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MULTIGRID FOR STRUCTURES ANALYSIS

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ABSTRACT

In structural analysis the amount of computational time necessary for a solution is proportional to the number of degrees of freedom times the bandwidth squared. In implicit time analysis, this solution must be calculated at each discrete point in time. If, in addition, the problem is nonlinear, then this solution must be iterated at each point in time. If the bandwidth is large, the size of the problem that can be analyzed is severely limited.

The multigrid method is a possible algorithm that can make this solution much more computationally efficient. This method has been used for years in computational fluid mechanics. It works on the fact that relaxation is very efficient on the high-frequency components of the solution (nearest-neighbor interactions) but is not very efficient on the low-frequency components of the solution (far interactions). The multigrid method relaxes the solution on a particular model until the residual stops changing, which indicates that the solution contains the higher frequency components. A coarse model is then generated and relaxed for the lower frequency components of the solution. These lower frequency components are then interpolated to the fine model.

In computational fluid mechanics the equations are usually expressed as finite differences. A coarse model is generated by just doubling the grid size and using a Green's integral theorem to obtain the forcing function on the coarse grid. Linear interpolation is used to transfer the lower frequency solution back to the fine grid.

In structural dynamics the equations are usually expressed as finite elements. Neighbor elements need not be connected. The process of condensing a fine model into a coarse model and interpolating the low-frequency solution to the fine model will be studied in this work.

OBJECTIVE

The objective of this work is to use an implicit time march solution to study nonlinear structural dynamics. The work will be done in three phases. The first phase, a beam structure, will have application in a multishaft, combined lateral, torsional, and axial rotordynamic analysis. The second phase, a plate structure, will have application in bladed disk vibration with coulomb damping. The third phase, a full three-dimensional structure, will have application in space structures.

To aid the reader, a symbols list has been included in the appendix.

IMPLICIT TIME MARCH SOLUTION OF NONLINEAR STRUCTURAL DYNAMICS

- **BEAM—MULTISHAFT, COMBINED LATERAL, TORSIONAL, AND AXIAL ANALYSIS**
- **PLATE—BLADE VIBRATION WITH COULOMB DAMPING**
- **THREE-DIMENSIONAL SPACE STRUCTURES ANALYSIS**

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NUMERICAL INTEGRATION

The numerical integration method is based on a Nordsieck-like method. The displacement, velocity, and acceleration are defined at an initial time. A modified Taylor series is used to calculate the displacement, velocity, and acceleration at the advanced time. The Lagrange remainder term, the time derivative of the acceleration, is calculated from the equations of motion at the advanced time. The constants α and β are determined so that the method is stable as time approaches infinity.

This method of integration for a first-order differential equation is Gear's method (Gear, 1971). Zeleznik showed that this method could be used on higher order equations (private communication with F.J. Zeleznik at NASA Lewis Research Center in 1979). Kascak (1980) showed that for a third-order integrator used on a linear second-order differential equation the method is unconditionally stable.

LET R(t) BE AN n ELEMENT VECTOR OF NODAL DISPLACEMENTS AND

$$V(t) = \dot{R} \qquad A(t) = \dot{V}$$

MODIFIED TAYLOR SERIES

$$R(t) = R(0) + V(0)t + \frac{1}{2} A(0)t^2 + \frac{1}{6} \alpha \dot{A}(\xi)t^3$$

$$V(t) = V(0) + A(0)t + \frac{1}{2} \beta \dot{A}(\xi)t^2$$

$$A(t) = A(0) + \dot{A}(\xi)t$$

WHERE

$$0 \leq \xi \leq t$$

AND α AND β ARE DETERMINED SO THAT THE METHOD IS NUMERICALLY STABLE AS $t \rightarrow \infty$

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NUMERICAL STABILITY

The numerical stability of the integration method can be examined by substituting the displacement, velocity, and acceleration into the linear equations of motion, and solving for the time derivative of the acceleration. As time approaches infinity the dominate term on each side of the equation has the stiffness matrix as a premultiplier. The time derivative of the acceleration is proportional to the initial acceleration divided by the time. If the time derivative is substituted into the modified Taylor series and if α is set to 3 and β is set to 2, then the acceleration is zero and the velocity is constant. The eigenvalues become zero and one.

$$MA + CV + KR = F$$

$$\left(tM + \frac{1}{2}\beta t^2 C + \frac{1}{6}\alpha t^3 K\right)\dot{A}(\xi) = F - \left(M + tC + \frac{1}{2}t^2 K\right)A(0) - (C + tK)V(0) - KR(0)$$

AS $t \rightarrow \infty$

$$\dot{A}(\xi) \approx -\left(\frac{3}{\alpha t}\right)A(0)$$

$$R = R(0) + V(0)t$$

$$V = V(0) + \left(1 - \frac{3}{2}\left(\frac{\beta}{\alpha}\right)\right)A(0)t$$

$$A = \left(1 - \left(\frac{3}{\alpha}\right)\right)A(0)$$

LET $\alpha = 3$ AND $\beta = 2$

$$\therefore R = R(0) + V(0)t \quad V = V(0) \quad A = 0$$

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ITERATIVE SOLUTION

If the initial displacement, velocity, and acceleration, and an initial estimate of the time derivative of the acceleration are given, then estimates of the advanced displacement, velocity, and acceleration are given by using the modified Taylor series. If a correction to the estimate of the time derivative of the acceleration is given, then new estimates of the displacement, velocity, and acceleration are given by the modified Taylor series. The correction to the time derivative of the acceleration can be found from the equations of motion.

GIVEN $R(0)$, $V(0)$, $A(0)$, AND $\dot{A}(\xi) \sim \dot{A}^{(0)}$

THEN
$$R^{(0)} = R(0) + V(0)t + \frac{1}{2} A(0)t^2 + \frac{1}{2} \alpha \dot{A}^{(0)} t^3$$

$$V^{(0)} = V(0) + A(0)t + \frac{1}{2} \beta \dot{A}^{(0)} t^2$$

$$A^{(0)} = A(0) + \dot{A}^{(0)} t$$

LET
$$\dot{A}(\xi) = \dot{A}^{(0)} + \Delta\dot{A}$$

THEN
$$R(t) = R^{(0)} + \frac{1}{6} \alpha \Delta\dot{A} t^3$$

$$V(t) = V^{(0)} + \frac{1}{2} \beta \Delta\dot{A} t^2$$

$$A(t) = A^{(0)} + \Delta\dot{A} t$$

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NONLINEAR EQUATIONS OF MOTION

The nonlinear equations of motion are the sum of both the static and dynamic forces for each element. As such, the equations are functions of the displacement, velocity, acceleration, and time. If the modified Taylor series is substituted into the equations of motion using the iterative form, then the equations of motion become a function of the correction to the time derivative of the acceleration.

$$0 = F(R, V, A, t)$$

WHERE F IS AN n ELEMENT VECTOR SUM OF THE STATIC AND DYNAMIC FORCES

THEN

$$0 = F\left(R^{(0)} + \frac{1}{6}\alpha\Delta\dot{A}t^3,\ V^{(0)} + \frac{1}{2}\beta\Delta\dot{A}t^2,\ A^{(0)} + \Delta\dot{A}t, t\right)$$

OR

$$0 = F(\Delta\dot{A})$$

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LINEARIZED EQUATIONS OF MOTION

To solve for the correction, the equations of motion are linearized about the estimated values. The instantaneous stiffness, damping, and mass are defined by the various partial derivatives with respect to displacement, velocity, and acceleration. If the linearization is done numerically, the stiffness, damping, and mass do not have to be calculated. The numerical differentiation of the correction to the time derivative of the acceleration is all that is needed.

This solution procedure is equivalent to the Newton-Raphson technique. The numerical differentiation and the solution of the linearized equations of motion are computationally time consuming, although straight forward. The multigrid technique could be potentially orders of magnitudes faster. The linearized equations of motion will be the basis for generating a coarse model from a fine model.

$$0 = F^{(0)} - B\Delta\dot{A}$$

WHERE

$$F^{(0)} = F(R^{(0)}, V^{(0)}, A^{(0)}, t)$$

$$B = \frac{1}{6}\alpha t^3 K + \frac{1}{2}\beta t^2 C + tM$$

$$K = -\frac{\partial F}{\partial R},\quad C = -\frac{\partial F}{\partial V},\quad M = -\frac{\partial F}{\partial A}$$

$$\therefore\ B\Delta\dot{A} = F^{(0)}$$

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STRUCTURAL CONDENSATION

If the linearized equation set is partitioned into nodes belonging to a coarse model (upper partition) and the nodes that are eliminated from the fine model (lower partition), then structural condensation can be used to solve for the coarse model. In addition, the structural condensation process can be used to interpolate the solution from the coarse model to the fine model. If the higher frequency part of the solution is found on the fine model and the lower frequency part of the solution is found on the coarse model, then the resultant forces must be zero. Thus the solution for the nodes eliminated from the fine model can be found.

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \Delta \dot{A}_1 \\ \Delta \dot{A}_2 \end{bmatrix} = \begin{bmatrix} F_1^{(f)} \\ F_2^{(f)} \end{bmatrix}$$

$$\left(B_{11} - B_{12} B_{22}^{-1} B_{21}\right) \Delta \dot{A}_1 = F_1^{(f)} - B_{12} B_{22}^{-1} F_2^{(f)}$$

$$\Delta \dot{A}_2 = B_{22}^{-1}\left(F_2^{(f)} - B_{21} \Delta \dot{A}_1\right)$$

$$\text{LET } B^{(c)} = B_{11} - B_{12} B_{22}^{-1} B_{21}, \quad F^{(c)} = F_1^{(f)} - B_{12} B_{22}^{-1} F_2^{(f)}$$

$$\therefore B^{(c)} \Delta \dot{A}_1 = F^{(c)}$$

$$\text{IF } F_2^{(f)} = 0 \Rightarrow \Delta \dot{A}_2 = -B_{22}^{-1} B_{21} \Delta \dot{A}_1 \text{ (INTERPOLATOR)}$$

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FINE-TO-COARSE AND COARSE-TO-FINE MODEL TRANSFORMATIONS

The fine-to-coarse model transformation is a rectangular matrix that averages the force from the fine model to the coarse model. The upper partition is an identity matrix, and the lower partition is defined in the structural condensation process. The coarse-to-fine transformation interpolates the correction of the time derivative of the acceleration from the coarse to fine model. In the symmetric case, the fine-to-coarse transformation is the transpose of the coarse-to-fine transformation.

FINE-TO-COARSE MODEL TRANSFORMATION

$$\Phi = \begin{bmatrix} I \\ \hline -B_{22}^{-1}B_{21} \end{bmatrix} \Rightarrow \Delta\dot{A} = \Phi\Delta\dot{A}_1$$

COARSE-TO-FINE MODEL TRANSFORMATION

$$\theta = \left[I \;\vdots\; -B_{12}B_{22}^{-1} \right] \Rightarrow F^{(c)} = \theta F^{(f)}$$

$$\therefore B\,\Delta\dot{A} = F^{(f)} \Rightarrow \theta B \Phi \Delta\dot{A}_1 = \theta F^{(f)}$$

OR

$$B^{(c)}\Delta\dot{A}_1 = F^{(c)}$$

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NONLINEAR CONDENSATION

The nonlinear condensation process transforms the independent variables from the coarse model to the fine model and the dependent variables from the fine to coarse model. Thus the resultant forces are relaxed on the coarse model. This would only require the inversion of a diagonal matrix. The corrections on the coarse model are then interpolated to the fine model. The linearization of the equations of motion are not needed in the solution process, but are needed only to define the transformations.

$$0 = F(\Delta\dot{A}) \Rightarrow 0 = \theta F(\Phi\Delta\dot{A}_1)$$

RELAXATION

$$0 = \theta F^{(f)} - D\Delta\dot{A}_1 \Rightarrow \Delta\dot{A}_1 = D^{-1}F^{(c)}$$

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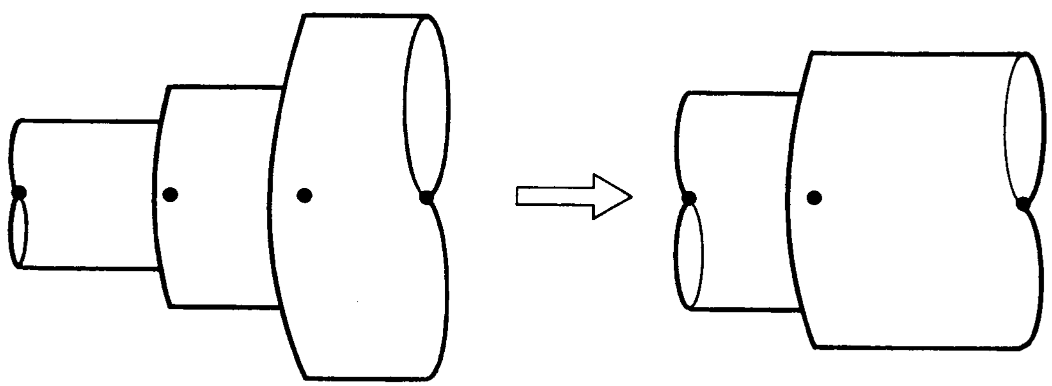
LOCAL STRUCTURAL CONDENSATION

The linearization of the equations of motion and the structural condensation process requires a considerable amount of computational time. Multigrid via relaxation is most efficient on nearest neighbor interactions. Thus only a partial linearization of the equations of motion is necessary. The equations of motion have to be linearized only with respect to the node under consideration and its nearest neighbors. Applying condensation to this local interaction model results in local structural condensation. In the case of a beam, this linearization results in a block tridiagonal matrix and the structural condensation results in a coarse model in which every other node is removed from the fine model.



B IS BLOCK TRIDIAGONAL—INCLUDES NEAREST NEIGHBOR INTERACTION, NEGLECTS FAR INTERACTION

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BEAM EXAMPLE

If the tridiagonal equation set is reordered so that the even numbers are in the upper partition for both the fine and coarse model and the odd numbers are in the lower partition for the fine model, then the structural condensation has a simple form. In the reordered equation set, the block matrices on the diagonal of the partitions are diagonal. The inversions of these block matrices are trivial.

$$\begin{bmatrix} Z_1W_1 & & & & & & & & \\ U_2Z_2W_2 & & & & & & & \bigcirc & \\ & U_3Z_3W_3 & & & & & & & \\ & & U_4Z_4W_4 & & & & & & \\ & & & U_5Z_5W_5 & & & & & \\ & & & & U_6Z_6W_6 & & & & \\ \bigcirc & & & & & U_7Z_7W_7 & & & \\ & & & & & & U_8Z_8W_8 & & \\ & & & & & & & U_9Z_9 & \end{bmatrix} \begin{bmatrix} \Delta\dot{A}_1 \\ \Delta\dot{A}_2 \\ \Delta\dot{A}_3 \\ \Delta\dot{A}_4 \\ \Delta\dot{A}_5 \\ \Delta\dot{A}_6 \\ \Delta\dot{A}_7 \\ \Delta\dot{A}_8 \\ \Delta\dot{A}_9 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{bmatrix}$$

$$B\Delta\dot{A} = F^{(f)}$$

$$\left[\begin{array}{cc|cc} Z_2 & \bigcirc & U_2W_2 & \bigcirc \\ Z_4 & & U_4W_4 & \\ \bigcirc & Z_6 & \bigcirc & U_6W_6 \\ & Z_8 & & U_8W_8 \\ \hline W_1 & \bigcirc & Z_1 & \\ U_3W_3 & & Z_3 & \bigcirc \\ U_5W_5 & & Z_5 & \\ \bigcirc & U_7W_7 & \bigcirc & Z_7 \\ & U_9 & & Z_9 \end{array}\right] \left[\begin{array}{c} \Delta\dot{A}_2 \\ \Delta\dot{A}_4 \\ \Delta\dot{A}_6 \\ \Delta\dot{A}_8 \\ \hline \Delta\dot{A}_1 \\ \Delta\dot{A}_3 \\ \Delta\dot{A}_5 \\ \Delta\dot{A}_7 \\ \Delta\dot{A}_9 \end{array}\right] = \left[\begin{array}{c} F_2 \\ F_4 \\ F_6 \\ F_8 \\ \hline F_1 \\ F_3 \\ F_5 \\ F_7 \\ F_9 \end{array}\right]$$

$$\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] \left[\begin{array}{c} \Delta\dot{A}_1 \\ \hline \Delta\dot{A}_2 \end{array}\right] = \left[\begin{array}{c} F_1^{(f)} \\ \hline F_2^{(f)} \end{array}\right]$$

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SOLUTION OF BEAM EXAMPLE

The solution for the nonidentity partition of both transformations is tridiagonal. For the fine-to-coarse transformation, the nonidentity partition is also lower triangular. For the coarse-to-fine transformation, the nonidentity partition is also upper triangular.

$$-B_{22}^{-1}B_{21} = \begin{bmatrix} T_1 & & & \\ S_2 & T_2 & & \\ & S_3 & T_3 & \\ & & S_4 & T_4 \\ & & & S_5 \end{bmatrix} \qquad T_L = -Z_{2L-1}^{-1}W_{2L-1} \qquad S_L = -Z_{2L-1}^{-1}U_{2L-1}$$

$$-B_{12}B_{22}^{-1} = \begin{bmatrix} X_1 & Y_1 & & & \\ & X_2 & Y_2 & & \\ & & X_3 & Y_3 & \\ & & & X_4 & Y_4 \end{bmatrix} \qquad X_L = -U_{2L}Z_{2L-1}^{-1} \qquad Y_L = -W_{2L}Z_{2L+1}^{-1}$$

$$B_{11} - B_{12}B_{22}^{-1}B_{21} = \begin{bmatrix} Z_1^{(c)} & W_1^{(c)} & & \\ U_2^{(c)} & Z_2^{(c)} & W_2^{(c)} & \\ & U_3^{(c)} & Z_3^{(c)} & W_3^{(c)} \\ & & U_4^{(c)} & Z_4^{(c)} \end{bmatrix} \qquad \begin{aligned} Z_L^{(c)} &= Z_{2L} + X_L W_{2L-1} + Y_L U_{2L+1} \\ U_L^{(c)} &= X_L U_{2L-1} \\ W_L^{(c)} &= Y_L W_{2L+1} \end{aligned}$$

$$\Delta\dot{A}_{2L} = \Delta\dot{A}_L^{(c)} \qquad \Delta\dot{A}_{2L-1} = S_L\Delta\dot{A}_{L-1}^{(c)} + T_L\Delta\dot{A}_L^{(c)}$$

$$F_L^{(c)} = F_{2L} + X_L F_{2L-1} + Y_L F_{2L+1}$$

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ACCELERATION PARAMETER

Normally the relaxation technique can be improved by using a weighted average of the previous and present calculated values of the corrections to the solution (overrelaxation). The rate of convergence of the high-frequency components can be improved at the expense of the low-frequency components. For this improvement, an estimate of the highest frequency eigenvalue is needed. The Rayleigh quotient is a good method to estimate the highest eigenvalue (at least in the symmetric case). In addition the highest eigenvalue should be a strong function of the nearest neighbors, therefore local linearization could be used in the Rayleigh quotient.

$\epsilon(\lambda)$ IS BASED ON LOCAL COEFFICIENTS

$$\lambda = \frac{(\Delta \dot{\vec{A}})^T D (\Delta \dot{\vec{A}})}{(\Delta \dot{\vec{A}})^T (\Delta \dot{\vec{A}})}$$

THIS IS THE RAYLEIGH QUOTIENT

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In summary, the multigrid method for structural dynamics is performed as follows. First relax the equations of motion on the fine grid to obtain the high-frequency components of the solution. Then calculate the norm of the residual on the fine model. Next check to see that the norm is small enough for a solution. If not, check to see if the norm has changed significantly from the previous iteration. If the norm has changed, then relax the solution until the norm stops changing. This indicates that the high-frequency components on this model have been found.

To find the lower frequency components of the solution, use the local structural condensation to generate a coarse model. On the coarse model, use relaxation to generate the lower frequency components of the solution. These lower frequency components are interpolated to the fine grid where the norm of the residual is calculated. Based on this norm, either a solution is found, more relaxation is needed, or a coarser model is needed. The process is repeated until a solution is found.

- **RELAX ON FINE GRID TO GET HIGH-FREQUENCY COMPONENT**
- **CALCULATE RESIDUAL ON FINE GRID**
- **CHECK RESIDUAL FOR SOLUTION**
- **CHECK CHANGE IN RESIDUAL FOR CHANGE IN GRID**
- **STATIC CONDENSE TO COARSE GRID**
- **RELAX ON COARSE GRID TO GET LOW-FREQUENCY COMPONENT**
- **INTERPOLATE LOW-FREQUENCY TO FINE GRID**

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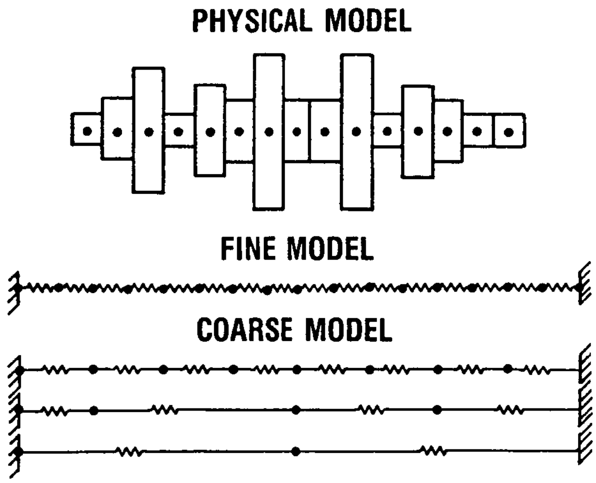

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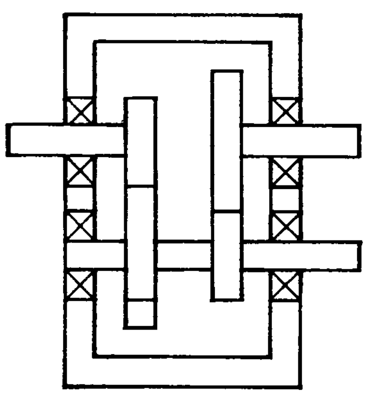
MULTIGRID ANALYSIS APPLIED TO TRANSMISSION DYNAMICS

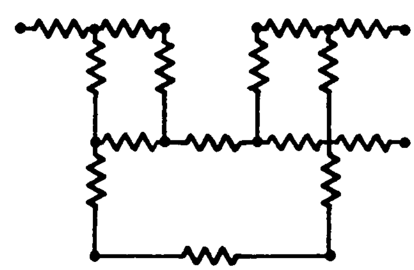
Complete transmission dynamic analyses are rare in the open literature. David and Mitchell (1986) have used a modal balance technique. The problem with modal techniques is that the nonlinearities cause the set of modes not to be closed. This results in side bands around the tooth passing frequency. Therefore, the solutions may not always include all of the important modes. Also, superfluous modes tend to overwhelm the solution technique. The time march multigrid method should eliminate these problems.

TRANSMISSION DYNAMIC ANALYSIS

Transmission dynamics is a case of nonlinear structural dynamics. Physically a transmission is composed of gears, shafts, bearings, seals, and a case. The case and the shafts can be modeled by finite element methods. The bearings and seals are modeled by special programs developed in tribology and other areas. Gear interactions are developed for some kinds of gears, but not for others. Thus a transmission can be modeled by a number of linear and nonlinear finite elements. As a first approximation, a transmission can be modeled as a beam structure. The transmission can be analyzed as a multishaft, combined lateral, torsional, and axial rotordynamic system.



PHYSICAL MODEL



NONLINEAR FINITE ELEMENT MODEL

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SPECIAL FEATURES

Special features complicate the dynamic analysis of transmissions. Gyroscopic and gear forces cause nonlinear lateral and torsional coupling. Gear-tooth passing frequencies are high-frequency forcing functions and, therefore, imply a need for a fine structural model. Gear-gear interactions cause the system to have a wide bandwidth.

- **GYROSCOPIC AND GEAR FORCES CAUSE NONLINEAR LATERAL AND TORSIONAL COUPLING**
- **GEAR-TOOTH PASSING FREQUENCIES ARE HIGH-FREQUENCY FORCING FUNCTIONS— IMPLIES NEED FOR FINE STRUCTURAL MODEL**
- **GEAR-GEAR INTERACTIONS CAUSE WIDE BAND WIDTH**

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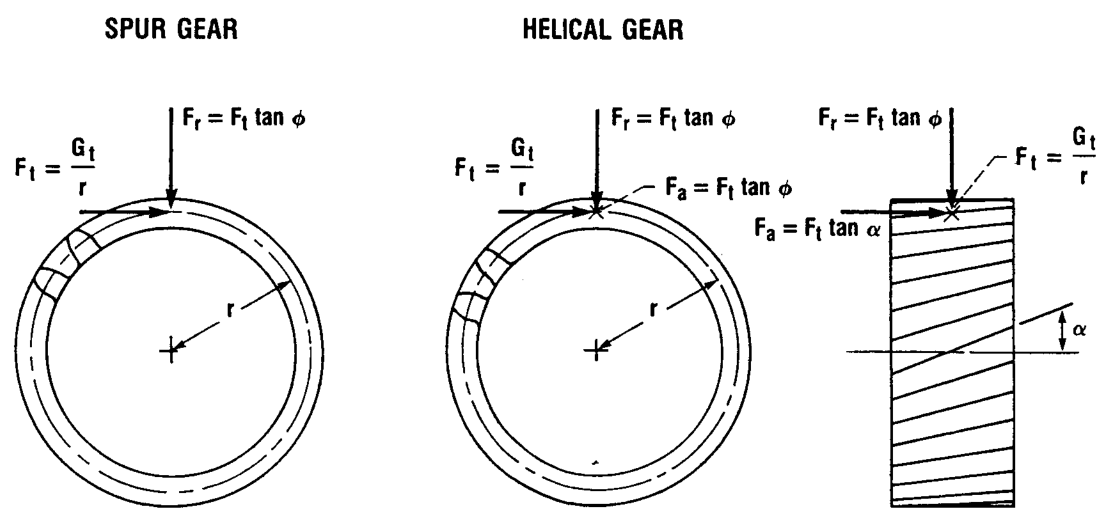
GEAR TOOTH INTERACTION

Consider gear tooth interaction. For any gear set, the line of force does not pass through the gear centers. In the case of spur gears any perturbation of the radial force will result in a perturbation of the tangential force and vice versa. In the case of helical or spiral gears any perturbation of the radial force will result in perturbations of both the axial and tangential force. These perturbations result in a nonlinear coupling between the axial, tangential, and radial directions.



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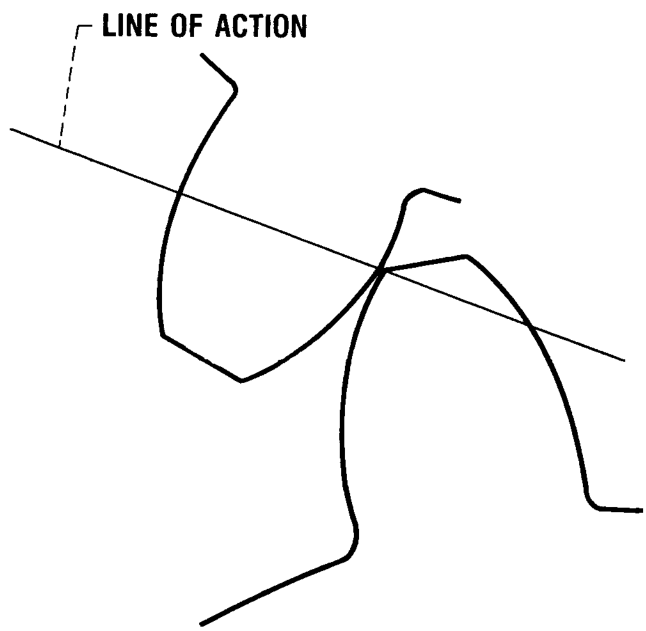
GEAR TOOTH MODEL

Consider the gear tooth interactions of a spur gear set. The contact point varies as the angle of the gear set varies. Machining errors cause the contact point to move. High torque can cause the teeth to bend. The number of teeth in contact varies as the torque varies. Negative torque can result in backlash. The force must be transmitted through the contact point. All these effects cause nonlinear time varying interactions between the spur gears set. For the other kind of gears the interaction is more complicated. Thus, gear tooth interactions cause high-frequency forcing functions on the structure.



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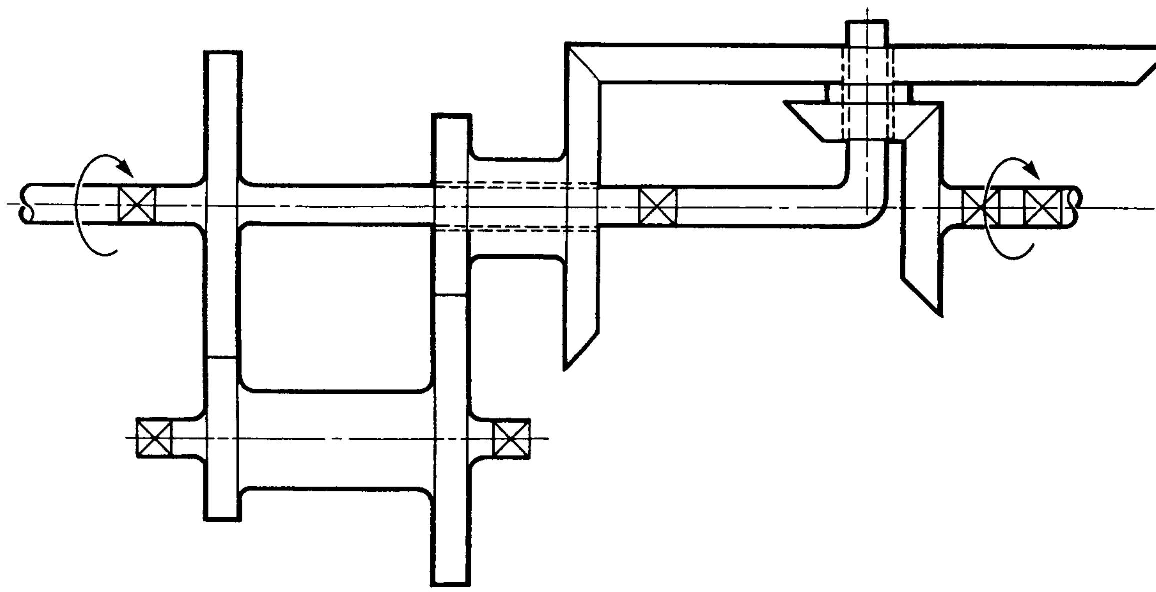
TYPICAL TRANSMISSION

In a typical transmission there are many gear sets. Each of these gear sets causes one location on the structure to interact with another point on the structure. Thus, far interactions are important and the structural model has a wide bandwidth.



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POTENTIAL IMPACT

The multigrid method, although used for years in fluid dynamics, now offers a new approach to nonlinear structural dynamics. The computing time does not depend on the cube of the number of degrees of freedom. Thus, dramatic reductions in computing time are possible. In addition, the relaxation process is applicable to parallel computation. Thus, the method is very attractive for future computers.

- **NEW APPROACH TO NONLINEAR STRUCTURAL DYNAMIC SIMULATION**

- **DRAMATIC REDUCTION IN COMPUTING TIME**

- **APPLICABLE TO PARALLEL COMPUTERS**

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APPENDIX - SYMBOLS

A	acceleration vector
B	linearized coefficient matrix
C	damping matrix
D	diagonal matrix
F	force vector
G	gear torque
K	stiffness matrix
M	mass matrix
n	number of degrees of freedom
R	displacement vector
r	pitch radius of gear
S	block matrix used in Φ (beam solution)
T	block matrix used in Φ (beam solution)
t	time
U	block matrix on lower diagonal of B
V	velocity vector
W	block matrix on upper diagonal of B
X	block matrix used in Θ (beam solution)
Y	block matrix used in Θ (beam solution)
Z	block matrix on diagonal of B
α	constant, modifying Taylor series
β	constant, modifying Taylor series
ε	weighting factor used in overrelaxation
Θ	coarse-to-fine transformation
θ	helical gear angle
λ	highest eigenvalue
ξ	value between 0 and t

Φ fine-to-coarse transformation

ϕ pressure angle

Subscripts:

a axial

L node number

r radial

t tangential

Superscripts:

c coarse model

f fine model

T transpose

• time derivative

(0) estimated value

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