

RAYCHAUDHURI EQUATION IN THE SELF-CONSISTENT  
EINSTEIN-CARTAN THEORY WITH SPIN-DENSITY

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(NASA-TM-101145) RAYCHAUDHURI EQUATION IN  
THE SELF-CONSISTENT EINSTEIN-CARTAN THEORY  
WITH SPIN-DENSITY (NASA) 18 p CSCL 20C

N88-24546

Unclas  
0146691

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ABSTRACT

We develop and discuss the physical implication of the Raychaudhuri equation for a spinning fluid in a Riemann-Cartan spacetime using the self-consistent Lagrangian based formulation for the Einstein-Cartan theory. We find in agreement with others that the spin-squared terms contribute to expansion (inflation) at early times and may lead to a bounce in the final collapse.

We clarify the relationship between the fluid's vorticity and spin angular velocity and investigate the effect of the interaction terms between the spin angular velocity and the spin in the Raychaudhuri equation. These results should prove useful for studies of systems with an intrinsic spin angular momentum in extreme astrophysical or cosmological problems.

## I. Introduction

There has been a long held belief that the generalization of the geometry of Riemannian spacetime to Riemann-Cartan (RC) spacetime, which involves the addition of torsion to the geometry, could avoid the initial/final spacetime singularity.<sup>1-4</sup> It follows that torsion could also play an important role in cosmological models with inflation. The basis of this belief stems from the identification of the torsion with the spin-density of matter, usually the nuclear spin, and indeed, proponents of Einstein-Cartan (EC) theories have shown that this idea has some merit.<sup>1</sup> With the development of an improved energy-momentum tensor for spinning fluids in general relativity (GR),<sup>5</sup> the investigation of spinning fluids within the context of GR has also gained added attention and may itself point the way for further studies in the EC theory.<sup>7</sup> It seems, however, apparent from the natural way the field equations and the conservation laws for energy plus angular momentum arise in the EC theory<sup>8</sup> and how the spin is directly identified with a natural object<sup>9</sup> in the geometry - the trace-free torsion - that the ideal place to investigate the effects of spin density in cosmological problems such as inflation, final collapse or even collapsing objects is in a RC spacetime. Indeed, using the improved energy momentum tensor for spinning fluids in EC theory,<sup>10</sup> Gasperini has obtained spin-dominated inflation in the EC theory for a spinning fluid with randomly oriented spin.<sup>11</sup> On the other hand, Bradas, Fennelly and Smalley have found, in general, power law inflation for an anisotropic Bianchi I cosmology within a RC spacetime.<sup>12</sup>

In this paper we propose to highlight several problems that have, up to now, caused difficulties in the implementation of spin density in gravitational physics. Since it seems obvious that spin density does lead

to inflation<sup>11-12</sup> and may give rise to bounce before the final collapse to a singularity,<sup>2-3</sup> we will investigate in a new light the relatively simple derivation of the Raychaudhuri equation<sup>13</sup> in EC theory using the improved energy-momentum tensor for a spinning fluid.<sup>10</sup> Although this topic has been investigated before in EC theory,<sup>2-3</sup> the major problems that we see involve a well-defined relationship between torsion and spin angular momentum and a general lack of awareness of the difference between the spin-density angular velocity related to the rotation of a local co-moving concurring set of tetrads and on the other hand, the angular velocity of the fluid flow, normally referred to as the vorticity which is associated with the rotation (twist) of the four-velocity congruences.<sup>13-15</sup>

In section II, we discuss the field equation of the self-consistent theory that gives the relationship between spin-density and torsion and in section III, we derive rigorously the Raychaudhuri equation in the EC theory using the improved energy-momentum tensor with spin density. By rigorous we mean here the derivation utilizing the field equations themselves instead of extending the definition of the vorticity to RC spacetime and then rewriting it as a Riemannian part plus torsion terms.<sup>16</sup> Although this latter extension may be carried out, it really leads to no useful results since the kinematical quantities associated with the flow of a fluid are defined the same way in both a RC and Riemannian spacetime. We discuss our conclusions in section IV.

## II. Spin-Density and Torsion

With the development of the self-consistent Lagrangian formulation of the Einstein-Cartan theory with spin density,<sup>5,10,17</sup> we have been able to make the relationship between spin density and torsions precise. We now

know that the spin density is only related to the trace free (proper) torsion,  $\hat{S}_{ij}{}^k = S_{ij}{}^k + (2/3) \delta_{[i}^k S_{j]}{}^x{}_x$ , thru the torsion field equation.<sup>9</sup>

$$\hat{S}_{ij}{}^k = \frac{1}{2} \kappa \rho S_{ij} u^k \quad (1)$$

where,  $S_{ij}{}^k = \Gamma_{[ij]}^k$  is the torsion,  $S_{ij}$  is the spin density,  $\kappa = 8\pi G$ ,  $G$  is the gravitational constant,  $\rho$  is the fluid density, and  $u^k$  is the four velocity. To arrive at (1), we have assumed, following a gravitational generalization<sup>8</sup> of the special relativistic treatment of spinning fluids by Halbwachs,<sup>18</sup> that the spin density tensor is described in terms of a set of tetrads in a co-moving frame in which the velocity four-vector is the fourth component of the tetrads. Necessarily this implies the Frenkel condition between the spin  $S^{ij} = \rho s^{ij}$  and the four velocity  $S^{ij} u_j = 0$ .<sup>19</sup> However, it appears that the position of this condition in the EC theory is not well understood. It is of course not the most general relationship between a spin-like tensor contracted with the four velocity. Let us suppose, for the moment, that the trace condition does not hold  $S^{ab} u_b \neq 0$ . Then we can decompose  $S^{ab}$  into a part  $\bar{S}^{ab}$  which satisfies the Frenkel condition and a remainder

$$S^{ab} = \bar{S}^{ab} + 2 u^{[a} S^{b]c} u_c \quad (2)$$

which does not. If we now assume the usual ad hoc relation between spin and torsion<sup>8</sup>  $S_{ab}{}^c = S_{ab} u^c$  then Eq. (2) becomes

$$S^{ab} = \bar{S}^{ab} + 2 u^{[a} S^{b]c} u_c \quad (3)$$

Thus, the part of  $S^{ab}$  that does not satisfy the Frenkel condition is related to the torsion vector. But in the self-consistent approach to perfect fluids in an Einstein-Cartan theory without spin-density<sup>17</sup> (in which Halbwachs' special relativistic treatment of fluids and a fortiori the Frenkel condition is not involved), the torsion takes the form

$$S_{ij}{}^k = (2/3) S_{rn}{}^u \delta_{[i}^r \delta_{j]}^k \quad (4)$$

That is, only the torsion vector is active in the theory since the proper torsion  $\hat{S}_{ij}{}^k$  vanishes (we will also see this condition later on). Therefore, the part of  $S^{ab}$  that does not satisfy the Frenkel condition is just simply not related to the spin, but instead is related to the non conservation of matter in the spin-less theory.<sup>10</sup> This of course is a contradiction of the ad hoc condition  $S_{ab}{}^c = S_{ab} u^c$  as can be seen if one substitutes Eq.(4) into Eq.(3). In view of these considerations, the Frenkel condition in the form of Eq.(1) takes a central position in our description of a fluid with intrinsic spin. These may be contrasted with the assertions of Kopczyński<sup>20</sup> for his quite different approach to spin in gravitational theories. The only distinction is the transposition of torsion vector terms from one side to the other side of Eq. (3) depending on the formulation of the theory. The self-consistent formulation may not be the most general formulation of an EC theory, but it seems to be the most general treatment of spin-density in a RC spacetime via the trace-free torsion given in Eq. (1).

Indeed, this conclusion concerning the trace-free torsion does not change even if the field equations are derived for spinning fluids which satisfy an equation of state in lieu of particle number conservation. In

these flows there may be sources of sinks or fluids such as when particle creation events occur.<sup>21</sup> In these fluids, we find that the proper torsion in Eq. (1) depends now on the thermodynamics through a parameter  $\alpha = \alpha(\rho)$  instead of just  $\rho$ , but the Frenkel condition remains valid.

### III. Raychaudhuri Equation

The most straight-forward way to obtain the Raychaudhuri equations in EC theory is to write down the metric field equation<sup>22</sup>

$$G_{jk}(\Gamma) - \nabla_{\ell}^* (T_{jk}^{\ell} - T_{k j}^{\ell} + T_{jk}^{\ell}) = \kappa T_{jk} \quad (5)$$

where  $G_{jk}(\Gamma)$  is the Einstein tensor in RC spacetime,

$$T_{jk}^{\ell} = S_{jk}^{\ell} + 2 \delta_{[j}^{\ell} S_{k]} \quad (6)$$

is the modified torsion,  $S_k = S_{kx}^x$  is the torsion vector,  $\nabla_{\ell}^* = \nabla_{\ell} + 2S_{\ell}$ , and  $T_{jk}$  is the improved perfect fluid energy-momentum tensor with spin-density<sup>10</sup>

$$T^{jk} = T_F^{jk} + T_S^{jk} \quad (7)$$

where the perfect fluid part is

$$T_F^{jk} = [\rho(1 + \epsilon) + p] u^j u^k + p g^{jk} \quad (8)$$

and the spin part is

$$T_S^{jk} = 2 \rho u^{(i_s j)k} \dot{u}_k + v_k^* [\rho u^{(i_s j)k}] - \rho \omega^{k(i_s j)} \dot{u}_k \quad (9)$$

where  $\varepsilon$  is the internal energy of the fluid. The angular velocity of the tetrad is given by

$$\omega_{ij} = a^{\mu i} \dot{a}_{\mu j} \quad (10)$$

and the over-dot represents the directional derivative along the four velocity, e.g.,  $\dot{u}_j = u^k \nabla_k u_j$ . Next we separate the left-hand-side of Eq. (5) into the Riemannian Ricci tensor plus torsion terms. This gives

$$\begin{aligned} R_{jk}(\{\}) + 2S_{jm}^{\ell} S_k^m{}_{\ell} + 2S_{j\ell}^m S_{km}^{\ell} \\ - S^{\ell m}{}_j S_{\ell mk} - 4S_j S_k = \kappa(T_{jk} - \frac{1}{2}g_{jk}T) \end{aligned} \quad (11)$$

In Riemannian spacetime the Raychaudhuri equations is obtained from<sup>13</sup>

$$\begin{aligned} R_{jk}(\{\})u^j u^k &= 2 u^i{}_{[;j;i]} u^j \\ &= \dot{u}^j{}_{;j} - \dot{\theta} - (1/3)\theta^2 - 2\sigma^2 + 2\Omega^2 \end{aligned} \quad (12)$$

where the expansion, shear and vorticity are defined, respectively<sup>22</sup>

$$\theta = u^j{}_{;j} \quad , \quad (13)$$

$$\sigma_{ij} = u_{(i;j)} + \dot{u}_{(i} u_{j)} - (g_{ij} + u_i u_j)\theta/3 \quad , \quad (14)$$



$$\Omega_{ij} = -u_{[i;j]} - \dot{u}_{[i}u_{j]} \quad (15)$$

Since these kinematical quantities have a physical interpretation in the flow of the fluid, they have the same form in a RC spacetime as in general relativity whereas the effects of the torsion on the expansion enter only through the field equation (11). Contracting Eq. (11) with  $u^j u^k$ , substituting Eq. (12) and the torsion field Eq. (1), we obtain the Raychaudhuri equation in RC spacetime

$$\begin{aligned} [\dot{u}^j]_{;j} - \dot{\theta} - (1/3)\theta^2 - 2\sigma^2 + (\Omega_{ij} + (\kappa/2) S_{ij}) (\Omega^{ij} + (\kappa/2) S^{ij}) \\ - (\kappa/2) \omega_{jk} S^{jk} + (8/3)S_\ell S^\ell = (\kappa/2) [\rho(1 + \epsilon) + 3p]. \end{aligned} \quad (16)$$

In many cosmological problems, the quantity of interest is the combination of parameters,  $\dot{\theta} + \theta^2/3$ , which can be directly related to the rate of change of the scale parameter in simple models. The spin-squared terms thus act oppositely to the pressure terms during the expansion or the contracting phases of the universe.

We note three significant differences occur in comparison with the standard EC<sup>23</sup> results for the Raychaudhuri equation:<sup>2,3,16</sup> the spin tensor enters with a factor of one half, there is a tetrad angular velocity spin term, and finally, a torsion-vector squared term.

The combination of spin terms which enter with a factor of one half vis-à-vis the vorticity terms is a direct result of the factor of one-half contained in the torsion field equation (1). This should be compared with the usual results obtained in an EC theory,<sup>3,13</sup> that is based upon the ad hoc Weyssenhoff condition. Nevertheless, Petti<sup>24</sup> has shown, based on

translational holonomy arguments in general relativity, that the factor of one-half in Eq.(1) is correct.<sup>25</sup>

In addition, we obtain two new additions to the Raychaudhuri equation in the self-consistent EC theory. The  $\omega_{ij} S^{ij}$  term represents the contribution of the spin kinetic energy in an RC spacetime. To understand the effect of this term, it is convenient to write it as a Riemannian plus torsion (or spin) correction terms. This gives

$$-\frac{1}{2}\kappa\omega_{jk}S^{jk} = -\frac{1}{2}\kappa\omega_{jk}^{(GR)}S^{jk} + \frac{1}{4}\kappa^2S_{jk}S^{jk}, \quad (17)$$

which doubles the effect of the spin-squared terms in cosmological models. In fact the presence of the spin-squared terms found in Eq.(16) implies a special significance for spin in RC spacetime. In the original Lagrangian formulation by Ray and Smalley<sup>5</sup> of spinning fluids in GR the internal energy was not considered as a function of the spin-density. As an example, we note the absence in the work of Bedran and Vasconcellos-Vaidya<sup>6</sup> of the general relativistic term given on the right-hand-side of Eq.(17) which does not, as a consequence, appear in their GR equivalent of Eq.(16). In an improved version,<sup>26</sup> paralleling the self-consistent formulation in RC spacetime,<sup>10</sup> such a term will be present in the GR version. There is, however, no spin-squared terms, so that symbolically, using a self-consistent GR version of a spinning fluid,

$$[\text{Eq.}(16)]_{\text{RC}} - [\text{Eq.}(16)]_{\text{GR}} = \frac{1}{2}\kappa^2S_{ij}S^{ij}. \quad (18)$$

This means that in the absence of vorticity in GR, the spin-squared terms do not directly influence the cosmology. On the other hand, in a RC

spacetime, the spin-squared terms can dominate in this situation. Thus the effect of spin is more pronounced in RC spacetime. This enhances the assertion that the obvious arena to investigate the effects of spin in gravitation is in RC spacetime because of the natural relationship between spin-density and torsion,<sup>9</sup> i.e. a one-to-one correspondence between a dynamical variable and a geometric object respectively.

In deriving Eq.(16), we have not assumed the mass conservation case,  $S_\ell = 0$ . The torsion-vector squared term  $S_k S^k$  thus represents a correction due to non-conservation of mass. Since  $S_\ell$  is proportional to  $U_\ell$ , this term is intrinsically negative<sup>9</sup> and could have been written,  $-3(S_\ell U^\ell)^2$ . Thus the effect of the torsion-vector squared term is similar to a positive pressure term. The mass conserving case is easily formed by setting the torsion vector  $S_\ell = 0$ .

#### IV. Discussion and Conclusions

If we assumed a randomly oriented spin-density for the mass-conserving case, the only spin term that survives an average overall direction is  $(1/2)\kappa^2 S^2$  where  $2S^2 = \langle S_{ij} S^{ij} \rangle$ .<sup>27</sup> Note that contributions arise both from the direct spin-squared term in Eq.(16) as well as from the expansion of the spin kinetic energy term given in Eq.(17). These terms, as shown by Eq.(16), tend to both increase expansion in the early universe<sup>6,11</sup> or oppose collapse, i.e., give rise to a bounce that could avert a spacetime singularity. Usually this latter result is predicted in considering only the effect of nuclear spins in the ultra high matter density conditions just before a final collapse.<sup>1</sup> Our formulation subscribes, however, a different macroscopic view of spin-density described by Kopczyński<sup>28</sup>, Bailey,<sup>29</sup> Israel,<sup>30</sup> and Bailey and Israel.<sup>31</sup> In this view, a continuous

medium may have an internal spin density. In this way objects such as protogalaxies, turbulent eddies or (primeval) black holes may form fluids which could have an internal spin density.

Equation (16) could be used to study star or galaxy formation from the collapse of matter clouds with intrinsic spin density as well as the inflationary formation or collapse of an universe with intrinsic spin density. The spin may not be randomly oriented for such examples. In this case there could be interaction terms between the vorticity and the spin which we have not yet considered as well as an interaction term between the spin density and spin angular velocity which could either hasten or slow down the collapse of the system depending on the relative sign of this term in Eq. (16).

Tsoubelis<sup>32</sup> has shown that spin density gives rise to a stationary spacetime because of the  $\omega_{ij}$  term in the energy-momentum tensor, Eq. (9). For emphasis we contrast this to the work of Prasana<sup>33</sup> and Arkuszewski,et al<sup>34</sup> which contends that the net effect of an EC calculation is to renormalize the fluid parameters. This supposedly allows one to match the matter boundary to the vacuum. Although one could include the spin-squared terms in Eq.(16) with renormalized density and pressure so that

$$\begin{aligned}\rho' &= \rho - \frac{1}{4}\kappa S^2, \\ p' &= p - \frac{1}{4}\kappa S^2,.\end{aligned}\tag{19}$$

[see refs. 1,3,6,11, and 35 for typical examples], this results would be incorrect here for two reasons. First the renormalization itself is not unique (which may not be important), but now the presence of the interaction term between the spin angular momentum and the spin-density can not be

the only factors governing the matching of boundary condition between matter and the vacuum. This now changes a negative conclusion to a positive conclusion concerning the influence of the spin density on the metric of spacetime as was shown by Tsoubelis. This puts the concept of intrinsic spin and angular momentum on an equal footing in gravitational physics and means that the collapse of a system with spin density can be treated with an interior solution of the field equations that can match to the correct external metric for a rotating star. Thus the generalized Raychaudhuri equation (16) emphasizes the importance of the spin angular velocity term in the consideration of spinning matter in gravitational theories.

Finally, we note that the torsion vector term in Eq.(16) tends to increase the rate of collapse. We have not explored its consequences in detail but only emphasize, as we did in section II, that it does not depend on the spin density of the matter. Preliminary investigations have shown, however, that the torsion vector may be important in the description of gravitational theories which include electromagnetic effects.<sup>36</sup> However, to proceed further, we will have to correctly include the effects of the electromagnetic field in the theory in order to study the formation of spinning galaxies and stars with magnetic fields. There has been preliminary results on the electromagnetic field for charged spinning fluids,<sup>37,38</sup> but the correct form of the thermodynamic laws for the self consistent approach has not been satisfactorily clarified in order to make this approach yet viable.<sup>39</sup> We are presently considering this problem.

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