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Wave interactions in a three-dimensional attachment line boundary layer.

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Abstract

The three-dimensional boundary layer on a swept wing can support different types of hydrodynamic instability. Here attention is focused on the so-called 'spanwise contamination' problem which occurs when the attachment line boundary layer on the leading edge becomes unstable to Tollmien-Schlichting waves. In order to gain insight into the interactions which are important in that problem a simplified basic state is considered. This simplified flow corresponds to the swept attachment line boundary layer on an infinite flat plate. The basic flow here is an exact solution of the Navier Stokes equations and its stability to two-dimensional waves propagating along the attachment line can be considered exactly at finite Reynolds number. This has been done in the linear and weakly nonlinear regimes by Hall, Malik and Poll (1984) and Hall and Malik (1986). Here the corresponding problem is studied for oblique waves and their interaction with two-dimensional waves is investigated. In fact oblique modes cannot be described exactly at finite Reynolds number so it is necessary to make a high Reynolds number approximation and use triple deck theory. It is shown that there are two types of oblique wave which, if excited, cause the destabilization of the two-dimensional mode and the breakdown of the disturbed flow at a finite distance from the leading edge. Firstly a low frequency mode closely related to the viscous stationary crossflow mode discussed by Hall (1986) and MacKerrell (1987) is a possible cause of breakdown. Secondly a class of oblique wave with frequency comparable with that of the two-dimensional mode is another cause of breakdown. It is shown that the relative importance of the modes depends on the distance from the attachment line.

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1. Introduction

Recent interest in the development of laminar flow wings has generated much interest in the different modes of instability which the three- dimensional boundary layer on a swept wing can support. There are essentially three different types of instability which such flows can support. Firstly these flows are susceptible to Tollmien-Schlichting instability waves induced by viscosity. These are important everywhere in the flow. Secondly the crossflow instability mechanism discussed by Gregory, Stuart and Walker (1955) is operational in three-dimensional boundary layers. This mechanism is inviscid in origin and is important sufficiently far from the leading edge of the wing where the spanwise velocity component of the flow is not negligible. Finally, if there are regions of concave curvature on the wing, the centrifugal instability mechanism will lead to counter-rotating Görtler vortices.

Clearly in some boundary layers all three mechanism will be operational and their interaction might induce the premature transition of the flow. However, it is clear that if the boundary layer does not remain laminar in the attachment line region then there is no point in trying to control the other mechanisms further away from the leading edge.

Experimental investigations of the attachment line instability problem have been made by Pfenninger and Bacon (1969), Gaster (1967) and Poll (1979). It was found that, if the input disturbances are small, instability waves propagate along the attachment line when the flow Reynolds number exceeds a certain critical value. Pfenninger and Bacon (1969) found that if the input disturbances were sufficiently large then there was some evidence of a subcritical response. The measurements made in these experiments were all close to the leading edge and showed no evidence of oblique Tollmien-Schlichting waves being present in the flow.

Here we shall first discuss the linear instability of the attachment line boundary layer on an infinite flat wall. We describe the instability using the approach of Smith (1979a,b) and therefore use triple deck theory. We find that in addition to the two-dimensional mode of Hall, Malik and Poll (1984) (hereafter referred to as HMP) there is possible, at any chordwise location, a family of unstable oblique waves. Each oblique mode is unstable only up to a finite value of X, the chordwise variable. This instability problem is discussed in §2.

We then consider the nonlinear interaction of a two-dimensional mode with an unstable oblique mode. This is done using the formulation of Hall and Smith (1984) for wave interactions in Blasius flow and leads to a pair of coupled amplitude equations which determine the evolution of the modes. These equations are found in §3 and in §4 we discuss the possible solution of these equations for different input waves. We show that there is a particular type of interaction which leads to both disturbance amplitudes becoming infinite at a finite chordwise location.

In §5 we briefly describe the numerical work needed to calculate the coefficients in the amplitude equations. We then discuss the nature of the solutions of the amplitude equations appropriate to the constants we calculate and the implications for the stability of the flow of interest.

2. Formulation of the stability problem

Consider the flow of a viscous incompressible fluid of kinematic viscosity ν adjacent to the flat plate defined by $y = 0$ with respect to Cartesian co-ordinates (x, y, z). The velocity field (u, v, w) corresponding to (x, y, z) satisfies the conditions

$$\begin{aligned} &u = v = w = 0, \qquad y = 0, \\ &u \to U_\infty \frac{x}{\ell}, \qquad w \to U_\infty, \qquad y \to \infty. \end{aligned} \tag{2.1a, b}$$

The Reynolds number R is defined by

$$R = \frac{U_\infty \ell}{\nu}, \tag{2.2}$$

and dimensionless variables $(X, Y, Z), (U, V, W)$ are defined by

$$\begin{aligned} (X, Y, Z) &= \left(\frac{x}{\ell}, \frac{y}{\ell}, \frac{z}{\ell}\right), \\ (u, v, w) &= U_\infty (U, V, W). \end{aligned} \tag{2.3a, b}$$

If the pressure is scaled on ρU_∞^2 and time, T, on U_∞/ℓ, then the Navier Stokes equations become

$$\underline{U}_T + (\underline{U} \bullet \nabla)\underline{U} = -\nabla P + \frac{1}{R}\nabla^2 \underline{U}. \tag{2.4}$$

In order to satisfy the no-slip condition the basic state has a boundary layer of thickness $\ell R^{-\frac{1}{2}}$ near $Y = 0$. If the boundary layer variable η is defined by

$$\eta = R^{\frac{1}{2}} Y,$$

the basic state can be expressed in the form

$$\underline{U} = (X\bar{u}(\eta), R^{-\frac{1}{2}}\bar{v}(\eta), \bar{w}(\eta))(1 + O(R^{-\frac{1}{2}})), \tag{2.5}$$

where

$$\begin{aligned} \bar{u} + \bar{v}' &= 0, \\ \bar{v}''' + \bar{v}'^2 - \bar{v}\bar{v}'' - 1 &= 0, \\ \bar{w}'' - \bar{v}\bar{w}' &= 0, \end{aligned} \tag{2.6a, b, c}$$

with

$$\bar{v}(0) = \bar{v}'(0) = \bar{w}(0) = 0,$$
$$\bar{v}'(\infty) = -1, \quad \bar{w}(\infty) = 1. \tag{2.7a, b}$$

It is known, Smith (1979a,b), that lower branch Tollmien-Schlichting instabilities of boundary layers are governed by triple deck theory. Here the interest is in oblique Tollmien-Schlichting waves proportional to E where

$$E = \exp\left[i\{\int^X \frac{\alpha(X)}{\epsilon^3} dX + \frac{\beta Z}{\epsilon^3} - \frac{\Omega T}{\epsilon^2}\}\right], \tag{2.8}$$

with $\epsilon = R^{-\frac{1}{8}} << 1$. The slowly varying wavenumber α is then expanded as

$$\alpha = \alpha_0 + \epsilon\alpha_1 + \cdots. \tag{2.9}$$

In the main part of the boundary layer the basic state is perturbed by writing

$$\underline{U} = (X\bar{u}, \epsilon^4\bar{v}, \bar{w})(1 + O(\epsilon^4)) + \delta(\epsilon U_0, \epsilon^2 V_0, \epsilon W_0)E + \cdots, \tag{2.10}$$

where U_0, V_0 and W_0 depend only on X and η whilst δ is assumed small compared to any power of ϵ. The corresponding pressure perturbation is $\epsilon^2 P_0 E$ where P_0 is a function of X only. The equations to determine U_0, V_0 in the main deck are:

$$i\alpha_0 U_0 + V_{0\eta} + i\beta W_0 = 0,$$
$$i\alpha_0 X\bar{u}U_0 + XV_0\bar{u}' + i\beta\bar{w}U_0 = 0, \tag{2.11a, b, c}$$
$$i\alpha_0 X\bar{u}W_0 + V_0\bar{w}' + i\beta\bar{w}W_0 = 0,$$

and the appropriate solution of this system is

$$U_0 = a(X)X\bar{u}', \qquad W_0 = a(X)\bar{w}',$$
$$V_0 = -ia(X)[\alpha_0 X\bar{u} + \beta\bar{w}], \tag{2.12a, b}$$

where $a(X)$ is an amplitude function to be determined. An investigation of the disturbed flow in the upper layer shows that all disturbance quantities decay exponentially there and that matching with the main solution requires

$$P_0\sqrt{\alpha_0^2 + \beta^2} = a[\alpha_0 X + \beta]^2. \tag{2.13}$$

It should be noted from (2.12) that when $\eta \to 0$

$$U_0 \simeq a(X)\lambda X, \quad W_0 \simeq a(X)\mu, \tag{2.14}$$

where from numerical calculations it is found that $\lambda = 1.236$, $\mu = 0.570$. Finally in the lower deck the pressure perturbation is still $\epsilon^2 P_0 E$ whilst the total velocity field can be written :

$$\begin{aligned} U &= \epsilon\zeta\lambda X + \cdots \quad + \delta\epsilon u_o E + \cdots, \\ V &= -\epsilon^6\zeta^2\lambda X/2 + \cdots \quad + \delta\epsilon^3 v_0 E + \cdots, \\ W &= \epsilon\zeta\mu + \cdots \quad + \delta\epsilon w_0 E + \cdots. \end{aligned} \tag{2.15a, b, c}$$

Here the lower deck variable ζ is defined by

$$\zeta = Y\epsilon^{-5}.$$

The equations to determine (u_0, v_0, w_0) in the lower deck can be written as

$$\mathcal{L}\begin{pmatrix} u_0 \\ v_0 \\ w_0 \\ P_0 \end{pmatrix} = 0, \tag{2.16}$$

where the matrix operator

$$\mathcal{L} \equiv \begin{pmatrix} \frac{d^2}{d\zeta^2} - i\alpha_0 X\lambda\zeta - i\beta\mu\zeta + i\Omega_0 & -\lambda X & 0 & -i\alpha_0 \\ i\alpha_0 & \frac{d}{d\zeta} & i\beta & 0 \\ 0 & -\mu & \frac{d^2}{d\zeta^2} - i\alpha_0 X\lambda\zeta - i\beta\mu\zeta + i\Omega_0 & -i\beta \end{pmatrix}. \tag{2.17}$$

The solution of the wall layer problem is then written as

$$\alpha_0 u_0 + \beta w_0 = b(X)\int_{\xi_0}^{\xi} Ai(s)ds, \tag{2.18}$$

$$\Delta v_0 = -i\int_{\xi_0}^{\xi} [\alpha_0 u_o + \beta w_0]ds, \tag{2.19}$$

where

$$\begin{aligned} \Delta &= \{i\lambda X\alpha_0 + i\beta\mu\}^{\frac{1}{3}}, \\ \xi &= \Delta\zeta + \xi_0, \\ \xi_0 &= -\frac{i\Omega}{\Delta^2}. \end{aligned} \tag{2.20}$$

The pressure P_0 can be related to the displacement function b using the X and Z momentum equations to give

$$i\{\alpha_0^2 + \beta^2\}P_0 = \Delta^2 Ai'(\xi_0)b. \tag{2.21}$$

The lower and main deck solutions are then found to match if

$$Ai'(\xi_0)(\alpha_0\lambda X+\beta\mu)^{\frac{5}{3}}=i^{\frac{1}{3}}\sqrt{\alpha_0^2+\beta^2}\chi_0(\alpha_0 X+\beta)^2, \tag{2.22}$$

where

$$\chi_0=\int_{\xi_0}^{\infty}Ai(s)ds.$$

The eigenrelation (2.22) determines the complex wavenumber α_0 for given β and Ω. For neutral stability it is known that

$$\xi_0\simeq-2.298i^{\frac{1}{3}},\quad \frac{Ai'(\xi_0)}{\chi_0}\simeq 1.001i^{\frac{1}{3}}. \tag{2.23a, b}$$

Thus the neutral values of α_0 and β are related by

$$1.001(\alpha_0\lambda X+\beta\mu)^{\frac{5}{3}}\simeq\sqrt{\alpha_0^2+\beta^2}(\alpha_0 X+\beta)^2. \tag{2.24}$$

The modes corresponding to HMP have $\alpha_0=0$ in which case β satisfies

$$\beta^{\frac{4}{3}}\simeq 1.001\mu^{\frac{5}{3}}.$$

At any given value of X there are in addition neutral three-dimensional modes with $\alpha_0\neq 0$. The above analysis fails if $\alpha_0 X+\beta$ or $\alpha_0\lambda X+\beta\mu$ become negative anywhere in the (α_0,β) plane so only eigenvalues of (2.24) above the lines $\alpha_0\lambda X+\beta\mu=0$, and $\alpha_0 X+\beta=0$ are acceptable. In Figure 1a α_0 is shown as a function of β for $X=0.1, 1., 10., 20., 30.$. The solutions in the second quadrant asymptote to the line $\alpha_0 X+\beta=0$ as $\beta\to-\infty$ whilst those in the fourth quadrant asymptote to the line $\alpha_0\lambda X+\beta\mu=0$ as $\beta\to 0$. Figure 1b shows the neutral values of Ω as a function of β for $X=0.1, 1., 10., 20., 30.$.

Finally in this section we notice that the two-dimensional mode of (2.22) which of course corresponds to $\alpha_0=0$ is neutrally stable at all values of X. The three-dimensional modes however are initially unstable on the attachment line $X=0$ and become stable beyond a critical value of X. Experimentally it appears that if the level of disturbances present in the flow is sufficiently small then it is the two-dimensional mode which is observed. In the next section we investigate the possibility that the two-dimensional mode might be destabilized by oblique modes which grow in the X direction. In Figure 2 we have shown typical growth rate curves for the three-dimensional modes.

In fact the small β solutions are related to the stationary modes of instability of the three-dimensional boundary layer discussed by Hall (1986) and MacKerrell (1987). These modes orient themselves such that the shear stress of the 'effective' velocity profile is zero;

the lower deck structure is then described by parabolic cylinder functions rather than Airy functions. Thus when α_0 tends to zero we find from (2.24) that

$$\mu\beta = -\alpha_0 \lambda X + 0(\alpha_0)^{\frac{9}{5}},$$

so that the neutral frequency tends to zero like $\alpha_0^{\frac{6}{5}}$. Some discussion of the time dependent version of the stationary modes discussed by Hall and MacKerrell has recently been given by Bassom (1987).

3. Weakly nonlinear theory

Suppose that the three-dimensional mode with $(\alpha, \beta, \Omega) = (\alpha_2, \beta_2, \Omega_2)$ is neutrally stable at $X = X_n$. We consider the interaction of this mode with the two-dimensional disturbance which propagates along the attachment line. We know from the work of Hall and Smith (1984) that in the absence of the two-dimensional mode the three-dimensional mode will evolve in a nonlinear, nonparallel manner in an $\epsilon^{\frac{3}{2}}$ neighbourhood of X_n. We therefore define $\tilde{X}$ by

$$\tilde{X} = \frac{(X - X_n)}{\epsilon^{\frac{3}{2}}}. \tag{3.1}$$

Later we can derive the 'quasi-parallel' evolution equations for $(X - X_n) > 0(\epsilon^{\frac{3}{2}})$ by taking the limit $\tilde{X} \to \infty$. In order that the two-dimensional mode in this neighbourhood should be of finite size we suppose that, with $\Omega = \Omega_1$, the neutral frequency for a two-dimensional wave, the spanwise wavenumber β_1 is expanded as

$$\beta_1 = \beta_{10} + \epsilon\beta_{11} + \epsilon^{\frac{3}{2}}\tilde{\beta} + \cdots, \tag{3.2}$$

where β_{10}, β_{11} are the first two terms in the expansion of the neutral spanwise wavenumber.

It is now convenient to represent the 'fast' dependence of the Tollmien-Schlichting waves in the X direction by multiple scales rather than the WKB formulation of §2. We therefore write

$$X^* = \frac{(X - X_n)}{\epsilon^3}. \tag{3.3}$$

Next we define

$$E_1 = \exp[i\{\frac{\beta_1 Z}{\epsilon^3} - \frac{\Omega_1 T}{\epsilon^2}\}],$$
$$E_2 = \exp[i\{\alpha_2 X^* + \frac{\beta_2 Z}{\epsilon^3} - \frac{\Omega_2 T}{\epsilon^2}\}],$$

where Ω_1, Ω_2 and β_2 expand as

$$\begin{aligned}\Omega_1 &= \Omega_{10} + \epsilon\Omega_{11} + O(\epsilon^2),\\ \Omega_2 &= \Omega_{20} + \epsilon\Omega_{21} + O(\epsilon^2),\\ \beta_2 &= \beta_{20} + \epsilon\beta_{21} + O(\epsilon^2).\end{aligned} \tag{3.4a, b, c}$$

Here Ω_{10}, Ω_{11} etc. are the neutral values appropriate to the location $X = X_n$ whilst β_1 is as given by (3.2). For the three- dimensional mode we further expand

$$\alpha_2 = \alpha_{20} + \epsilon\alpha_{21} + O(\epsilon^2), \tag{3.4d}$$

where α_{20}, α_{21} are the first two terms in the expansion of the neutral value of α_2 at $X = X_n$. In the lower deck we write the velocity in the form

$$\begin{aligned}U &= \epsilon\zeta\lambda X + \cdots \quad + \epsilon\hat{U},\\ V &= -\epsilon^6\zeta^2\lambda X/2 + \cdots \quad + \epsilon^3\hat{V},\\ W &= \epsilon\zeta\mu + \cdots \quad + \epsilon\hat{W},\end{aligned} \tag{3.5}$$

and then expand the disturbance velocity field $(\hat{U}, \hat{V}, \hat{W})$ together with the corresponding pressure perturbation as

$$(\hat{U}, \hat{V}, \hat{W}, \hat{P}) = \epsilon^{\frac{3}{4}}\underline{S}_1 + \epsilon^{\frac{3}{2}}\underline{S}_2 + \epsilon^{\frac{7}{4}}\underline{S}_3 + \epsilon^{\frac{9}{4}}\underline{S}_4 + \cdots. \tag{3.6}$$

Here the term $\underline{S}_1$ corresponds to the fundamental modes proportional to E_1 and E_2. The second order term $\underline{S}_2$ corresponds to first harmonic and mean flow correction terms generated by the interaction of the TS waves. The third order term $\underline{S}_3$ again contains the fundamentals generated because the correction terms in (3.4) are $0(\epsilon)$. Finally the fourth order term $\underline{S}_4$ contains fundamental and other terms driven by the interaction of $\underline{S}_1$ and $\underline{S}_2$.

Clearly the function $\underline{S}_1$ satisfies the linearized problem of §2 so we write

$$\underline{S}_1 = A\underline{S}_{11}E_1 + B\underline{S}_{12}E_2 + C.C. \tag{3.7}$$

where $C.C.$ denotes 'complex conjugate' and A, B are functions of $\tilde{X}$ to be found at higher order. The functions $\underline{S}_{ij}$ are defined by $\underline{S}_{ij} = (U_{ij}, V_{ij}, W_{ij}, P_{ij})$ for $i, j \geq 1$ and $\underline{S}_{11}, \underline{S}_{12}$ satisfy (2.16) with $(\alpha_0, \beta, \Omega)$ replaced by $(0, \beta_{10}, \Omega_{10})$ and $(\alpha_{20}, \beta_{20}, \Omega_{20})$ respectively. Thus for example we can show that

$$\alpha_{20}U_{12}+\beta_{20}W_{12}=\int_{\xi_{20}}^{\xi_2}Ai(s)\,ds,$$

$$\xi_{20}=-\frac{i\Omega_{20}}{\Delta_2^2},\quad \xi_2=\Delta_2\zeta+\xi_{20},\quad \Delta_2=\{i\lambda X_n\alpha_{20}+i\beta_{20}\mu\}^{\frac{1}{3}}.$$

At next order we find that the first harmonics and mean flow corrections can be written as

$$\begin{aligned}\underline{S}_2 =&\{A^2\underline{S}_{21}E_1^2+B^2\underline{S}_{22}E_2^2+AB\underline{S}_{23}E_1E_2+A\bar{B}\underline{S}_{24}E_1E_2^{-1}\}+C.C.\\ &+\mid A\mid^2\underline{S}_{25}+\mid B\mid^2\underline{S}_{26},\end{aligned}\tag{3.8}$$

where $\bar{B}$ denotes the complex conjugate of B. We find that $\underline{S}_{21},\underline{S}_{22}$ satisfy the differential system

$$\mathcal{L}(2\alpha_{n0},2\beta_{n0},2\Omega_{n0})\begin{pmatrix}U_{2n}\\ V_{2n}\\ W_{2n}\\ P_{2n}\end{pmatrix}=\begin{pmatrix}i\alpha_{n0}U_{1n}^2+V_{1n}\frac{dU_{1n}}{d\zeta}+i\beta_{n0}W_{1n}U_{1n}\\ 0\\ i\alpha_{n0}U_{1n}W_{1n}+V_{1n}\frac{dW_{1n}}{d\zeta}+i\beta_{n0}W_{1n}^2\end{pmatrix},\tag{3.9}$$

for $n=1,2$ and $\alpha_{10}=0$. These equations must be solved subject to

$$U_{2n}=V_{2n}=W_{2n}=0,\quad \zeta=0.\tag{3.10}$$

The functions $\underline{S}_{23}$ and $\underline{S}_{24}$ satisfy similar equations but with $(2\alpha_{n0},2\beta_{n0},2\Omega_{n0})$ replaced by $(\alpha_{10}\pm\alpha_{20},\ \beta_{10}\pm\beta_{20},\ \Omega_{10}\pm\Omega_{20})$ for $n=3,4$ and the right-hand-side of (3.9) replaced respectively by

$$\begin{pmatrix}i\alpha_{20}U_{11}U_{12}+V_{12}\frac{dU_{11}}{d\zeta}+V_{11}\frac{dU_{12}}{d\zeta}+i[\beta_{10}W_{12}U_{11}+\beta_{20}W_{11}U_{12}]\\ 0\\ i\alpha_{20}U_{11}W_{12}+V_{12}\frac{dW_{11}}{d\zeta}+V_{11}\frac{dW_{12}}{d\zeta}+i[\beta_{10}+\beta_{20}]W_{11}W_{12}\end{pmatrix},$$

and

$$\begin{pmatrix}-i\alpha_{20}U_{11}\bar{U}_{12}+\bar{V}_{12}\frac{dU_{11}}{d\zeta}+V_{11}\frac{d\bar{U}_{12}}{d\zeta}+i[\beta_{10}\bar{W}_{12}U_{11}-\beta_{20}W_{11}\bar{U}_{12}]\\ 0\\ -i\alpha_{20}U_{11}\bar{W}_{12}+\bar{V}_{12}\frac{dW_{11}}{d\zeta}+V_{11}\frac{d\bar{W}_{12}}{d\zeta}+i(\beta_{10}-\beta_{20})W_{11}\bar{W}_{12}\end{pmatrix}.$$

The mean flow corrections for $\underline{S}_{25}$ and $\underline{S}_{26}$ have $V_{25}=V_{26}=0$ whilst for $n=1,2$

$$\begin{aligned}\frac{d^2U_{2n+4}}{d\zeta^2}&=V_{1n}\frac{d\bar{U}_{1n}}{d\zeta}+\bar{V}_{1n}\frac{dU_{1n}}{d\zeta}+i\beta_{n0}(\bar{W}_{1n}U_{1n}-W_{1n}\bar{U}_{1n}),\\ \frac{d^2W_{2n+4}}{d\zeta^2}&=i\alpha_{n0}(\bar{U}_{1n}W_{1n}-U_{1n}\bar{W}_{1n})+V_{1n}\frac{d\bar{W}_{1n}}{d\zeta}+\bar{V}_{1n}\frac{dW_{1n}}{d\zeta},\end{aligned}\tag{3.11a, b}$$

which must be solved subject to

$$U_{2n+4} = W_{2n+4} = 0, \quad \zeta = 0,$$
$$\frac{dU_{2n+4}}{d\zeta} = \frac{dW_{2n+4}}{d\zeta} = 0, \quad \zeta = \infty. \tag{3.12a, b}$$

In the main and upper decks the mean flow corrections and the first harmonic functions are not forced by the fundamentals and therefore satisfy similar equations to those discussed in §2; the matching of the main deck solutions for the first harmonics with the lower deck solution produces boundary conditions at $\zeta = \infty$ for $\underline{S}_{21}, \underline{S}_{22}, \underline{S}_{23}, \underline{S}_{24}$.

At next order in the lower deck problem we obtain only fundamental terms driven by the variation of the mean state. In fact the solution of this linear problem when matched with the main deck solution determines the $0(\epsilon)$ terms in the expansion of the neutral wavenumber and frequencies. Since the solution at this order has no effect on the amplitude equations for A and B we give no details of it here.

The interaction of the fundamental term $\underline{S}_1$ with the mean flow correction and first harmonic term generates fundamental terms in $\underline{S}_4$. In addition further fundamental terms are produced by the evolution of the amplitude functions A and B and the basic state on the $\tilde{X}$ length scale. If we write $\underline{S}_4$ in the form

$$\underline{S}_4 = \underline{S}_{41}E_1 + \underline{S}_{42}E_2 + C.C. + \cdots,$$

where $\cdots$ represents other terms forced by the interactions, then after some manipulation we find that $\underline{S}_{41}$ and $\underline{S}_{42}$ satisfy

$$\mathcal{L}(0, \beta_{10}, \Omega_{10}) \begin{pmatrix} U_{41} \\ V_{41} \\ W_{41} \\ P_{41} \end{pmatrix} = \begin{pmatrix} [A \mid A \mid^2 \Phi_1 + A \mid B \mid^2 \Phi_2 + i\mu\tilde{\beta}\zeta U_{11} A \\ +\lambda\tilde{X}V_{11}A + \lambda X_n \zeta U_{11}\frac{dA}{d\tilde{X}} + P_{11}\frac{dA}{d\tilde{X}}] \\ -U_{11}\frac{dA}{d\tilde{X}} - iW_{11}\tilde{\beta}A \\ [A \mid A \mid^2 \Phi_3 + A \mid B \mid^2 \Phi_4 + i\tilde{\beta}\mu\zeta W_{11} A \\ +\lambda\zeta X_n W_{11}\frac{dA}{d\tilde{X}} + i\tilde{\beta}P_{11}A] \end{pmatrix},$$

$$\mathcal{L}(\alpha_{20}, \beta_{20}, \Omega_{20}) \begin{pmatrix} U_{42} \\ V_{42} \\ W_{42} \\ P_{42} \end{pmatrix} = \begin{pmatrix} [B \mid B \mid^2 \Phi_5 + B \mid A \mid^2 \Phi_6 + \lambda X_n \zeta \frac{dB}{d\tilde{X}} U_{12} \\ +i\alpha_{20}\tilde{X}\lambda\zeta B U_{12} + \lambda\tilde{X}BV_{12} + P_{12}\frac{dB}{d\tilde{X}}] \\ -U_{12}\frac{dB}{d\tilde{X}} \\ [B \mid B \mid^2 \Phi_7 + B \mid A \mid^2 \Phi_8 + \lambda\zeta X_n W_{12}\frac{dB}{d\tilde{X}} \\ +\lambda\zeta\tilde{X}W_{12} i\alpha_{20} B] \end{pmatrix}.$$

Here the functions Φ_1, Φ_2, Φ_3 and Φ_4 are defined by

$$\Phi_1 = i\alpha_{10}(U_{25}U_{11} + U_{21}\bar{U}_{11}) + V_{11}U_{25}' + V_{21}\bar{U}_{11}' + \bar{V}_{11}U_{21}' + i\beta_{10}(W_{25}U_{11} - W_{21}\bar{U}_{11} + 2\bar{W}_{11}U_{21}),$$

$$\begin{aligned}\Phi_2 =& i\alpha_{10}(U_{12}U_{24} + U_{26}U_{11} + U_{23}\bar{U}_{12}) + V_{12}U'_{24} + V_{23}\bar{U}'_{12} + V_{24}U'_{12} \\ &+ \bar{V}_{12}U'_{23} + V_{11}U'_{26} + i(\beta_{10} + \beta_{20})\bar{W}_{12}U_{23} + i(\beta_{10} - \beta_{20})W_{12}U_{24} \\ &+ i\beta_{20}W_{24}U_{12} - i\beta_{20}W_{23}\bar{U}_{12} + i\beta_{10}W_{26}U_{11}, \\ \Phi_3 =& i\alpha_{10}(U_{25}W_{11} + 2\bar{U}_{11}W_{21} - U_{21}\bar{W}_{11}) + V_{11}W'_{25} + V_{21}\bar{W}'_{11} \\ &+ \bar{V}_{11}W'_{21} + i\beta_{10}(W_{25}W_{11} + \bar{W}_{11}W_{21}), \\ \Phi_4 =& i(\alpha_{10} + \alpha_{20})\bar{U}_{12}W_{23} + i(\alpha_{10} - \alpha_{20})U_{12}W_{24} - i\alpha_{20}U_{23}\bar{W}_{12} \\ &+ i\alpha_{20}U_{24}W_{12} + i\alpha_{10}U_{26}W_{11} + V_{12}W'_{24} + V_{23}\bar{W}'_{12} + V_{24}W'_{12} \\ &+ \bar{V}_{12}W'_{23} + V_{11}W'_{26} + i\beta_{10}(W_{12}W_{24} + W_{26}W_{11} + W_{23}\bar{W}_{12}),\end{aligned} \tag{3.13a, b, c, d}$$

with $\alpha_{10} = 0$. The corresponding expressions for Φ_5, Φ_6, Φ_7, and Φ_8 are obtained from (3.13) with $(\alpha_{10}, \beta_{10}, \Omega_{10})$ and $(\alpha_{20}, \beta_{20}, \Omega_{20})$ interchanged and then α_{10} set equal to zero together with

$$\{A, B \text{suffixes } 11, 12, 21, 22, 23, 24, 25, 26\}$$

replaced by

$$\{B, A \text{suffixes } 12, 11, 22, 21, 23, 24, 26, 25\}.$$

Finally the terms with suffix 24 are replaced by their complex conjugate. The disturbance velocities (U_{41}, V_{41}, W_{41}) and (U_{42}, V_{42}, W_{42}) must of course vanish at $\zeta = 0$ and the functions $\underline{S}_{41}, \underline{S}_{42}$ must match with the corresponding functions in the main deck. The latter matching conditions completely specify inhomogeneous differential systems for $\underline{S}_{41}$ and $\underline{S}_{42}$. Since the homogeneous form of these systems have a solution it follows that we must apply solvability conditions to the systems for $\underline{S}_{41}, \underline{S}_{42}$. In order to write down these conditions we must introduce the differential systems adjoint to those which determine the fundamentals. We first note that if we define

$$F = \alpha_0 u_0 + \beta w_0,$$

in (2.18) then the eigenvalue problem which leads to (2.22) can be written as

$$\begin{aligned}F''' - i[\alpha_0\lambda X + \beta\mu]\zeta F' + i\Omega F' = 0, \\ F(0) = F'(\infty) = 0, \\ F(\infty) = CF''(0),\end{aligned} \tag{3.14}$$

where

$$C = \frac{-i[\alpha_0\lambda X + \beta\mu]}{(\alpha_0^2 + \beta^2)^{\frac{1}{2}}(\alpha_0 X + \beta)^2}.$$

The system adjoint to (3.14) is

$$\begin{aligned} &p' = 0, \\ &r'' - i[\alpha_0\lambda X + \beta\mu]\zeta r - i\Omega r = p, \\ &q(0) = r(\infty) = 0, \quad r(0) = Cp(\infty). \end{aligned} \tag{3.15}$$

It is easily seen that we can take $p = 1$ above and then solving the equation for r using variation of parameters and the boundary conditions give (2.22) again. It then follows that if $\Omega = \Omega(\alpha_0, \beta)$ is an eigenvalue of (3.14) then the system

$$\begin{aligned} &G''' - i(\alpha_0 X\lambda + \mu\beta)\zeta G' + i\Omega G' = R, \\ &G(0) = G'(\infty) = 0, \quad G(\infty) - CG''(0) = \gamma, \end{aligned}$$

will have a solution if

$$\int_0^\infty rRd\zeta = \gamma.$$

Thus it follows that the differential systems for $\underline{S}_{41}, \underline{S}_{42}$ will have a solution if

$$\begin{aligned} \frac{dA}{d\tilde{X}} &= \lambda_1\tilde{\beta}A - a_1 A \mid A \mid^2 - b_1 A \mid B \mid^2, \\ \frac{dB}{d\tilde{X}} &= \lambda_2\tilde{X}B - a_2 B \mid B \mid^2 - b_2 B \mid A \mid^2 . \end{aligned} \tag{3.16a, b}$$

Here the coefficients λ_1, λ_2 are defined by

$$\begin{aligned} \lambda_1 &= \frac{-2i}{3\beta_{10}}[2 + H_1]/[X_n G_1 + \frac{2X_n\lambda H_1}{3\beta_{10}\mu}], \\ \lambda_2 &= -\left[\alpha_{20}G_2 + \frac{2\alpha_{20}\lambda H_2}{3(\alpha_{20}X_n\lambda + \beta_{20}\mu)}\right]/\left[X_n G_2 + \frac{\alpha_{20}}{\alpha_{20}^2 + \beta_{20}^2} + \frac{2X_n\lambda H_2}{3(\alpha_{20}X_n\lambda + \beta_{20}\mu)}\right], \end{aligned} \tag{3.17a, b}$$

where for $n = 1, 2, G_n, H_n$ are defined by

$$\begin{aligned} G_n &= \frac{2}{\alpha_{n0}X_n + \beta_{n0}} - \frac{5\lambda}{3(\alpha_{n0}X_n\lambda + \beta_{n0}\mu)}, \\ H_n &= \xi_{0n}Ai(\xi_{0n})(\frac{\xi_{0n}}{Ai'(\xi_{0n})} + \frac{1}{\chi_0(\xi_{0n})}), \end{aligned}$$

with

$$\xi_{0n} = \frac{-i^{\frac{1}{3}}\Omega_{n0}}{(\alpha_{n0}X_n\lambda + \beta_{n0}\mu)^{\frac{2}{3}}}, \qquad \alpha_{10} = 0.$$

Finally the constants a_1, b_1, a_2, b_2 are defined by

$$a_1 = \frac{\beta_{10}\int_0^\infty r_1\Phi_3' d\zeta}{\beta_{10}\lambda X_n \int_0^\infty \zeta r_1 W_{11}' d\zeta - 2iW_{11}(\infty)X_n},$$
$$b_1 = \frac{\beta_{10}\int_0^\infty r_1\Phi_4' d\zeta}{\beta_{10}\lambda X_n \int_0^\infty \zeta r_1 W_{11}' d\zeta - 2iW_{11}(\infty)X_n},$$
$$a_2 = \frac{\int_0^\infty r_2[\alpha_{20}\Phi_5' + \beta_{20}\Phi_7']d\zeta}{\lambda X_n \int_0^\infty \zeta r_2[\alpha_{20}U_{12}' + \beta_{20}W_{12}']d\zeta - 2i[\alpha_{20}U_{12} + \beta_{20}W_{12}]_\infty \frac{X_n}{\alpha_{20}X_n+\beta_{20}}},$$
$$b_2 = \frac{\int_0^\infty r_2[\alpha_{20}\Phi_6' + \beta_{20}\Phi_8']d\zeta}{\lambda X_n \int_0^\infty \zeta r_2[\alpha_{20}U_{12}' + \beta_{20}W_{12}']d\zeta - 2i[\alpha_{20}U_{12} + \beta_{20}W_{12}]_\infty \frac{X_n}{\alpha_{20}X_n+\beta_{20}}},$$

where r_1 and r_2 correspond to 'r' in (3.15) with $(\alpha, \beta, \Omega) = (0, \beta_{10}, \Omega_{10})$ and $(\alpha, \beta, \Omega) = (\alpha_{20}, \beta_{20}, \Omega_{20})$ respectively.

4. The generalization and solution of the amplitude equations

Firstly we note that the three-dimensional wave can also be 'de-tuned' by varying β_1 by an amount $\epsilon^{\frac{3}{2}}\tilde{\tilde{\beta}}$ from the neutral value. In that case (3.16) becomes

$$\begin{aligned}\frac{dA}{d\tilde{X}} &= \lambda_1\tilde{\beta}A - a_1 A \mid A\mid^2 - b_1 A \mid B \mid^2, \\ \frac{dB}{d\tilde{X}} &= \lambda_3\tilde{\tilde{\beta}}B + \lambda_2\tilde{X}B - a_2 B \mid B\mid^2 - b_2 B \mid A \mid^2,\end{aligned} \qquad (4.1a, b)$$

where λ_3 is defined by an expression similar to (3.17a). Secondly we note that (3.16) apply in a $\epsilon^{\frac{3}{2}}$ neighbourhood of the position where the three-dimensional wave is neutrally stable. Following Hall and Smith (1984) it can be shown that (3.16) apply over a longer length scale if $\lambda_2\tilde{X}$ in (4.1b) is replaced by $\lambda_2\tilde{\tilde{X}}$ where $\tilde{\tilde{X}}$ is then treated as a constant in the amplitude equations. This result can be found directly from (4.1) by letting $\tilde{X} \to \infty$ and introducing a length scale shorter than $\tilde{X}$ in order to retain the derivative terms. The resulting amplitude equations have a 'quasi-parallel' nature and correspond to the calculation of Smith (1979b).

We now define ρ and σ by

$$\rho = \mid A \mid^2, \quad \sigma = \mid B \mid^2$$

in which case (3.16) and the generalization of this system for $\tilde{X} >> 1$ can be written

$$\begin{aligned}\rho_{\tilde{X}} &= 2\rho\{\lambda_{1r}\tilde{\beta} - a_{1r}\rho - b_{1r}\sigma\}, \\ \sigma_{\tilde{X}} &= 2\sigma\{\lambda_{2r}\tilde{X} - a_{2r}\sigma - b_{2r}\rho\},\end{aligned} \qquad (4.2a, b)$$

and

$$\rho_{\tilde{X}} = 2\rho\{\lambda_{1r}\tilde{\beta} - a_{1r}\rho - b_{1r}\sigma\},$$
$$\sigma_{\tilde{X}} = 2\sigma\{\lambda_{3r}\tilde{\tilde{\beta}} + \lambda_{2r}\tilde{\tilde{X}} - a_{2r}\sigma - b_{2r}\rho\}. \quad (4.3a, b)$$

The precise nature of the solutions of (4.2), (4.3) depends sensitively on the constants appearing in these equations. We shall see in the next section that the constants a_{1r} and a_{2r} are positive almost everywhere so we first discuss such a situation in detail. In fact a_{1r} is always positive and this result is entirely consistent with the finite Reynolds number calculations of Hall and Malik (1986) for the two-dimensional mode.

A matter of some importance is the question of whether ρ or σ in (4.2) or (4.3) can become infinite at a finite value of $\tilde{X}$. This would mean that three-dimensionality could destroy the stable equilibrium states of Hall and Malik (1986). We seek a singularity of either system as $\tilde{X} \to \tilde{X}_0$ by writing

$$\rho = \frac{\rho_0}{(\tilde{X}_0 - \tilde{X})} + \cdots,$$
$$\sigma = \frac{\sigma_0}{(\tilde{X}_0 - \tilde{X})} + \cdots,$$

in which case ρ_0, σ_0 satisfy

$$\frac{1}{2} = -a_{1r}\rho_0 - b_{1r}\sigma_0,$$
$$\frac{1}{2} = -a_{2r}\sigma_0 - b_{2r}\rho_0, \quad (4.4a, b)$$

and ρ_0 and σ_0 must of course both be positive. It follows immediately that no such singularity is possible if a_{1r}, a_{2r}, b_{1r} and b_{2r} are all positive. In fact it is easily shown that with a_{1r} and a_{2r} positive the only case when the singularity can occur is when b_{1r} and b_{2r} are negative and

$$a_{1r}a_{2r} < b_{1r}b_{2r}. \quad (4.5)$$

This condition effectively identifies an important class of three-dimensional waves which can have a significant effect on the two-dimensional equilibrium states of Hall and Malik (1986). In order to see why this is the case it is necessary for us to discuss the solutions of (4.2) and (4.3) in more detail. We continue to discuss the solution for the case when a_{1r} and a_{2r} are both positive.

In fact we begin with a discussion of (4.3) and return to (4.2) later. It is easily shown from (4.3) that ρ and σ have the possible equilibrium states:

$$
\begin{aligned}
&a. \quad \rho = \sigma = 0, \\
&b. \quad \rho = \lambda_{1r}\tilde{\beta}a_{1r}^{-1},\ \sigma = 0, \\
&c. \quad \rho = 0,\ \sigma = [\lambda_{3r}\tilde{\tilde{\beta}} + \lambda_{2r}\tilde{\tilde{X}}]a_{2r}^{-1}, \\
&d. \quad \rho = [\lambda_{1r}\tilde{\beta} - b_{1r}\sigma]a_{1r}^{-1},\ \sigma = \{\lambda_{1r}\tilde{\beta}b_{2r} - [\lambda_{3r}\tilde{\tilde{\beta}} + \lambda_{2r}\tilde{\tilde{X}}]a_{1r}\}/\{b_{1r}b_{2r} - a_{1r}a_{2r}\}.
\end{aligned}
\tag{4.6a, b, c, d}
$$

The solutions b and c correspond to 'pure' 2D and 3D modes respectively whilst d is a mixed mode. If the detuning parameters $\tilde{\beta}$ and $\tilde{\tilde{\beta}}$ are held fixed whilst $\tilde{\tilde{X}}$ is varied we can determine the evolution of the equilibrium amplitudes as the disturbance develops away from the attachment line $X_n = 0$. Note that λ_{2r} from (3.17b) is negative. The stability of the different equilibrium solutions can be checked by a routine stability analysis. Before discussing the nature of the solutions we note that in all the cases we computed, a_{1r} and a_{2r} are almost always positive in which case nonlinear effects are stabilizing if either the 2D or 3D mode exist separately. We further assume that the detuning parameter $\tilde{\beta}$ has been chosen such that $\lambda_{1r}\tilde{\beta}a_{1r} > 0$ so that in the absence of a 3D wave a stable finite amplitude wave propagating along the attachment line is possible. If a_{1r} and a_{2r} are positive then there are four possible combinations of signs for b_{1r} and b_{2r}. The bifurcation properties for these four cases are summarized below:

Case I $a_{1r}, a_{2r}, b_{1r}, b_{2r} > 0$.

The different possible solutions in this case are shown in Figures 3a,b for the 'sub-cases' $b_{1r}b_{2r} > a_{1r}a_{2r}$ and $b_{1r}b_{2r} < a_{1r}a_{2r}$ respectively. Sufficiently far upstream we see that only the pure 3D mode is a possible stable mode whilst sufficiently far downstream only the 2D mode is a possible equilibrium flow. In the case $b_{1r}b_{2r} < a_{1r}a_{2r}$ there is a short interval where the mixed mode is the only possible stable state.

Case II $a_{1r}, a_{2r} > 0$, $b_{1r}, b_{2r} < 0$.

The solutions in this case are shown in Figures 3c,d for the 'sub-cases' $b_{1r}b_{2r} > a_{1r}a_{2r}$ and $b_{1r}b_{2r} < a_{1r}a_{2r}$ respectively. In the first case the only stable solution is the 2D mode beyond the position where the mixed mode bifurcates from it. However, a phase plane analysis shows that a sufficiently large disturbance to this state is unstable. Thus there is a threshold type of response where a small disturbance to the 2D mode decays whilst a sufficiently large one will grow. The size of the 'sufficiently large disturbance' decreases to zero as $\tilde{\tilde{X}}$ decreases to the point where the mixed mode bifurcates. Before this point there are no stable modes and any disturbance will grow, in this case and the threshold amplitude case the growing disturbances terminate in the finite $\tilde{X}$ singularity discussed previously.

Case III $a_{1r}, a_{2r}, b_{2r} > 0$, $b_{1r} < 0$.

Here the situation is as illustrated in Figure 3e. Dependent on the value of $\tilde{\tilde{X}}$ either the mixed or 2D mode is stable. A phase plane analysis shows that each stable state is stable to an arbitrarily large disturbance so there is no threshold amplitude type of response.

Case IV $a_{1r}, a_{2r}, b_{1r} > 0$, $b_{2r} < 0$.

The situation is now virtually the same as Case III except that the mixed mode loses stability to the 3D mode when $\tilde{\tilde{X}}$ decreases so that the 3D mode is stable as $\tilde{\tilde{X}} \to -\infty$. Again there is no threshold amplitude type of response at any value of $\tilde{\tilde{X}}$. This is illustrated in Figure 3f.

Thus we see that apart from the case $a_{1r}, a_{2r} > 0, b_{1r}, b_{2r} < 0$ with $b_{1r}b_{2r} > a_{1r}a_{2r}$ there is always a stable equilibrium state available at any value of $\tilde{\tilde{X}}$. Furthermore in the latter situation at sufficiently negative values of $\tilde{\tilde{X}}$ the stable state is never the 2D mode. However, as the disturbance develops with increasing $\tilde{\tilde{X}}$ ultimately only the 2D mode is stable. In the exceptional case a sufficiently large initial disturbance will terminate in a singularity at a finite value of $\tilde{X}$.

We now turn to the case where a_{1r} and a_{2r} are not both positive. We shall see in the next section that this situation is unusual and occurs when the constant a_{2r} becomes negative so that nonlinear effects destabilize the three-dimensional mode. The situation in this case can be investigated following the previous discussion. The main result is that (4.1) then always permits a solution which becomes infinite at a finite value of $\tilde{X}$. The singularity has the same structure as that discussed above with the only change being that, dependent on the other constants, it is possible for B alone to become infinite. The equilibrium solutions of the amplitude equations and thus instability characteristics can similarly be investigated for the case $a_{2r} < 0$. Here the three-dimensional mode bifurcates to the right and is always unstable. In some situations the mixed mode exists and it is possible for the two-dimensional mode to be stable to small perturbations. However sufficiently large perturbations always destabilize the flow so that we conclude that when $a_{2r} < 0$ the presence of sufficiently large amplitude perturbations will always lead to the finite-$\tilde{X}$ singularity being set up. We conclude that there are just two situations where the ultimate state set up after a wave interaction between two and three-dimensional modes will not be a stable two-dimensional mode. These exceptional circumstances correspond to when $a_{1r}, a_{2r} > 0, b_{1r}, b_{2r} < 0$ with $a_{1r}a_{2r} < b_{1r}b_{2r}$ or whenever $a_{2r} < 0$.

A similar type of discussion for (4.2) is not possible because there are no equilibrium states for this system for all $\tilde{X}$. However, for large values of $\tilde{X}$ it is easy to show that there is a solution with $\rho = \lambda_{1r}\tilde{\beta}a_{1r}^{-1}$, $\sigma = 0$ and that this solution is stable. There are no other equilibrium states so that, unless limit cycle solutions of (4.2) exist, or a singularity develops we expect any initial disturbance to evolve into a pure 2D mode at large $\tilde{X}$. Numerical investigation of (4.2) showed no evidence of limit cycle behaviour and that in the exceptional case a finite $\tilde{X}$ singularity develops and the 2D equilibrium state is then never set up. It remains for us to discuss the values of $a_{1r}, a_{2r}, b_{1r}, b_{2r}$ found in our calculations so that the above results can be applied to the instability of attachment line flow.

5. Results and discussion

We have seen in the previous section that the nature of the solutions of the amplitude equations depends crucially on the constants $a_{1r}, b_{1r}, a_{2r}, b_{2r}$. These constants can be found only after the differential systems for the fundamentals, adjoint, first harmonic, mean flow correction functions have been solved numerically. These systems were solved using finite differences in the manner described in Hall and Smith (1984); the reader is referred to that paper for a more detailed description of the method. It was found to be convenient to map the region $0 < \zeta < \infty$ into $[0, 1]$ using the transformation

$$\eta = \frac{2}{\pi}\tan^{-1}\zeta,$$

which aids the convergence of the velocity field at large ζ. The other significant difference between our calculations and those of Hall and Smith is that here the spanwise momentum equation has a solution with the velocity component tending to a constant rather than decaying algebraically to zero. In order to illustrate how this can be taken into account we consider the equations for $(U_{21}, V_{21}, W_{21}, P_{21})$. By combining the X^* and Z momentum equations we can show that $F = \alpha_{20}U_{21} + \beta_{20}W_{21}$, and $G = \mu U_{21} - \lambda X_n W_{21}$ satisfy

$$\begin{aligned} &F''' - [-i\Omega_{20} + i(\lambda X_n\alpha_{20} + \mu\beta_{20})\zeta]F' = 0, \\ &G'' - [-i\Omega_{20} + i(\lambda X_n\alpha_{20} + \mu\beta_{20})\zeta]G = i(\alpha_{20}\mu - \lambda X_n\beta_{20})P_{21}. \end{aligned} \qquad (5.1a, b)$$

The first of these equations is to be solved such that $F(0) = F'(0)$ and $F \to$ constant when $\zeta \to \infty$ whilst the second equation is solved subject to $G(0) = 0, G(\infty) \sim \zeta^{-1}$. Thus the combination $\mu U_{21} - \lambda X_n W_{21}$ decays algebraically when $\zeta \to \infty$. Having solved for F and G they can be combined to determine U_{21} and W_{21}, then the equation of continuity is solved to determine V_{21}. The equation for the first harmonic functions can be integrated using the same procedure. Finally in our discussion of the numerical scheme we note that the convergence of our scheme was checked when appropriate by varying the step length over ζ_∞ the approximation to ∞ in the ζ direction.

The constants a_1, b_1, a_2 and b_2 were calculated for $X_n = 0.1, 1., 5.$ and 10. The results are normalized by making $\alpha_{10}U_{11}'' + \beta_{10}W_{11}''$ and $\alpha_{20}U_{12}'' + \beta_{20}W_{12}''$ both equal to unity at $\zeta = 0$. Some typical value of these constants are shown in Table 1 below. We see that it is possible for either of the two exceptional cases of the previous section to occur. In Figures 4 and 5 we have plotted the neutral values of the α_{20}, β_{20} and indicated where the exceptional cases occur. The first exceptional case with $a_{1r}, a_{2r} > 0$ is denoted by $\cdots$ whilst the other exceptional case is denoted by $- - - -$.

We see that at $X_n = 0.1$ an interaction of the two-dimensional mode with the three-dimensional mode with $\alpha/\beta >\sim 1.27$ will cause a singularity in the disturbance amplitudes to occur. Thus at $X_n = 0.1$ three-dimensional waves propagating at an angle of more than about 50° to the attachment line will cause the catastrophic breakdown of the two-dimensional mode.

A further band of modes with $\alpha_{20} < 0$ which leads to the first exceptional case is also seen to exist. These correspond to low frequency three-dimensional modes. In the limit as $\alpha_{20} \to 0$ these modes have zero effective shear stress and correspond to the stationary viscous crossflow modes of Hall (1986) and MacKerrell (1987). We conclude that near the attachment line the stimulation of oblique waves propagating at an angle greater than about 50° or the stimulation of the viscous crossflow modes of Hall and MacKerrell will cause a new larger amplitude disturbance flow structure to develop.

When $X_n = 1$ only the destabilizing band of wavenumbers corresponding to the low frequency modes remains and the interval over which they exist has decreased. However when $X_n = 5$ the stationary viscous crossflow modes become subcritically unstable so that the stationary viscous crossflow modes cause the finite $\tilde{X}$ singularity to develop at almost all of the possible negative values of α_{20}. In addition there is a very short band of oblique modes propagating at an angle of about 80° to the attachment line which leads to the singularity being set up. This band of unstable wavelengths no longer occurs at $X_n = 10$. but the stationary viscous crossflow modes are now subcritically unstable for almost all of the possible values of α_{20} with $\alpha_{20} < 0$.

Without prohibitively expensive numerical calculations we cannot confirm that the results discussed above show the overall trend of the possible interactions when X_n increases. In fact some further investigation showed that the small band of destabilizing oblique modes at X_n appears and disappears as X_n varies. However our calculations do suggest that at small values of X_n there is a wide range of possible oblique modes and a small band of low frequency modes which, if excited, will cause a catastrophic breakdown of the disturbance flowfield. Further away from the attachment line the oblique modes become less important and it is the lower frequency modes which become the dominant mechanism.

Clearly our analysis cannot predict what kind of flow will be set up once the singularity appears. However we note that other modes, notably the inviscid stationary crossflow vortex mode of Gregory, Stuart and Walker (1955) might then become important.

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α_{20}	β_{20}	Ω_{20}	a_{1r}	b_{1r}	a_{2r}	b_{2r}
-0.1614	0.0427	0.0615	-15.5854	-32081339.	-12895.	-80.4869
-0.3162	0.1142	0.2019	-15.5801	-159114.	-757.2574	100.4104
-0.3747	0.1933	0.3674	-15.5801	41.1670	-177.9206	300.6889
-0.3725	0.2523	0.4879	-15.5801	1314.9614	-89.1333	202.8252
-0.3443	0.3110	0.6040	-15.5801	666.9944	-52.8080	-45.7879
0.0490	0.5000	1.0095	-15.5801	-74.2458	-14.8748	2.3777
0.3011	0.4668	1.0375	-15.5801	10.6631	-13.9080	-73.8144
0.5251	0.3795	0.9864	-15.5801	334.7004	-15.7352	113.9694
0.9849	0.2020	0.8797	-15.5801	368.3667	-19.8390	692.2803
3.4675	-0.1164	1.1672	-15.5801	675.7219	1.1880	-2153.2899
5.0859	-0.2706	1.3970	-15.5801	231.6547	2.0798	-2059.4528
10.6798	-0.7970	2.0859	-15.5801	36.0614	0.7039	-1666.8360

Table 1

Typical neutral values for X_n 0.1

<u>Legends for figures</u>

Figure 1a The neutral eigenvalues satisfying (2.24)

Figure 1b The neutral eigenvalues satisfying (2.24)

Figure 2 The imaginary part of α_0 satisfying (2.24), as a function of X for a range of values of Ω .

Figure 3a The equilibrium solutions for a_{1r}, a_{2r}, b_{1r}, $b_{2r} > 0$ with $b_{1r}b_{2r} > a_{1r}a_{2r}$.

Figure 3b The equilibrium solutions for a_{1r}, a_{2r}, b_{1r}, $b_{2r} > 0$ with $b_{1r}b_{2r} < a_{1r}a_{2r}$.

Figure 3c The equilibrium solutions for a_{1r}, $a_{2r} > 0$, b_{1r}, $b_{2r} < 0$ with $b_{1r}b_{2r} > a_{1r}a_{2r}$.

Figure 3d The equilibrium solutions for a_{1r}, $a_{2r} > 0$, b_{1r}, $b_{2r} < 0$ with $b_{1r}b_{2r} < a_{1r}a_{2r}$.

Figure 3e The equilibrium solutions for a_{1r}, a_{2r}, $b_{2r} > 0$ and $b_{1r} < 0$.

Figure 3f The equilibrium solutions for a_{1r}, a_{2r}, $b_{1r} > 0$ and $b_{2r} < 0$.

Figure 4 The different bifurcation solutions for $X_n = 0.1,\ 1$.

Figure 5 The different bifurcation solutions for $X_n = 5,\ 10$.

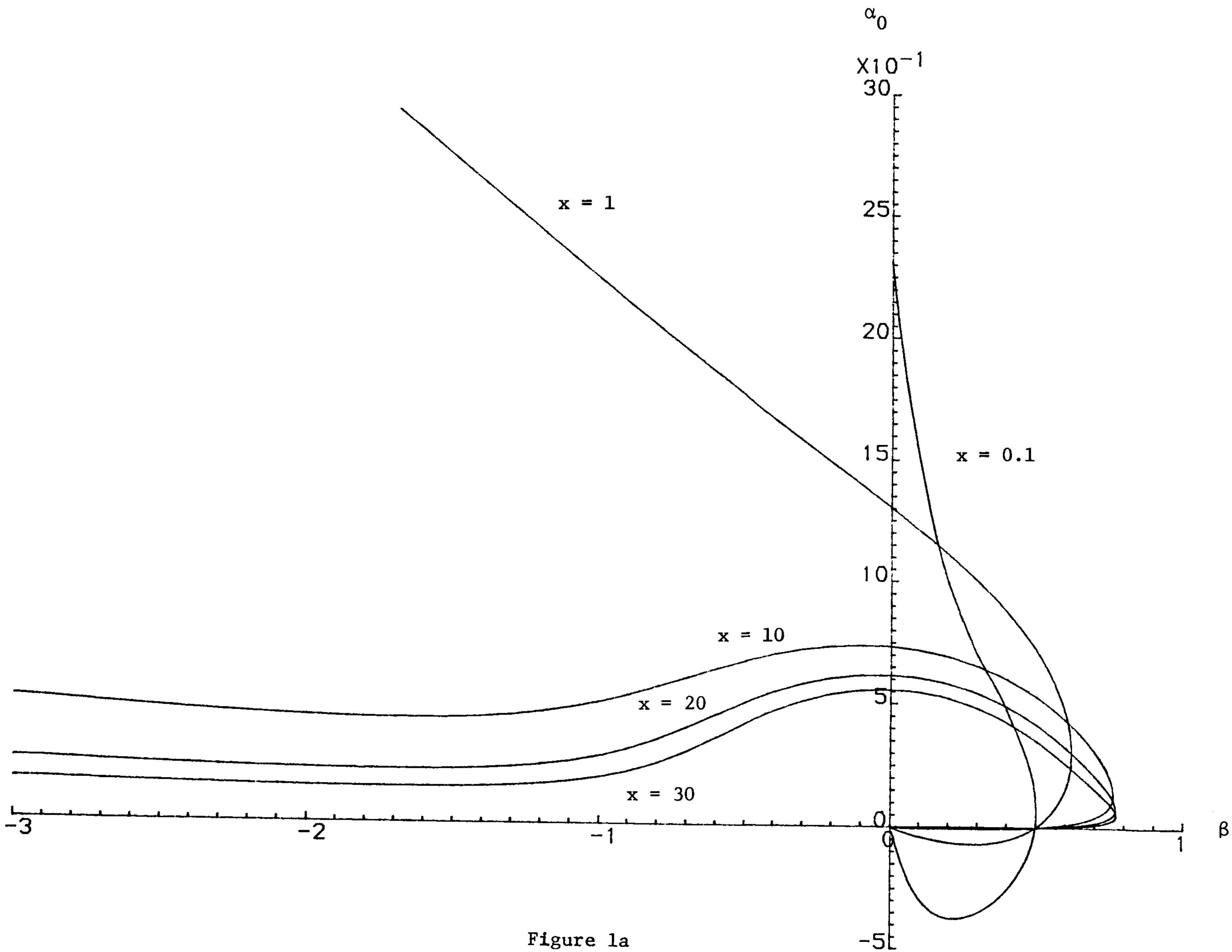
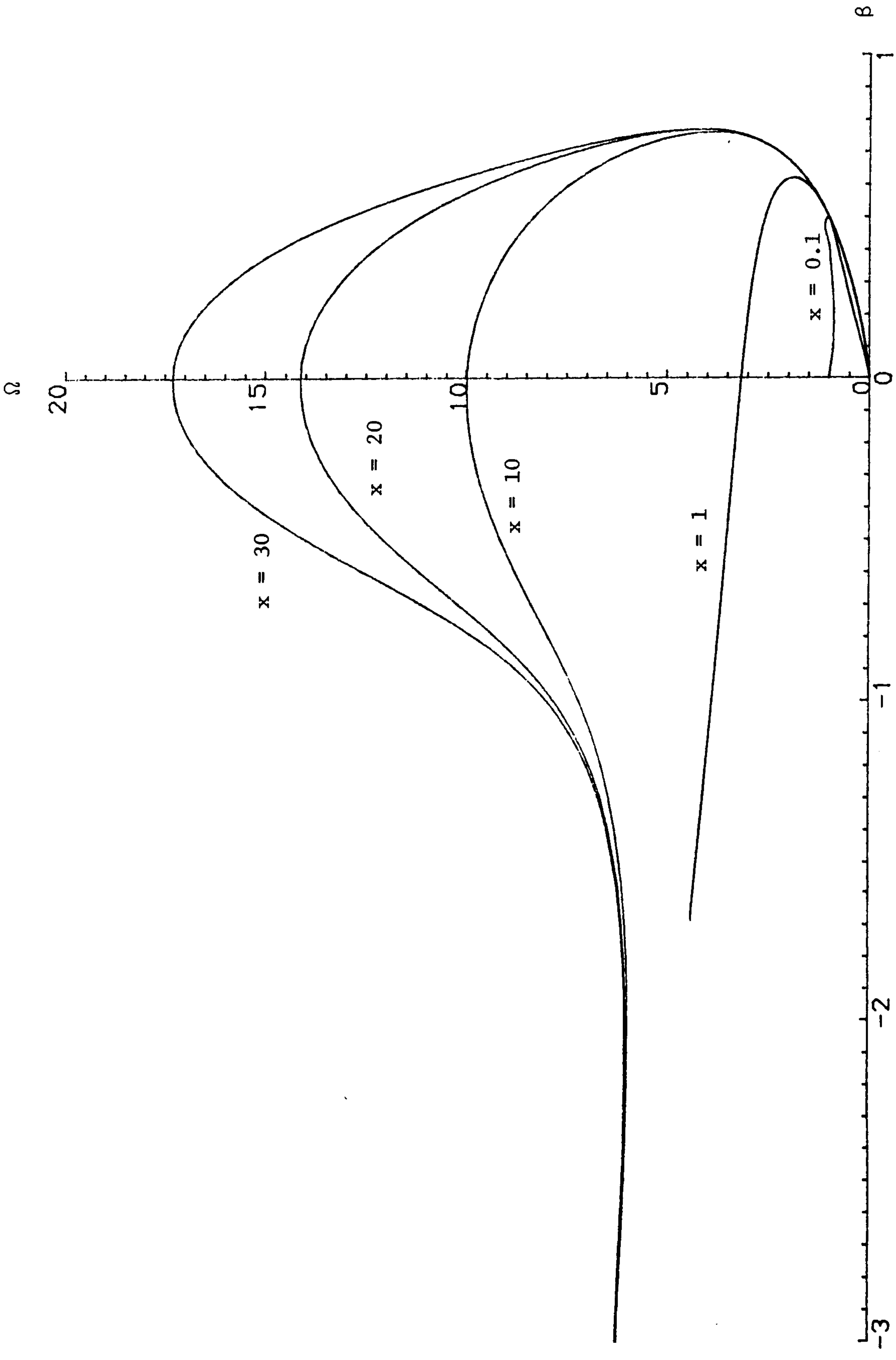


Figure 1a



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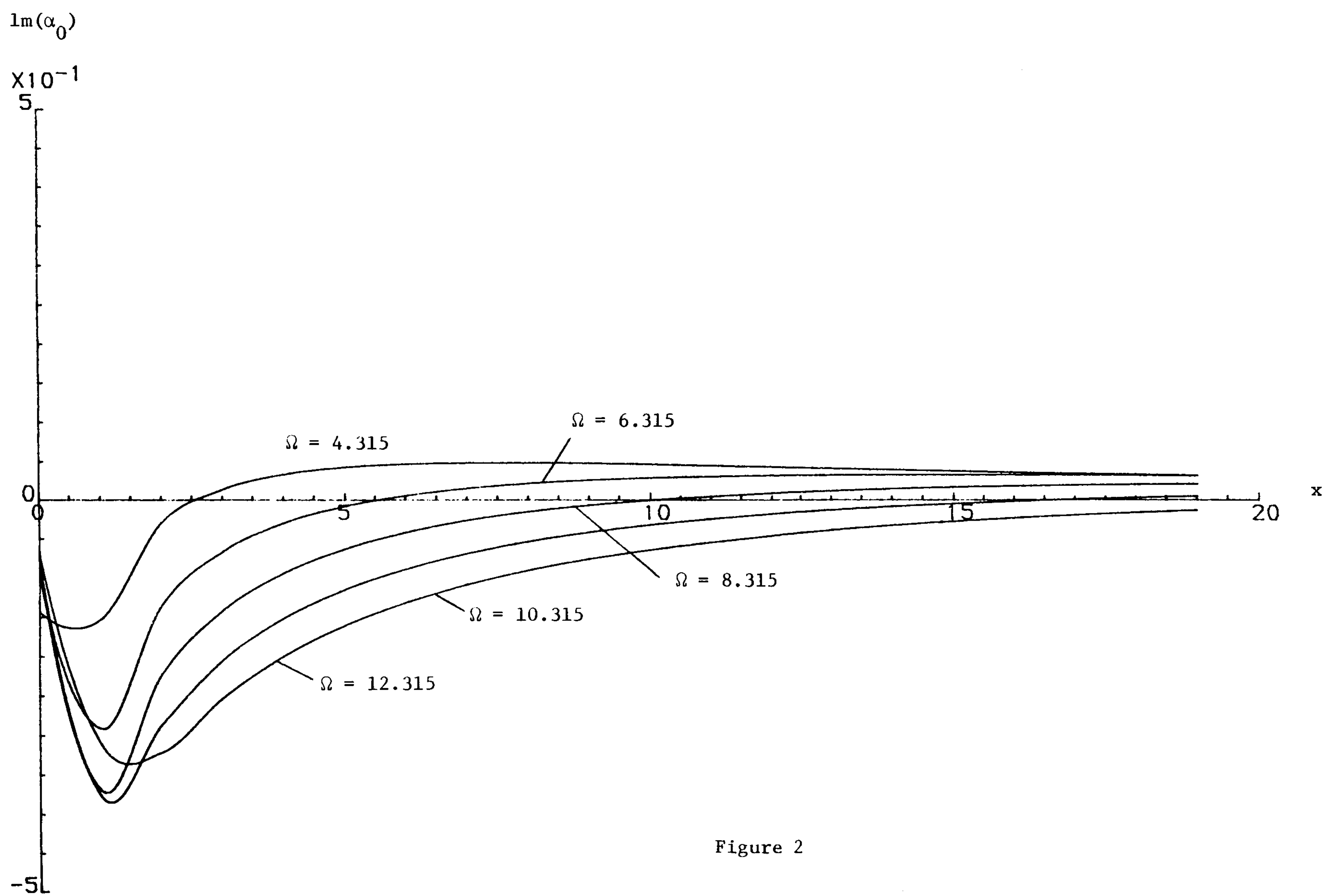

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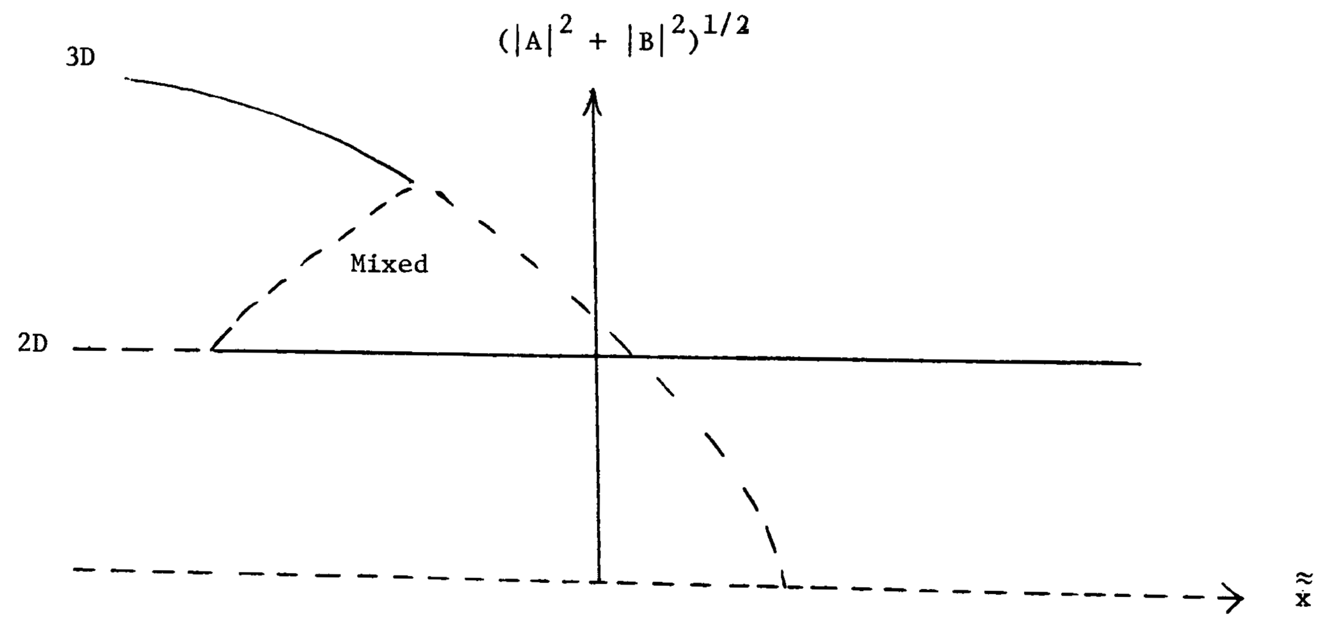

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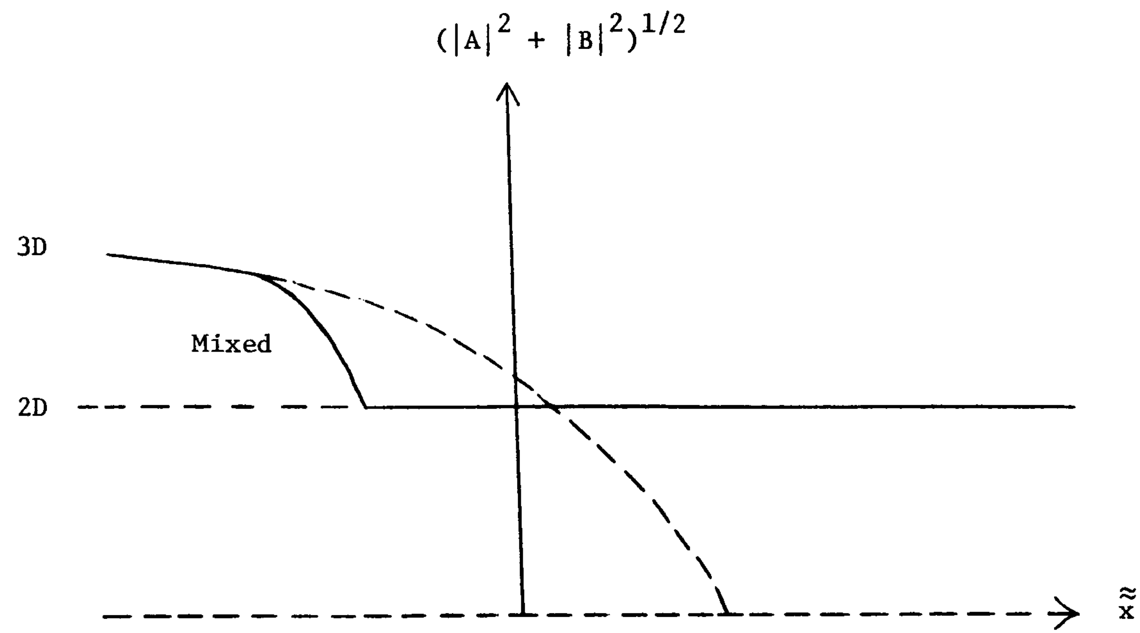


Figure 3b

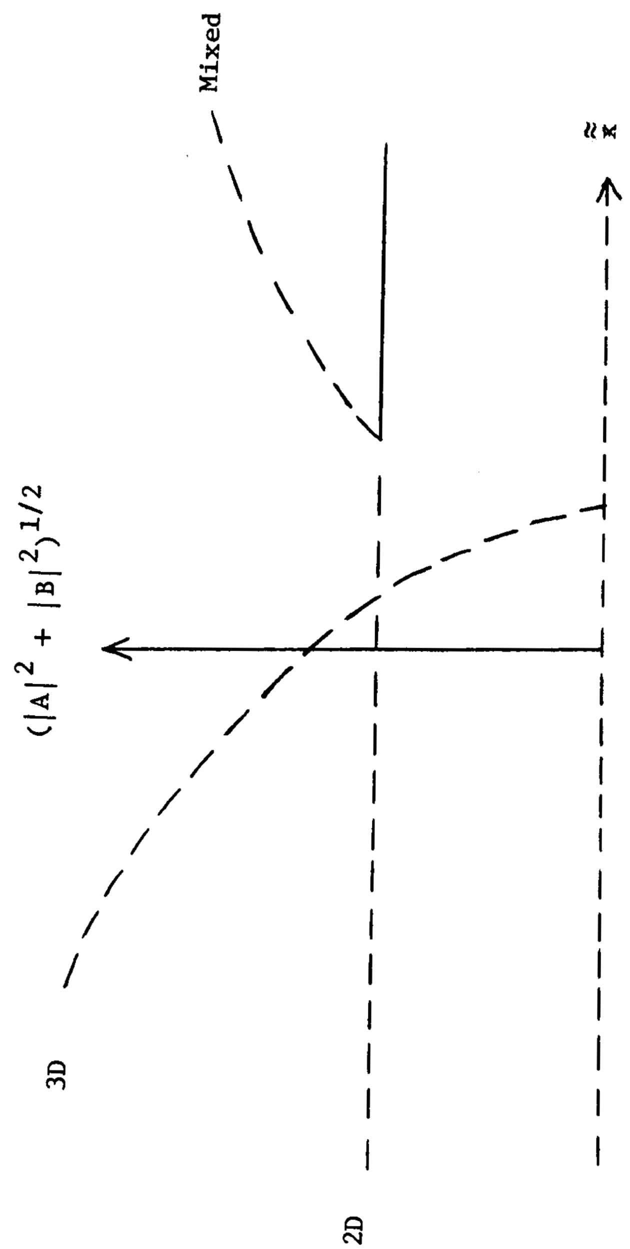


Figure 3c

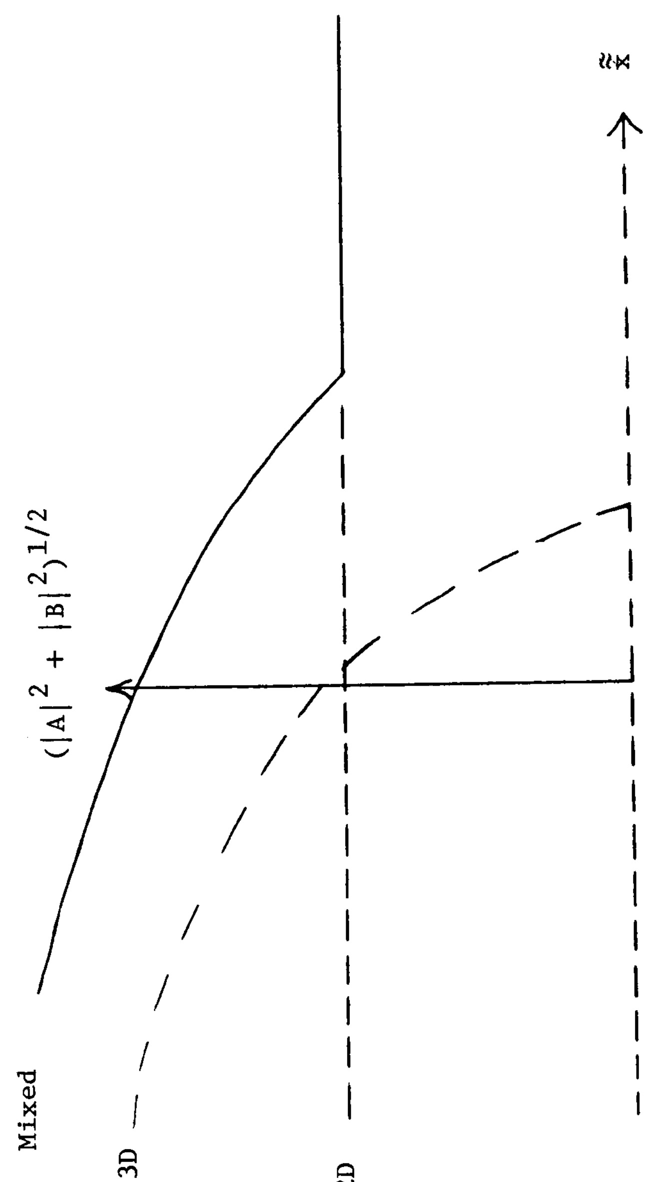
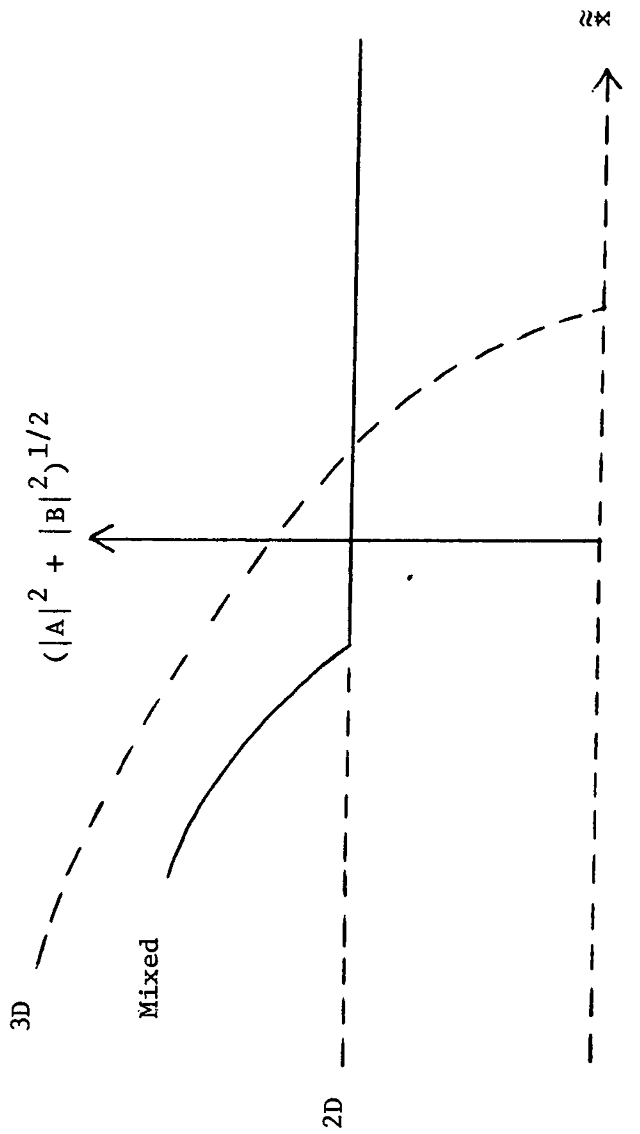


Figure 3d



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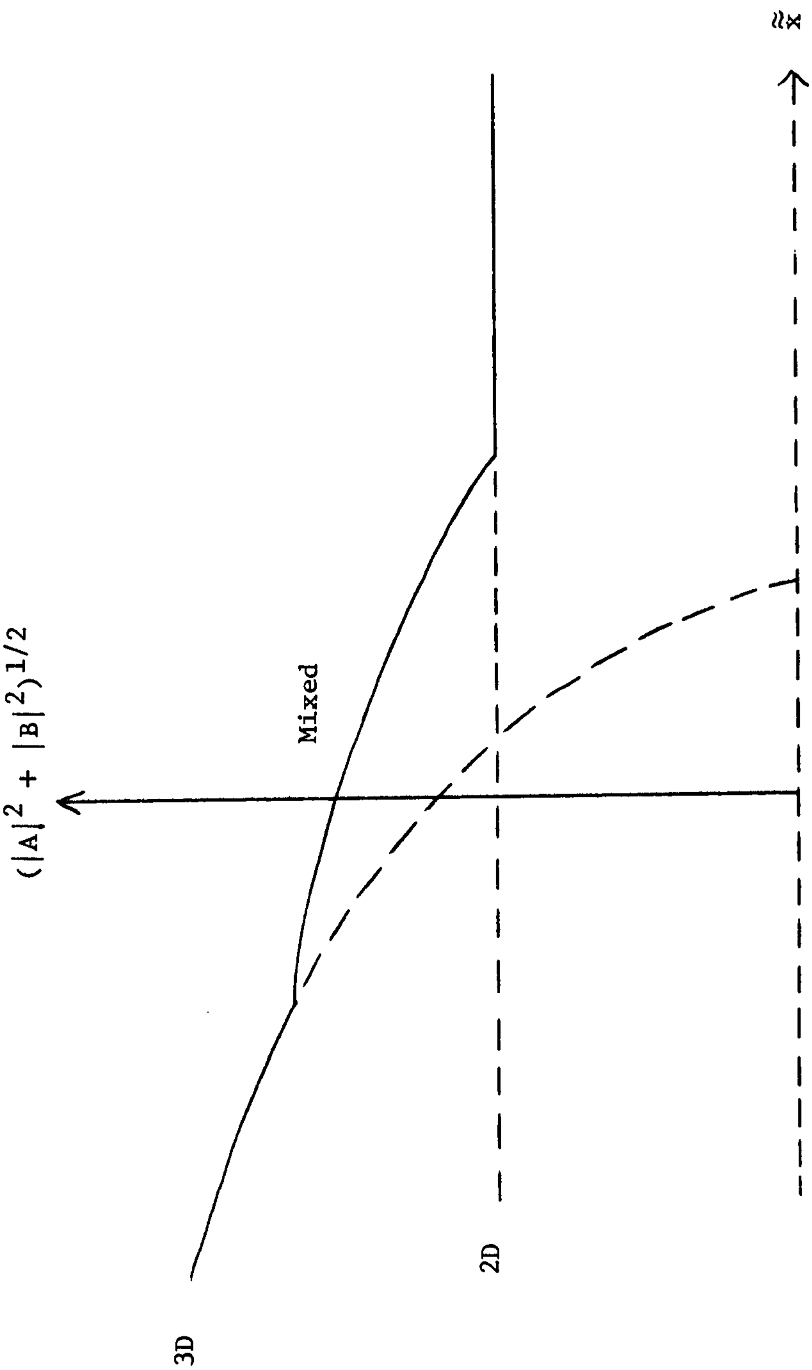

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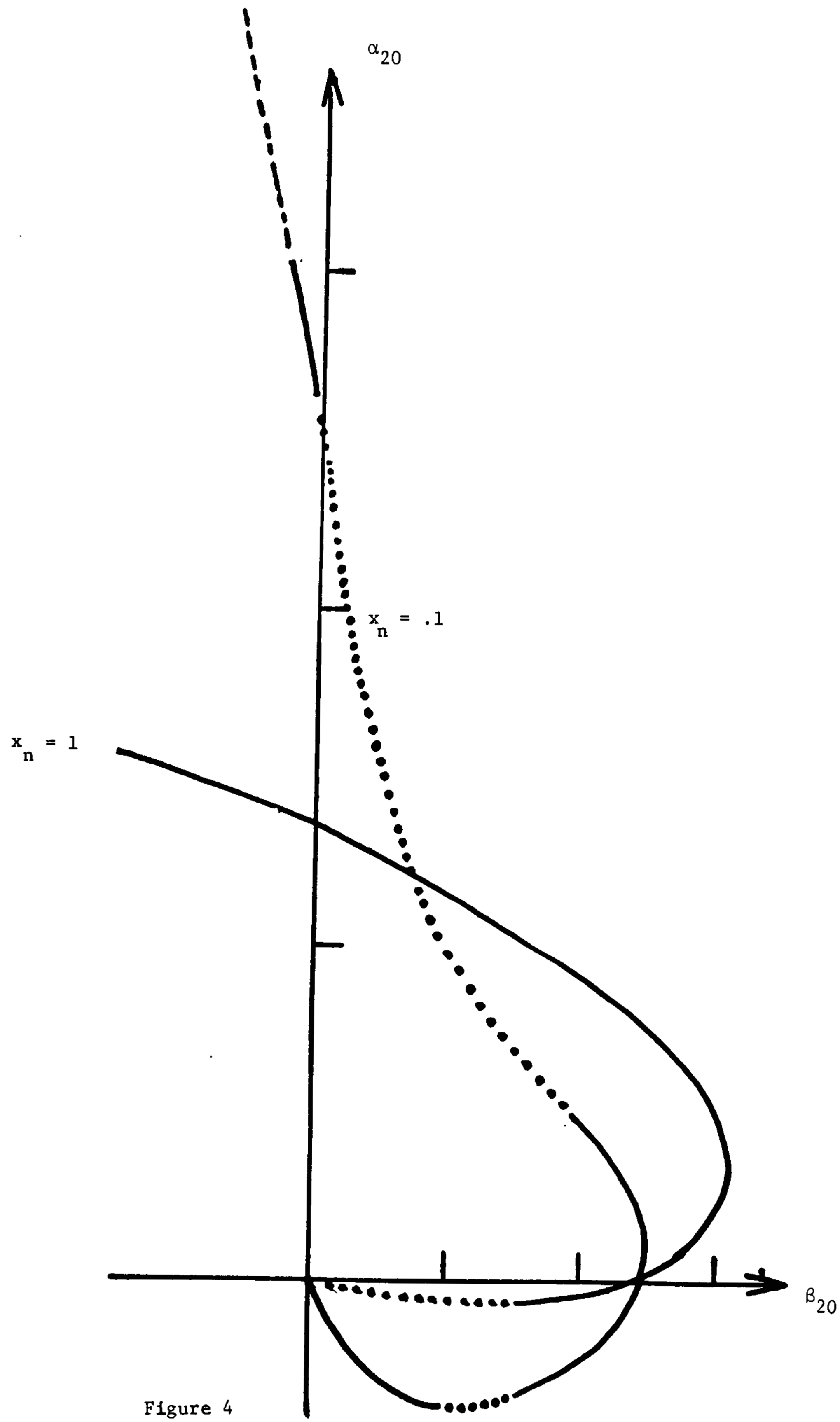

Figure 4

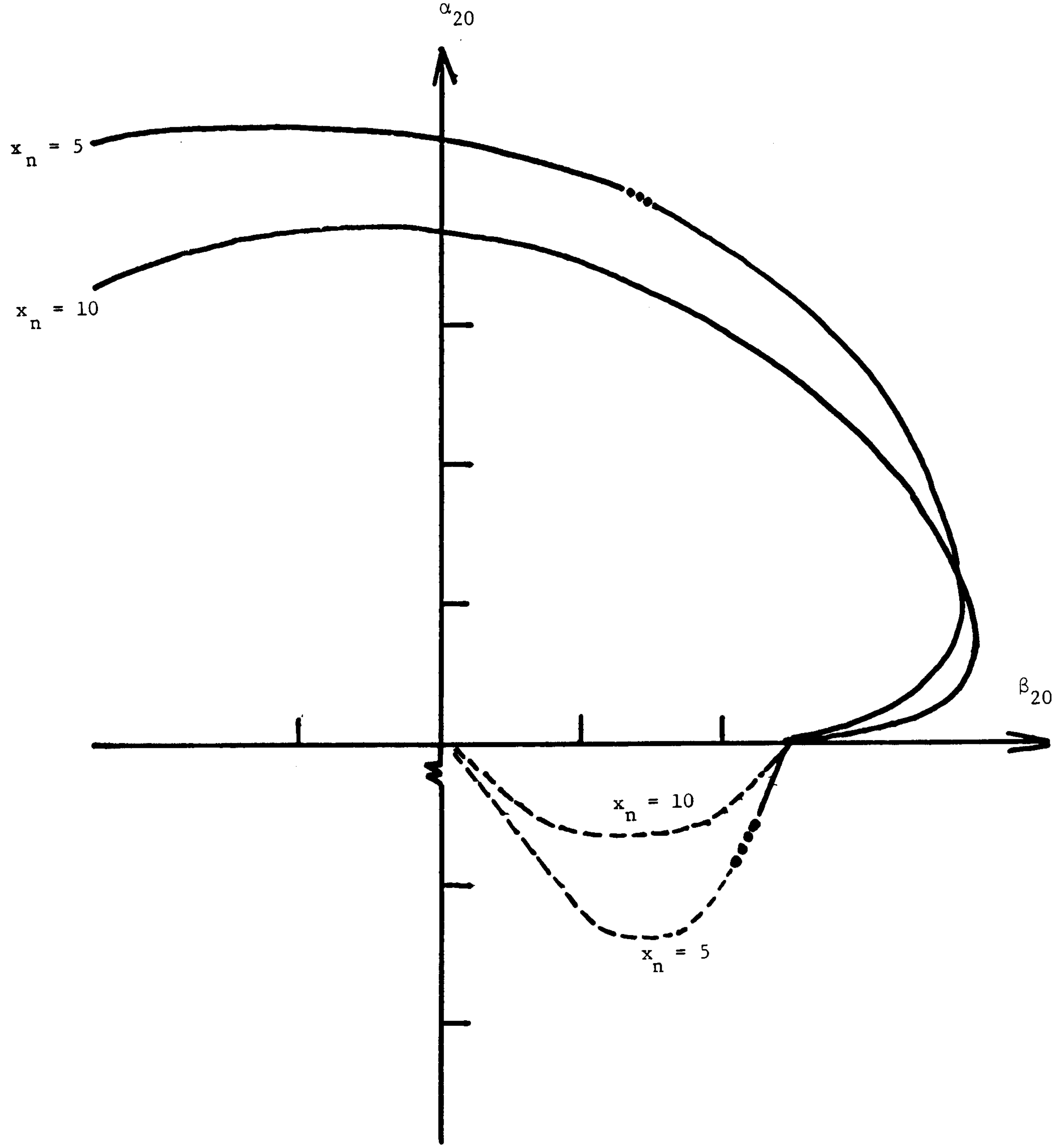

Figure 5

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Final Report

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16. Abstract

The three-dimensional boundary layer on a swept wing can support different types of hydrodynamic instability. Here attention is focused on the so-called 'spanwise contamination' problem which occurs when the attachment line boundary layer on the leading edge becomes unstable to Tollmien-Schlichting waves. In order to gain insight into the interactions which are important in that problem, a simplified basic state is considered. This simplified flow corresponds to the swept attachment line boundary layer on an infinite flat plate. The basic flow here is an exact solution of the Navier-Stokes equations and its stability to two-dimensional waves propagating along the attachment line can be considered exactly at finite Reynolds number. This has been done in the linear and weakly nonlinear regimes by Hall, Malik, and Poll (1984) and Hall and Malik (1986). Here the corresponding problem is studied for oblique waves and their interaction with two-dimensional waves is investigated. In fact oblique modes cannot be described exactly at finite Reynolds number so it is necessary to make a high Reynolds number approximation and use triple deck theory. It is shown that there are two types of oblique wave which, if excited, cause the destabilization of the two-dimensional mode and the breakdown of the disturbed flow at a finite distance from the leading edge. Firstly, a low frequency mode closely related to the viscous stationary crossflow mode discussed by Hall (1986) and MacKerrell (1987) is a possible cause of breakdown. Secondly, a class of oblique wave with frequency comparable with that of the two-dimensional mode is another cause of breakdown. It is shown that the relative importance of the modes depends on the distance from the attachment line.

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