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DEPARTMENT OF MECHANICAL ENGINEERING & MECHANICS
COLLEGE OF ENGINEERING & TECHNOLOGY
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA 23529

NUMERICAL METHODS FOR INCOMPRESSIBLE VISCOUS FLOWS WITH ENGINEERING APPLICATIONS

By

M. E. Rose, Co-Principal Investigator

and

R. L. Ash, Principal Investigator

Final Report
For the period ended April 16, 1986

Prepared for the
National Aeronautics and Space Administration
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Task Authorization No. 32
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M. E. Rose[1] and R. L. Ash[2]

ABSTRACT

A numerical scheme has been developed to solve the incompressible, three-dimensional Navier-Stokes equations using velocity-vorticity variables. This report summarizes the development of the numerical approximation schemes for the divergence and curl of the velocity vector fields and the development of compact schemes for handling boundary and initial-boundary value problems.

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INTRODUCTION

This report is intended to document the research which was conducted in support of the development, a velocity-vorticity, Navier-Stokes solver (Ref.1). The numerical techniques developed in this effort have been used subsequently in the study of three-dimensional vortex breakdown (Ref. 2).

Approximation Schemes for div $\underline{u} = 0$, curl $\underline{u} = \hat{\zeta}$

Let the planes $\underline{x}$ = const, describe a Cartesian grid in R^3, and denote by e_ℓ a volume element with center at $\underline{x}_\ell$. Suppose that the fundamental domain, D, is the union of such elements. As Fig. 1 shows, each element, e, has faces, γ, edges, σ, and vertices, ν, and we identify these by a point associated with each. In addition, $|e|$, $|\gamma|$, $|\sigma|$ denote the respective volume, area, and length such that, if $|\sigma| = O(h)$, then $|\gamma| = O(h^2)$, and $|e| = O(h^3)$.

Consistent with this geometric construction, let $\underline{u}(e)$, $\underline{u}(\gamma)$, and $\underline{u}(\sigma)$ denote the average values of $\underline{u}$ on e, γ, and σ, respectively. The values $\underline{u}(\nu)$ associated with the vertices are sometimes called box-variables and are often useful for quadrature evaluations of $\underline{u}(e)$, $\underline{u}(\gamma)$, and $\underline{u}(\sigma)$. For simplicity of notation, these will also designate the quadrature evaluation of these average values in terms of box-variables.

Referring to Figure 1, consider the faces of an element $e_{i,j,k}$, where the index i is associated with the x^1 or x-axis, j is associated with the x^2 or y-axis, and k is associated with the x^3 or z-axis. The defining relations for the average and difference operators on the $\gamma_{i\pm1/2,j,k}$ faces are

$$\mu_1 \underline{u}(\gamma_{i,j,k}) \doteq [\underline{u}(\gamma_{i+1/2,j,k}) + \underline{u}(\gamma_{i-1/2,j,k})]/2 \qquad (1)$$

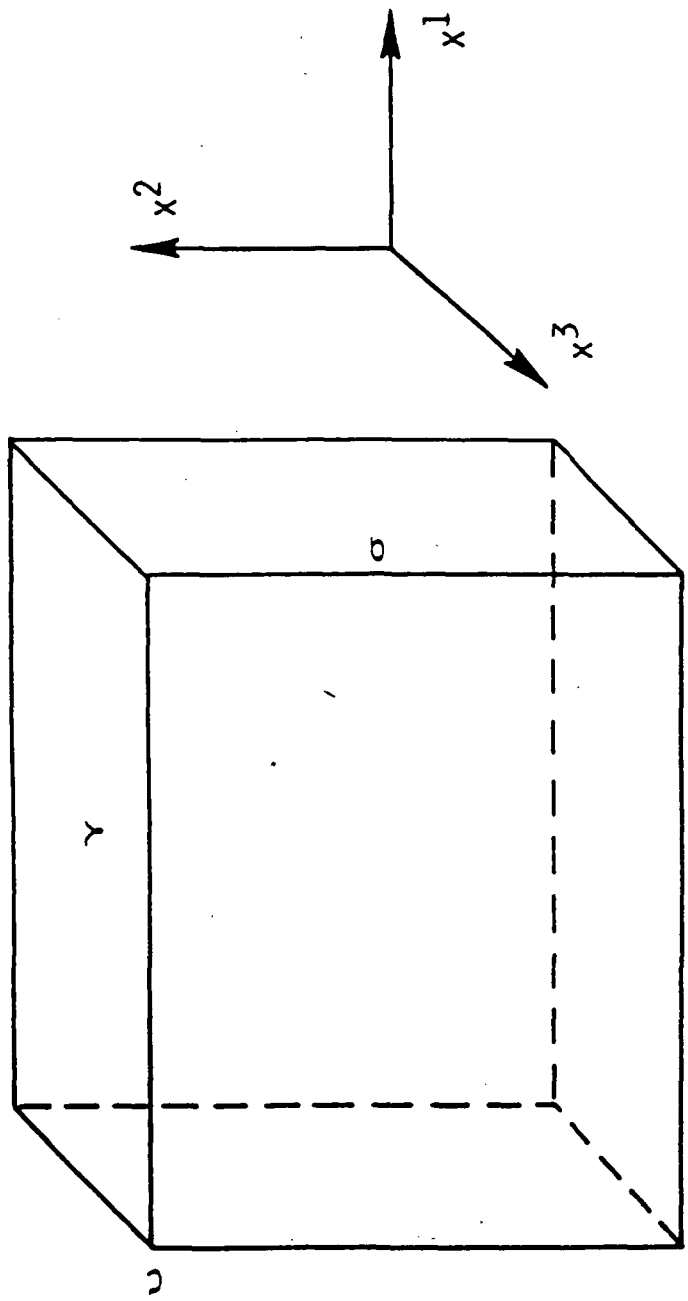

Figure 1. Cartesian element e, with faces γ, edges σ, and vertices ν.

$$\Delta_1 \underline{u}(\gamma_{i,j,k}) \doteq [\underline{u}(\gamma_{i+1/2,j,k}) - \underline{u}(\gamma_{i-1/2,j,k})]/2; \tag{2}$$

the operators μ_2,Δ_2 and μ_3,Δ_3 are similarly defined. The operators $\delta_i \underline{u}(\gamma_{i,j,k})$ are determined by

$$\Delta_i \doteq h^i \delta_i, \quad i = 1,2,3 \tag{3}$$

where $h^i = \frac{\Delta x^i}{2}$. The defining relations for the average and difference operators relating sides and edges, and edges and vertices are similar.

With these operator definitions in mind, we can construct approximation schemes for both div $\underline{u}$ = 0 and curl $\underline{u} = \hat{\underline{\zeta}}$ as follows:
First, define the volume average of div $\underline{u}$ over an element e as

$$\text{div}_e \, \underline{u} \doteq \frac{1}{|e|} \int \text{div} \, \underline{u} \, de. \tag{4}$$

Gauss's theorem evaluates $\text{div}_e \, \underline{u}$ in terms of $\underline{u}(\gamma) \cdot \underline{n}$ on ∂e, where $\underline{n}$ is the unit outward normal, i.e.,

$$\text{div}_e \, \underline{u} \doteq \frac{1}{|e|} \oint_{\partial e} \underline{u} \cdot \underline{n} \, d\,|\gamma|$$

$$= \frac{1}{|e|} \sum_{\gamma \in \partial e} \underline{u}(\gamma) \cdot \underline{\gamma} \tag{5}$$

where γ is the oriented area. By suitably arranging the order of summation and using Eqs. (2) and (3), Eq. (5) can be written as

$$\mathrm{div}_e\, \underline{u} = \delta_1 u_1(\gamma) + \delta_2 u_2(\gamma) + \delta_3 u_3(\gamma). \tag{6}$$

Using box-variables to evaluate the average values on the right hand side of Eq. (6), one obtains

$$\mathrm{div}_e\, \underline{u} = (\delta_1 \mu_2 \mu_2 \mu_3 u_1 + \delta_2 \mu_1 \mu_3 u_2 + \delta_3 \mu_1 \mu_2 u_3). \tag{7}$$

In a similar fashion, define the surface average of the normal component of curl u over a surface γ as

$$\underline{n} \cdot \mathrm{curl}_\gamma\, \underline{u} = \frac{1}{|\gamma|} \int_\gamma \underline{n} \cdot \mathrm{curl}_\gamma\, \underline{u}\, d|\gamma|. \tag{8}$$

Stokes' theorem evaluates $\underline{n} \cdot \mathrm{curl}_\gamma\, \underline{u}$ in terms of $\underline{u}(\sigma) \cdot \underline{\sigma}$ on ∂_γ, where $\underline{\sigma}$ is the unit tangent vector, i.e.,

$$\underline{n} \cdot \mathrm{curl}_\gamma\, \underline{u} = \frac{1}{|\gamma|} \oint_{\sigma \in \partial\gamma} \underline{u}(\sigma) \cdot \underline{\sigma}. \tag{9}$$

For a Cartesian element the opposite edges $\sigma_\pm$ on a face γ have equal lengths, and the associated unit tangent vectors have different signs. This allows one to rewrite the summation in Eq. (9) in terms of an operator acting on the edges. For example, the component on the x^1-face is given by

$$(\underline{n} \cdot \mathrm{curl}_\gamma\, \underline{u})|_1 = \delta_2 \mu_3(\sigma_3) - \delta_3 u_2(\sigma). \tag{10}$$

Using box variables to evalulate the average values on the right hand side of Eq. (10), one obtains

$$(\underline{n} \cdot \text{curl}_\gamma \underline{u})|_1 = (\partial_2\mu_3\mu_3 - \partial_3\mu_2\mu_2)\cdot \tag{11}$$

With these defining relations, the general form for the (normal) vorticity components on the faces of an element, e, can be written as

$$(\underline{n} \cdot \text{curl}_\gamma \underline{u})|_1 = \delta_j(\mu_k u_k) - \delta_k(\mu_j u_j), \ (i,j,k) \tag{12}$$

where (i,j,k) indicates an even permutation of (1,2,3).

Using the definitions (7) and (12), Fix and Rose (Ref. 3) have shown that the Cauchy-Rieman type equations described in (Ref. 1) can be solved by the least-squares solution of

$$\text{div}_e \underline{u} = 0 \qquad \text{in} \quad D \tag{12}$$

$$\text{curl}_\gamma \underline{\mu} = \hat{\zeta} \qquad \text{on} \quad \Gamma \tag{13a}$$

$$\underline{n} \cdot \underline{u} = n \cdot u_\Gamma \tag{13b}$$

when $\text{div}_e \hat{\underline{\zeta}} = 0$.

Compact Schemes for Boundary and Initial-Boundary Value Problems

The following discussion will outline the general development of the schemes that were used in developing the numerical schemes for (Ref. 1). These schemes will be seen to provide domain decomposition extensions of conventional boundary integral and boundary element methods. As a result, both boundary value and initial-boundary value problems can be handled.

Consider a vector function, V, having s components. A typical feature of boundary value problems for systems of elliptic equations involving V on a domain, D, is that s-s' components of V on ∂D are determined, by means of the solution operator on D, by s' prescribed components of V on ∂D, $s' < s$. We may call the s' prescribed components the primary variables and the s-s' components the complementary variables. In certain simple cases the relationship between these variables can be described by means of simple integral equations.

By constructing an approximate solution operator on D it may be possible to determine the relationship between the primary and complementary variables at N points of interpolation on ∂D. We may expect that these approximate values will converge, as $N \to \infty$, to the solution values $V_{\partial D}$ under reasonable precautions about the construction. This is the basis of discrete boundary integral methods.

Returning to the continuous problem on D, suppose D is partitioned into volume elements, $D = \{e\}$. With arbitrary values of the primary variables chosen on the boundary of each element (but consistent with the values prescribed on $\partial e \cap \partial D$) we can solve the boundary value problem in each element; the solution will be identical to the solution values of the boundary value problem in D in corresponding volume elements if they both have the same values on the boundary of each element and, thus, is continuous across interelement boundaries.

This suggests the following discrete approximation method: In each element, e, choose the center point of each face, γ, of e as an interpolation point and, using an appropriate solution operator in e, obtain the discrete boundary integral relationship between the primary and complementary variables at the interpolation points on the boundary of ∂e. We call this a compact equation on the element. Next impose continuity conditions

at the interpolation points in D and use values prescribed by the problem on D when the interpolation point lies on ∂D. Then, solve the resulting algebraic problem. This, in effect, provides a domain decomposition extension of boundary element methods. (In this construction one may incorporate the continuity conditions quite simply by identifying the left and right limits at an interpolation point by their common value.)

We call this construction a compact scheme. It requires for its development only an element-by-element description of the discrete integral equations which relate the primary and complementary variables at the interpolation points on the boundary of each element e. As shown below, this idea may be applied to time dependent problems as well, (cf. Refs. 4-6).

The weak-element method (Ref. 7) implements this construction by using as an approximation basis a manifold of solutions of the differential equation (or an approximation to it) in each element. Necessarily, then, the compact scheme which results is consistent with the differential equation in each element. This construction also leads immediately to a discrete energy estimate which approximates that which applies to the differential equation on D. Thus the convergence of the scheme is assured and leads to second order accurate results.

We will now indicate how a simple Galerkin method can be used to obtain compact schemes for general volume elements.

A Boundary Value Problem

As an example, we will discuss the Poisson equation $\nabla^2 v = f$. Consistent with earlier notations, let f(e) indicate the value at the center of e and $\nabla(\gamma)$ the value at the interpolation point on a face γ. Define the bilinear boundary operator $B_e(v,w)$ by

$$B_e(v,w) \doteq \int_{\partial e} \left(w \frac{\partial v}{\partial n} - v \frac{\partial w}{\partial n} \right) d|\gamma| \tag{14}$$

so that Green's theorem can be written

$$B_e(v,w) = \oint_e (w\nabla^2 v - v\nabla^2 w)\, d|e|. \tag{15}$$

If w is any solution of the homogeneous problem ($\nabla^2 w = 0$), then

$$B_e(v,w) = \int_e wf\, d|e| \doteq (w,f)_e . \tag{16}$$

Second order accurate quadrature approximations to Eqs. 14 and 15 yield

$$B_e^h(v,w) = \sum_{\gamma \in \partial e} \left[w(\gamma) \frac{\partial v(\gamma)}{\partial n} - v(\gamma) \frac{\partial w(\gamma)}{\partial n} \right] |\gamma| \tag{17}$$

$$(w,f)_e^h = w(e)f(e)|e|. \tag{18}$$

Suppose e has ℓ faces: let w_i, $i = 1,2,\ldots,\ell$ denote, say, the first ℓ harmonic polynomials (i.e., $\nabla^2 w_i = 0$). Compact equations on e are given by

$$B_e^h (v,w_i) = w_i,f)_e^h, \quad i = 1,2,\ldots,\ell \tag{19}$$

and provide an $O(h^2)$ truncation error.

These equations establish an algebraic relationship between the 1

values $v(\gamma)$ and $\frac{\partial v(\gamma)}{\partial n}$ on the faces of ∂e. The coefficients in the equation are determined by evaluating $w_i(\gamma)|\gamma|$ and $\frac{\partial w_i(\gamma)}{\partial n}|\gamma|$ on the faces, $i = 1,2,\ldots,\ell$. For the Poisson equation $\nabla^2 v = f$, application of these ideas is more straightforward if it is written as the system,

$$\nabla \cdot \underline{p} = f \tag{20a}$$

$$\underline{p} = \nabla v. \tag{20b}$$

seen, in the case of a Cartesian grid, use of the functions

$$w = (1,\ x,\ y,\ z,\ x^2 - y^2,\ x^2 - z^2) \tag{21}$$

in Eq. (19) leads to the compact equations

$$\delta_x p_x + \delta_y p_y + \delta_z p_z = f \tag{22a}$$

$$\mu_x p_x = \delta_x v,\ \mu_y p_y = \delta_y v,\ \mu_z p_z = \delta_z v \tag{22b,c,d}$$

$$\mu_x v - \frac{1}{2} h_x^2 \delta_x p_x = \mu_y v - \frac{1}{2} h_y^2 \delta_y p_y = \mu_z v - \frac{1}{2} h_z^2 \delta_z p_z \tag{22e,f}$$

In order to show the convergence of this scheme, it is possible to construct a discrete energy estimate. Recall that the weak-element method (Ref. 7) constructs an approximate solution v^e on each element as

$$v^e = \sum_{i=1}^{\ell} c_i v_i + \hat{v}, \tag{23}$$

where $\hat{v}$ is a particular solution of $\nabla^2 v = f(e)$. Equation (19) determines the coefficients by a Galerkin construction on each element. Since v^e is a solution of $\nabla^2 v^e = f(e)$ on e, it satisfies the energy equation

$$\oint_{\partial e} v^e \frac{\partial v^e}{\partial n} d|\gamma| = (v^e, f)_e + \int_e |\nabla v^e|^2 d|e|. \tag{24}$$

Using the second order accurate quadrature formulas to evaluate the integral terms, one can approximate Eq. (24) as

$$\sum_{\gamma \in \partial e} v^e(\gamma) \frac{\partial v^e(\gamma)}{\partial n} |\gamma| = (v^2, f)_e + \int_e (|\nabla v^e|^2) d|e| \tag{25}$$

Recalling the continuity conditions imposed by the compact construction, and summing over elements in D, we obtain

$$\oint_{\gamma \in \partial D} v^e(\gamma) \frac{\partial v^e(\gamma)}{\partial n} |\gamma| = \sum_{e \in D} \{(v^e, f)_e + \int_e (|\nabla v^e|^2) d|e|\}. \tag{26}$$

This is a discrete approximation to energy estimates for the solution in D, viz.,

$$\oint_{\partial D} v \frac{\partial v}{\partial n} d|\gamma| = \int_D (vf + |\nabla v|^2) d|e|. \tag{27}$$

This discrete energy estimate, together with the obvious fact that the approximation v^h is consistent with the differential equation, implies by standard arguments that the scheme converges and, in fact, with second order accuracy.

It is, of course, possible to extend these ideas to more general boundary value problems of the form $Lv = f$. For sufficiently small volume elements, L can be approximated by an operator with constant coefficients in each element, which we call L_e. In this case Green's theorem can be written as (cf. Eq. (15)).

$$B_e(v,w) = \int_e (wL_e v - vL_e^* w)\, d|e| \tag{28}$$

where L_e^* is the adjoint of L_e. Solutions of the adjoint equation $L^* w = 0$, are easily generated in the form

$$w = \exp [\underline{\alpha} \cdot \underline{x}], \tag{29}$$

where $\underline{\alpha}$ satisfies the characteristic polynomial equation $L_e^*(\underline{\alpha}) = 0$. The compact equations which result,

$$B_e(v,w) = (w, f)_e \tag{30}$$

may now involve exponential factors. It is possible to avoid the use of exponentials by using polynomial solutions of $L^* w = 0$ which can be generated by

$$w_i = \frac{\partial^i}{\partial \alpha_{(j)}^i} (\exp[\underline{\alpha} \cdot \underline{x}])|_{\alpha=0}, \quad i = 0,1,2,\ldots; \quad j = 1,2,3. \tag{31}$$

The weak-element construction just described is based upon using a projection on a manifold of solutions of the differential equation in each element. This same idea can be applied to time-dependent problems as well.

Initial-Boundary Value Problems

As an example, we will base our discussion on the diffusion equation

$$v_t = \nabla^2 v, \quad \underline{x} \in D, \quad 0 < t < T \tag{32a}$$

with initial and boundary conditions

$$v(\underline{x}, 0) = g(\underline{x}), \quad t = 0, \tag{32b}$$

$$v(\underline{x}_\Gamma, t) = v_\Gamma(\underline{x}_\Gamma), \quad \underline{x}_\Gamma \in \partial D. \tag{32c}$$

From the discussion in (Ref. 1), it is sufficient to solve this problem in a time strip S^m: $|t - t_m| \leq \tau$; i.e., with initial data $v(\underline{x}, t_{m-1/2})$ and boundary data v on $\partial D \times S^m$, we seek to determine $\frac{\partial v(\underline{x}_\Gamma, t)}{\partial n}$ on $\partial D \times S^m$ as well as $v(\underline{x}, t_{m+1/2})$, $x \in D$.

Introducing a domain decomposition $D = \{e\}$, we consider the same type of problem on each cylinder set $e \times S^m$. The general solution can be written as

$$v(\underline{x}, t; \underline{\alpha}) = \int A(\underline{\alpha}) \exp(\underline{\alpha} \cdot \underline{x} - \beta t)\, d\underline{\alpha}, \tag{33}$$

where β satisfies the dispersion relation

$$\beta = |\underline{\alpha}|^2, \tag{34}$$

and $A(\underline{\alpha})$ is determined from the initial and boundary conditions in $e \times S^m$. If e has ℓ faces, one can seek an approximate solution which interpolates to the initial and boundary conditions on the element in the form

$$v(\underline{x}, t) = \sum_{i=0}^{\ell} A(\underline{\alpha}_1) \exp(\underline{\alpha}_i \cdot \underline{x} - \beta_i t). \tag{35}$$

Once again, the relationship between the primary and complementary variables for the discrete problem can be determined by a Galerkin procedure using an appropriate form of Green's theorem. Let w indicate a solution of the adjoint equation

$$L^*(w) = w_t + \nabla^2 w = 0. \tag{36}$$

The application of Green's theorem to this problem leads to the relation (cf. Eq. (30)).

$$\frac{d}{dt} (w, v)_e = B_e (v, w). \tag{37}$$

The approximation which results by interpolation is then (cf. Eq. 19)).

$$\frac{d}{dt} (w,v)_e^h = B_e^h(v,w). \tag{38}$$

A time average of Eq. (38) on S^m produces the equation

$$\delta_t(w, v)_e^h = B_e^h(v^m, w^m) \tag{39}$$

where v^m, w^m indicate time averages over the strip S^m and

$$\delta_t v(t_m) = (v(t_m + \tau) - v(t_m - \tau))/\Delta t \tag{40a}$$

$$\mu_t v(t_m) = (v(t_m + \tau) + v(t_m - \tau))/2. \tag{40b}$$

Once again, the choice of $s' + 1$ solutions v_i of the adjoint Eq. (36) will then determine the $s' + 1$ complementary solution values at points of interpolation in $e \times S^m$ in terms of the $s' + 1$ primary solution values. The result is a set of compact equations for the problem. Write Eq. (32a) as the first order system

$$\begin{aligned} v_t &= \frac{\partial P}{\partial x} \\ p &= \frac{\partial v}{\partial x}. \end{aligned} \tag{41}$$

In this case, e_i is the interval $|x - x_i| < h$. The discrete mixed initial value problem on $e \times S^m$ can be stated as: given $v(e, t_{m-1/2})$ as initial data and $v^m(x_\Gamma)$, $x_\Gamma \in \partial e$, as boundary data, determine (i) $\frac{\partial v^m(x_\Gamma)}{\partial x}$, $x_\Gamma \in \partial e$, and (ii) $v(e, t_{m+1/2})$.

Three elementary polynomial solutions of the adjoint Eq. (36) are

$$w_i \doteq (w_0, w_1, w_2) = \left(1, x, t + \frac{x^2}{2}\right), \tag{42}$$

where the origin is taken at the center of e x S^m. The compact Eq. (39) then leads to

$$\delta_t v(e, t_m) = B_e^h(v^m, w_0^m)/\Delta x \tag{43a}$$

$$0 = B_e^h(v^m, w_1^m)/\Delta x \tag{43b}$$

$$\mu_t v(e, t_m) = B_e^h(v^m, w_2^m)/\Delta x. \tag{43c}$$

These simplify to

$$\delta_t v = \delta_x p \tag{44a}$$

$$\mu_x p = \delta_x v \tag{44b}$$

and $$\mu_t v = \mu_x v - \frac{h^2}{2}\delta_x p, \tag{44c}$$

where we have suppressed the reference to e x S^m. The compact scheme results by requiring that v and p be continuous across endpoints of the

intervals interior to D, using prescribed values of v on ∂D. (Note that the space operators in Eq. (44) apply to the face values of (v,p) on the cylinder e x S^m, while the time operators apply to the values of v on the upper and lower bases.)

It is possible to obtain an energy estimate for the system described by Equations (44). Multiply Eq. (44a) by $\mu_t v$ and use both Eqs. (44b,c); the resulting equation is

$$\frac{1}{2}\partial_t v^2 + (\partial_x v)^2 + \frac{h^2}{2}(\delta_x p)^2 = \delta_x(vp). \tag{45}$$

Summing over the elements in D, the discrete energy estimate for the approximation is

$$\frac{1}{2}\delta_t \sum_e v^2 \Delta x + \sum_e [(\delta_x v)^2 + \frac{h^2}{2}(\delta_x p)^2]\Delta x = vp|_{\delta D}. \tag{46}$$

This expression corresponds to that for the continuous problem

$$\frac{1}{2}\frac{d}{dt}\int_D v^2\, dx + \int_D (v_x)^2\, dx = vp|_{\partial D}. \tag{47}$$

The compact equations (44) are obviously consistent with the differential equation and, in view of the discrete energy estimate Eq. (46), the Lax equivalence theorem implies the convergence of the compact scheme. This same argument holds in three dimensions using an arbitrary partition of the domain D, $D = \{E\}$.

Finally, it is straightforward to extend these ideas to the advection-diffusion equation

$$v_t + \nabla \cdot (\underline{a}v) = \nu \nabla^2 v, \tag{48}$$

where $\underline{a}$ and ν are coefficients, which in the context of the physical problems of Ref. 1 are associated with the velocity and viscosity of a fluid, respectively. Once again, writing this as a first order system yields the equations

$$v_t = \nabla \cdot \underline{p} \tag{49a}$$

$$\underline{p} = \nu \nabla v - \underline{a}v \tag{49b}$$

The corresponding adjoint equations are.

$$-w_t = \nabla \underline{q} + \nu^{-1} \underline{a}\, \underline{q} \tag{50a}$$

$$\underline{q} = \nu \nabla w \tag{50b}$$

The resulting form of Green's theorem is now

$$\frac{d}{dt}(w,v)_e = B_e(v,w) = \int_{\partial e} (w\underline{p} - v\underline{q}) \cdot \underline{n} d\gamma . \tag{51}$$

If, in the time-strip S^m, the coefficient $\underline{a}$ in each element e is frozen as

$\underline{a} = \underline{a}^m$, then the elementary solution of Eq. (50) is

$$w = \exp\,[\underline{\alpha} \cdot \underline{x} - \beta t] \tag{52a}$$

where

$$\beta = \nu\,|\underline{\alpha}|^2 + \underline{a}^m \cdot \underline{x} \tag{52b}$$

and, again, the origin is taken as the centerpoint of $e \times S^m$. The discrete form of Eq. (50) is obtained by using appropriate interpolated values in the quadrature approximation to B_e and taking the time average over S^m.

The values $\alpha = 0$, $\alpha = -\nu^{-1}a^m$ in Eq. (51) lead to steady-state solutions, while $x - a^m t$ is a simple time-dependent polynomial solution. Thus the appropriate approximation basis for this simple advection-diffusion equation is

$$w_i = (w_0, w_1, w_2) = \left(1,\ x - a^m t,\ \exp\left[-\frac{a^m x}{\nu}\right]\right). \tag{53}$$

The compact equations which result are

$$\delta_t v = B_e^h\,(v^m,\ w_0^m)/\Delta x \tag{54a}$$

$$-\mu_x a^m \mu_t v = B_e^h\,(v^m, w_1^m)/\Delta x \tag{54b}$$

$$[(\theta^m)^{-1}\ \sinh\,\theta^m]\delta_t v = B_e^h(v^m, w_2^m)/\Delta x \tag{54c}$$

where

$$\theta^m = a^m h/\nu . \tag{54d}$$

When Eqs. (54a) and (54c) are combined (using (43)), (44c) can be replaced by

$$0 = B_e^h(v^m,(\theta^m)^{-1} \sinh \theta_m - w_2^m) . \tag{55}$$

Expanding the right hand sides of Eqs. (54a,b and d), one obtains the compact scheme

$$\delta_t v = \delta_x p \tag{56a}$$

$$-\mu_x a^m \mu_t v = \mu_x p - \nu\delta_x v \tag{56b}$$

$$\mu_x p - h_\rho \delta_x p = \nu\delta_x v - \mu_x a^m(c_x \mu_x v - h_\rho \delta_x v) \tag{56c}$$

where

$$\rho = \coth \theta^m - (\theta^m)^{-1} \tag{56d}$$

and

$$c_x = 1 - \Delta_x a^m(\mu_x a^m)^{-1}\rho . \tag{56e}$$

The coefficient ρ given by (55d) controls the weighting given to upwind terms in the compact scheme. It can be consistently approximated by

$$\rho(\theta) = \theta/3, \qquad |\theta| < 3$$
$$= \operatorname{sgn}\theta, \qquad |\theta| > 3 \tag{57}$$

whose use allows us to employ an exponential-type scheme without having, in fact, to calculate exponential terms.

The extension to three dimensions results by using the basis

$$w_i = (1,\ x - a_1 t,\ y - a_2 t,\ z - a_3 t,\ \exp(-a_1 x/\nu),\ \exp(-a_2 y/\nu),\ \exp(-a_3 z/\nu)). \tag{58}$$

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