N88-21508

A THEORY OF VISCOPLASTICITY ACCOUNTING FOR INTERNAL DAMAGE

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E-3956-10

A constitutive theory for use in structural and durability analyses of high-temperature isotropic alloys is presented. Constitutive equations based upon a potential function are determined from conditions of stability and physical considerations. The theory is self-consistent; terms are not added in an ad hoc manner. It extends a proven viscoplastic model by introducing the Kachanov-Rabotnov concept of net stress. Material degradation and inelastic deformation are unified; they evolve simultaneously and interactively. Both isotropic hardening and material degradation evolve with dissipated work which is the sum of inelastic work and internal work. Internal work is a continuum measure of the stored free energy resulting from inelastic deformation.

INTRODUCTION

The nucleation, growth and coalescence of voids and microcracks are physical phenomena that degrade a material's continuity. This degradation results in a loss of strength, and is the eventual cause of failure. Continuous damage mechanics applies whenever the distribution of defects does not include one or more dominating macroscopic cracks; otherwise, fracture mechanics applies. The subject of this paper falls under the topic of continuous damage mechanics; applications to fracture mechanics are not discussed.

A constitutive theory applicable to structural and durability analyses of high-temperature isotropic alloys is developed. A set of constitutive equations based on a single potential function is determined from stability conditions and physical considerations. A specific potential function from a proven viscoplastic theory is extended to account for internal damage by introducing the Kachanov - Rabotnov (refs. 1 and 2) concept of a net stress. Internal damage and inelastic deformation are unified in this approach; they evolve simultaneously and interactively. The theory is self-consistent in that it is derived from a potential function; terms are not added in an ad hoc manner. Other viscoplastic theories that incorporate continuous damage mechanics have been proposed. The evolutionary equations for material degradation in the theories of Chaboche (ref. 3), Bodner (ref. 4), and Walker and Wilson (ref. 5) are phenomonologically determined, whereas, the Perzyna theory (ref. 6) is micromechanistically based. In this paper the evolutionary equation for material degradation is derived from a potential function.

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Continuous damage mechanics deals with, an infinitesimal volume element of material (called a particle), whose dimensions are large enough to contain many material defects, yet small enough to be treated as a mathematical point in a continuum sense. Consider a face on such an element whose unit normal is given by n_i.[1] Let A denote its surface area in a flawless (or undamaged) state, and let A' denote its net surface area in the presence of material defects (or in a damaged state); thus $A' \leq A$. The internal damage associated with this particle, and in the direction of this unit normal, is defined by

$$\omega = \frac{A - A'}{A} \tag{1}$$

which is bounded by the interval $0 \leq \omega < 1$ where $\omega = 0$ in an undamaged state. Whenever the orientations of material defects have preferred directions, damage becomes a function of these directions resulting in an entity of tensorial nature (refs. 7 to 9); otherwise, damage is isotropic and can be represented by a scalar. In this paper, damage is taken to be isotropic as a simplifying assumption. Kachanov (ref. 1) calls the quantity $\psi = 1 - \omega$ the continuity of the material.

Consider once again a face on an infinitesimal material volume element. In an undamaged state, traction is the ratio of the force transmitted through the surface F_i to the surface area A. It is related to the unit normal n_i by a homogeneous linear operator σ_{ij} called the applied (or Cauchy) stress, that is[2]

$$\frac{F_i}{A} = \sigma_{ij} n_j \tag{2}$$

In a damaged state, traction becomes the ratio of the force transmitted through the surface F_i to the net surface area A'. It is related to the unit normal n_i by a homogeneous linear operator σ'_{ij} called the net (or Kachanov-Rabotnov) stress; thus

$$\frac{F_i}{A'} = \sigma'_{ij} n_j \tag{3}$$

Combining equations (1) to (3) results in

$$\sigma'_{ij} = \frac{\sigma_{ij}}{(1 - \omega)} \tag{4}$$

which relates the net stress to the applied stress.

Like the classical theories of creep and plasticity, strain ε_{ij} is given by the sum

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[1]All scalar, vector and tensor fields are defined at particles whose spatial coordinates are x_i at the instant t in a Cartesian reference frame.

[2]Repeated indices are summed over in the usual manner.

$$\varepsilon_{ij} = \varepsilon^{e}_{ij} + \varepsilon^{p}_{ij} \tag{5}$$

where ε^{e}_{ij} is the elastic strain and ε^{p}_{ij} is the inelastic (or plastic) strain. From a thermodynamic viewpoint, the elastic change in strain is the reversible portion of a change in strain, while the inelastic change in strain is the irreversible portion of that change in strain. Small displacements and rotations are assumed.

The elastic response of polycrystalline metals is given by the relationship

$$\varepsilon^{e}_{ij} = \frac{(1 + \nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij} + \alpha\ \Delta T\ \delta_{ij}}{E(1 - \omega)} \tag{6}$$

where ν is the Poisson ration, α is the mean coefficient of thermal expansion, ΔT is the temperature change and δ_{ij} is Kronecker's delta. Since E is Young's modulus in an undamaged state, and $E' = E(1 - \omega)$ can be considered as Young's modulus in a damaged state, we obtain the following expression:

$$\omega = \frac{(E - E')}{E} \tag{7}$$

This is a useful measure of internal damage, because it can be readily determined by experiment (ref. 10).

CONSTITUTIVE THEORY

Much of the essential structure in the classical theory of plasticity derives, not so much from thermodynamic concepts, but from concepts of material stability as described by Drucker (ref. 11). A single postulate of stability is sufficient to unify the description of inelastic behavior of time-independent materials under isothermal conditions. A dual postulate of stability has been applied by Ponter (ref. 12) to time-dependent materials whose hereditary behavior can be represented in terms of internal state variables ξ_α $(\alpha = 1,2,\ldots,n)$ and their conjugate thermodynamic forces f_α. In that work, small isothermal changes in stress at constant internal state are assumed to obey the inequality

$$d\sigma_{ij}\ d\dot{\varepsilon}^{p}_{ji} \geq 0 \tag{8}$$

where f_α and T are constant; whereas small isothermal changes in internal state at constant stress are assumed to satisfy the inequality

$$df_\alpha\ d\dot{\xi}_\alpha \geq 0 \tag{9}$$

where σ_{ij} and T are constant. In contrast, a thermodynamic counterpart to the second-order inequality in equation (9) is the restriction of positive internal dissipation

$$f_\alpha \dot{\xi}_\alpha \geq 0 \tag{10}$$

which is derived from the second law.

Since changes in inelastic strain rate and internal variable rates are path independent in the complete state space, the inequalities in equations (8) and (9) can be integrated along a straight-line path between two arbitrary states $(\sigma^1_{ij}, f^1_\alpha, T)$ and $(\sigma^2_{ij}, f^2_\alpha, T)$ resulting in the following inequality:

$$\left(\sigma^2_{ij} - \sigma^1_{ij}\right)\left(\dot{\varepsilon}^{p2}_{ji} - \dot{\varepsilon}^{p1}_{ji}\right) + \left(f^2_\alpha - f^1_\alpha\right)\left(\dot{\xi}^2_\alpha - \dot{\xi}^1_\alpha\right) \geq 0 \tag{11}$$

Along a constant stress path (i.e. under conditions of creep) only the last term in this inequality remains, and we can easily show that a sufficient condition for its satisfaction is

$$\dot{\xi}_\alpha = \frac{\partial \Omega}{\partial f_\alpha} \tag{12}$$

where $\Omega(\sigma_{ij}, f_\alpha, T)$ is convex and positive definite in σ_{ij}, f_α. Here we assumed that equation (12) is not constrained just to constant stress conditions, but is valid in general. Rice (ref. 13), Martin (ref. 14), and others have shown, using thermodynamic arguments, that if the kinetic (or evolutionary) law can be expressed as equation (12), then the flow law given by

$$\dot{\varepsilon}^p_{ij} = \frac{\partial \Omega}{\partial \sigma_{ij}} \tag{13}$$

is a derived result.

The criteria for stability and the resulting kinetic and flow laws lead to a vital theorem (ref. 12):

> "The stress and state histories are uniquely defined for time $t > t_0$ by the initial conditions at $t = t_0$ and the loading history."

The existence of this theorem is essential if this is to be a meaningful constitutive theory for use in structural analyses.

Following the lead of Ponter and Leckie (ref. 15) and Ponter (ref. 12), we adopted an additional constitutive assumption, that is

$$\frac{\dot{f}_\alpha}{h(f_\alpha)} = -\dot{\xi}_\alpha = -\frac{\partial \Omega}{\partial f_\alpha} \tag{14}$$

in which h is a hardening function of the internal force f_α. The physical origin of equation (14) in describing the local response of a crystallographic slip system, and the limitations that result in transferring from a local format to a global one, are discussed by Ponter and Leckie. An additional motive for adopting equation (14) comes from considering conditions in the neighborhood of a stress free state, as depicted in figure 1. In particular, for a "J_2-type" material (considered in the following section), the surface ($\Omega(0,f_\alpha,T)$ = constant) is a sphere of radius $|f_\alpha|$ in thermodynamic force space. The gradient vector $\partial\Omega/\partial f_\alpha$ at each point on the surface where Ω = constant is directed along the outward normal. By considering the constitutive assumption $\dot{f}_\alpha/h = -\dot{\xi}_\alpha$, the thermodynamic restriction in equation (10) can be expressed as $f_\alpha \dot{f}_\alpha/h \leq 0$, which for positive h constrains the vector $\dot{f}_\alpha/h$ to be contained within a half-sphere in thermodynamic force space. (See fig. 1.) The Ponter-Leckie constitutive assumption equation (14) selects the direction of $\dot{f}_\alpha/h$ so that its projection on f_α is a maximum; that is, the Ponter-Leckie constitutive assumption ensures that state recovery occurs under maximum internal dissipation in the neighborhood of a stress free state.

The extended normality structure expressed in equations (13) and (14) provides the basis for the present development. Moreover, this structure is assumed to hold under nonisothermal conditions.

A SPECIAL POTENTIAL FUNCTION

The governing differential equations of a theory of viscoplasticity that accounts for internal damage are taken to be associated with the normality structure of a potential function Ω as discussed in the previous section. The independent arguments of this potential function are the applied stress σ_{ij}, an internal stress β_{ij}, a threshold strength Z, the damage ω, and the temperature T; thus, $\Omega(\sigma_{ij},\beta_{ij},Z,\omega)$, where the temperature dependence is implicit. From a thermodynamic viewpoint, the internal stress and the threshold strength are averaged thermodynamic forces, and damage is an averaged internal variable (or thermodynamic displacement). The internal stress and threshold strength are associated with kinematic and isotropic hardening behaviors, whereas damage is associated with material degradation.

Moderate states of hydrostatic pressure have virtually no influence on the inelastic response of metals.[3] The stress dependence of Ω can therefore be expressed in terms of the deviatoric applied stress

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{15}$$

and the deviatoric internal stress

$$B_{ij} = \beta_{ij} - \frac{1}{3}\beta_{kk}\delta_{ij} \tag{16}$$

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[3]Moderate states of hydrostatic pressure have a strong influence on the formation and growth of material defects and, therefore, on the damage. This effect, however, is accounted for in the degradation function, not in the potential function.

where their difference

$$\Sigma_{ij} = S_{ij} - B_{ij} \tag{17}$$

is the effective stress associated with inelastic deformation. Under conditions of full isotropy, the invariants

$$I_2 = \frac{1}{2} B_{ij}B_{ji}$$
$$I_3 = \frac{1}{3} B_{ij}B_{jk}B_{ki} \tag{18}$$

and

$$J_2 = \frac{1}{2} \Sigma_{ij}\Sigma_{ji}$$
$$J_3 = \frac{1}{3} \Sigma_{ij}\Sigma_{jk}\Sigma_{ki} \tag{19}$$

provide a complete description of the stress dependence of Ω.

For the chosen potential function, the flow law is

$$\dot{\varepsilon}^p_{ij} = \frac{\partial\Omega}{\partial\sigma_{ij}} \tag{20}$$

and the evolutionary laws are taken to be

$$\frac{\dot{B}_{ij}}{2h_b} = - \frac{\partial\Omega}{\partial B_{ij}} \tag{21}$$

$$\frac{\dot{Z}}{h_z} = - \frac{\partial\Omega}{\partial Z} \tag{22}$$

and

$$\frac{\dot{\omega}}{D} = \frac{\partial\Omega}{\partial\omega} \tag{23}$$

in accordance with the results of the previous section. Here h_b and h_z are the hardening functions for the internal stress and the threshold strength, and D is a degradation function. Equation (20) is the flow law of Rice (ref. 13). Equations (21) and (22) are the evolutionary laws of Ponter and Leckie (ref. 15) and equation (23) is the proposed evolutionary law for damage.

Since damage is an internal variable, whereas internal stress and threshold strength are thermodynamic forces, the sign for equation (23) is different from that for equations (21) and (22). The reason for this difference is a Legendre transformation, like those used in equilibrium thermodynamics. These equations form the foundation for a theory of viscoplasticity that incorporates internal damage. A specific model is obtained by choosing a particular form for the potential function.

The potential function considered for this model is

$$\Omega = \int K^2 \frac{1}{2\mu} f(F)\ dF + \int K^2 g(G)\ dG + \int z(Z)\ dZ \tag{24}$$

where the stress dependence enters through the functions $F(\Sigma_{ij})$ and $G(B_{ij})$. This extends the function used by Robinson (ref. 16) to include isotropic effects. The fact that equation (24) is a sum of integrals is consistent with Rice's formulation (ref. 13). In his definition of the potential function, each integrand denotes the rate of change of a thermodynamic displacement (or internal variable), which is integrated with respect to its conjugate thermodynamic force.

In the spirit of von Mises (ref. 17), the stress dependence of F and G relies only on the second invariants; in particular,[4]

$$F = \frac{J_2}{K^2} - 1 \tag{25}$$

and

$$G = \frac{I_2}{K^2} \tag{26}$$

Equation (25) is a Bingham-Prager (refs. 18 and 19) yield condition with K denoting the yield strength in shear. Inelastic strain only occurs when $F > 0$; an elastic domain is defined by the inequality $F \leq 0$. The boundary between these two regions, $F = 0$, is a sphere in deviatoric stress space; it is the threshold or quasi-static yield surface. The origin of this sphere is at B_{ij}, and its radius is K. The inelastic domain, at a fixed inelastic state, consists of a nested family of spherical surfaces in deviatoric stress space; each is a surface of constant F, and thus of constant Ω. Viscoplasticity differs from classical plasticity in that stress states that lie outside the quasi-static yield surface are admissible; they are not admissible in classical plasticity.

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[4]Many theories of viscoplasticity take $F = J_2/K^2$ instead of equation (25); thus, there is no elastic domain. The only influence that this choice for F would have on the resulting theory is that the inequality would be removed from the flow function.

If the stress dependence of F and G is to be a net stress dependence in the sense of Kachanov (ref. 1) and Rabotnov (ref. 2), then K must be a linear homogeneous function of damage,

$$K = \kappa(Z)(1 - \omega) \tag{27}$$

where κ is the quasi-static yield strength in shear. Although damage, by definition, influences the state of stress, this influence is manifested by a reduction in strength as exemplified in equations (7) and (27). In equation (27) the reduction in strength due to material degradation competes with the process of hardening which enhances strength.

Given the potential function (eq. (24)), the flow law of equation (20) becomes Prager's flow equation (ref. 20),

$$2\mu \, \dot{\varepsilon}^{p}_{ij} = f(F)\Sigma_{ij} \tag{28}$$

where μ is the viscosity and f is the flow function. (This is derived in the appendix.) The Bingham-Prager yield condition (eq. (25)), constrains the flow function so that it is zero in the elastic domain. Coaxiality between the effective stress and the inelastic strain rate is implied in equation (28). Stability (in the sense of eq. (8)) constrains the flow function $f(F)$ to be nondecreasing with increasing values of F. Most theories of viscoplasticity use the general form of this flow equation.

Given the potential function (eq. (24)) the evolutionary law for internal stress (eq. (21)), becomes a Bailey-Orowan type relationship (refs. 21 and 22), that is

$$\dot{B}_{ij} = 2h_b(G)\dot{\varepsilon}^{p}_{ij} - r_b(G)B_{ij} \tag{29}$$

where h_b and r_b are the kinematic functions for hardening and thermal recovery. (Equation (29) is derived in the appendix.) The first term in this equation, for constant h_b, is Prager's rule for kinematic hardening (ref. 20). To model dynamic recovery of the internal stress, Robinson (refs. 23 and 24) presents a kinematic hardening function that exhibits an analytical discontinuity whenever there is a reversal in stress.

The second term in equation (29) accounts for the thermal recovery of the internal stress state. This is an anelastic response since it continues until the internal stress has relaxed to zero, regardless of whether the current deformation state is elastic or inelastic.

The experimental results of Mitra and McLean (ref. 25) verify the Bailey-Orowan hypothesis that inelastic deformation occurs as a result of two competing mechanisms: a hardening process that progresses with inelastic deformation, and a thermal recovery process that progresses with time. Whenever these two mechanisms balance such that $\dot{B}_{ij} = 0$, the internal stress is in a steady state. Stability (in the sense of eq. (9)) constrains the function $g(G) = r_b(G)/2h_b(G)$ to be nondecreasing with increasing values of G.

The general form of this evolutionary equation for the internal stress is used in many viscoplastic models.

Data from metals that strain-age indicate that the evolution of the quasi-static yield strength (defined in eq. (27)) depends on the history of thermo-mechanical loading (ref. 26). These data suggest an evolution such that

$$\dot{\kappa} = \Gamma \dot{Z} - \Theta(Z)\dot{T} \tag{30}$$

where the parameter Γ reflects the temperature dependence of the quasi-static yield strength in an annealed state, and the function Θ represents the change in quasi-static yield strength resulting from a change in temperature. The evolution of the quasi-static yield strength given by equation (30) is path independent whenever the following equation is satisfied:

$$\Theta(Z) = -Z \frac{d\Gamma}{dT} \tag{31}$$

If equation (30) is to be self-consistent, then $\dot{\kappa}$ must be path independent in the annealed state; therefore

$$\Theta(Z_a) = -Z_a \frac{d\Gamma}{dT} \tag{32}$$

where $0 < Z_a \leq Z$. This constraint must always be satisfied; it is like an initial condition for the functional dependence of Θ.

Given the potential function (eq. (24)) and the equation of evolution for the quasi-static yield strength (eq. (30)), the evolutionary law for the threshold strength (eq. (22)) becomes a Bailey-Orowan type relationship (refs. 21 and 22),

$$Z = h_z(Z) \left(\Gamma - \int_0^t \frac{\partial \Theta}{\partial Z} \dot{T} \, dt \right) \frac{\dot{W}}{\kappa} - r_z(Z) \tag{33}$$

where

$$\dot{W} = \sigma_{ij} \dot{\epsilon}^p_{ji} - \frac{\beta_{ij} \dot{B}_{ji}}{2h_b} (G) \tag{34}$$

and h_z and r_z are the isotropic functions for hardening and thermal recovery. (These equations are derived in the appendix.) Equation (33) implies that the path of thermomechanical loading influences the rate of isotropic hardening; under isothermal conditions it reduces to

$$\dot{Z} = \frac{h_z(Z)\dot{W}}{Z} - r_z(Z) \tag{35}$$

The first term in equations (33) and (35) implies that isotropic hardening progresses with dissipated work. This dissipated work, as defined in equation (34), is the sum of the inelastic work and internal work.[5] The internal work can be thought of as a continuum measure of the free energy stored in the material that arises from inelastic deformation. With the exception of the viscoplastic model of Bodner and Partom (ref. 27) (where kinematic hardening is not present, and isotropic hardening evolves with inelastic work), all viscoplastic models that incorporate isotropic hardening, to the best of our knowledge, assume that this process progresses with inelastic path length $\int(\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ji})^{1/2}\,dt$. This is an assumption that our theoretical derivation does not support.

The second term in equations (33) and (35) accounts for the anelastic thermal recovery (or annealing) of the threshold strength. This function must be constrained so that recovery terminates when the annealed value of threshold strength is obtained. Stability (in the sense of eq. (9)) constrains the function $z(Z) = r_Z(Z)/h_Z(Z)$ to be nondecreasing with increasing values of Z.

Given the potential function (eq. (24)), the evolutionary law for damage (eq. (23)) becomes

$$\omega = D(\omega)\frac{\dot{W}}{1 - \omega} \tag{36}$$

where D is the degradation function. (Equation (36) is derived in the appendix.)

Since materials do not degrade in states of sufficient hydrostatic compression, in general the degradation function ought to switch off the evolution of internal damage when a critical state of hydrostatic compression is reached. Equation (36) implies that damage evolves with dissipated work, as defined in equation (34). Hereditary effects are included through the dependence of dissipated work on inelastic strain and internal stress; thus, equation (36) has the potential to account for time-dependent effects in a natural way. Stability (in the sense of eq. (9)) is satisfied if $D(\omega)$ does not increase with increasing values of ω; but this is not observed. Initially the dissipation function is virtually a constant, and the material response is stable for all practical purposes. However, near the end of life, the value of the dissipation function explodes, thereby leading to material instability or failure. This is not to say that this theory is undesirable, for it is precisely this instability that continuum damage mechanics attempts to characterize.

Many researchers have used inelastic work as a parameter to characterize fatigue damage (e.g. refs. 28 to 31). The equation of damage evolution given in equation (36) differs from these earlier, largely empirical, energy criteria by including the influence of internal work. Albeit this is a lesser

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[5] In accordance with equations (14) and (21), the quantities $-\dot{B}_{ij}/2h_b$ denote the rates of change in the thermodynamic displacements (or internal variables) conjugate to the thermodynamic forces β_{ij}. Therefore, the quantity $-\beta_{ij}\dot{B}_{ji}/2h_b$ can be interpreted as the rate of change in the internal work.

effect than that of inelastic work, nevertheless, it is believed not to be a negligible one, especially when time-dependent effects are present. Additional insight is gained from an equivalent expression for the dissipated work rate (eq. (34)), that is

$$\dot{W} = \Sigma_{ij}\dot{\varepsilon}^{p}_{ji} + r_b(G)\,\frac{I_2}{h_b}\,(G) \tag{37}$$

which is obtained from equations (29) and (34). The first term in this relationship evolves with inelastic deformation and is a measure of fatigue damage. (See fig. 2.) The second term evolves with time at internal stress (I_2 is the second invariant of internal stress) and is a measure of the interactive creep damage. Thus, the interaction between fatigue and creep damage is specified. The temperature dependence for this measure of creep damage is accounted for, to a large extent, in the thermal recovery function r_b.

CONCLUDING REMARKS

A theory of viscoplasticity has been derived from conditions of stability and physical arguments, for an initially isotropic continua that exhibits internal damage. This material degradation was incorporated through the Kachanov-Rabotnov concept of a net stress. Damage was assumed to be an internal variable that evolves isotropically according to a Ponter-Leckie type constitutive assumption. A potential function was considered that extends the Robinson viscoplastic model by including the effects of isotropic hardening and material degradation. The yield strength was not considered to be an independent variable; rather, it was assumed to evolve with changes in threshold strength and temperature.

We determined that inelastic strain evolves according to a Prager type flow equation, and that Bailey-Orowan type kinetic equations govern the evolution of both internal stress and threshold strength. The internal stress hardens like a Prager hardening rule, whereas the threshold strength hardens with dissipated work - not inelastic path length - at a rate that depends on thermal history. Internal damage was shown to evolve with dissipated work leading to a loss of material stability. Dissipated work is the sum of inelastic work and internal work. Internal work is a continuum measure of the free energy stored in a material due to inelastic deformation.

APPENDIX

This appendix provides the derivations for the flow and evolutionary equations given in equations, (28), (29), (33), and (36).

By using equations, (15), (17), (19), (24), and (25), the flow law (eq. (20)) can be written as

$$\dot{\varepsilon}^{p}_{ij} = \frac{\partial\Omega}{\partial F}\frac{\partial F}{\partial J_2}\frac{\partial J_2}{\partial \Sigma_{uv}}\frac{\partial \Sigma_{vu}}{\partial S_{mn}}\frac{\partial S_{nm}}{\partial \sigma_{ij}} \tag{A1}$$

by the chain rule, where

$$\frac{\partial\Omega}{\partial F} = \frac{K^2 f(F)}{2\mu} \tag{A2}$$

$$\frac{\partial F}{\partial J_2} = \frac{1}{K^2} \tag{A3}$$

$$\frac{\partial J_2}{\partial \Sigma_{uv}} = \Sigma_{uv} \tag{A4}$$

$$\frac{\partial \Sigma_{vu}}{\partial S_{mn}} = \delta_{vm}\delta_{un} \tag{A5}$$

and

$$\frac{\partial S_{nm}}{\partial \sigma_{ij}} = \delta_{ni}\delta_{mj} - \frac{1}{3}\delta_{nm}\delta_{ij} \tag{A6}$$

Combining these equations results in

$$2\mu\, \dot{\varepsilon}^{p}_{ij} = f(F)\ \Sigma_{ij} \tag{A7}$$

which is the flow equation (28).

By utilizing equations (16) to (19) and (24) to (26), the evolutionary law for internal stress (eq. (21)) can be written as

$$\dot{B}_{ij} = -2h_b(G)\left(\frac{\partial\Omega}{\partial F}\frac{\partial F}{\partial J_2}\frac{\partial J_2}{\partial \Sigma_{uv}}\frac{\partial \Sigma_{vu}}{\partial B_{mn}} + \frac{\partial\Omega}{\partial G}\frac{\partial G}{\partial I_2}\frac{\partial I_2}{\partial B_{mn}}\right)\frac{\partial B_{nm}}{\partial \beta_{ij}} \tag{A8}$$

by the chain rule, where

$$\frac{\partial \Sigma_{vu}}{\partial B_{mn}} = -\delta_{vm}\delta_{un} \tag{A9}$$

$$\frac{\partial\Omega}{\partial G} = K^2 g(G) \tag{A10}$$

$$\frac{\partial G}{\partial I_2} = \frac{1}{K^2} \quad \text{(A11)}$$

$$\frac{\partial I_2}{\partial B_{mn}} = B_{mn} \quad \text{(A12)}$$

and

$$\frac{\partial B_{nm}}{\partial \beta_{ij}} = \delta_{ni}\delta_{mj} - \frac{1}{3}\delta_{nm}\delta_{ij} \quad \text{(A13)}$$

Combining equations (A2) to (A4) with equations (A8) to (A13), we obtain

$$\dot{B}_{ij} = h_b(G)f(F)\frac{\Sigma_{ij}}{\mu} - 2h_b(G)g(G)B_{ij} \quad \text{(A14)}$$

which when joined with the flow equation (eq. (A7)) results in

$$\dot{B}_{ij} = 2h_b(G)\dot{\varepsilon}^p_{ij} - r_b(G)B_{ij} \quad \text{(A15)}$$

where $r_b(G)$ is defined to be $2h_b(G)g(G)$. This is the evolutionary equation for internal stress given in equation (29).

We define the following expression:

$$\dot{W} = -K\left(\frac{\partial \Omega}{\partial F}\frac{\partial F}{\partial K} + \frac{\partial \Omega}{\partial G}\frac{\partial G}{\partial K}\right) \quad \text{(A16)}$$

From equations (18), (19), (25), and (26), we obtain

$$\frac{\partial F}{\partial K} = -\frac{\Sigma_{ij}\Sigma_{ji}}{K^3} \quad \text{(A17)}$$

$$\frac{\partial G}{\partial K} = -\frac{B_{ij}B_{ji}}{K^3} \quad \text{(A18)}$$

which when substituted into equation (A16), along with equations (A2) and (A10), gives

$$\dot{W} = \frac{f(F)\Sigma_{ij}\Sigma_{ji}}{2\mu} + g(G)B_{ij}B_{ji} \quad \text{(A19)}$$

By substituting equations (A7) and (A15) into this relationship, it becomes

$$\dot{W} = S_{ij}\dot{\varepsilon}^p_{ji} - \frac{B_{ij}\dot{B}_{ji}}{2h_b}(G) \quad \text{(A20)}$$

which can be expressed as

$$\dot{W} = \sigma_{ij}\dot{\varepsilon}^p_{ji} - \frac{\beta_{ij}\dot{B}_{ji}}{2h_b}(G) \tag{A21}$$

because of equations (15) and (16), and the fact that $\dot{\varepsilon}^p_{ij}$ and $\dot{B}_{ij}$ are deviatoric. This is the rate of dissipated work given in equation (34).

From equations (24) to (27), the evolutionary law for threshold strength (eq. (22)) can be written as

$$\dot{Z} = -h_Z(Z)\left[\left(\frac{\partial\Omega}{\partial F}\frac{\partial F}{\partial K} + \frac{\partial\Omega}{\partial G}\frac{\partial G}{\partial K}\right)\frac{\partial K}{\partial Z} + z(Z)\right] \tag{A22}$$

by the chain rule, where

$$\frac{\partial K}{\partial Z} = \frac{K}{\kappa}\frac{\partial\kappa}{\partial Z} \tag{A23}$$

and where (from eq. (30)),

$$\frac{\partial\kappa}{\partial Z} = \Gamma - \int_0^t \frac{\partial\Theta}{\partial Z}\dot{T}\,dt \tag{A24}$$

Joining equations (A16) and (A22) to (A24) results in

$$\dot{Z} = h_Z(Z)\left(\Gamma - \int_0^t \frac{\partial\Theta}{\partial Z}\dot{T}\,dt\right)\frac{\dot{W}}{\kappa} - r_Z(Z) \tag{A25}$$

where $r_Z(Z)$ is defined to be $h_Z(Z)z(Z)$. This is the evolutionary equation for threshold strength given in equation (33).

By using equations (24) to (27), the evolutionary law for damage (eq. (23)) can be written as

$$\dot{\omega} = D(\omega)\left(\frac{\partial\Omega}{\partial F}\frac{\partial F}{\partial K} + \frac{\partial\Omega}{\partial G}\frac{\partial G}{\partial K}\right)\frac{\partial K}{\partial\omega} \tag{A26}$$

by the chain rule, where

$$\frac{\partial K}{\partial\omega} = -\frac{K}{1-\omega} \tag{A27}$$

Combining equations (A16), (A26), and (A27) results in

$$\dot{\omega} = D(\omega)\frac{\dot{W}}{1-\omega} \tag{A28}$$

which is the evolutionary equation for damage given in equation (36).

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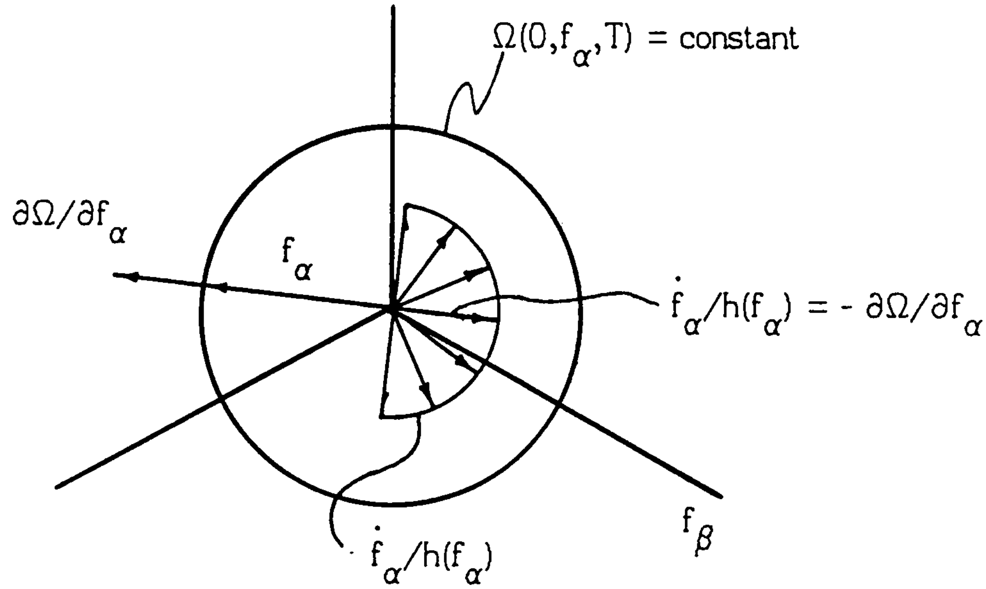

Figure 1. - In the stress free state $\dot{f}_\alpha/h(f_\alpha)$ provides maximum internal dissipation during state recovery.

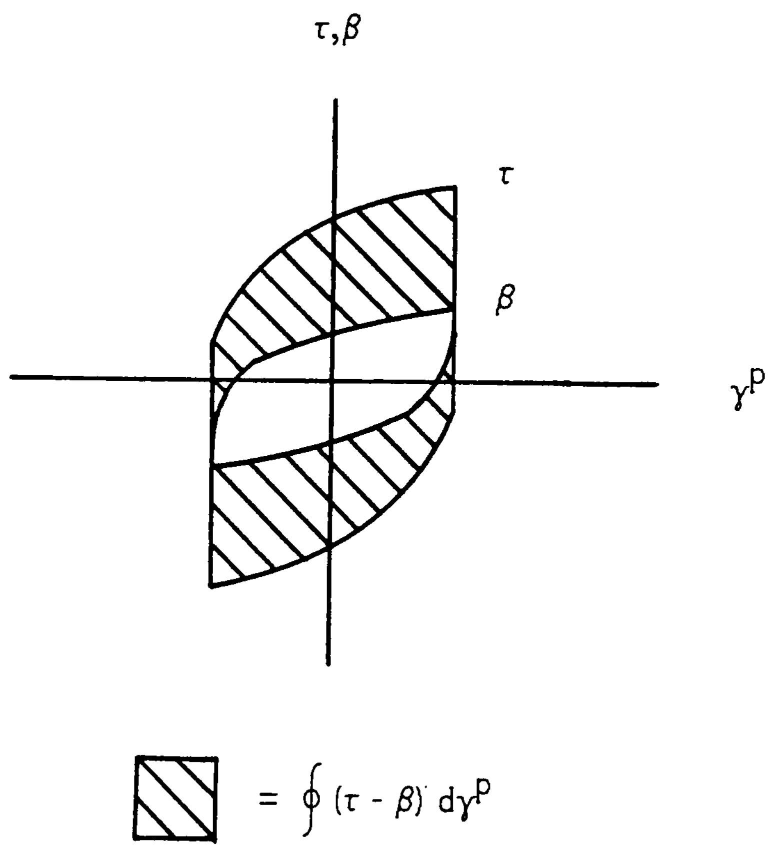

Figure 2. - Dissipated work in cyclic shear in the absence of thermal recovery.