MHD BENDING WAVES IN A CURRENT SHEET

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ABSTRACT

Transverse MHD bending waves are considered in an isothermal and compressible two-dimensional current sheet of finite thickness in which the magnetic field changes direction and strength. The general form of the wave equation is obtained. It is shown that rotation of the magnetic field across the current sheet prevents the existence of singular points so that continuous spectrum solutions and the concomitant wave decay disappear. Instead, normal modes exist and closed integral solutions for arbitrary current sheet structure are found.

The results are discussed in terms of small-scale waves on the heliospheric current sheet.

Subject headings: sun:solar wind - hydromagnetics - surface waves

1. INTRODUCTION

Bending waves are an essentially transverse displacement of an equilibrium interface or layer in which, to first approximation, gradients along the layer are negligible in comparison to gradients across the layer. Recently, Bertin and Coppi (1985) applied the theory of MHD bending waves to a current sheet in which the magnetic field reverses direction. They suggested that these waves might provide an explanation for the well-known sector structure in the interplanetary magnetic field polarity (Wilcox and Ness, 1965) which has often been interpreted as a warping of the heliospheric current sheet (HCS) dividing opposite magnetic polarity regions in the interplanetary medium (Hundhausen, 1972, op. cit.; Suess and Hildner, 1985).

It is now generally accepted that warping of the HCS does exist but that it is a direct consequence of the large-scale structure of the solar magnetic field as that field is carried outward into the interplanetary medium by the solar wind (Hoeksema, et al., 1982, 1983; Hoeksema, 1984). The large-scale warping is therefore understandable in the kinematic limit where the internal dynamics of the current disk is altogether ignorable. A surface plot of the current sheet made using this approximation is shown in Figure 1. The plot was made under the assumption that the interplanetary field is the projection of a "source surface" potential magnetic field model of the corona ending at 2.5 solar radii with a constant solar wind velocity of 400 km/s. The epoch chosen is the time the Giotto spacecraft passed by Halley's comet - approximately the same instant that both objects passed through the current sheet. In situ verification has shown that these projections are valid to within one day at 1 AU.

Nevertheless, important reasons exist for studying bending waves on the HCS. There are small-scale "ripples" on the HCS that have so far eluded complete understanding. Consequently, the fine-scale structure of the HCS is a topic of active investigation (Behannon, et al., 1981). It is worth noting that the analysis by Bertin and Coppi (1985) explicitly invoked a WKB approximation that, a priori, made the results far more relevant to these small-scale fluctuations in the first place and seemingly invalidated their discussion in terms of the large-scale sector structure

Here we re-analyze bending waves in the ideal MHD limit in a two-dimensional current sheet of finite thickness - implicitly invoking the WKB approximation by ignoring gradients along the current sheet. The unperturbed state will be in equilibrium with its surroundings, isothermal, and compressible. The waves will also be assumed to be isothermal but incompressible; thus, we consider the Alfven type of purely transverse waves. These are the same assumptions made by Bertin and Coppi (1985). With no additional complexity, we are able to consider any two-dimensional, equilibrium current sheet in which the magnetic field changes direction and/or strength, with a current sheet across which the field reverses direction (a "neutral sheet") in an arbitrary manner being a special case.

Our approach is a generalization of the often-studied problem of hydromagnetic surface waves associated with a layer between otherwise homogeneous domains in static equilibrium that has been analyzed in many contexts (see Lee and Roberts, 1986, op cit.; Hollweg, 1986; Roberts, 1983). These earlier studies have shown that, for a layer in which the magnetic field vector does not change direction and the field strength has no local extremum, a singular point exists where the wave frequency equals the local Alfven frequency. The physical consequence of this is the decay of a surface disturbance propagating along the layer. The decay results from "mode conversion" (sometimes called "phase mixing") of the collective surface disturbance into local oscillations within the interface. That is, there is no normal mode solution for these wave numbers - a continuous spectrum of modes exists except in the limit that the thickness of the layer goes to zero. This behavior has been analyzed most accurately and clearly by Lee and Roberts (1986).

Under particular circumstances it is also possible to have normal mode solutions for a limited region in (ω,k) space (where ω is the circular frequency and k is the wavenumber) in a compressible medium (Hopcraft and Smith, 1986). What we show here is that in a current sheet in which the field vector undergoes rotation in the plane of the sheet, in addition to an amplitude change, the region of (ω,k) space permitting normal mode solutions expands and that continuous mode solutions can sometimes disappear altogether, along with the concomitant wave decay. This is particularly interesting in reference to the HCS, wherein it is observed that the magnetic field normally reverses direction without the amplitude going to zero (Behannon, et al., 1981).

Our formalism assumes waves with no compressive component in an otherwise compressible medium of finite sound speed, as opposed to assuming the limit of infinite sound speed as outlined by Priest (1982, eqn. (4.60)) or assuming an incompressible medium. This allows us to reduce the order of the wave equation for normal modes and eventually to find, for the first time, a closed integral solution for arbitrary current sheet structure for those conditions when normal modes exist. Our wave equation reduces to a known equation in various limits, giving us confidence in our results.

In the following, we explicitly formulate the bending wave problem in section 2, present the MHD bending wave equations in section 3, and discuss our results in an astrophysical context and in terms of small-scale waves on the HCS in section 4.

2. FORMULATION OF THE PROBLEM

In this section, we describe the model and geometry of the current sheet, discuss the type of bending waves which can be excited in the current sheet, and present the governing magnetohydrodynamic (MHD) equations needed to describe these waves. We consider the ideal MHD equations obtained by neglecting diffusion of the magnetic field, displacement currents and electrostatic forces and by making the assumption that the gas pressure is a scalar. We begin with the current sheet model, then present the perturbations in the current sheet and the MHD equations.

(a) The Model and Geometry of a Current Sheet

A current sheet is characterized by changing strength and direction of the magnetic field and by the pressure balance across the sheet. From the theoretical point of view, both the magnetic field and the gas pressure can change across the current sheet in an arbitrary way (Bertin and Coppi, 1985), however, for a current sheet in static equilibrium the total (gas + magnetic) pressure has to be constant. On the other hand, in situ observations of the HCS (Behannon et al. 1981), across which the field changes polarity, give restrictions on the behaviour of the magnetic field across the current sheet; the data show that the magnetic field predominantly rotates across the sheet but also that some variations of the strength of the field can be seen. In the latter case, gas pressure variations across the sheet are needed in order to maintain the pressure balance.

In the approach presented in this paper, we consider an isothermal current sheet in static equilibrium. We allow for rotation of the magnetic field across the current sheet and for arbitrary changes in the field strength; the latter requires changes of density across the sheet to account for the pressure variations. We impose perturbations that bend the current sheet and look for the effects on the behaviour of MHD bending waves when gradients of the physical parameters across the sheet are taken into account. Variations of the physical parameters along the current sheet are neglected in this approach as they are assumed to be on characteristic scales much larger than those considered across the current sheet. We introduce a coordinate system with the x and y-axis in the plane of the sheet (the y-axis is fixed in the direction of the magnetic field on the "upper" edge of the sheet) and with the z-axis normal to the sheet (see Figure 2a); all the physical parameters that describe the medium inside the sheet vary across the sheet and show dependence on z alone.

Having defined the coordinate system, we can, with complete generality, describe the rotation and gradient of the magnetic field $\vec{B}_o$ inside the current sheet in the following way:

$$\vec{B}_o(z) = B_{ox}(z)\hat{x} + B_{oy}(z)\hat{y} = B_o(z)[\hat{x} \sin \alpha(z) + \hat{y} \cos \alpha(z)], \tag{2.1}$$

where α is the angle between the direction of the magnetic field and the y-axis (see Figure 2b), and B_o (z) is an arbitrary function of z. The static equilibrium is defined by

$$\nabla p_o - \frac{1}{4\pi}[(\nabla \times \vec{B}_o) \times \vec{B}_o] = 0, \tag{2.2}$$

which can easily be reduced to the pressure balance of the current sheet (along the z-axis) with its surroundings

$$p_o + \frac{B_o^2}{8\pi} = p_e^{(1)} + \frac{(B_e^{(1)})^2}{8\pi} = p_e^{(2)} + \frac{(B_e^{(2)})^2}{8\pi} = \text{const}, \tag{2.3}$$

where p_o is the gas pressure inside the current sheet, and $p_e^{(1)}$, $p_e^{(2)}$, $\vec{B}_e^{(1)}$ and $\vec{B}_e^{(2)}$ are the gas pressure and magnetic field outside, on opposite sides of the current sheet. Both latter quantities are assumed to be constant in the vicinity of the current sheet as we look at wavelength larger than the thickness of the current sheet. $\vec{B}_e^{(1)}$ and $B_e^{(2)}$ need not be in the same direction in the x-y plane.

(b) Perturbations of the Current Sheet

To develop a formalism for describing small-amplitude MHD bending waves of arbitrary frequency, we consider perturbations of the velocity and the magnetic field normal to the current sheet; these perturbations bend the current sheet and are transverse MHD bending waves propagating in any directions along the current sheet. The amplitude of the density compression is negligible for small-amplitude transverse waves and to the first-order approximation the driven compressional wave may also be neglected. In the approach presented here, we neglect density and pressure perturbations. We allow for arbitrary propagation along the current sheet (the wave vector $\vec{k} = k_x\hat{x} + k_y\hat{y}$, see Figure 3) and consider the most general form of velocity $\vec{U}(x,y,z,t)$ and magnetic field $\vec{B}(x,y,z,t)$ perturbations:

$$\vec{U}(x,y,z,t) = \vec{U}_p(z)\exp[-i(\omega t - k_x x - k_y y)], \qquad (2.4)$$

and

$$\vec{B}(x,y,z,t) = \vec{B}_p(z)\exp[-i(\omega t - k_x x - k_y y)], \tag{2.5}$$

where ω is the wave frequency.

As shown by equations (2.4) and (2.5), the velocity $\vec{U}_p(z)$ and the magnetic field $\vec{B}_p(z)$ perturbations have three components, however, only the z-component of the perturbations excite MHD bending waves which are purely transverse waves propagating along the current sheet. Purely transverse waves that show oscillations of the velocity and magnetic field only in the xy-plane can also be excited by the x- and y-components of equations (2.4) and (2.5); in this case, however, the wave is not a bending wave. Note also that this transverse wave cannot be excited by the perturbations along the magnetic field as there is no the restoring force acting in that direction.

(c) The MHD Equations

Our model of the current sheet and the type of the perturbations chosen reduce the ideal MHD equations to the following linearized form:

$$\nabla \cdot (\rho_o \vec{U}) = 0, \tag{2.6}$$

$$\frac{\partial \vec{U}}{\partial t} - \frac{1}{4\pi\rho_o} [(\nabla \times \vec{B}) \times \vec{B}_o + (\nabla \times \vec{B}_o) \times \vec{B}] = 0, \tag{2.7}$$

$$\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{U} \times \vec{B}_o) = 0, \tag{2.8}$$

where ρ_o is the density inside the current sheet. Note that the equation of motion is simplified by the static equilibrium condition (equation 2.2) and that the perturbations of $\vec{U}$ and $\vec{B}$ can be calculated from equations (2.7) and (2.8) alone; the continuity equation (2.6) and the solenoidal condition ($\nabla \cdot \vec{B} = 0$) give additional restrictions on these perturbations.

The set of MHD equations (2.6) through (2.8) fully describes the propagation of purely transverse waves in a compressible medium and allows us to generalize the approach developed for incompressible ideal MHD (Bertin and Coppi, 1985; Lee and Roberts, 1986).

In order to consider the effect of gradients inside the current sheet on the behaviour of MHD bending waves the wave equation for either the velocity $U_{pz}(z)$ or the magnetic field $B_{pz}(z)$ perturbation has to be derived and solved. We will address these problems in the next section.

3. THE MHD BENDING WAVE EQUATIONS AND THEIR SOLUTIONS

In the following, we derive the wave equations for MHD bending waves, show general solutions and discuss limiting cases.

(a) The Wave Equations

In the standard approach, the velocity and magnetic field perturbations defined by equations (2.4) and (2.5) are substituted into the equation of motion (2.7) and the induction equation (2.8). Then, with the help of the continuity equation

$(\nabla\cdot\vec{U}_p = 0)$, the wave equation for the velocity perturbation U_{pz} is obtained in the form

$$\frac{d}{dz}\left(D_A\frac{dU_{pz}}{dz}\right) = k^2 D_A U_{pz} \tag{3.1}$$

where $D_A \equiv \omega^2 - (\vec{k}\cdot\vec{V}_A)^2$ and where $\vec{V}_A = [B_{ox}\hat{x} + B_{oy}\hat{y}]/(4\pi\rho_o)^{1/2}$. This equation describes transverse MHD bending waves in a fully incompressible medium and has been extensively studied by many authors (e.g., Tataronis and Grossmann, 1973; Roberts, 1984; Bertin and Coppi, 1985; Lee and Roberts, 1986).

In the approach presented here, we consider a compressible medium and combine the continuity equation (2.6) with the induction equation. It gives

$$i\omega B_{px} + iU_{px}(\vec{k}\cdot\vec{B}_o) - U_{pz}\left(\frac{dB_{ox}}{dz}\right) + U_{pz}B_{ox}\lambda_r = 0, \tag{3.2}$$

$$i\omega B_{py} + iU_{py}(\vec{k}\cdot\vec{B}_o) - U_{pz}\left(\frac{dB_{oy}}{dz}\right) + U_{pz}B_{oy}\lambda_r = 0, \tag{3.3}$$

$$i\omega B_{pz} + iU_{pz}(\vec{k}\cdot\vec{B}_o) = 0, \tag{3.4}$$

where $\lambda_r = (d\rho_o/dz)/\rho_o$.

To calculate the wave equation for U_{pz} we eliminate B_{px}, B_{py}, and B_{pz} in terms of U_{pz} from equations (3.2), (3.3), and (3.4) using the x and y components of the equation of motion. After some algebra, we get

$$i\omega B_{px,py} = \Big[\Big(\frac{dB_{ox,oy}}{dz}\Big) - \frac{k_{x,y}V_A^2}{\omega^2}(\vec{k}\cdot\vec{B}_o)\lambda_b + \Big(\frac{\omega^2 B_{ox,oy} - (\vec{k}\cdot\vec{B}_o)k_{x,y}V_A^2}{\omega^2 B_o^2 - (\vec{k}\cdot\vec{B}_o)^2 V_A^2}\Big)\lambda_r B_o^2\Big]U_{pz}, \quad (3.5)$$

where $\lambda_b = (dB_o/dz)/B_o$.

Having obtained B_{px} and B_{py}, we may eliminate them from the z-component of the equation of motion and obtain the wave equation for U_{pz} in the form

$$D_A U_{pz} + \frac{1}{\rho_o}\frac{d}{dz}[((\lambda_b - \lambda_r)D_A - (\vec{k}\cdot\vec{V}_A)^2\lambda_r)\frac{V_A^2}{\omega^2}\rho_o U_{pz}] = 0, \quad (3.6)$$

where again $D_A = \omega^2 - (\vec{k}\cdot\vec{V}_A)^2$ is the dispersion relation for Alfven waves. To further reduce this equation, we use the pressure balance across the current sheet (equation 2.3) and find

$$\lambda_r = -\frac{V_A^2}{V_S^2}\lambda_b, \quad (3.7)$$

where V_S^2 $(= p_o/\rho_o)$ is the sound speed. Thus, the MHD bending wave equation for U_{pz} can finally be written as

$$\omega^2 D_A U_{pz} + \frac{1}{\rho_o}\frac{d}{dz}[\rho_o V_A^2(D_A - \omega^2\beta_A^2)\lambda_b U_{pz}] = 0, \quad (3.8)$$

where $\beta_A = V_A/V_S$.

Following the same procedure, one may obtain the MHD bending wave equation for B_{pz} in the form

$$\omega^2 D_A B_{pz} + \frac{1}{\rho_o}(\vec{k}\cdot\vec{B}_o)\frac{d}{dz}\left[(\vec{k}\cdot\vec{B}_o)\rho_o V_A^2(D_A - \omega^2\beta_A^2)\lambda_b B_{pz}\right] = 0, \tag{3.9}$$

which is slightly different than for U_{pz}. It is important to know the equation for B_{pz} and its solution as the magnetic field can usually be determined more easily than the velocity from observational data.

Both wave equations presented above may be simplified to the well-known dispersion relation for the Alfven waves when the gradient of magnetic field disappears ($\lambda_b = 0$) and no layer exists. The general solutions can also be easily found in analytical forms (see next subsection), however, in order to calculate variations of the velocity or magnetic field perturbations across the current sheet the explicit form for the magnetic field has to be given (see subsection 3e for detailed discussion). The rotation of the magnetic field (being forceless in the equation of motion) does not play as important role in equations (3.8) and (3.9) as gradients of density and magnetic field; the latter effects introduce forces into the equation of motion and determine the wave behavior.

Comparison of the wave equations (3.8) and (3.9) to equation (3.1) shows that now we are dealing with first order differential equations instead of second. Note also that the wave equations (3.8) and (3.9) are valid for a compressible medium and have a form that is simple to deal with.

(b) Singular Points

In the wave equations for MHD bending waves (equations 3.8 and 3.9) the derivative with respect to z can be explicitly calculated and the wave equations can be written in the following forms:

$$D_{AS}\delta_o \frac{dU_{pz}}{dz} = -\left[8\pi\rho_o\omega^2 D_A + D_{AS}\frac{d\delta_o}{dz} + \delta_o\frac{dD_{AS}}{dz}\right]U_{pz}, \qquad (3.10)$$

and

$$D_{AS}\delta_o \frac{dB_{pz}}{dz} = -\left[8\pi\rho_o\omega^2 D_A + D_{AS}\frac{d\delta_o}{dz} + \delta_o\frac{dD_{AS}}{dz} - \frac{D_{AS}\delta_o}{(\vec{k}\cdot\vec{B}_o)}\frac{d}{dz}(\vec{k}\cdot\vec{B}_o)\right]B_{pz}, \qquad (3.11)$$

where $\delta_o = d(B_o^2)/dz$, and

$$D_{AS} \equiv D_A + \omega^2\beta_A^2. \qquad (3.12)$$

Both equations (3.10) and (3.11) show two singular points where $D_{AS} = 0$ or $\delta_o = 0$; for incompressible medium ($\beta_A \to 0$) the first singular point occurs when $D_A = 0$ (see equation 3.1). This case is discussed by Tataronis and Grossmann (1973) who showed that the existence of singular points leads to a continuous spectrum of frequencies, in which case the normal mode analysis fails, and the solution is given by an integral over some range of wave frequencies in addition to a possible sum of discrete modes. They also showed that the continuous spectrum of frequencies is responsible for decaying of MHD waves by phase mixing (see also Lee and Roberts, 1986).

We begin with the first singular point and calculate the phase velocity ($U_A = \omega/k$) for which the condition $D_{AS} = 0$ is fulfilled. Introducing the angle ϕ between the magnetic field and the wave vector (see Figure 3), we obtain

$$U_A^2 = \frac{V_S^2 \, V_A^2 \cos^2\phi}{V_S^2 + V_A^2}. \tag{3.13}$$

Note that for $\phi = 0$ this expression describes the phase velocity of purely acoustic waves (or MHD slow waves when $V_A >> V_S$) guided by an intense magnetic flux tube and is obtained when a thin flux tube approximation is considered (Defouw, 1976; Musielak, et al., 1987).

We solve equations (3.10) and (3.11) at the point $D_{AS} = 0$ (a singular point solubility criterium) and find that for all layers in which $|\vec{B}_o|$ goes uniformly through a minimum and shows no rotation there are no real wave frequencies for which the solutions can be obtained and only continuous spectrum solutions exist. In general, however, when rotation is taken into account the critical condition for existence of the solutions is found as

$$\omega^2 = -(2V_S^2 + V_A^2)\lambda_b^2 - 2(V_S^2 + V_A^2)\lambda_a\lambda_b \tan\phi, \tag{3.14}$$

where $\lambda_a = d\phi/dz$. This condition shows that non-zero and negative gradient of rotation of the magnetic field ($\lambda_a < 0$) introduces a possibility of finding a real wave frequency for which the condition (3.14) is satisfied ($|\lambda_a \tan\phi| > |\lambda_b|$); then, we may define a real critical frequency ω_c (defined by the RHS of equation 3.14) that satisfies $D_{AS} = 0$ and normal mode solutions can be recovered.

To consider the second singular point, we assume that the strength of magnetic field disappears at one particular point in the current sheet and that the first derivative of the square of the field strength with respect to z becomes zero in the vicinity of this point. The second derivative, however, remains non-zero leading to smooth and continous variations of B_o across the current sheet. In this case, $\delta_o = 0$ and using again singular point solubility criterium, we find

$$\omega^2 = - \frac{1}{8\pi\rho_o} \frac{d\delta_o}{dz}. \qquad (3.15)$$

If the second derivative of the magnetic field with respect to z is positive, then no real wave frequency can satisfy equation (3.15). The situation is different when the magnetic field exhibits a maximum inside the current sheet and the second derivative of the field becomes negative; it leads to the critical frequency ω_c defined by the RHS of equation (3.15) for which normal modes are the expected solutions of the wave equations.

(c) Absence of Singular Points

As described in the previous subsection the general MHD bending wave equations show two singular points which lead to limits on normal mode solutions. For an incompressible medium and non-rotating magnetic field, these two points are not removable and exist for all wave frequencies that satisfy the condition $D_A = 0$ in any layer (e.g. Tataronis and Grossmann, 1973; Lee and Roberts, 1986). In the case considered in this paper, there are two agents; compressibility and rotation of the magnetic field vector which may remove singular points from the wave equations.

Rotation of the magnetic field across the current sheet means that $B_o(z)$ need not be zero somewhere in a "neutral" sheet and removes the second singular points from the wave equations. Observational data (Behannon et al., 1981) show that zero strength of the magnetic field apparently rarely exists for the HCS, mainly because of rotation, although variations of the magnetic field across the HCS often show a minimum.

Now, we look for physical conditions necessary to remove the first singular point. Suppose that the condition (3.13) is <u>not</u> satisfied anywhere in the current sheet. Then, the phase velocity of transverse bending waves is equal to one of the local Alfven velocities characteristic of the current sheet and condition (3.13) is never fulfilled in the sheet, at least as long as the sound speed is less than the Alfven velocity ($V_A > V_S$). The latter inequality is not satisfied in the current sheet only in the vicinity of the point where the magnetic field disappears (i.e., a neutral sheet) or if the field variations across the sheet show a deep minimum. However, as shown above both latter cases can be prevented when the magnetic field rotates. It allows the condition $V_A > V_S$ to be satisfied everywhere in the current sheet (see also Behannon et al., 1981) and leads to absence of the first singular point. Note that when $V_A < V_S$ the first singular point is not removable from the wave equations. Absence of the singular points leads to normal mode solutions for any wave frequency.

(d) General Solutions

Having discussed the singular points of the wave equations and their removal, we now present the general solutions of these equations valid for absence of the singular points ($D_{AS} \neq 0$ and $\delta_o \neq 0$). To integrate the wave equations, we separate the variables and the physical parameters in equations (3.10) and (3.11), and obtain the solution for U_{pz} in the form

$$U_{pz} = U^o_{pz} \frac{C_{AS}C_o}{D_{AS}\delta_o} \exp\left[- \omega^2 \int \frac{D_A}{V_A{}^2 \lambda_b D_{AS}} dz\right], \tag{3.16}$$

the solution for B_{pz} is given by

$$B_{pz} = B^o_{pz} \frac{C_{AS}C_o}{D_{AS}\delta_o} \frac{(\vec{k}\cdot\vec{B}_o)}{C_B} \exp\left[- \omega^2 \int \frac{D_A}{V_A{}^2 \lambda_b D_{AS}} dz\right], \tag{3.17}$$

where U^o_{pz}, B^o_{pz}, C_{AS}, C_B and C_o are constant of integrations. The integrals can be evaluated explicitly only for very special cases that are discussed in the next subsection.

(e) Simplifying the Problem: Limiting Cases

In order to obtain some simple analytical solutions, we consider limiting cases for which both MHD bending waves equations can be significantly reduced. The simplest solutions of equations (3.8) and (3.9) are found either when the gradient of magnetic field is neglected or when the dispersion relation for Alfven waves is satisfied (see two next cases). Simple solutions can also be obtained when the sound speed is much larger than the Alfven velocity (see third and fourth cases discussed below). The problem would also be simplified if one could assume constant Alfven velocity across the current sheet; however, in the approach presented here (an isothermal medium) the pressure balance for the current sheet does not allow us to make this assumption.

(i) $\vec{B}_o$ rotates, $B_o(z)$ = constant

In this simple case, both density and magnetic field gradients disappear ($\lambda_b = \lambda_r = 0$), leading to constant density across the current sheet. The rotation of the magnetic field does not introduce additional forces into the equation of motion as

$$(\nabla \times \vec{B}_o) \times \vec{B}_o = 0, \qquad (3.18)$$

and the solutions for both U_{pz} and B_{pz} are given by the dispersion relation for Alfven waves

$$\omega^2 - (\vec{k}\cdot\vec{V}_A)^2 = 0. \qquad (3.19)$$

which is valid for any perturbations of the velocity and magnetic field.

Purely transverse Alfven waves are the only MHD bending waves for a current sheet consisting of an arbitrary rotation of the magnetic field through any angle, including 180° for a neutral sheet. To calculate the phase velocity U_A for these waves, we introduce the angle θ between the wave vector $\vec{k}$ and the y-axis (the axis is fixed in the direction of the magnetic field at the upper edge of the current sheet) and obtain

$$U_A = \pm V_A \cos (\alpha(z) - \theta). \tag{3.20}$$

For propagation along the y-axis ($\theta = 0^\circ$) and arbitrary direction of the magnetic field with respect to the y-axis one gets $U_A = V_A \cos \alpha$; and, for a magnetic field parallel to the y-axis and arbitrary direction of the wave propagation equation (3.21) gives $U_A = V_A \cos \theta$. In addition, one may introduce the angle ϕ between the magnetic field and wave-vector ($\phi = \alpha - \theta$) to further simplify the expression for the phase velocity. All latter results are well-known solutions for Alfven waves in a homogeneous medium with uniform magnetic field.

(ii) $\vec{B}_o$ rotates, $D_A = 0$

In this very special case, the dispersion relation for transverse Alfven waves is satisfied and the solution for U_{pz} (equation 3.16) reduces to the form

$$U_{pz} = U_{pz}^o \frac{C_{AS} C_o V_S^2}{2 B_o^2 \lambda_b \omega^2 V_A^2} . \tag{3.21}$$

This solution is valid only for one particular point in the current sheet where $D_A = 0$ for each wave frequency; note that this point is not a singular point for the wave equations (3.8 and 3.9) and normal mode solutions exist. In general the Alfven velocity varies across the sheet and the dispersion relation is not satisfied. This means that the global solution (3.16) for U_{pz} must equal this value at the critical layer where $D_A = 0$.

(iii) $\vec{B}_o$ rotates, $B_o(z)$ is arbitrary, $V_S \gg V_A$

The later assumption ($\beta_A \to 0$) is equivalent to neglecting the density gradient in the continuity equation (see Bertin and Coppi 1985) and leads to the incompressible fluid approximation ($\nabla \cdot \vec{U} = 0$). Formally, this Boussinesq type of assumption makes the approach of studying MHD bending waves inconsistent, mainly because the considered medium is compressible and the density gradient in the continuity equation cannot be neglected. Referring to the results previously obtained, we discuss this case in order to present the solution of the wave equation for U_{pz} given by equation (3.16). For $\beta_A \to 0$ the behaviour of MHD bending waves is described either by the wave equation (3.1) or by equation (3.16) reduced by the assumption $D_{AS} = D_A$. Now, the first critical point is defined by the condition $D_A = 0$ and cannot be removed by the rotation alone. This critical point is always present in equation (3.16), reduced for an incompressible medium, and an integral over some range of wave frequencies appears in addition to a sum of discrete modes (Tataronis and Grossman, 1973; Priest, 1982). Simple solutions are found only for real wave frequencies that do not satisfy $D_A = 0$ anywhere in the current sheet (however the condition can be satisfied outside the sheet) and according to equation (3.16), we obtain

$$U_{pz} = U^{o}_{pz} \frac{C_A C_o}{2 B_o^2 \lambda_b D_A} \exp\left[- \omega^2 \int \frac{dz}{\lambda_b V_A^2}\right], \tag{3.22}$$

where C_A is a constant of integration; this integral can be calculated when simple variations of the magnetic field across the current sheet are assumed (see the next case).

(iv) $\vec{B}_o$ rotates, $B_o(z) = B_e \tanh(z/d)$ and $V_S \gg V_A$

In order to evaluate the integral in equation (3.24), we have to calculate the Alfven velocity as well as λ_b for this specific variation of the magnetic field (Figure 4). From the pressure balance, and assuming $\vec{B}_e^{(1)} = \vec{B}_e^{(2)} = B_e$ and $p_e^{(1)} = p_e^{(2)} = p_e$ we obtain

$$\rho_o = \rho_e \frac{\beta^2_{Ae} + 2 \cosh^2(z/d)}{2 \cosh^2(z/d)}, \tag{3.23}$$

where $\beta_{Ae} = V_{Ae}/V_S$, $V_{Ae} = B_e/(4\pi\rho_e)^{1/2}$ and 2d is a thickness of the current sheet. It can also be shown that

$$(\rho_o - \rho_e) \cosh^2(z/d) = \frac{1}{2} \rho_e \beta^2_{Ae} = \text{const.} \tag{3.24}$$

Using equation (3.25), the Alfven velocity is found in the form

$$V_A^2 = 2 V_{Ae}^2 \frac{\sinh^2(z/d)}{\beta^2_{Ae} + 2 \cosh^2(z/d)}. \tag{3.25}$$

which can be further simplified by the assumption $V_S \gg V_A$ ($\beta_{Ae} \to 0$). Then, a simple analytical formula is found for the integral in equation (3.22) and the solution is given by

$$U_{pz} = U^{o}_{pz} \frac{C_A d_o}{2D_A d} \frac{\cosh^2(z/d)}{\sinh^{\kappa}(z/d)} \coth(z/d) \exp[-\tfrac{1}{2} \kappa \cosh^3(z/d)] \tag{3.26}$$

where d_o is a constant of integration and $\kappa = \omega^2/V_A^2 d^2$. The solution given by equation (3.26) is valid only for those frequencies which do not satisfy the dispersion relation $D_A = 0$ anywhere in the current sheet; otherwise, the continuous spectrum of frequencies is the expected solution of the problem.

4. DISCUSSION

We have presented the normal mode analysis for ideal MHD, incompressible (the Alfven type of purely transverse) bending waves on a two-dimensional current sheet. The current sheet is of completely general structure, in equilibrium with its homogeneous surroundings. The magnetic field may undergo both amplitude and directional changes within the sheet to simulate what is observed to occur in the heliospheric current sheet (HCS). The behavior of MHD incompressible waves is fully described by the wave equations (3.8) and (3.9) which are the most important results of this paper. In the limit of no field rotation and infinite sound speed, we recover the results of Lee and Roberts (1986) and others; that the normal mode analysis fails, there is a continuous spectrum of solutions for any given wave number, and that waves undergo decay through "phase mixing" or "mode conversion" - ultimately involving dissipative and nonlinear processes that are beyond the scope of our work but which have analyzed by others (Steinolfson, et al., 1986, op cit.).

A new result is that in the presence of field rotation, finite sound speed, and/or a local maximum in field strength, it is possible to recover conditions under which normal modes exist. Furthermore, there are specific conditions under which the singularity in the wave equation that leads to continuous spectrum solutions can be removed by using a critical point solubility criterium such that a different class of normal modes is recovered. There are even conditions under which continuous spectrum solutions disappear altogether.

These results mean that disturbances on a current sheet need not always decay through mode conversion - which leads us to speculate on the physical processes in the HCS. First, we note that the HCS at 1 AU apparently generally attains a field reversal through field rotation and hence may not support decay through mode conversion. Second, "ripples" are commonly observed on the HCS. These ripples are, therefore, probably Alfven-like normal modes which are undergoing no further decay. However, near the sun the establishment of the HCS does not necessarily require a field rotation. Thus waves present near the sun - and it is reasonable to assume waves are present in that they are generally observed at 1 AU - would undergo mode conversion and decay, ultimately leading to dissipation and local magnetic diffusion such that the magnetic field relaxes into a configuration having a field rotation and the waves would no longer decay and diffusion ceases. However, this scenario is not completely in agreement with the data for a combination of reasons. First, Steinolfson, et al. (1986) report that viscous decay is more efficient than ohmic decay under coronal conditions. In view of this, we would expect heating in the vicinity of the HCS. However, Borinni, et al. (1981) report just the opposite - the proton temperature tends to be a minimum in the vicinity of the HCS. The only way to circumvent this discrepancy is if dissipation in the interplanetary medium behaves differently than in the corona so that most of it occurs via a decay process involving reconnection of the magnetic field.

Returning to the analysis of Bertin and Coppi (1985), it is clear that their idea of MHD bending waves on the HCS now deserves considerable attention, even though they misapplied their idea in attempting to explain the warping of the HCS. Data show that warping of the HCS can be explained as the imprint of coronal structure. However, bending waves analyzed for small-scale waves, when the WKB approximation universally used in their anlaysis is valid, offer the possibility of understanding the character and evolution of the ripples known to exist on the HCS. Furthermore, our analysis, which includes all types of configurations describable by a two-dimensional layer whether or not the field actually reverses, offers tools for defining when surface waves will decay via mode conversion in the general astrophysical context. Finally, the new closed integral solution we have found is an analytic tool that simplifies finding solutions under those conditions for which normal modes exist.

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FIGURE CAPTURES

Figure 1. Topology of the heliospheric current sheet using Wilcox Solar Observatory line-of-sight magnetic field data, a potential field model of the corona with a source surface radius of 2.5 R_o, and a 400 km/s solar wind speed. At position A, Halley's Comet intersects the current sheet on 13.1 March 1986 - at the same time and place as the ESA Giotto spacecraft. The Earth was at position B at the same time.

Figure 2. Sketch of the model layer with definition of coordinate system (2a) and variations of the magnetic field in the current sheet plane (2b).

Figure 3. Definition of the pertinent symbols relating the directions of the local magnetic field and the wave vector with respect to the coordinate system.

Figure 4. Variations of the magnetic field $\vec{B}_o(z) = \vec{B}_e \tanh (z/d)$ in the current sheet plane; 2d is a thickness of the current sheet. The semi-circular dashed lines show the path of the tip of the $\vec{B}_o(z)$ vector with two arbitrary intermediate vectors shown as $\vec{B}_o(z_1)$ and $\vec{B}_o(z_2)$.

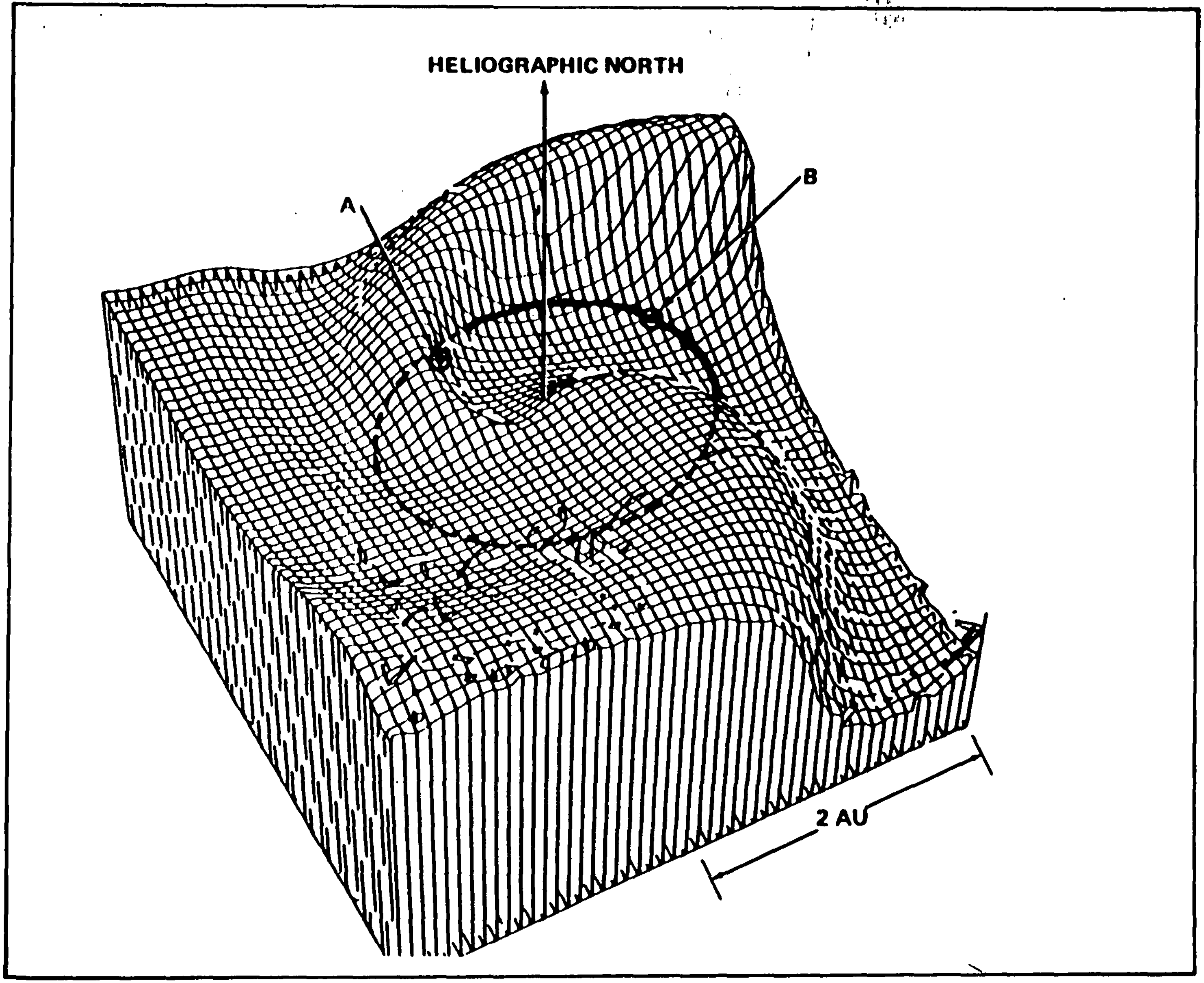


Figure 1

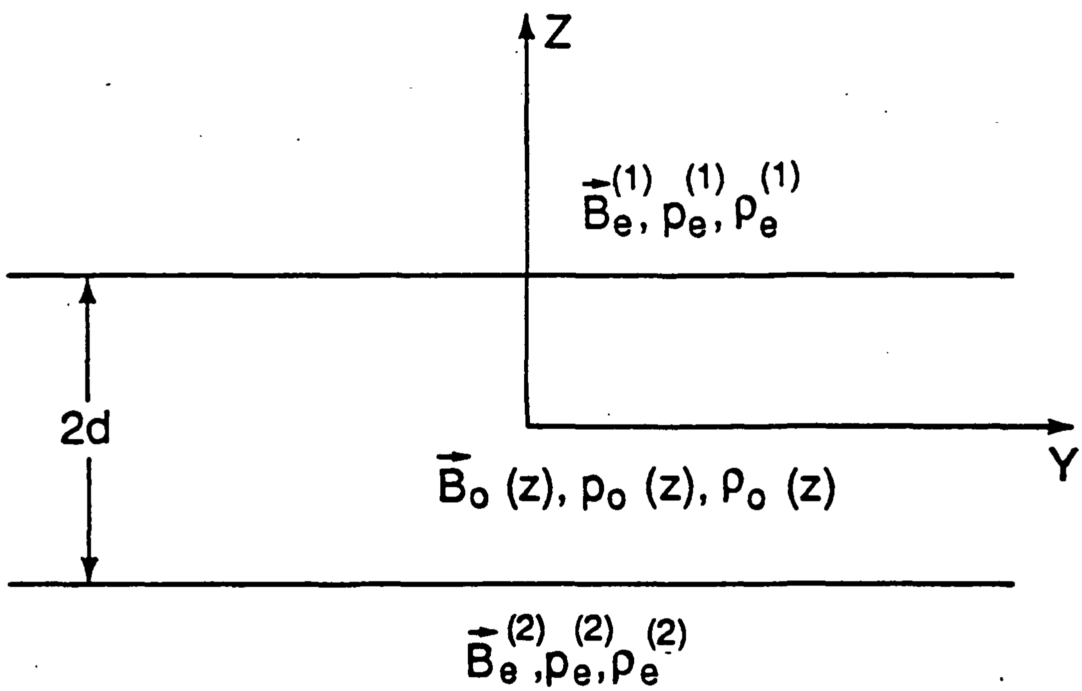
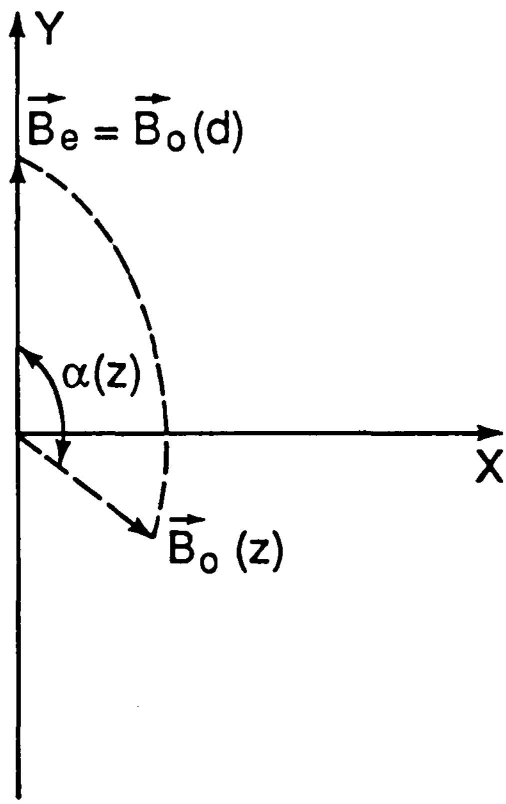


Figure 2 a



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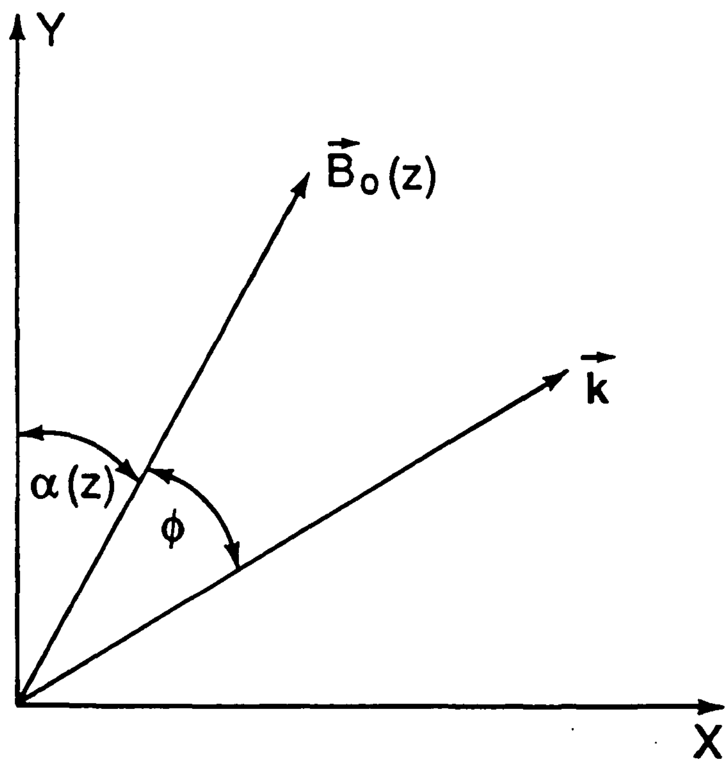

Figure 3

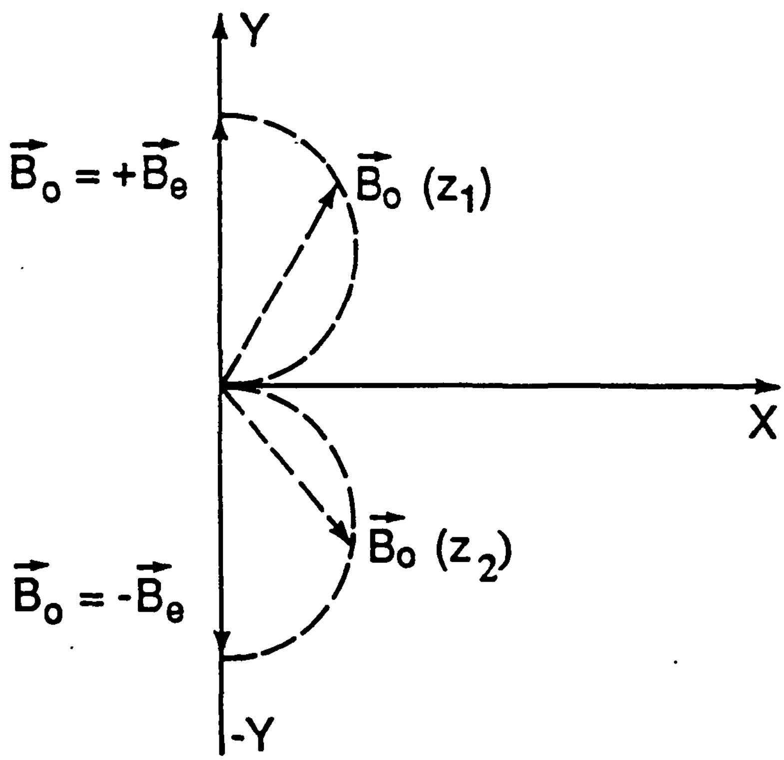

Figure 4