	0.941913715	1.83	0.990346805	2.32	0.998965513
1.35	0.943762196	1.84	0.990735948	2.33	0.999016195
1.36	0.945561437	1.85	0.991111030	2.34	0.999064570
1.37	0.947312398	1.86	0.991472488	2.35	0.999110733
1.38	0.949016035	1.87	0.991820748	2.36	0.999154777
1.39	0.950673296	1.88	0.992156223	2.37	0.999196790
1.40	0.952285120	1.89	0.992479318	2.38	0.999236858
1.41	0.953852439	1.90	0.992790429	2.39	0.999275064
1.42	0.955376179	1.91	0.993089940	2.40	0.999311486
1.43	0.956857253	1.92	0.993378225	2.41	0.999346202
1.44	0.958296570	1.93	0.993655650	2.42	0.999379283
1.45	0.959695026	1.94	0.993922571	2.43	0.999410802
1.46	0.961053510	1.95	0.994179334	2.44	0.999440826
1.47	0.962372900	1.96	0.994426275	2.45	0.999469420
1.48	0.963654065	1.97	0.994663725	2.46	0.999496646

2.47	0.999522566	2.96	0.999971618	3.45	0.999998934
2.48	0.999547236	2.97	0.999973334	3.46	0.999999008
2.49	0.999570712	2.98	0.999974951	3.47	0.999999077
2.50	0.999593048	2.99	0.999976474	3.48	0.999999141
2.51	0.999614295	3.00	0.999977910	3.49	0.999999201
2.52	0.999634501	3.01	0.999979261	3.50	0.999999257
2.53	0.999653714	3.02	0.999980534	3.51	0.999999309
2.54	0.999671979	3.03	0.999981732	3.52	0.999999358
2.55	0.999689340	3.04	0.999982859	3.53	0.999999403
2.56	0.999705837	3.05	0.999983920	3.54	0.999999445
2.57	0.999721511	3.06	0.999984918	3.55	0.999999485
2.58	0.999736400	3.07	0.999985857	3.56	0.999999521
2.59	0.999750539	3.08	0.999986740	3.57	0.999999555
2.60	0.999763966	3.09	0.999987571	3.58	0.999999587
2.61	0.999776711	3.10	0.999988351	3.59	0.999999617
2.62	0.999788809	3.11	0.999989085	3.60	0.999999644
2.63	0.999800289	3.12	0.999989774	3.61	0.999999670
2.64	0.999811181	3.13	0.999990422	3.62	0.999999694
2.65	0.999821512	3.14	0.999991030	3.63	0.999999716
2.66	0.999831311	3.15	0.999991602	3.64	0.999999736
2.67	0.999840601	3.16	0.999992138	3.65	0.999999756
2.68	0.999849409	3.17	0.999992642	3.66	0.999999773
2.69	0.999857757	3.18	0.999993115	3.67	0.999999790
2.70	0.999865667	3.19	0.999993558	3.68	0.999999805
2.71	0.999873162	3.20	0.999993974	3.69	0.999999820
2.72	0.999880261	3.21	0.999994365	3.70	0.999999833
2.73	0.999886985	3.22	0.999994731	3.71	0.999999845
2.74	0.999893351	3.23	0.999995074	3.72	0.999999857
2.75	0.999899378	3.24	0.999995396	3.73	0.999999867
2.76	0.999905082	3.25	0.999995697	3.74	0.999999877
2.77	0.999910480	3.26	0.999995980	3.75	0.999999886
2.78	0.999915587	3.27	0.999996245	3.76	0.999999895
2.79	0.999920418	3.28	0.999996493	3.77	0.999999903
2.80	0.999924987	3.29	0.999996725	3.78	0.999999910
2.81	0.999929307	3.30	0.999996942	3.79	0.999999917
2.82	0.999933390	3.31	0.999997146	3.80	0.999999923
2.83	0.999937250	3.32	0.999997336	3.81	0.999999929
2.84	0.999940898	3.33	0.999997515	3.82	0.999999934
2.85	0.999944344	3.34	0.999997681	3.83	0.999999939
2.86	0.999947599	3.35	0.999997838	3.84	0.999999944
2.87	0.999950673	3.36	0.999997983	3.85	0.999999948
2.88	0.999953576	3.37	0.999998120	3.86	0.999999952
2.89	0.999956316	3.38	0.999998247	3.87	0.999999956
2.90	0.999958902	3.39	0.999998367	3.88	0.999999959
2.91	0.999961343	3.40	0.999998478	3.89	0.999999962
2.92	0.999963645	3.41	0.999998582	3.90	0.999999965
2.93	0.999965817	3.42	0.999998679	3.91	0.999999968
2.94	0.999967866	3.43	0.999998770	3.92	0.999999970
2.95	0.999969797	3.44	0.999998855	3.93	0.999999973

3.94	0.999999975	4.43	1.000000000
3.95	0.999999977	4.44	1.000000000
3.96	0.999999979	4.45	1.000000000
3.97	0.999999980	4.46	1.000000000
3.98	0.999999982	4.47	1.000000000
3.99	0.999999983	4.48	1.000000000
4.00	0.999999985	4.49	1.000000000
4.01	0.999999986	4.50	1.000000000
4.02	0.999999987	.	.
4.03	0.999999988	.	.
4.04	0.999999989	.	.
4.05	0.999999990	5.00	1.000000000
4.06	0.999999991		
4.07	0.999999991		
4.08	0.999999992		
4.09	0.999999993		
4.10	0.999999993		
4.11	0.999999994		
4.12	0.999999994		
4.13	0.999999995		
4.14	0.999999995		
4.15	0.999999996		
4.16	0.999999996		
4.17	0.999999996		
4.18	0.999999997		
4.19	0.999999997		
4.20	0.999999997		
4.21	0.999999997		
4.22	0.999999998		
4.23	0.999999998		
4.24	0.999999998		
4.25	0.999999998		
4.26	0.999999998		
4.27	0.999999998		
4.28	0.999999999		
4.29	0.999999999		
4.30	0.999999999		
4.31	0.999999999		
4.32	0.999999999		
4.33	0.999999999		
4.34	0.999999999		
4.35	0.999999999		
4.36	0.999999999		
4.37	0.999999999		
4.38	0.999999999		
4.39	0.999999999		
4.40	1.000000000		
4.41	1.000000000		
4.42	1.000000000		

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15. Supplementary Notes

16. Abstract

The effect of atmospheric turbulence on the bit error rate of a space to ground near infrared laser communications link is investigated, for a link using binary pulse position modulation and an avalanche photodiode detector. Formulas are presented for the mean and variance of the bit error rate as a function of signal strength. Because these formulas require numerical integration, they are of limited practical use. Approximate formulas are derived which are easy to compute and sufficiently accurate for system feasibility studies, as shown by numerical comparison with the exact formulas. A very simple formula is derived for the bit error rate as a function of signal strength, which requires only the evaluation of an error function. It is shown by numerical calculations that, for realistic values of the system parameters, the increase in the bit error rate due to turbulence does not exceed about thirty percent for signal strengths of four hundred photons per bit or less. The increase in signal strength required to maintain an error rate of one in ten million is about one or two tenths of a db.

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