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Mixed Finite Element Formulation Applied to Shape Optimization

by

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ABSTRACT

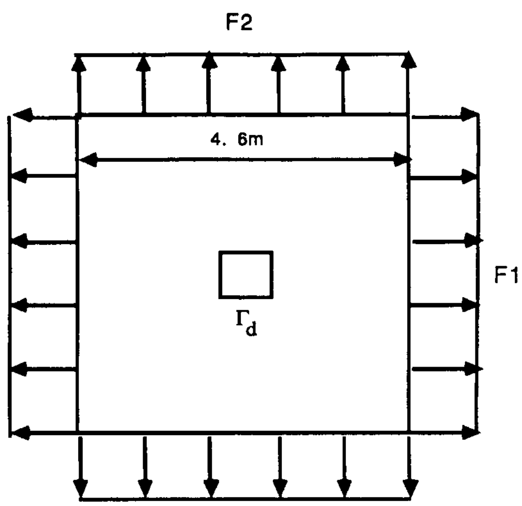
This study is concerned with development of a variational formulation and a procedure for computational solution for the shape optimal design of a two-dimensional linear elastic body, using a mixed finite element discretization. Shape optimal design is a problem that has interested many researchers in the last fifteen years. The subject has been surveyed in a number of review articles, see e.g. Haftka and Grandhi [11].

Zienkiewicz and Campbell [see , e.g. 12], were among the first to approach this problem using finite element methods. Subsequently this method has been applied widely to problems in shape optimal design[see, e.g.2,3,13- 15] , but only with mixed success. The finite element method based on the displacement formulation has two main disadvantages, (1) the increase of finite element error that results from mesh distortion during shape redesign, and (2) in some situations, a lack of sufficient precision in the prediction of stresses and strains at the boundary and internal nodes. There are some methods one can consider to overcome these difficulties. Some investigators have applied the Boundary Element Method[see, e.g.16 - 18]. While the BEM has proved to be very useful and looks promising in certain applications of shape optimal design, for problems that require numerous evaluations of state variables in the domain (objective function = $\max_{x \in \Omega} F(\mathbf{u},\varepsilon)$, for example) the BEM loses some of its advantages, also at the current stage of development it lacks the generality provided by FEM in structural analysis. Within the FEM applications, the domain method [see, e.g. 19], where sensitivity expressions are defined in terms of domain integrals rather than boundary integrals (thereby avoiding the evaluation of state variables at the boundary), provides for improved accuracy in the numerical calculation of sensitivities. Also recently, Haber proposed an Eulerian - Lagrangian formulation based on the mutual Reissner energy [see, e.g. 20], where the shape optimization problem can be formulated in an arbitrary initial domain as a means to overcome the difficulties inherited from shape redesign .

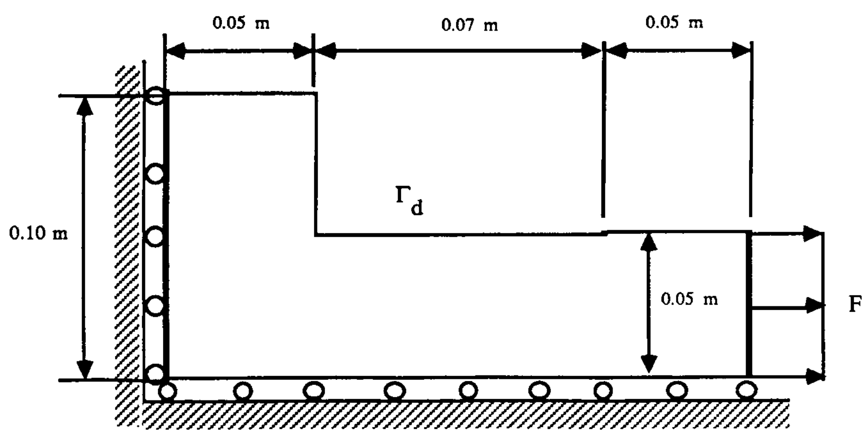
In this work another approach is considered. With the development of automatic mesh generation and optimization techniques[see, e.g.7] the first of the cited disadvantages of FEM is avoided. Mixed finite element methods[see, e.g.8] that may provide for accurate computation of stresses and strains at the element nodes appears to be a natural approach to resolve the other difficulty. These considerations are brought together in the developments reported here, to demonstrate a more effective approach to the overall treatment of shape optimal design.

EXAMPLES OF SHAPE OPTIMIZATION PROBLEMS

For simplicity let us consider plane linearly elastic structures to find the optimal shape using the mixed finite element formulation together with an automatic mesh generation method based on the elliptic differential equations. Two model problems shown in the figure are solved by the present method to demonstrate its effectiveness. It is noted that the fillet optimal shape design problem (Model B) is a standard one, but is one of the most difficult problems because of sharp design change at the left edge of the design boundary.



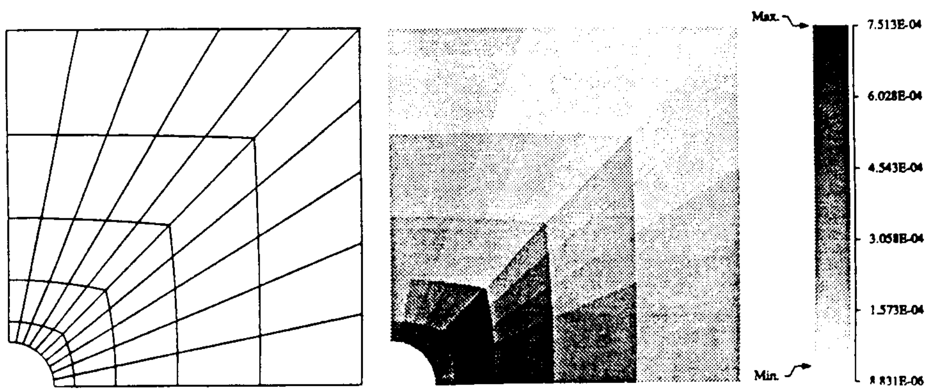
Model A



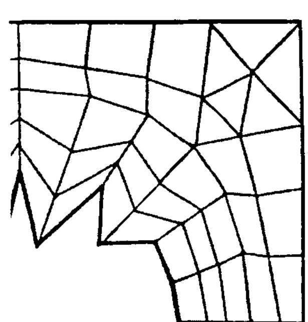
Model B

DIFFICULTIES OF SHAPE OPTIMIZATION PROBLEMS

The difficulty of the shape optimization problem is large design changes lead to significant changes of the corresponding finite element model of the structure during an optimization process. If the final, that is, optimal shape is known, it is possible to set the initial finite element model very close to the optimum. In this case, design change does not imply large change of finite element models, and then it is possible to avoid distorted finite elements which yield significant approximation errors and sometimes even negative values of the Jacobian of the isoparametric transformation. It is natural that the optimal shape is not known *a priori*, and then it is necessary to establish a shape optimization method to deal with large design change. Finite element approximation errors are strongly dependent upon the size and shape of finite elements. Errors are generally very large in regions where stresses are rapidly varying. In most shape optimization problems, stresses are varying rapidly at the end points of the design boundary where shape also changes rapidly. Furthermore, if shape change is large, this almost automatically yields distortion of finite elements, i.e. generation of unnecessary approximation errors. Figure (a) shows the distribution of finite element approximation error in a similar problem to Model A, while Figure (b) indicates a pathology in shape optimization.



(a) Finite Element Approximation Error Distribution



(b) Oscillation of the Design Boundary (by. C. Fleury)

FORMULATION OF THE DESIGN PROBLEM

The optimal shape design problem is defined by minimizing the maximum value of a local criterion $F(\mathbf{u},\varepsilon)$ stated by the displacement **u** and the linearized strain tensor ε, subject to the state equations which represent equilibrium, constitutive relation, and boundary conditions. Here D represents the design variable to describe the shape of the design boundary.

$$\mathrm{Min}_D \quad \mathrm{Max}_{\mathbf{x}\in\Omega} \quad F(\mathbf{u}(\mathbf{x}),\varepsilon(\mathbf{x}))$$

subject to

the resource constraint $\int_\Omega d\Omega - A \leq 0$

the equilibrium equations $\mathrm{div}\,\Sigma + \mathbf{f} = \mathbf{0}\,, \quad \Sigma = \Sigma^T \quad \text{in } \Omega$

the strain - displacement relations $\varepsilon = \frac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla) \quad \text{in } \Omega$

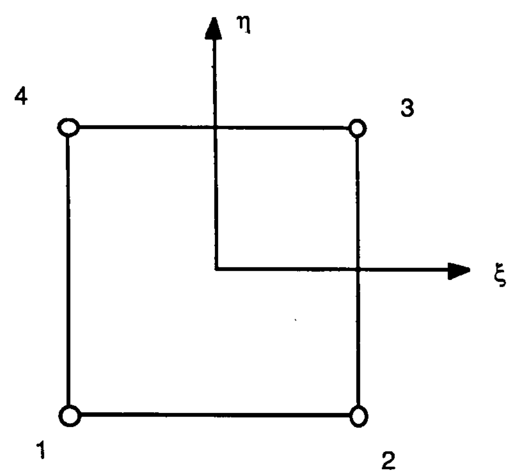
the stress - strain relation $\Sigma = \mathbb{E} : \varepsilon \quad \text{in } \Omega$

the traction boundary condition $\mathbf{n}\,\Sigma = \mathbf{t} \quad \text{on } \Gamma_t$

the displacement boundary condition $\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_u$

MIXED FINITE ELEMENT FORMULATION

If the displacement method is applied in finite element analysis, strain and stress components are computed at each Gaussian integration point to form element stiffness matrices and load vectors. However, values of stress components must be obtained at the nodes on the design boundary for the shape optimization problems. Thus an extrapolation method must be introduced to obtain nodal values of stress components. For example, if the least squares method is applied to obtain nodal values, it cannot provide sufficiently accurate values in the region that stress gradient is high. Furthermore, it becomes very inaccurate if distorted irregular finite elements exist in a finite element model. To avoid such problems, we here apply a mixed formulation that computes nodal values of displacements, strains, and stresses, directly without applying an extrapolation method.



QUAD4

8 degrees of freedom per node $\{u_x,u_y,\varepsilon_{xx},\varepsilon_{yy},\gamma_{xy},\sigma_{xx},\sigma_{yy},\sigma_{xy}\}^T$

$$u_x = \sum_{\alpha=1}^{4} u_{x\alpha} N_\alpha(\xi,\eta), \text{ etc}$$

$$\varepsilon_{xx} = \sum_{\alpha=1}^{4} \varepsilon_{xx\alpha} N_\alpha(\xi,\eta), \text{ etc}$$

$$\sigma_{xx} = \sum_{\alpha=1}^{4} \sigma_{xx\alpha} N_\alpha(\xi,\eta), \text{ etc}$$

$$N_\alpha(\xi,\eta) = \frac{1}{4}(1+\xi_\alpha\xi)(1+\eta_\alpha\eta), \quad \alpha=1,\dots,4$$

OPTIMALITY CONDITIONS

Transferring the original shape optimization problem to the upper-bound formulation $\text{Min}_D\ \beta$ subject to the above constraints and the additional one $F(\mathbf{u},\varepsilon) - \beta \leq 0$ in Ω, the Lagrange multiplier method implies the necessary condition for optimality as shown in below.

Equilibrium Equations

$$\int_\Omega \{\frac{1}{2}(\nabla\delta\mathbf{v}+\delta\mathbf{v}\nabla):\Sigma - \delta\mathbf{v}.\mathbf{f}\}d\Omega - \int_{\Gamma_t}\delta\mathbf{v}.\mathbf{t}\ d\Gamma$$

$$+ \int_\Omega [\delta\mathbf{e}:(\mathbb{E}\varepsilon - \Sigma) + \delta T:\{\frac{1}{2}(\nabla\mathbf{u}+\mathbf{u}\nabla) - \varepsilon\}]d\Omega = 0, \quad \mathbf{u}=\mathbf{0} \quad \text{on } \Gamma_u.$$

Adjoint Problems

$$\int_\Omega \{\frac{1}{2}(\nabla\mathbf{v}+\mathbf{v}\nabla):\delta\Sigma + \lambda(\frac{\partial F}{\partial\delta\varepsilon}:\delta\varepsilon + \frac{\partial F}{\partial\mathbf{u}}.\delta\mathbf{u})\}d\Omega$$

$$+ \int_\Omega \mathbf{e}:(\mathbb{E}:\delta\varepsilon - \delta\Sigma) + T:(\frac{1}{2}(\nabla\delta\mathbf{u}+\delta\mathbf{u}\nabla) - \delta\varepsilon)\}\ d\Omega = 0$$

Optimality Condition due to Variation by Shape Change

$$\int_{\Gamma_d} (\Sigma:\mathbf{e}+\Lambda)(\theta.\mathbf{n})\ d\Gamma = 0, \quad \Lambda(\int_\Omega d\Omega - A) = 0, \quad 0\leq\Lambda, \quad \int_\Omega d\Omega - A \leq 0$$

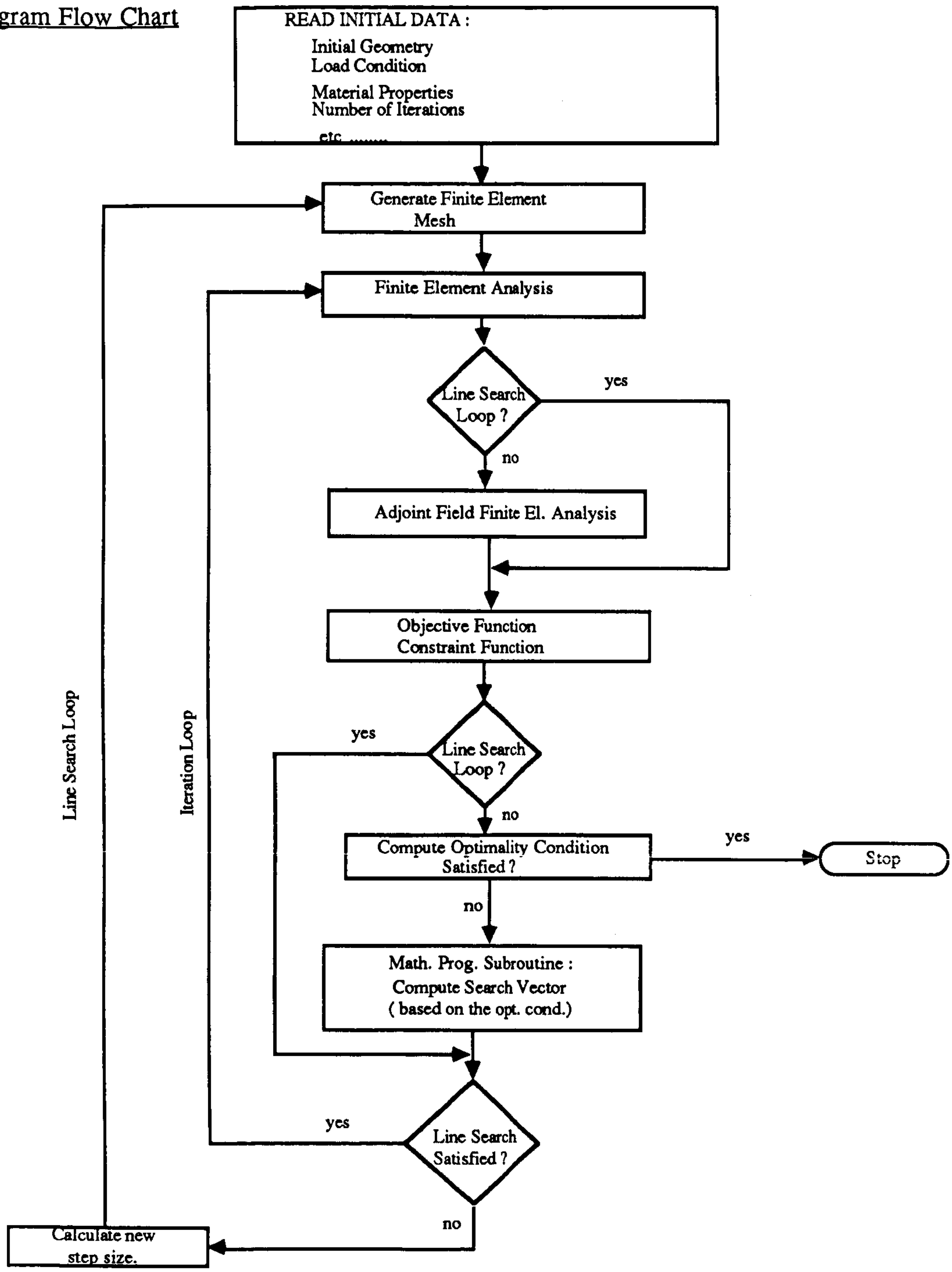
Normality Condition

$$\int_\Omega \lambda d\Omega = 1, \quad \lambda(F(\mathbf{u},\varepsilon) - \beta) = 0, \quad 0\leq\lambda, \quad F(\mathbf{u},\varepsilon) - \beta\leq 0, \quad \text{in } \Omega$$

SOLUTION PROCEDURE : OPTIMIZATION ALGORITHM

Solution procedure for the shape optimization is described in the following flow chart.

Program Flow Chart



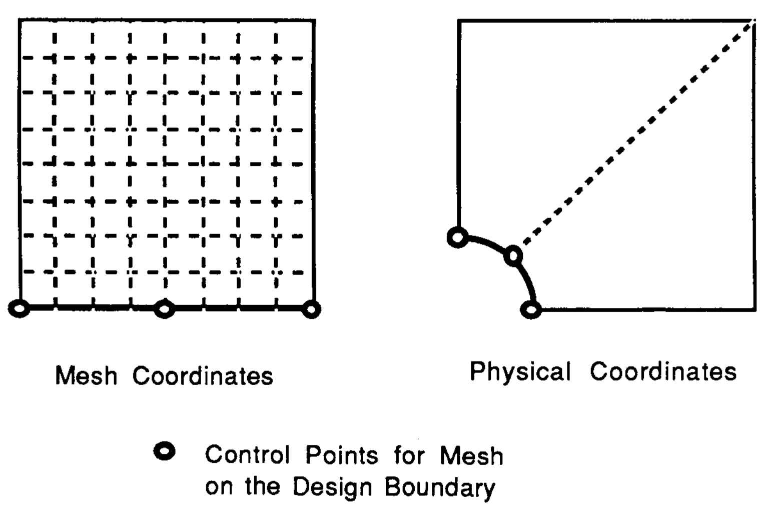
MESH GENERATION METHOD : ELLIPTIC MESH GENERATOR

At each design iteration, a new finite element mesh is reconstructed using the elliptic mesh generation method. In most of shape optimization practice, finite element meshes are not regenerated, but are modified during the optimization process. Because of the method applied for modification of the initial finite element mesh, unnecessary element distortion is, in general, generated in each iteration. Element distortion can become so large it can actually destroy accuracy of the finite element approximations, and then yield unsatisfactory results in shape optimization. To avoid this difficulty, it is better to regenerate a finite element mesh at each design stage despite of expense required. Here the elliptic mesh generation method is used that generates almost orthogonal meshes using only the data of the boundary of the domain occupied by a structure. If the boundary is represented by a set of spline curves, it is possible to represent the design boundary by several spline functions defined by the location θ_α, $\alpha=1,...,\alpha_{max}$, of the so-called control points without loss of generality.

Elliptic Mesh Generation Method

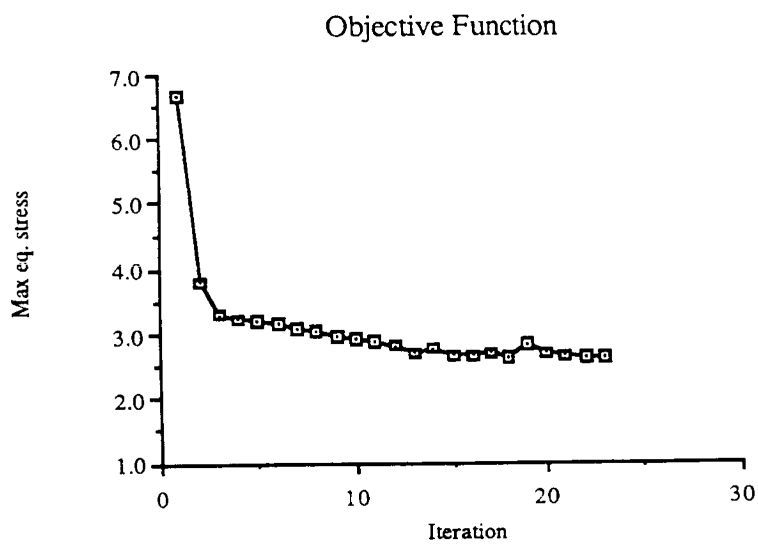
$$\nabla(\mathbf{D}\nabla\xi) = -P(\xi,\eta) \quad \text{and} \quad \nabla(\mathbf{D}\nabla\eta) = -Q(\xi,\eta)$$

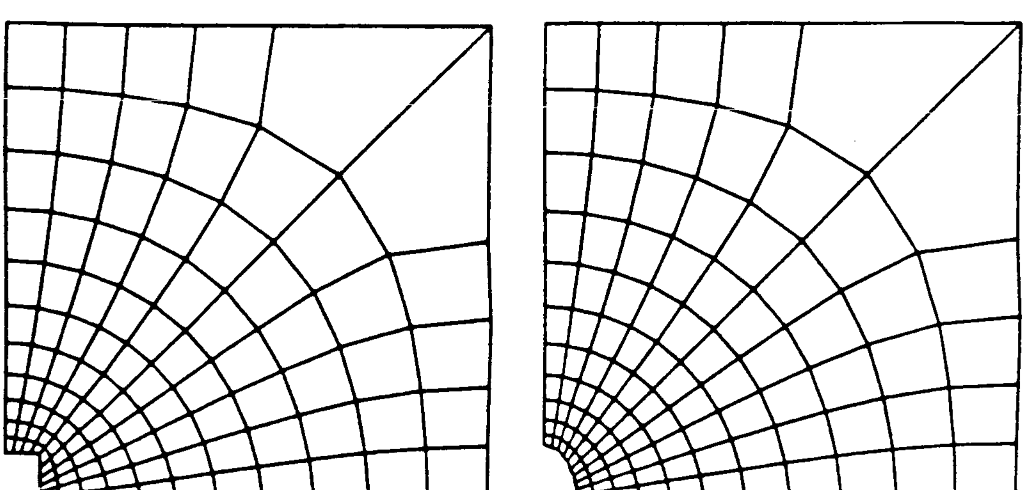
where $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}$, (x,y) are the physical coordinates, and (ξ,η) are the mesh coordinates.



EXAMPLE : MODEL A

Model Problem A is to find the optimally shaped hole in a biaxially loaded linearly elastic thin plate. Starting from a rectangular hole, the optimal shape of the hole is obtained for the loading condition which yields an elliptic hole as the optimum. Applying the symmetry condition, only a quarter part of the plate is discretized by 110 QUAD4 elements together with 11 control points on the design boundary for shape optimization. The minimum admissible area of the hole prescribed by the user is restricted to $A=0.16m^2$. Iteration history and the optimal shape are given in the figure. The first three redesigns rapidly reduce the maximum value of the von Mises equivalent stress from 6.6 to 3.3, while the optimum obtained after 23 iterations is about 2.8.



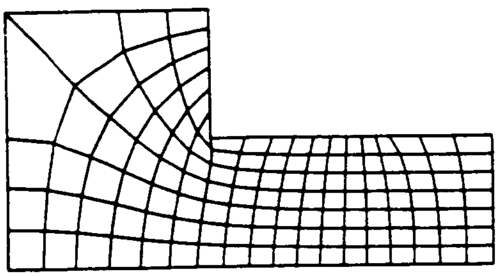


Iteration 1

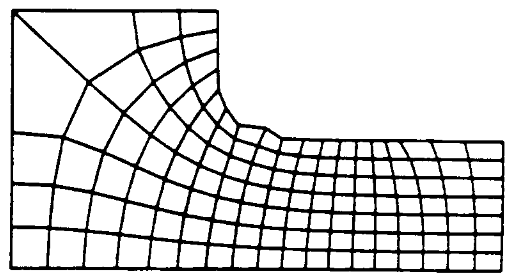
Iteration 23

EXAMPLE : MODEL B

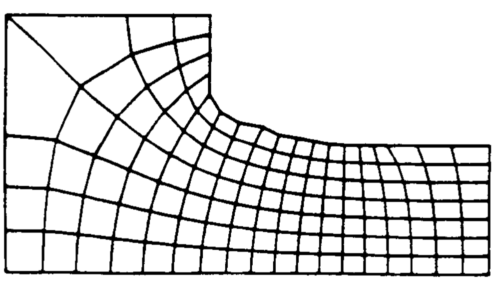
For finding the optimal shape for a linearly elastic fillet, the domain is discretized by 126 QUAD4 elements together with 14 control points on the design boundary. In this case, the objective function F is the von Mises equivalent stress. Assuming that the maximum allowable area of the fillet is restricted to $A=1.135 \times 10^{-2} m^2$, an optimal shape is obtained. The maximum value of the equivalent stress becomes very stable after 15 design iterations, and its minimum is achieved at the 24th iteration. This means that optimization and finite element remeshing is performed 24 times. First 10 iterations give rapid reduction of the maximum value of the stress. The maximum von Mises stress in the final fillet obtained is less than half of the maximum stress of the initial structure.



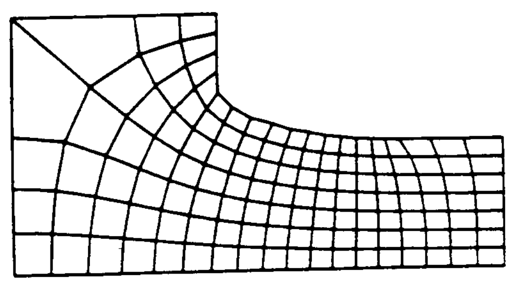
Iteration 1



Iteration 3



Iteration 11



Iteration 24

FINAL REMARKS

The development presented introduces a general form of mixed formulation for the optimal shape design problem, and the associated optimality conditions are easily obtained without resorting to highly elaborate mathematical developments. Also the physical significance of the adjoint problem comes out to be clearly defined with this formulation.

In the examples presented, an elliptical automatic mesh generator assuring an orthogonal finite element mesh at the domain boundary [see, e.g. 7] was used at each shape redesign . Although this procedure might seem to be computationally a very expensive procedure, actually it guarantees a good accuracy for the discrete model with an increase on computational time of less than 5% of the actual time required for the finite element analysis.

The numerical examples presented demonstrates the stability of the procedure. Problems commonly encountered in shape optimization arising from the development of instabilities in the design boundary definition were largely avoided. As is to be expected, however, this improvement is accomplished at the expense of the increase in cost of computation as compared to the simple displacement formulation.

SHAPE OPTIMIZATION

1. **Mixed Finite Element Methods for Analysis**
2. **Automatic Remeshing Scheme by Elliptic Mesh Generation Methods**
3. **Optimality Conditions are Obtained by the Upper Bound Method**
4. **Demonstration by Numerical Examples**

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